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# Atomic Quasi-Injective Modules

By

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**1. Quasi-injective modules.** $R$ is an associative ring with an indentity element, all $R$-modules considered here are left unitary modules.

Given an $R$-module $M$ and submodules $N_1$, $N_2$ of $M$, $N_2$ is called an essential extension of $N_1$ or $N_1$ is essential in $N_2$ if

(1) $N_1 \subset N_2$ and (2) $Rx \cap N_1 \neq 0$ for every nonzero $x$ in $N_2$.

Condition (2) is equivalent to that every nonzero submodule in $N_2$ has nonzero intersection with $N_1$. If $N_2$ is an essential extension of $N_1$ then the left ideal $(N_1 : x) = \{r \in R \mid rx \in N_1\}$ is essential in $R$ for every $x$ in $N_2$ (considering $R$ as a left $R$-module).

Every submodule of $M$ has a maximal essential extension in $M$. If the singular submodule $M^{\blacktriangle}$ of $M$ is zero then for every submodule $N$, $N^s$ is the unique maximal essential extension of $N$ in $M$, where

$$N^s = \{m \in M \mid (N : m) \text{ is essential in } R\}$$
$$M^{\blacktriangle} = \{m \in M \mid (0 : m) \text{ is essential in } R\}.$$

The verification of $N^s$ being the unique maximal essential extension of $N$ in $M$ in the case of $M^{\blacktriangle} = 0$ is straight forward.

In this paper we assume every $R$-module has zero singular submodule unless stated otherwise. Of course if $N_1$ is essential in $N_2$ then $N_1^{\blacktriangle} = 0$ if and only if $N_2^{\blacktriangle} = 0$.

In [2] we call an $R$-module $M$ quasi-injective if for any submodule $N$ of $M$ and any $f \in \mathrm{Hom}_R(N, M)$, $f$ can be extended to an element of $\mathrm{Hom}_R(M, M)$. An injective module of course is quasi-injective.

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On the other hand a simple module is quasi-injective but not necessary injective.

Given an $R$-module $M$, denote the minimal injective extension (injective hull, injective envelope) of $M$ by $\bar{M}$ and the total ring of $R$-endomorphisms of $\bar{M}$ by $K$. $M$ is quasi-injective if and only if $M$ is a $K$-submodule of $\bar{M}$ [2, 1.1 Th] (considering $\bar{M}$ as a right $K$-module). In this case $\mathrm{Hom}_R(M, M)=K$. We also know that $K$ is a regular ring (von Neumann) and self injective as a left module over itself. Furthermore $K$ is a division ring if and only if every nonzero submodule of $M$ is essentital in $M$ [2, 1.7 Th].

**Definition 1.1.** A submodule $N$ of an $R$-module $M$ is called a closed submodule if $N=N^s$.

Sometimes we call $N^s$ the closure of $N$ in $M$. $(N^s)^s=N^s$. For submodules $N_1$, $N_2$ of $M$, $N_1\cap N_2=0$ if and only if $N_1^s\cap N_2^s=0$. For any submodule $N$ of $M$ there exists a closed submodule $B$ of $M$ such that $B\cap N=0$ and $(B+N)^s=M$.

In [2] we proved the following two valuable theorems about quasi-injective modules.

**Theorem 1.2.** $N_1, \cdots, N_n$ are submodules of a quasi-injective module. $(N_1+\cdots+N_n)^s=N_1^s+\cdots+N_n^s$.

*Proof*: In [2, 1.4 Th], we proved the case where $N_i$ are closed.

For any submodules $N_1$, $N_2$, $(N_1+N_2)^s\supset N_1^s+N_2^s$. If $z\in N_1^s+N_2^s$, $z=x+y$ where $x\in N_1^s$ and $y\in N_2^s$. The intersetion $L$ of the essential left ideals $L_1$ and $L_2$ again is essential where $L_1x\subset N_1$, $L_2y\subset N_2$. $Lz\subset N_1+N_2$, $z\in(N_1+N_2)^s$.

$(N_1+N_2)^s\subset(N_1^s+N_2^s)^s=N_1^s+N_2^s$. Thus $(N_1+N_2)^s=N_1^s+N_2^s$. The rest follows by induction method.

**Theorem 1.3.** Let $M$ be quasi-injective $R$-module (singular submodule not necessary zero) and let $K=\mathrm{Hom}_R(M, M)$. If $N$ is an annihilator $K$-submodule of $M$ so that $N^{lr}=N$ ($N^l$ the annihilator of $N$ in $R$), and if $x\in M$, then

$$(N+xK)^{lr}=N+xK.$$

*Proof*: 2.1 Theorem [2].

**Corollary 1.4.** If $x_1, \cdots, x_n \in M$ then $(x_1K + \cdots x_nK)^{lr} = x_1K + \cdots + x_nK$.

*Proof*: 2.2 Corollary [2].

**Proposition 1.5.** $M$ is quasi-injective $R$-module. A submodule $N$ of $M$ is closed if and only if $N$ is a direct summand of $M$.

*Proof*: $N$ is a closed submodule. Let $B$ be a closed submodule so that $B \cap N = 0$ and $(B+N)^s = M$. By Theorem 1.2, $(B+N)^s = B+N = M$, $N$ is a direct summand of $M$.

Suppose $M = N \oplus C$, $x \in N^s$, $x = n + c$ for some $n \in N$ and $c \in C$. $(N : x)$ is essential in $R$ and $(N : x)x \subset N$. Hence $(N : x)c = 0$ and $c = 0$ by the assumption $M^{\blacktriangle} = 0$. $N = N^s$.

Combining this with the fact that a submodule of a quasi-injective module is itself quasi-injective if it is closed [2, 1.6], we conclude that a direct summand of a quasi-injective module is also quasi-injective. Of course the converse is not true.

**Definition 1.6.** A closed submodule $N \neq 0$ of an $R$-module $M$ is called an atom if $Rx \cap Ry \neq 0$ for all nonzero $x$ and $y$ in $N$.

A closed submodule $N \neq 0$ is an atom if and only if $N = B^s$ for any nonzero submodule $B$ in $N$.

Also immediately it follows that an atom is a minimal nonzero closed submodule. In the case of injective modules the atoms and the nonzero indecomposable injective submodules are the same objects as described in [3] by E. Matlis.

If $N$ is essential in $M$ then the closed submodules of $M$ and the closed submodules of $N$ (closed in $N$) have a one-to-one correspondence given by:

$C \to C \cap N$, where $C$ is closed in $M$,

$A \to A^s$, where $A$ is closed in $N$ and $A^s$ is the closure of $A$ in $M$. $A$ is an atom in $N$ if and only if $A^s$ is an atom in $M$.

For any module $N$, let $\overline{N}$ be the injective hull of $N$, and $M$ be a quasi-injective extension of $N$ in $\overline{N}$ ($M$ can be any $K$-submodule of $\overline{N}$ containing $N$, $K = \mathrm{Hom}_R(\overline{N}, \overline{N})$). If $\overline{N}$ is the sum of its own atoms such as in the case where $R$ is a left Noetherian ring [3, Th 2.5], then $\overline{N}$ is an essential extension of the sum of all atoms of $M$ which in terms is an essential extension of the sum of all atoms of $N$.

**Theorem 1.7.** $M$ is quasi-injective $R$-module, $C$ is an atom of $M$, $K=\mathrm{Hom}_R(M, M)$, then

$$A=\{k\in K \mid Mk\subset C\} \text{ is a minimal left ideal of } K;$$
$$A=Ke,\ e^2=e\neq 0, \text{ and } C=Me.$$

Conversely if $A=Ke$, $e^2=e\neq 0$, a minial left ideal of $K$, then $C=Me$ is an atom of $M$.

*Proof*: This theorem follows easily from the fact that an atom is an indecomposable direct summand. However we are going to present another proof which bases on the property of quasi-injective module.

For any nonzero elements $a$ and $b$ in $A$, it will suffice to show that there exists a $k\in K$ such that $a=kb$. Notice that $A\neq 0$. The kernels $H_a$, $H_b$ of $a$ and $b$ respectively are closed $R$-submodules of $M$. There exist nonzero closed submodules $S_1$ and $S_2$ where

$$M=H_a\oplus S_1,\ M=H_b\oplus S_2, \text{ and}$$
$$0\neq S_1a\cap S_2b\subset C.$$

Choose $s_1$ in $S_1$ and $s_2$ in $S_2$ so that $s_1a=s_2b\neq 0$. The mapping $t$ defined by $(rs_1)t=rs_2$, for all $r\in R$ is a $R$-isomorphism of $Rs_1$ onto $Rs_2$. Extend $t$ to an element $k'$ of $K$. Since $Rs_1$ is essential in $S_1$, $Rs_1(a-k'b)=0$ implies $S_1(a-k'b)=0$. If we denote the projection of $M$ onto $S_1$ relative to $H_a$ by $f$ then $a=kb$ where $k=fk'$.

So we have proved that $A$ is a minimal left ideal of $K$. $A=Ke$, $e^2=e\neq 0$, and $C=Me$ are obvious.

Conversely if $A$ is a minimal nonzero left ideal, $A=Ke$. $Me$ is a direct summand of $M$ and hence it is closed. If $Me=N\oplus S$, $N\neq 0$, then the left ideal $B=\{k\in K \mid Mk\subset N\}$ must be equal to $A$. This shows $S$ must be zero and $Me$ is an atom.

**Corollary 1.8.** There is a one-to-one correspondence between the atoms of $M$ and the nonzero minimal left ideals (nonzero minimal right ideals) of $K$.

*Proof*: For any two atoms $C_1$ and $C_2$, either $C_1 \cap C_2 = 0$ or $C_1 = C_2$. The correspondence between the atoms of $M$ and the nonzero minimal left ideals of $K$ is clearly one-to-one. $K$ is a regular ring, $Ke$ is a minimal left ideal where $e$ is an idempotent, if and only if $eK$ is a minimal right ideal. The corollary is completed.

**Corollary 1.9.** $xK$ is a simple $K$-submodule of $M$ for any nonzero $x$ in an atom.

*Proof*: $C$ is an atom. $C = Me$, where $e$ is an idempotent of $K$ and $eK$ is a minimal right ideal. $eK$ and $xK$ are isomorphic as $K$-modules.

**Proposition 1.10.** $M$ is quasi-injective, $C$ is an atom of $M$, and $C_1, \cdots, C_n$ are submodules. Then either $C \cap (C_1 + \cdots + C_n) = 0$ or $C \subset C_1^s + \cdots + C_n^s$.

*Proof*: If $B = C \cap (C_1 + \cdots + C_n) \neq 0$ then $B^s \subset (C_1^s + \cdots + C_n^s)$ by Theorem 1.2. But $B^s = C$.

Let $\mathcal{F} = \{C_i\}$ be the collection of all atoms of a quasi-injective module $M$ and let $C = \sum_{C_i \in \mathcal{F}} C_i$. By the above proposition $C$ can be written as a direct sum of a subset $T$ of $\mathcal{F}$

$$C = \sum_{C_i \in T} \oplus C_i .$$

Furthermore by the Azumaya's generalization of the Krull-Remark-Schmidt Theorem [1], the direct sum decomposition of $C$ into atoms is unique up to an automorphism of $C$. That is, if

$$C = \sum_{C_j' \in T'} \oplus C_j', \ C_j' \text{ atom,}$$

then for each $C_i$ in the first decomposition there exists a unique $C_j'$ in the second decomposition that $C_i$ and $C_j'$ are isomorphic.

**2. Atomic closed submodules.** We say that the lattice of closed submodules of an $R$-module $M$ is atomic if each nonzero closed submodule of $M$ contains an atom. If $R$ is a left Noetherian ring then any $R$-module is atomic.

As in section 1, let $M$ be quasi-injective and $C$ the sum of all atoms of $M$. $C$ can be written as a direct sum of atoms.

From now on, we assume that the lattice of closed submodules of $M$ are atomic.

**Theorem 2.1.** $C$ is also quasi-injective and $C^s=M$.

*Proof*: $C^s=M$ is obvious. Otherise there exists an atom $B$ such that $B\cap C=0$.

Let $x\in C$, $x=c_1+\cdots+c_n$ for some $n$ and $c_i$ belongs to atom $C_i$ for $i=1,\cdots,n$. For any $k$ in $K=\mathrm{Hom}_R(M,M)$, $xk\in C_1k+\cdots+C_nk$. Since $C_ik$ is either zero or an atom for each $i$, $xk\in C$ and $C$ is $K$ invariant. From the first part $C^s=M$ implies the injective hull $\overline{C}$ of $C$ coincides with the injective hull $\overline{M}$ of $M$. Thus

$$\mathrm{Hom}_R(\overline{C},\overline{C})=\mathrm{Hom}_R(\overline{M},\overline{M})=\mathrm{Hom}_R(M,M)=\mathrm{Hom}_R(C,C)=K.$$

By [2, 1.3 Th], $C$ is also quasi-injective.

**Corollary 2.2.** $\mathrm{Hom}_R(M,M)=\mathrm{Hom}_R(\sum_i\oplus C_i,\ \sum_j\oplus C_j)$, where $C_i$, $C_j$ are atoms.

**Theorem 2.3.** Considering $C$ as a right $K$-module, $C$ is completely reducible.

*Proof*: From Corollary 1.10, $xK$ is a simple $K$-module if $x$ is an element in an atom. $y\in C$, $y=x_1+\cdots+x_n$, $x_i\in C_i$ atom. $y\in x_1K+\cdots+x_nK$. Thus $C=\sum xK$ where $x$ runs through all nonzero elements of atoms of $M$. $C=\sum\oplus xK$ follows from the fact that $xK$ is simple.

The concepts of dimensionality and linear independency of $C$ relative to $K$ are defined as usual. Since every $K$-submodule of $C$ is a direct summand of $C$. $C$ is quasi-injective $K$-module.

For any element $x$ of $C$, $x=x_1k_1+\cdots+x_nk_n$ with some $n$ and $x_i\in C_i$, $k_i\in K$, where $C_i$ are atoms. $k\in K$, $xk=0$ if and only if $x_ik_ik=0$ for all $i=1,\cdots,n$. If $x\neq 0$, we might assume $x_1k\neq 0$, $(Rx_1)k_1k=0$ implies $C_1k_1k=0$. $C_1k_1=Me$ is an atom of $M$. $C_1k_1k=Mek=0$, $ek=0$, and $k\in(1-e)K$. Therefore if $x\neq 0$ in $M$ then the annihilator of $x$ in $K$ is contained in $(1-e)K$ for some nonzero idempotent $e$ of $K$. $(1-e)K$ is not an essential right ideal of $K$. Therefore the singular submodule of $C$ relative to $K$ is zero.

Let $C=\sum\oplus C_i$ be a direct sum decomposition of $C$ into atoms. For each $C_i$ in the decomposition let $e_i$ be a projection of $M$ onto $C_i$. For all $y \in C$, $ye_i = 0$, for each $i$, if and only if $y = 0$. Every nonzero $x$ in $M$ there exists an $r$ in $R$, $0 \neq rx \in C$ because $C^s = M$. Therefore there exists an $e_i$ such that $rxe_i \neq 0$, $0 \neq xe_i \in C$. $C$ is essential in $M$ as $K$-modules. $C \subset M \subset \overline{C}$, where $\overline{C}$ is the $K$-injective hull of $C$. Since $S = \mathrm{Hom}_K(C, C) = \mathrm{Hom}_K(\overline{C}, \overline{C})$, $\mathrm{Hom}_K(M, M) \subset \mathrm{Hom}_K(C, C) = S$. In othere words the total ring of $K$-endomorphisms of $M$ is a subring of the total ring of $K$-endomorphisms of $C$. $\mathrm{Hom}_K(M, M) = S$ if and only if $M$ is also $K$-quasi-injective. $S$ is a regular ring and self injective as a right module over itself by our previous results.

**Lemma 2.4.** Let $M$ be a quasi-injective module, $C_1, \cdots, C_n$ atoms of $M$, and $x_i \in C_i$, $i = 1, \cdots, n$. If $A_1, \cdots, A_n$ are left ideals of $R$ that $A_i x_i \neq 0$, for each $i = 1, \cdots, n$. Then for each $s \in S = \mathrm{Hom}_K(C, C)$ there exists an essential left ideal $L$ of $R$ such that

$$Lsx_i \subset A_i x_i, \quad i = 1, \cdots, n.$$

*Proof*: For each $i$, $C_i$ is a direct summand of $M$. $C_i$ is $S$ invariant. There exists an essential left ideal $L_i$ of $R$ such that $L_i s x_i \subset A_i x_i$, for each $i$. Let $L = \bigcap_{i=1}^{n} L_i$, $L$ is still essential and will do the job.

**Theorem 2.5.*** $M$ is quasi-injective $R$-module and $\{x_1, \cdots, x_n\}$ is a finite subset of generator of $C$ as $K$-module. For each $s \in S = \mathrm{Hom}_K(C, C)$ there exist $r$, $\bar{r}$ in $R$ such that

$$rsx_i = \bar{r}x_i, \quad i = 1, \cdots, n.$$

*Proof*: Without loss of generality we can suppose that each $x_i$ in $\{x_1, \cdots, x_n\}$ belongs to some atom $C_i$, $i = 1, \cdots, n$, and the set $\{x_1, \cdots, x_n\}$ is $K$-linearly independent.

$$\text{Let} \quad N_i = \sum_{j \neq i}^{n} x_j K, \quad j = 1, \cdots, n.$$

By corollary 1.4, $N_i^{lr} = N_i$, for each $i$. Therefore

$$N_i^l x_i \neq 0 \text{ and } N_i^l x_j = 0, \quad i \neq j.$$

* Corollary 2.6 in [2] proved the case where $M$ is irreducible.

From Lemma 2. 4 there exists an essential left ideal $L$ of $R$ where

$$Lsx_i \subset N_i^l x_i, \quad i=1, \cdots, n.$$

If $sx_i \neq 0$ for some $i$ we can choose $r \in L$ such that $rsx_i = \bar{r}_i x_i \neq 0$. Now $rsx_i = \bar{r}_i x_i$, $\bar{r}_i \in N_i^l$, $i=1, \cdots, n$. The theorem follows if we let $r = \bar{r} + \cdots + \bar{r}_n$.

Theorem 2. 5 is a generalization of the classical density theorem for completely reducible modules.

For any nonempty subset of $M$ its annihilator in $R$ is a closed left ideal since $M^{\blacktriangle}=0$. Thus if either *d.c.c.* or the *a.c.c.* holes for the closed left ideals of $R$ then $C$ must be finite dimnsional over $K$ [2]. If this is the case then for each nonzero $s$ in $S$, $Rs \cap R \neq 0$.

**3. Quotient rings.** If $R$ is a subring of a ring $Q$, then $Q$ is called a left quotient ring of $R$ if $Rq \cap R \neq 0$ for each nonzero $q \in Q$. The idea of this type of quotient ring was introduced by R. E. Johnson and has been studied through different approaches (see [2] for references).

If $M$ is a faithful $R$-module, i.e., $rM=0$, if and only if $r=0$ for all $r$ in $R$. Then $C$ is automatically $R$-faithful since $C$ is essential in $M$ as $K$-modules and the singular submodule of $C$ relative to $K$ is zero. Therefore if $M$ is $R$-faithful then $R$ can be considered as a subring of $S=\mathrm{Hom}_K(C, C)$. We have just seen that if $C$ is finite dimensional over $K$ then $S$ is a left quotient ring of $R$ providing $M$ is a faithful $R$-module.

We assume now $M$ is a faithful $R$-module. If $C$ is finite dimensional over $K$ then any proper $K$-submodule of $C$ has nonzero annihilator in $R$ and $S$ is a left quotient ring of $R$. In the remaining of this paper we are going to show that $S$ is a left quotient ring of $R$ if and only if every proper $K$-submodule of $C$ has nonzero annihilator in $R$.

An element $s$ in $S$ is said to be finite rank if $sC$ is finite dimensional over $K$. The collection of all finite rank elements of $S$ forms a two sided ideal $\Phi$ in $S$. $s\Phi=0$, $\Phi t=0$, if and only if $s=0$, $t=0$. for all $s$, $t$ in $S$.

If $0\neq\varphi\in\Phi$, then $C$ can be written as $C=W\oplus D$, where $D$ is the kernel of $\varphi$ and $W=x_1K\oplus\cdots\oplus x_nK$, for some $n$ and $x_i$ belongs to some atom $C_i$, $i=1, \cdots, n$.

Let $N_i=x_1K\oplus\cdots\oplus x_{i-1}K\oplus x_{i+1}K\oplus\cdots\oplus x_nK\oplus D$, $i=1, \cdots, n$.

Suppose we assume that each $N_i$ has nonzero annihilator $N_i^l$ in $R$, then it is obvious that $N_i^{lr}=N_i$ for all $i=1, \cdots, n$.

As before there exists an essential left ideal $L$ in $R$ that

$$L\varphi x_i\subset N_i^l x_i, \quad i=1, \cdots, n.$$

Choose $r$ in $L$ such $r\varphi x_i\neq 0$ for any fixed $i$. There exist $\bar{r}_i\in N_i^l$, $i=1, \cdots, n$, such that $r\varphi x_i=\bar{r}_i x$. Let $\bar{r}$ be the sum of all $\bar{r}_i$, then $(r\varphi-\bar{r})C=0$ and $r\varphi=\bar{r}\neq 0$.

So far we have shown that $R\varphi\cap R\neq 0$ for each nonzero $\varphi\in\Phi$ The left ideal $(R:\varphi)=\{r\in R\,|\,r\varphi\in R\}$ is essential in $R$ for each $\varphi\in\Phi$. For a nonzero $s$ in $S$, let $\varphi\in\Phi$, that $0\neq\varphi s\in\Phi$. Let $r, \bar{r}$ in $R$ so that $r(\varphi s)=(r\varphi)s=\bar{r}\neq 0$. $(R:r\varphi)(r\varphi)s=(R:r\varphi)\bar{r}$. $(R:r\varphi)\bar{r}\neq 0$ since the faithfulness of $M$ and $M^{\blacktriangle}=0$ guarantee $R^{\blacktriangle}=0$. $S$ is a left quotient ring of $R$.

**Theorem 3.1.** $S$ is a left quotient ring of $R$ if and only if every proper $K$-submodule of $C$ has nonzero annihilator in $R$.

*Proof*: We have proved the if part. The only if part is obvious since every proper $K$-submodule of $C$ has nonzero annihilator in $S$.

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