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On the generalized Hopf homomorphism and the higher composition, Part I

Dedicated to Prof. A. Kobori for his 60th birthday

By

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§ 1. Introduction

Much progress has been made in the study of homotopy groups of spheres since H. Freudenthal defined the "Suspension" in his paper "Über die Klassen der Sphärenabbildungen", Composito. Math. 5 (1937), 299–314. Though many topologists have studied to compute the homotopy groups of spheres, the problem is still open.

The present paper attempts to define and study new generalized Hopf homomorphisms $\bar{H}_k: \pi_{i+1}(S^{n+1}) \rightarrow \pi_{i+1}(S^{kn+1})$ $(k=1, 2, \cdots)$. The difference among the miscellaneous Hopf homomorphisms will be studied. The higher composition than the secondary composition will also be constructed and its properties will be stated in this paper.

Sections 2 and 3 of the paper are devoted to the contruction of higher compositions. Their properties are stated in these sections. Main tool is the secondary composition.

In Section 4 we will give the formula of the Hopf homomorphism H defined in [10] for the higher composition.

In Section 5 we define the generalized Hopf homomorphism by use of the structure of the suspension space of the reduced product complex.

Section 6 is the application of Section 2 and the preparation for the forthcoming paper [11].

In [11] the author will compute the $(n+i)$-th homotopy groups of the n-sphere for $i=21$ and 22.

In this paper we use the following notations;

$[X, Y]$: *the set of all homotopy classes of maps*: $X \rightarrow Y$,
EX : *the suspension space of* X,
CX : *the cone of* X,
Ef : *the suspension of a map* f,
$E\alpha$: *the suspension of a homotopy class* α,
$\bar{\alpha}$: (*the set of*) *the extensions of* α,
$\tilde{\beta}$: (*the set of*) *the coextensions of* β (for the definition, see [10]),
$X \bigcup_{\alpha} CY$: *the mapping cone attached by* $\alpha: Y \rightarrow X$,
ΩX : *the loop space of* X,
π_i^n or $\pi_i(S^n:2)$: *the 2-components of* $\pi_i(S^n)$.

I would like to thank Prof. H. Toda who read the manuscript and gave me the benefit of many helpful conversations.

§ 2. The tertiary compositions

We assume given a sequence of maps

$$A \overset{a}{\longleftarrow} B \overset{b}{\longleftarrow} C \overset{c}{\longleftarrow} D$$

such that $a \circ b$ and $b \circ c$ are null-homotopic. According to [10], we define a homotopy class $\{a, b, c\} \in [ED, A]$ which is defined modulo left multiplication by the subgroup $a_*[ED, B]$ and right multiplication by the subgroup $(Ec)^*[EC, A]$. This double coset will depend only on the homotopy classes of a, b, c, and be denoted by $\{\alpha, \beta, \gamma\}$ where α, β, γ stand for the homotopy classes of a, b, c, respectively.

We recall some properties concerning the secondary compositions. (For the proof see [9], [10]).

Proposition 2.1.

(0) *If one of* α, β, *or* γ *is* 0, *then* $\{\alpha, \beta, \gamma\} \equiv 0$.
(1) $\{\alpha, \beta, \gamma\} \circ E\delta \subset \{\alpha, \beta, \gamma \circ \delta\}$
(2) $\{\alpha, \beta, \gamma \circ \delta\} \subset \{\alpha, \beta \circ \gamma, \delta\}$
(3) $\{\alpha \circ \beta, \gamma, \delta\} \subset \{\alpha, \beta \circ \gamma, \delta\}$
(4) $\alpha \circ \{\beta, \gamma, \delta\} \subset \{\alpha \circ \beta, \gamma, \delta\}$.

Proposition 2. 2.

$$-\{\alpha,\beta,\gamma\}\circ E\delta = \alpha\circ\{\beta,\gamma,\delta\}$$

Proposition 2. 3.

If sums are defined, then

(1) $\{\alpha,\beta,\gamma\}+\{\alpha,\beta,\gamma'\}\supset\{\alpha,\beta,\gamma+\gamma'\}$.

(2) $\{\alpha,\beta,\gamma\}+\{\alpha,\beta',\gamma\}=\{\alpha,\beta+\beta',\gamma\}$, $\gamma=E\gamma'$.

(3) $\{\alpha,\beta,\gamma\}+\{\alpha',\beta,\gamma\}\supset\{\alpha+\alpha',\beta,\gamma\}$, $\beta=E\beta'$, $\gamma=E\gamma'$.

Assume that we have a commutative diagram

$$\begin{array}{ccccccc}
A_1 & \xleftarrow{\alpha_1} & B_1 & \xleftarrow{\beta_1} & C_1 & \xleftarrow{\gamma_1} & D_1 \\
\downarrow f & & \downarrow g & & \downarrow h & & \downarrow i \\
A_2 & \xleftarrow[\alpha_2]{} & B_2 & \xleftarrow[\beta_2]{} & C_2 & \xleftarrow[\gamma_2]{} & D_2
\end{array}$$

where $\alpha_i\circ\beta_i=\beta_i\circ\gamma_i=0$ $(i=1,2)$. Then we have

Proposition 2. 4.

$f_*\{\alpha_1,\beta_1,\gamma_1\}$ and $(Si)^*\{\alpha_2,\beta_2,\gamma_2\}$ are equal as double cosets of $\alpha_{2*}[SD_1,B_2]$ and $(S\gamma_1)^*[SC_1,A_2]$.

Let α, β and γ be same as above. Consider the commutative diagram

$$\begin{array}{ccccc}
A & \xleftarrow{\bar{\alpha}} & B\bigcup_{\beta} CC & \xleftarrow{\tilde{\gamma}} & ED \\
\downarrow i & & \downarrow p & & \downarrow 1_{ED} \\
A\bigcup_{\alpha} CB & \xleftarrow[-\tilde{\beta}]{} & EC & \xleftarrow[E\gamma]{} & ED
\end{array}$$

where p is a mapping shrinking B to a point (See Lemma 2.7).

Proposition 2. 5.

The set of all the compositions $\tilde{\beta}\circ E\gamma$ coincides with $-i_*\{\alpha,\beta,\gamma\}$.

Similarly, we have

Proposition 2. 6.

For an extension $\bar{\beta}$ of β, there exists an element λ of $[ED, A]$ such that $p^*\lambda=\alpha\circ\bar{\beta}$. The set of $\{\lambda\}$ of such elements forms a coset of $[EC, A]\circ E\gamma$ which is a subset of $\{\alpha,\beta,\gamma\}$. Furthermore, any element λ of $\{\alpha,\beta,\gamma\}$ satisfies the relation $p^*\lambda=\alpha\circ\bar{\beta}$ for some choice of $\bar{\beta}$.

Assume that $\alpha\circ\beta=\beta\circ\gamma=\gamma\circ\delta=0$ in the diagram

$$X \xleftarrow{\alpha} Y \xleftarrow{\beta} Z \xleftarrow{\gamma} U \xleftarrow{\delta} V.$$

Before defining the tertiary composition, we prepare

Lemma 2.7. ([7 ; Propositions 5.11])
The following diagram is commutative

$$\begin{array}{ccc} Y & \xleftarrow{\bar{\beta}} & Z\cup_{\gamma} CU \\ \downarrow i & & \downarrow p \\ Y\cup_{\beta} CZ & \xleftarrow[-\tilde{\gamma}]{} & EU . \end{array}$$

Proof is given in [7].

So we obtain the commutative diagram

$$\begin{array}{ccccccc} X & \xleftarrow{\alpha} & Y & \xleftarrow{\bar{\beta}} & Z\cup_{\gamma} CU & \xleftarrow{\tilde{\delta}} & EV \\ \downarrow 1_X & & \downarrow i & & \downarrow p & & \downarrow 1_{EV} \\ X & \xleftarrow[\bar{\alpha}]{} & Y\cup_{\beta} CZ & \xleftarrow[-\tilde{\gamma}]{} & EU & \xleftarrow[E\delta]{} & EV . \end{array} \tag{2.1}$$

We assume furthermore that

(i) $\{\alpha, \beta, \gamma\} \ni 0$ *and* $\{\beta, \gamma, \delta\} = 0$,

or

(ii) $\{\alpha, \beta, \gamma\} = 0$ *and* $\{\beta, \gamma, \delta\} \ni 0$

The case of (i).

There exist $\bar{\alpha}$ and $\tilde{\gamma}$ such that $\bar{\alpha}\circ\tilde{\gamma}=0$. By Proposition 2.6 there exists $\bar{\beta}$ such that $\alpha\circ\bar{\beta}=p^*(\bar{\alpha}\circ\tilde{\gamma})=0$. We have $\bar{\beta}\circ\tilde{\delta}=0$ for any $\tilde{\delta}$ by the assumption $\{\beta, \gamma, \delta\}=0$.

By Proposition 2.5 we have

$$-\tilde{\gamma}\circ E\delta \in i_*\{\beta, \gamma, \delta\} = 0 .$$

Therefore we can define the secondary compositions

$$\{\alpha, \bar{\beta}, \tilde{\delta}\} \quad \text{and} \quad \{\bar{\alpha}, -\tilde{\gamma}, E\delta\}$$

By Proposition 2.4 and (2.1), these two secondary compositions coincide as double cosets of

(2.2) $\bar{\alpha}_*[E^2V, Y\bigcup_{\beta} CZ]$ *and* $(E\tilde{\delta})^*[EZ\bigcup_{\beta\gamma} CEU, X]$.

We define the tertiary composition $\{\alpha, \beta, \gamma, \delta\}$ as $\{\alpha, \bar{\beta}, \tilde{\delta}\} = \{\bar{\alpha}, -\tilde{\gamma}, E\delta\}$ which is the double coset of (2.2) under the conditions $\{\alpha, \beta, \gamma\} \ni 0$ and $\{\beta, \gamma, \delta\}=0$.

The case of (ii).

By the similar arguements, we can define $\{\alpha, \beta, \gamma, \delta\}$ as double cosets of (2.2).

Note that under the conditions $\{\alpha, \beta, \gamma\} = \{\beta, \gamma, \delta\}=0$ the above two cases coincide.

Proposition 2.9.

(0) *If one of* α, β, γ *or* δ *is* 0, *then*
$\{\alpha, \beta, \gamma, \delta\} \equiv 0$.

(i) *If* $\{\alpha, \beta, \gamma\} \ni 0$ *and* $\{\beta, \gamma, \delta\circ\varepsilon\}=0$,
or $\{\alpha, \beta, \gamma\}=0$ *and* $\{\beta, \gamma, \delta\circ\varepsilon\} \ni 0$, *then*
$\{\alpha, \beta, \gamma, \delta\}\circ E^2\varepsilon \subset \{\alpha, \beta, \gamma, \delta\circ\varepsilon\}$.

(ii) *If* $\{\alpha, \beta, \gamma\} \ni 0$ *and* $\{\beta, \gamma\circ\delta, \varepsilon\}=0$,
or $\{\alpha, \beta, \gamma\circ\delta\} = \{\alpha, \beta, \gamma\}=0$ *and* $\{\beta, \gamma, \delta\circ\varepsilon\} \ni 0$, *then*
$\{\alpha, \beta, \gamma, \delta\circ\varepsilon\} \subset \{\alpha, \beta, \gamma\circ\delta, \varepsilon\}$.

(iii) *If* $\{\alpha\circ\beta, \gamma, \delta\} \ni 0$ *and* $\{\gamma, \delta, \varepsilon\} = \{\beta\circ\gamma, \delta, \varepsilon\}=0$,
or $\{\alpha, \beta\circ\gamma, \delta\}=0$ *and* $\{\gamma, \delta, \varepsilon\} \ni 0$, *then*
$\{\alpha\circ\beta, \gamma, \delta, \varepsilon\} \subset \{\alpha, \beta\circ\gamma, \delta, \varepsilon\}$.

(iv) *If* $\{\beta, \gamma, \delta\} \ni 0$ *and* $\{\gamma, \delta, \varepsilon\}=0$,
or $\{\beta, \gamma, \delta\} = \{\alpha\circ\beta, \gamma, \delta\}=0$ *and* $\{\gamma, \delta, \varepsilon\} \ni 0$, *then*
$\alpha\circ\{\beta, \gamma, \delta, \varepsilon\} \subset \{\alpha\circ\beta, \gamma, \delta, \varepsilon\}$.

Before we prove this proposition, we prepare the following

Lemma 2.10.

(i) $\tilde{\gamma}\circ E\delta \subset \widetilde{\gamma\circ\delta}$

(ii) $\beta\circ\bar{\gamma} \subset \overline{\beta\circ\gamma}$

(iii) $\bar{\beta}_1+\bar{\beta}_2=\overline{\beta_1+\beta_2}$

The proof is left to the reader.

Proof of Proposition 2.9.

The proof of (i) and (iv) is clear.

(0) $\alpha=0$ or $\delta=0$, then the proof is similar to those of (0)

of Proposition 2. 1. Let $\beta=0$, then in the definition of $\{\alpha, \beta, \gamma, \delta\}$ we may choose $\bar{\beta}=0$. By use of Proposition 2. 1, we can prove the proposition. The case $\gamma=0$ is similar.

(ii) We have

$$\begin{aligned}\{\bar{\alpha}, \tilde{\gamma}, E(\delta\circ\varepsilon)\} &\subset \{\bar{\alpha}, \tilde{\gamma}\circ E\delta, E\varepsilon\}\\ &\subset \{\bar{\alpha}, \widetilde{\gamma\circ\delta}, E\varepsilon\} \quad \text{by (i) of Lemma 2. 10.}\end{aligned}$$

This proves (ii).

(iii) We have

$$\begin{aligned}\{\alpha\circ\beta, \tilde{\gamma}, \varepsilon\} &\subset \{\alpha, \beta\circ\tilde{\gamma}, \varepsilon\}\\ &\subset \{\alpha, \overline{\beta\circ\gamma}, \varepsilon\} \qquad \text{by (ii) of Lemma 2. 10.}\end{aligned}$$

This proves (iii).

Proposition 2. 11.

(i) $\{\alpha, \beta, \gamma, \delta_1+\delta_2\} \subset \{\alpha, \beta, \gamma, \delta_1\} + \{\alpha, \beta, \gamma, \delta_2\}$

(ii) $\{\alpha, \beta, \gamma_1+\gamma_2, \delta\} = \{\alpha, \beta, \gamma_1, \delta\} + \{\alpha, \beta, \gamma_2, \delta\}$

(iii) $\{\alpha, \beta_1+\beta_2, \gamma, \delta\} = \{\alpha, \beta_1, \gamma, \delta\} + \{\alpha, \beta_2, \gamma, \delta\}$

(iv) $\{\alpha_1+\alpha_2, \beta, \gamma, \delta\} \subset \{\alpha_1, \beta, \gamma, \delta\} + \{\alpha_2, \beta, \gamma, \delta\}$.

Proof.

(i) and (iv) follow immediately from Proposition 2. 3.

We have

$$\begin{aligned}\{\alpha, \beta_1, \gamma, \delta\} + \{\alpha, \beta_2, \gamma, \delta\} &= \{\alpha, \bar{\beta}_1, \tilde{\delta}\} + \{\alpha, \bar{\beta}_2, \tilde{\delta}\}\\ &= \{\alpha, \bar{\beta}_1+\bar{\beta}_2, \tilde{\delta}\} \quad \text{by Proposition 2. 3}\\ &= \{\alpha, \overline{\beta_1+\beta_2}, \tilde{\delta}\} \quad \text{by (iii) of Lemma 2. 10}\\ &= \{\alpha, \beta_1+\beta_2, \gamma, \delta\}.\end{aligned}$$

So we obtain (iii). The proof of (ii) is similar. Q.E.D.

In the diagram

$$X \xleftarrow{\alpha} X \xleftarrow{\beta} Z \xleftarrow{\gamma} U \xleftarrow{\delta} V \xleftarrow{\varepsilon} W$$

we assume that

$$\{\alpha, \beta, \gamma\} = \{\gamma, \delta, \varepsilon\} = 0 \quad \text{and} \quad \{\beta, \gamma, \delta\} \ni 0,$$
$$\text{or} \quad \{\alpha, \beta, \gamma\} \ni 0, \ \{\gamma, \delta, \varepsilon\} \ni 0 \quad \text{and} \quad \{\beta, \gamma, \delta\} = 0.$$

Proposition 2.12.

Under the above conditions we have

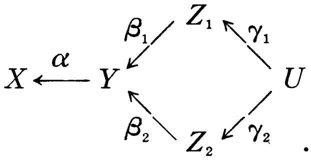
$$-\{\alpha, \beta, \gamma, \delta\}\circ E^2\varepsilon = \alpha\circ\{\beta, \gamma, \delta, \varepsilon\}\ .$$

Proof.

$$\begin{aligned}\{\alpha, \beta, \gamma, \delta\}\circ E^2\varepsilon &= \{\alpha, \bar{\beta}, \tilde{\delta}\}\circ E^2\varepsilon \\ &= -\alpha\circ\{\bar{\beta}, \tilde{\delta}, E\varepsilon\} \qquad \text{by Proposition 2.2} \\ &= \alpha\circ\{\beta, \gamma, \delta, \varepsilon\}\,.\end{aligned}$$

§ 3. Generalized Toda bracket

In [1] M. G. Barratt defined a generalized Toda bracket $\left\{\alpha, \begin{matrix}\beta_1, \gamma_1\\ \beta_2, \gamma_2\end{matrix}\right\}$ under the relations $\alpha\circ\beta_1=\alpha\circ\beta_2=\beta_1\circ\gamma_1+\beta_2\circ\gamma_2=0$, where



.

However this higher composition is a special case of the secondary composition $\{\alpha, \beta, \gamma\}$, where $\alpha\in[Y, X]$, $\beta=\beta_1+\beta_2\in[Z, Y]=[Z_1, Y]+[Z_2, Y]$ and $\gamma=\gamma_1+\gamma_2\in[U, Z_1]+[U, Z_2]\subset[U, Z]$ ($Z=Z_1\vee Z_2$, one point union of Z_1 and Z_2). So all the properties which hold for usual $\{\alpha, \beta, \gamma\}$ are true for this composition, e.g.,

(i) $\left\{\alpha_1+\alpha_2, \begin{matrix}\beta, \gamma\\ \beta', \gamma'\end{matrix}\right\}\subset\left\{\alpha_1, \begin{matrix}\beta, \gamma\\ \beta', \gamma'\end{matrix}\right\}+\left\{\alpha_2, \begin{matrix}\beta, \gamma\\ \beta', \gamma'\end{matrix}\right\}$

(ii) $\left\{\alpha, \begin{matrix}\beta_1+\beta_2, \gamma\\ \beta_1'+\beta_2', \gamma'\end{matrix}\right\}=\left\{\alpha, \begin{matrix}\beta_1, \gamma\\ \beta_1', \gamma'\end{matrix}\right\}+\left\{\alpha, \begin{matrix}\beta_2, \gamma\\ \beta_2', \gamma'\end{matrix}\right\}$

(iii) $\left\{\alpha, \begin{matrix}\beta, \gamma_1+\gamma_2\\ \beta', \gamma_1'+\gamma_2'\end{matrix}\right\}\subset\left\{\alpha, \begin{matrix}\beta, \gamma_1\\ \beta', \gamma_1'\end{matrix}\right\}+\left\{\alpha, \begin{matrix}\beta, \gamma_2\\ \beta', \gamma_2'\end{matrix}\right\}$

It is clear that this composition is a double coset of $\alpha_*[EU, Y]$ and $(E\gamma_1)^*[X, EZ_1]+(E\gamma_2)^*[X, EZ_2]$.

§ 4. The Hopf homomorphism and the composition

Consider tertiary compositions for spheres:

$$S^n \xleftarrow{\alpha} S^q \xleftarrow{E\beta} S^r \xleftarrow{E\gamma} S^s \xleftarrow{E\delta} S^t .$$

In the above we assume that $\{\alpha, E\beta, E\gamma\} \ni 0$ and $\{\beta, \gamma, \delta\} = 0$, or $\{\alpha, E\beta, E\gamma\} = 0$ and $\{\beta, \gamma, \delta\} \ni 0$. Under these assumptions we may define the tertiary composition $\{\alpha, E\beta, E\gamma, E\delta\}$ as $\{\bar{\alpha}, -E\tilde{\gamma}, E^2\delta\} = \{\alpha, E\bar{\beta}, E\tilde{\delta}\}$. Denote by H the generalized Hopf homomorphism defined in [10], then by Proposition 2.3 of [10], we obtain

$$\begin{aligned} H\{\alpha, E\beta, E\gamma, E\delta\} &= H\{\alpha, E\bar{\beta}, E\tilde{\delta}\} \\ &\subset \{H\alpha, E\bar{\beta}, E\tilde{\delta}\} \\ &= \{H\alpha, E\beta, E\gamma, E\delta\} . \end{aligned}$$

We have proved

Proposition 4.1.

$$H\{\alpha, E\beta, E\gamma, E\delta\} \subset \{H\alpha, E\beta, E\gamma, E\delta\} .$$

Next we assume that $\alpha\circ\beta = \beta\circ\gamma = \gamma\circ\delta = 0$ and $\{E\alpha, E\beta, E\gamma\} \ni 0$ and $\{E\beta, E\gamma, E\delta\} = 0$, or $\{E\alpha, E\beta, E\gamma\} = 0$ and $\{E\beta, E\gamma, E\delta\} \ni 0$. Then we can define $\{E\alpha, E\beta, E\gamma, E\delta\}$ as $\{E\alpha, E\bar{\beta}, E\tilde{\delta}\} = \{E\bar{\alpha}, -E\tilde{\gamma}, E^2\delta\}$

Proposition 4.2.

$$H\{E\alpha, E\beta, E\gamma, E\delta\} \subset \Delta^{-1}\{\alpha, \beta, \gamma\}\circ E^3\delta .$$

Proof.

$$\begin{aligned} H\{E\alpha, E\beta, E\gamma, E\delta\} &= H\{E\bar{\alpha}, -E\tilde{\gamma}, E^2\delta\} \\ &= \Delta^{-1}(\bar{\alpha}\circ\tilde{\gamma})\circ E^3\delta \qquad \text{by Proposition 2.6 of [10]} \\ &\subset \Delta^{-1}\{\alpha, \beta, \gamma\}\circ E^3\delta \qquad \text{Q.E.D.} \end{aligned}$$

For the generalized Toda bracket, the following proposition is obvious.

Proposition 4.3.

$$H\left\{\alpha, \begin{matrix} E\beta, & E\gamma \\ E\beta', & E\gamma' \end{matrix}\right\} \subset \left\{H\alpha, \begin{matrix} E\beta, & E\gamma \\ E\beta', & E\gamma' \end{matrix}\right\} .$$

§ 5. The generalized Hopf homomorphism

The reduced product complex of S^n $(n \geq 1)$ is a CW-complex S^n_∞ with a cell structure

$$S^n \cup e^{2n} \cup \cdots \cup e^{kn} \cup \cdots,$$

so there is one cell in every dimension which is a multiple of n. We study at first the structure of the suspension space of this complex, $E(S^n_\infty)$.

Theorem 5. 1.

$E(S^n_\infty)$ is homotopy equivalent to $\bigvee_{i=1}^{\infty} S^{in+1}$.

Proof.

Set $T_k = \bigvee_{i=1}^{k} S^n \times \cdots \times S^n \times \underset{(i)}{*} \times S^n \times \cdots \times S^n$, $(S^n)^k = S^n \times \cdots \times S^n$ the product of k n-spheres and $(S^n)^k - T_k = e_0^{kn}$ the open kn-cell. Let f_k be the identification map: $(S^n)^k \to S^n_k$, where S^n_k is the kn-skelton of S^n_∞, and denote by $\bar{f}_k$ the restriction of f_k to T_k, i.e.,

$$\bar{f}_k : T_k \to S^n_{k-1}.$$

Consider the closed kn-cell $E_1^{kn} = E^n_+ \times \cdots \times E^n_+$ contained in e_0^{kn}, where E^n_+ is the upper hemi-sphere of S^n and the base point $*$ belongs to $S^n - E^n_+$.

Choose a differentiable imbedding of $A_k = (S^n)^k - \operatorname{Int} E_1^{kn}$ in E^{kn+1}, the closed $(kn+1)$-cell, so that $\partial A_k = A_k \cap S^{kn} = S^{kn-1}$. Obviously T_k is the deformation retract of A_k and we denote by r_k the deformation retraction of A_k onto T_k.

Set $F_k = \bar{f}_k \circ r_k | \partial A_k : S^{kn-1} \to S^n_{k-1}$, then F_k is homotopic to the attaching map of e^{kn} in $S^n_k = S^n_{k-1} \cup e^{kn}$.

We want to show that

$$EF_k \simeq 0 : S^{kn} \to E(S^n_{k-1}).$$

This is equivalent to that EF_k can be extended to E^{kn+1}.

Let $S^{kn} = E^{kn}_+ \cup E^{kn}_-$, where E^{kn}_+ (E^{kn}_-) is the upper (lower) hemisphere of S^{kn}. Then we may divide E^{kn+1} into two submanifolds V^{kn+1}_+ and V^{kn+1}_- such that V^{kn+1}_+ (V^{kn+1}_-) contains E^{kn}_+ (E^{kn}_-) respectively and $V^{kn+1}_+ \cap V^{kn+1}_- = A_k$.

We define a map $G_k : A_k \cup S^{kn} \to E(S^n_{k-1})$ as follows:

$$G_k | A_k = \bar{f}_k \circ r_k : A_k \to S^n_{k-1},$$
$$G_k | S^{kn} = EF_k : S^{kn} \to E(S^n_{k-1}).$$

Set $E(S^n_{k-1}) = V_+ \cup V_-$, where V_+ (V_-) is the upper (lower) half of $E(S^n_{k-1})$. By the definition of the suspension of F_k, we see that

$$G_k(A_k \cup E^{kn}_+) \subset V_+$$

and

$$G_k(A_k \cup E^{kn}_-) \subset V_- .$$

G_k can be extended to V^{kn+1}_+ (V^{kn+1}_-), as V_+ (V_-) is contractible. The map thus obtained is an extension of EF_k to E^{kn+1}. It follows immediately the theorem. Q.E.D.

Remark.

This theorem is also obtained easily by use of Satz 20 of [8].

Hereafter we fix the homotopy equivalence between $E(S^n_\infty)$ and $\bigvee_i S^{in+1}$ (preserving orientations).

Let $i: S^n_\infty \to \Omega S^{n+1}$ be the canonical injection, Ω_0 be the one to one correspondence : $\pi_{i+1}(S^{n+1}) \to \pi_i(\Omega S^{n+1})$ and p_k be the projection : $\bigvee_i S^{in+1} \to S^{kn+1}$.

Now we define generalized Hopf homomorphisms $\bar{H}_k$ $(k=1, 2, \cdots)$ as follows :

$$\bar{H}_k = p_{k*} \circ E \circ i_*^{-1} \circ \Omega_0 : \ \pi_{i+1}(S^{n+1}) \to \pi_{i+1}(S^{kn+1}) .$$

Proposition 5. 2.

$$\bar{H}_1 \circ E = identity.$$

$$\bar{H}_k \circ E = 0 \qquad (k \geq 2) .$$

Proof.

We have the commutative diagram :

$$(5.1) \qquad \begin{array}{ccc} \pi_i(S^n) & \xleftarrow{\ E\ } & \pi_{i+1}(S^{n+1}) \\ & {\scriptstyle i'_*}\searrow \quad \swarrow{\scriptstyle \Omega_0} & \\ & \pi_i(\Omega S^{n+1}) & \end{array}$$

where $i': S^n \to \Omega S^{n+1}$ is the natural injection and the relation $i | S^n = i'$ holds.

For an element $E\alpha \in \pi_{i+1}(S^{n+1})$, we obtain

$$\begin{aligned} (E \circ i_*^{-1} \circ \Omega_0)(E\alpha) &= (E \circ i_*^{-1} \circ i'_*)(\alpha) \qquad \text{by (5. 1)} \\ &= E\alpha \in \pi_{i+1}(S^{n+1}) \end{aligned}$$

Hence

$$\bar{H}_1(E\alpha) = p_{1*}\circ(E\circ i_*^{-1}\circ\Omega_0)\circ E\alpha$$
$$= p_{1*}(E\alpha)$$
$$= E\alpha$$

and

$$\bar{H}_k(E\alpha) = p_{k*}\circ(E\circ i_*^{-1}\circ\Omega_0)\circ E\alpha$$
$$= p_{k*}(E\alpha)$$
$$= 0 \qquad \text{for} \quad k\geq 2\,.$$

Proposition 5.3.

$$\bar{H}_k(\alpha\circ E\beta) = \bar{H}_k(\alpha)\circ E\beta\,.$$

Proof.

We have the formula

$$\Omega_0(\alpha\circ E\beta) = \Omega_0(\alpha)\circ\beta\,.$$

Therefore we obtain

$$\bar{H}_k(\alpha\circ E\beta) = (p_{k*}\circ E\circ i_*^{-1}\circ\Omega_0)(\alpha\circ E\beta)$$
$$= p_{k*}((E\circ i_*^{-1})\circ(\Omega_0\alpha\circ\beta))$$
$$= p_{k*}(i_*^{-1}(\Omega_0\alpha)\circ E\beta)$$
$$= \bar{H}_k(\alpha)\circ E\beta\,. \qquad \text{Q.E.D.}$$

I. M. James defined in [6] the Hopf-James homomorphism H_k as follows:

$$H_k=\Omega_0^{-1}\circ i_*\circ h_{k*}\circ i_*^{-1}\circ\Omega_0 :\ \pi_{i+1}(S^{n+1}) \rightarrow \pi_{i+1}(S^{kn+1})\,,$$

where h_k is the map $(S^n_\infty, S^n_k, S^n_{k-1})\rightarrow(S^{kn}_\infty, S^{kn}, *)$ and $S^n_k-S^n_{k-1}$ is mapped onto $S^{kn}-*$ by degree 1 by h_k.

We will study the differences between two such homomorphisms. Let us consider a condition on a map that

$(5.2)_k$: *it is a map*: $(E(S^n_\infty), E(S^n_k), E(S^n_{k-1}))\rightarrow(E(S^{kn}_\infty), E(S^{kn}), *)$ *and* $E(S^n_k)-E(S^n_{k-1})$ *is mapped homeomorphically onto* $E(S^{kn})-*$ *by it.*

Obviously the map Eh_k satisfies the condition $(5.2)_k$.

Let h'_k and h''_k be maps: $(S^n_\infty, S^n_k, S^n_{k-1})\rightarrow(S^{kn}_\infty, S^{kn}, *)$ such that Eh'_k and Eh''_k satisfy the condition $(5.2)_k$. Then we have the following commutative diagram:

$$
(5.3)\qquad
\begin{array}{ccc}
\pi_{i+1}(S^{n+1}) & & \\
\downarrow \Omega_0 & & \\
\pi_i(\Omega S^{n+1}) & \xrightarrow{E} & \pi_{i+1}(E\Omega S^{n+1}) \\
\uparrow i_* & & \uparrow (Ei)_* \\
\pi_i(S^n_\infty) & \xrightarrow{E} & \pi_{i+1}(ES^n_\infty) \\
\downarrow h'_{k*},\ h''_{k*} & & \downarrow (Eh'_k)_*,\ (Eh''_k)_* \\
\pi_i(S^{kn}_\infty) & \xrightarrow{E} & \pi_{i+1}(ES^{kn}_\infty) \\
\downarrow i_* & & \downarrow (Ei)_* \\
\pi_i(\Omega S^{kn+1}) & \xrightarrow{E} & \pi_{i+1}(E\Omega S^{kn+1}) \\
\uparrow \Omega_0 & \swarrow a_k & \\
\pi_{i+1}(S^{kn+1}) & &
\end{array}
$$

where a_k is the natural map: $E\Omega S^{kn+1} \to S^{kn+1}$.

We have the following exact sequence (see [8]):

$$[E(S^n_\infty/S^n_k), ES^{kn}_\infty] \xrightarrow{p^*} [ES^n_\infty, ES^{kn}_\infty] \xrightarrow{i^*} [ES^n_k, ES^{kn}_\infty].$$

By use of $(5.2)_k$ we obtain that

$$i^*(\{Eh'_k\} - \{Eh''_k\}) = 0 \quad \text{in} \quad [ES^n_k, ES^{kn}_\infty],$$

whence there exists an element γ of $[E(S^n_\infty/S^n_k), ES^{kn}_\infty]$ such that $p^*\gamma = \{Eh'_k\} - \{Eh''_k\}$,

i.e., $$\{Eh'_k\} - \{Eh''_k\} \in p^*[E(S^n_\infty/S^n_k), ES^{kn}_\infty].$$

Accordingly, for an element $\bar{\alpha} \in \pi_i(S^n_\infty)$ we have

$$
(5.4)\quad
\begin{aligned}
\{Eh'_k\}\circ E\bar{\alpha} - \{Eh''_k\}\circ E\bar{\alpha} &= (\{Eh'_k\} - \{Eh''_k\})\circ E\bar{\alpha} \\
&\in [\bigvee_{j\geq k+1} S^{jn+1}, ES^{kn}_\infty]\circ \pi_{i+1}(\bigvee_{j\geq k+1} S^{jn+1}).
\end{aligned}
$$

Define homomorphisms H'_k and H''_k : $\pi_{i+1}(S^{n+1}) \to \pi_{i+1}(S^{kn+1})$ as follows:

$$
\begin{aligned}
H'_k &= \Omega_0^{-1}\circ i_*\circ h'_{k*}\circ i_*^{-1}\circ \Omega_0 \\
H''_k &= \Omega_0^{-1}\circ i_*\circ h''_{k*}\circ i_*^{-1}\circ \Omega_0 .
\end{aligned}
$$

Then for an element α of $\pi_{i+1}(S^{n+1})$ we obtain by use of the commutative diagram (5.3) and (5.4)

Theorem 5.4.

$$H'_k(\alpha) \equiv H''_k(\alpha) \qquad \mathrm{mod}\, f_* \pi_{i+1}(\bigvee_{j \geq k+1} S^{jn+1}),$$

where f is a map: $\bigvee_{j \geq k+1} S^{jn+1} \to S^{kn+1}$.

Similarly we get

Theorem 5.5.

$$\bar{H}_k(\alpha) \equiv H_k(\alpha) \qquad \mathrm{mod}\, g_* \pi_{i+1}(\bigvee_{j \geq k+1} S^{jn+1}),$$

where $g: \bigvee_{j \geq k+1} S^{jn+1} \to S^{kn+1}$.

Corollary 5.6.

$$\bar{H}_k = H_k \qquad \text{if} \quad i < (k+1)n.$$

Next consider the Hilton-Hopf homomorphism $H_0: \pi_{i+1}(S^{n+1}) \to \pi_{i+1}(S^{2n+1})$.

The following correspondence defines the Hilton-Hopf homomorphism:

$$\Omega S^{n+1} \xrightarrow{\Omega\varphi_{n+1}} \Omega(S^{n+1} \vee S^{n+1}) \xleftarrow{j} \Omega S^{n+1} \times \Omega S^{n+1} \times \Omega S^{2n+1} \times \Omega S^{3n+1} \times \cdots \xrightarrow{p} \Omega S^{2n+1},$$

where φ_{n+1} is the map: $S^{n+1} \to S^{n+1} \vee S^{n+1}$ which pinches the equator of S^{n+1}, j is a singular homotopy equivalence and p is the projection on the third factor.

It is well known that $p_* \circ j_*^{-1} \circ (\Omega\varphi_{n+1})_*: H_{2n}(\Omega S^{n+1}) \cong H_{2n}(\Omega S^{2n+1})$. Since $i: S^{2n}_\infty \to \Omega S^{2n+1}$ is a singular homotopy equivalence, there exists a map $h: S^n_\infty \to S^{2n}_\infty$ such that $h_*: H_{2n}(S^n_\infty) \cong H_{2n}(S^{2n}_\infty)$ and $\Omega_0^{-1} \circ i_* \circ h_* \circ i_*^{-1} \circ \Omega_0 = H_0: \pi_{i+1}(S^{n+1}) \to \pi_{i+1}(S^{2n+1})$. It is easily verified that Eh satisfies the condition $(5.2)_2$.

Hence we obtain the following theorem, as H_2 defined in the above is the Hopf homomorphism H defined in [10].

Theorem 5.7.

$$H(\alpha) \equiv H_0(\alpha) \ \mathrm{mod}\, f'_* \pi_{i+1}(\bigvee_{j \geq 3} S^{jn+1}) \qquad \text{for} \quad \alpha \in \pi_{i+1}(S^{n+1}),$$

where $f': \bigvee_{j \geq 3} S^{jn+1} \to S^{2n+1}$.

In particular,

Corollary 5.8.

If $i < 7n$, *then*

$$H(\alpha) \equiv H_0(\alpha) \quad \bmod \sum_{k=3}^{6} f_{k*}\pi_{i+1}(S^{kn+1}),$$

where $f_k : S^{kn+1} \to S^{2n+1}$.

Remark.

For the higher Hilton-Hopf homomorphism $\tilde{H}_k$ defined in [2], the similar result to Theorem 5.5 holds. $(\tilde{H}_2 = H_0)$.

Lemma 5.9.

$$EH_0(\alpha\circ\beta) = EH_0(\alpha)\circ E\beta + E^n\alpha\circ E^p\alpha\circ EH_0(\beta)$$
$$\textit{for } \alpha\in\pi_p(S^n) \textit{ and } \beta\in\pi_i(S^p).$$

This is the conclusion from Theorem 6.2 of [2] and Theorem 1 of [3].

Corollary 5.10.

If $i<7n$, $p<3n$ *and* $i<3p$, *then for* $\alpha\in\pi_{p+1}(S^{n+1})$ *and* $\beta\in\pi_{i+1}(S^{p-1})$

$$H(\alpha\circ\beta) = H(\alpha)\circ\beta + E^n\alpha\circ E^p\alpha\circ H(\beta)$$
$$\bmod \sum_{k=3}^{6} f_{k*}\pi_{i+1}(S^{kn+1}) + \mathrm{Ker}\,(E : \pi_i(S^{2n}) \to \pi_{i+1}(S^{2n+1})),$$

where $f_k : S^{kn+1} \to S^{2n+1}$.

This follows immediately from Corollary 5.8 and Lemma 5.9.

Problem 5.11.

Is it true or not that

$$H(\alpha\circ\beta) = H(\alpha)\circ\beta + E^{n-1}\alpha\circ E^{p-1}\alpha\circ H(\beta)\,?$$

Hereafter the notations of generators of π^n_{n+i} are refered to [10].

Example 5.12.

$$H(\sigma'\circ\omega_{14}) = \eta_{13}\circ\omega_{14} = \varepsilon^*_{13} \textit{ for } \sigma'\in\pi^7_{14} \textit{ and } \omega_{14}\in\pi^{14}_{30}.$$

Proof.

Let $n=6$, $p=13$ and $i=29$ in Corollary 5.10, then

$$H(\sigma'\circ\omega_{14}) \equiv H(\sigma')\circ\omega_{14} + E^6\sigma'\circ E^{13}\sigma'\circ H(\omega_{14}) \quad \bmod G$$
$$= \eta_{13}\circ\omega_{14} + 4\sigma^2_{13}\circ\nu_{27} \qquad \text{by Lemmas 5.14 and 12.15 of [10]}$$

where $G = f_{3*}\pi_{30}(S^{19}) + f_{4*}\pi_{30}(S^{25}) + \mathrm{Ker}\,(E : \pi_{29}(S^{12}) \to \pi_{30}(S^{13}))$
$= \{\nu^2_{23}\circ\zeta_{19}\} = 0$

and $4\sigma_{13}^2\circ\nu_{27}=0$.
By Proposition 3.1 of [10] we have

$$\eta_2\#\omega_{14} = \eta_{16}\circ\omega_{17} = \omega_{16}\circ\eta_{32}$$
$$= \varepsilon_{16}^* \quad \text{by Lemma 12.15 of [10].}$$

As $E^3: \pi_{30}(S^{13})\to\pi_{33}(S^{16})$ is a monomorphism, we obtain

$$H(\sigma'\circ\omega_{14}) = \varepsilon_{13}^*.$$

This relation will be used in [11].

§6. Some elements given by the higher compositions

This section is an application of §2 and §4 and the preparation for the forthcoming paper [11].

In Lemma 12.18 of [10] Toda obtained the element λ of π_{31}^{13} such that

$$H(\lambda) = \nu_{25}^2 \quad\text{and}\quad E^3\lambda-2\nu_{16}^* = \pm\Delta\nu_{33}$$

by use of Lemma 11.17 of [10].

Consider the secondary composition $\{\nu_{20}, 2\nu_{23}, \nu_{26}\}$ which is well defined since the order of ν_n^2 is 2 for $n\geq 5$. By (3.10) of [10] $\{\nu_{20}, 2\nu_{23}, \nu_{26}\}$ and $(-1)^3\{2\nu_{20}, \nu_{23}, 2\nu_{26}\}$ have a common element. We have

$$\begin{aligned}\{2\nu_{20}, \nu_{23}, 2\nu_{26}\} &\subset \{2\nu_{20}, 2\nu_{23}, \nu_{26}\} && \text{by Proposition 2.1}\\ &= E^{17}\varepsilon' && \text{by the definition of } \varepsilon' \text{ in [10]}\\ &= 2(\nu_{20}\circ\sigma_{23}) && \text{by (7.10) of [10]}\\ &= 0 && \text{by (7.20) of [10]}\end{aligned}$$

Therefore we have

$$\{\nu_{20}, 2\nu_{23}, \nu_{26}\}\equiv 0 \mod \{\nu_{20}\circ\sigma_{23}\} = 0.$$

Let a cell complex $K=S^{23}\cup e^{27}$ have the characteristic class $2\nu_{23}$ of e^{27}. By Proposition 1.7 of [10], there exists an extension $\text{Ext}(\nu_{20})\in[K, S^{20}]$ of ν_{20}, and a coextension $\text{Coext}(\nu_{20})\in\pi_{30}(K)$ of ν_{26}, such that $\text{Ext}(\nu_{20})\circ\text{Coext}(\nu_{26})=0$. By Proposition 2.6, there exists an element γ of $\pi_{27}(S^{13})$ such that $p^*\gamma=\sigma_{13}\circ\text{Ext}(\nu_{20})$. The set $\{\gamma\}$ of such elements forms a subset of $\{\sigma_{13}, \nu_{20}, 2\nu_{23}\}$. It is easily checked that secondary composition $\{\sigma_{13}, \nu_{20}, 2\nu_{23}\}$ contains 0. Thus the following tertiary composition is defined

$$\{\sigma_{13}, \mathrm{Ext}(\nu_{20}), \mathrm{Coext}(\nu_{26})\} = \{\sigma_{13}, \nu_{20}, 2\nu_{23}, \nu_{26}\} \subset \pi_{31}(S^{13}).$$

By Lemma 4.2,

$$H\{\sigma_{13}, \nu_{20}, 2\nu_{23}, \nu_{27}\} = \Delta^{-1}\{\sigma_{12}, \nu_{19}, 2\nu_{22}\}\circ\nu_{28} = \nu_{25}^2,$$

since we have

$$H\{\sigma_{12}, \nu_{19}, 2\nu_{22}\} = 2\nu_{23} = H\Delta\nu_{25}.$$

So we choose an element λ_0 from this tertiary composition $\{\sigma_{13}, \nu_{20}, 2\nu_{23}, \nu_{26}\}$.

As $H(\lambda_0)=H(\lambda)=\nu_{25}^2$, we obtain

$$\begin{aligned} &\lambda \equiv \lambda_0 \mod E\pi_{30}^{12} = \{\xi_{13}, 2\lambda, \eta_{13}\circ\bar{\mu}_{14}\}, \\ &H(\lambda) = H(\lambda_0) = \nu_{25}^2. \end{aligned} \tag{6.1}$$

Lemma 6.1.

There exists an element α of π_{26}^5 such that $E\alpha\equiv\eta_6\circ\bar{\kappa}_7 \mod \nu_6^2\circ\pi_{27}^{12} + \pi_{13}^6\circ\kappa_{13}$, $H(\alpha)=\nu_9\circ\kappa_{12}$ and $2\alpha=0$.

Proof.

We choose an element α from $\{\nu_5^2, 2\iota_{11}, \kappa_{11}\}$. We have

$$\begin{aligned} H(\alpha) &\in H\{\nu_5^2, 2\iota_{11}, \kappa_{11}\} \\ &= -\Delta^{-1}(\nu_5^2\circ 2\iota_{11})\circ\kappa_{12} && \text{by Proposition 2.6 of [10]} \\ &\equiv \nu_9\circ\kappa_{12} \mod 2\nu\circ\kappa_{12} = 0 \end{aligned}$$

and

$$\begin{aligned} 2\alpha &\in \{\nu_5^2, 2\iota_{11}, \kappa_{11}\}\circ 2\iota_{26} \\ &= -\nu_5^2\circ\{2\iota_{11}, \kappa_{11}, 2\iota_{25}\} && \text{by Proposition 2.2} \\ &\ni -\nu_5^2\circ\kappa_{11}\circ\eta_{25} && \text{by Corollary 3.7 of [10]} \\ &= \nu_5^2\circ\eta_{11}\circ\kappa_{12} && \text{by (10.23) of [10]} \\ &= 0 && \text{by (5.9) of [10].} \end{aligned}$$

That is $2\alpha\equiv 0 \mod \nu_5^2\circ 2\pi_{26}^{11} + \nu_5^2\circ\pi_{26}^{11}\circ 2\iota_{26} = 0$.
By the definition of $\bar{\kappa}_9$ we have

$$\begin{aligned} \eta_8\circ\bar{\kappa}_9 &\in \eta_8\circ\{\nu_9, \mathrm{Ext}(\eta_{12}), \mathrm{Coext}(\kappa_{13})\} \\ &= \{\eta_8, \nu_9, \mathrm{Ext}(\eta_{12})\}\circ\mathrm{Coext}(\kappa_{14}) && \text{by Proposition 2.2} \\ &\subset \{\nu_8^2, 2\iota_{14}, \kappa_{14}\}, \end{aligned}$$

since we have by Lemma 5.12 of [10]

$$\{\eta_8, \nu_9, \mathrm{Ext}(\eta_{12})\} | S^{14} = \{\eta_8, \nu_9, \eta_{12}\} = \nu_8^2 .$$

Therefore we obtain

$$\begin{aligned} \eta_7 \circ \bar{\kappa}_8 &\in \{\nu_7^2, 2\iota_{13}, \kappa_{13}\} \\ &\supset E\{\nu_6^2, 2\iota_{12}, \kappa_{12}\} . \end{aligned}$$

In the exact sequence

$$\pi_{29}^{13} \xrightarrow{\Delta} \pi_{27}^{6} \xrightarrow{E} \pi_{28}^{7} ,$$

we have by Proposition 12.20 and Lemma 5.14 of [10]

$$\Delta\pi_{29}^{13} = \{\Delta(\sigma_{13}\circ\mu_{20})\} = \{\Delta(\eta_{13}\circ\rho_{14})\} = \{(\Delta\circ H)(\sigma'\circ\rho_{14})\} = 0 .$$

So E is a monomorphism and we obtain

$$\eta_6\circ\bar{\kappa}_7 \in \{\nu_6^2, 2\iota_{12}, \kappa_{12}\} . \qquad \text{Q.E.D.}$$

Choose elements $\sigma^{*'''} \in \pi_{34}^{12}$, $\sigma^{*''} \in \pi_{36}^{14}$, $\sigma^{*'} \in \pi_{37}^{15}$ and $\sigma_{16}^{*} \in \pi_{38}^{16}$ from the following secondary compositions

$$\begin{aligned} \sigma^{*'''} &\in \{\sigma_{12}, \nu_{19}, \zeta_{22}\}_1 \\ \sigma^{*''} &\in \{\sigma_{14}, 8\sigma_{21}, \sigma_{28}\}_1 \\ \sigma^{*'} &\in \{\sigma_{15}, 4\sigma_{22}, \sigma_{29}\}_1 \end{aligned}$$

and

$$\sigma_{16}^{*} \in \{\sigma_{16}, 2\sigma_{23}, \sigma_{39}\}_1 .$$

Denote that $\sigma_n^* = E^{n-16}\sigma_{16}^*$ for $n \geq 16$ and $\sigma^* = E^{\infty}\sigma_{16}^*$

Lemma 6.2.

(1) $H(\sigma^{*'''}) \equiv \zeta_{23} \bmod 2\zeta_{23}$ *and* $8\sigma^{*'''} \equiv 4\sigma_{12}\circ\rho_{19} \bmod 8\sigma_{12}\circ\rho_{19}$

(2) $H(\sigma^{*''}) = \nu_{27}^3 + \eta_{27}\circ\varepsilon_{28}$ *and* $2\sigma^{*''} \equiv \rho_{14}\circ\sigma_{29} \bmod \sigma_{14}\circ E\pi_{35}^{20} + \pi_{29}^{14}\circ 2\sigma_{29}$

(3) $H(\sigma^{*'}) = \bar{\nu}_{29} + \varepsilon_{29}$ *and* $2\sigma^{*'} \equiv E\sigma^{*''} \bmod \sigma_{15}\circ E\pi_{36}^{21} + \pi_{30}^{15}\circ\sigma_{30}$

(4) $H(\sigma_{16}^*) \equiv \sigma_{31} \bmod 2\sigma_{31}$, $2\sigma_{16}^* \equiv E\sigma^{*'} \bmod \sigma_{16}\circ E\pi_{37}^{22} + \pi_{31}^{16}\circ\sigma_{31}$

and $16\sigma_{16}^* = 2\sigma_{16}\circ\rho_{23}$.

Proof.

By Proposition 2.6. and (7.21) of [10], we have

$$\begin{aligned} H(\sigma^{*'''}) &\in H\{\sigma_{12}, \nu_{19}, \zeta_{22}\}_1 \\ &= -\Delta^{-1}(\sigma_{11}\circ\nu_{18})\circ\zeta_{23} \\ &\equiv \zeta_{23} \qquad \bmod 2\zeta_{23} . \end{aligned}$$

And $$8\sigma^{*'''} \in \{\sigma_{12}, \nu_{19}, \zeta_{22}\}_1 \circ 8\iota_{34}$$
$$= \sigma_{12} \circ E\{\nu_{18}, \zeta_{21}, 8\iota_{32}\} \quad \text{By Proposition 2.2.}$$

Consider the stable secondary compositions $\langle \nu, \zeta, 8\iota \rangle$.
For an odd integer x,

$$x\langle 8\iota, \nu, \zeta \rangle = \langle 8\iota, \nu, \langle 8\iota, \nu, \sigma \rangle\rangle \quad \text{see p. 90 of [10]}$$
$$= \langle\langle 8\iota, \nu, 8\iota \rangle, \nu, \sigma \rangle + \langle 8\iota, \langle \nu, 8\iota, \nu \rangle, \sigma \rangle$$
$$= \langle 8\iota, 8\sigma, \sigma \rangle \quad \text{by Lemma 5.13 of [10] and } G_4 = 0$$
$$\ni 4\rho .$$

Next we have

$$\langle\langle 2\sigma, 8\iota, \nu \rangle, 8\iota, \nu \rangle - \langle 2\sigma, \langle 8\iota, \nu, 8\iota \rangle, \nu \rangle$$
$$- \langle 2\sigma, 8\iota, \langle \nu, 8\iota, \nu \rangle\rangle \equiv 0 \quad \bmod 0 ,$$

whence $\langle \zeta, 8\iota, \nu \rangle = 16\rho$.

By use of these relations and Jacobi identity for the stable secondary composition, we have

$$\langle 8\iota, \nu, \zeta \rangle + \langle \nu, \zeta, 8\iota \rangle + \langle \zeta, 8\iota, \nu \rangle \equiv 0 \quad \bmod 8G_{15} ,$$
so $$\langle \nu, \zeta, 8\iota \rangle \equiv \langle 8\iota, \nu, \zeta \rangle \equiv 4\rho \quad \bmod 8G_{15} .$$

Thus we have proved the relation

$$8\sigma^{*'''} \equiv 4\sigma_{12} \circ \rho_{19} \quad \bmod 8\sigma_{12} \circ \rho_{19} .$$

By the definition we have

$$H(\sigma^{*''}) \in H\{\sigma_{14}, 8\sigma_{21}, \sigma_{28}\}_1$$
$$= -\Delta^{-1}(\sigma_{13} \circ 8\sigma_{20}) \circ \sigma_{29} \quad \text{by Proposition 2.6 of [10]}$$
$$= \eta_{27}^2 \circ \sigma_{29} \quad \text{by (10.10) of [10]}$$
$$= \nu_{27}^3 + \eta_{27} \circ \varepsilon_{28} \quad \text{by Lemma 6.4 of [10].}$$

We have

$$2\sigma^{*''} \in \{\sigma_{14}, 8\sigma_{21}, \sigma_{28}\}_1 \circ 2\iota_{36}$$
$$\subset \{\sigma_{14}, 8\sigma_{21}, 2\sigma_{28}\} \quad \text{by Proposition 2.1}$$
$$\supset \{\sigma_{14}, 8\sigma_{21}, 2\iota_{28}\} \circ \sigma_{29} \quad \text{by Proposition 2.1}$$
$$\ni \rho_{14} \circ \sigma_{29} .$$

Thus $\rho_{14} \circ \sigma_{29} \equiv 2\sigma^{*''} \bmod \sigma_{14} \circ E\pi_{35}^{20} + \pi_{29}^{14} \circ 2\sigma_{29}$.
Similarly we have

$$
\begin{aligned}
H(\sigma^{*\prime}) &\in H\{\sigma_{15}, 4\sigma_{22}, \sigma_{29}\}_1 \\
&= -\Delta^{-1}(\sigma_{14}\circ 4\sigma_{21})\circ\sigma_{30} && \text{by Proposition 2.6 of [10]} \\
&= \eta_{29}\circ\sigma_{30} && \text{by (10.10) of [10]} \\
&= \bar{\nu}_{29}+\varepsilon_{29} && \text{by Lemma 6.4 of [10],} \\
2\sigma^{*\prime} &\in \{\sigma_{15}, 4\sigma_{22}, \sigma_{29}\}_1\circ 2\iota_{37} \\
&\subset \{\sigma_{15}, 8\sigma_{22}, \sigma_{29}\}_1 && \text{by Proposition 2.1} \\
&\supset E\{\sigma_{14}, 8\sigma_{21}, \sigma_{28}\}_1 \\
&\ni E\sigma^{*\prime\prime}.
\end{aligned}
$$

So we obtain $E\sigma^{*\prime\prime}\equiv 2\sigma^{*\prime} \bmod \sigma_{15}\circ\pi_{37}^{22}+\pi_{30}^{15}\circ\sigma_{30}$.

$H(\sigma_{16}^{*})\equiv\sigma_{31} \bmod 2\sigma_{31}$ is obtained in the similar way by use of Proposition 2.6 of [10].

We have

$$
\begin{aligned}
2\sigma_{16}^{*} &\in \{\sigma_{16}, 2\sigma_{23}, \sigma_{30}\}_1\circ 2\iota_{31} \\
&\subset \{\sigma_{16}, 4\sigma_{23}, \sigma_{30}\}_1 && \text{by Proposition 2.3} \\
&\supset E\{\sigma_{15}, 4\sigma_{22}, \sigma_{29}\}_1 \\
&\ni E\sigma^{*\prime},
\end{aligned}
$$

whence $2\sigma_{16}^{*}\equiv E\sigma^{*\prime} \bmod \sigma_{16}\circ E\pi_{37}^{22}+\pi_{31}^{16}\circ\sigma_{31}$
$= \{\sigma_{16}\circ\rho_{23}, \sigma_{16}\circ\bar{\varepsilon}_{23}, \rho_{16}\circ\sigma_{31}, \Delta\sigma_{33}\}$.

Also we have

$$
\begin{aligned}
16\sigma_{16}^{*} &\ni \{\sigma_{16}, 2\sigma_{23}, \sigma_{30}\}_1\circ 16\iota_{38} \\
&= \sigma_{16}\circ E\{2\sigma_{22}, \sigma_{29}, 16\iota_{36}\}_1 \\
&\subset \sigma_{16}\circ\{2\sigma_{23}, \sigma_{30}, 16\iota_{37}\}_1 \\
&\subset \sigma_{16}\circ\{\sigma_{23}, 2\sigma_{30}, 16\iota_{37}\}_1 && \text{by Proposition 2.1} \\
&\supset 2\sigma_{16}\circ\{\sigma_{23}, 2\sigma_{30}, 8\iota_{37}\}_1 && \text{by Proposition 2.1} \\
&\ni 2\sigma_{16}\circ\rho_{23},
\end{aligned}
$$

where the indeterminacy is $\sigma_{16}^2\circ E\pi_{37}^{29}+\sigma_{16}\circ\pi_{38}^{23}\circ 16\iota_{38}=0$,

i.e., $16\sigma_{16}^{*}=2\sigma_{16}\circ\rho_{23}$.

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