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Extremal length and Kuramochi boundary

Dedicated to Professor A. Kobori on his 60th birthday

By

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1. Introduction

The notion of extremal length was first introduced by Ahlfors and Beurling [1] and various equivalent or extended definitions of extremal length were considered by J. Hersch [4], J. Jenkins [5] and others. Let Γ be a family of locally rectifiable curves given on a domain D of a Riemann surface R, and $\rho(z)|dz|$ be a conformal metric on D with non-negative covariant $\rho(z)$ such that $\int_\gamma \rho|dz|$ is defined (possibly ∞) for all $\gamma\in\Gamma$. According to Jenkins, $\rho(z)$ will be said to be admissible for the problem of extremal length of Γ (or briefly "admissible") when $A(\rho)=\iint_D \rho^2(z)dx\,dy\leqq 1$, and, putting $L_\rho(\Gamma)=\inf_{\gamma\in\Gamma}\int_\gamma \rho|dz|$ for each admissible $\rho(z)|dz|$, the extremal length of Γ is defined as the square of $\sup_\rho L_\rho(\Gamma)$ when the supremum is taken over all admissible conformal metrics. This definition of extremal length is equivalent to that of Ahlfors-Beurling and that of Hersch when $\rho(z)$ is limited in the same class of measurability.

By this definition of extremal length we know at once that extremal length of the family Γ_0 of all locally rectifiable curves whose ρ-length are infinite for a certain admissible conformal metric $\rho(z)|dz|$ is infinite, because $\lambda(\Gamma)^{1/2}=\sup_\rho L_\rho(\Gamma_0)\geqq L_\rho(\Gamma_0)=\infty$.

Especially, if we take the family Γ_1 of divergent curves in $R-K$ (K: compact domain in R) which start from ∂K and along which a function $f(z)$ $\left(\iint_{R-D}|f'|^2 dxdy<\infty\right)$ of the class C^1 has not finite limit, the extremal length of Γ_1 is infinite, because each curve of Γ_1 has infinite length when it is measured by the metric $\rho(z)|dz|=|f'(z)||dz|$ (Ohtsuka [7]).

Secondly, we consider a family Γ_2 of divergent curves and a meromorphic function $f(z)$ in R, for which $\iint_R\left(\frac{|f'|}{1+|f|^2}\right)^2 dxdy$ $=M(f)<\infty$ $(z=x+iy)$. The zero points of $f(z)$ in R are at most enumerable and do not cluster in R. And extremal length of the family of curves which pass through a zero point is infinite. So, extremal length of the family of all curves which pass through a certain zero point of $f(z)$ is infinite. For the remaining curves of Γ_2, the existence of non-zero limit of $f(z)$ along a curve is equivalent to the existence of finite limit of $\frac{1}{f(z)}$ along the curve. So, if $f(z)$ has not a limit along a curve c, $\int_c\left|\left(\frac{1}{f}\right)'\right||dz|=\int_c\frac{|f'|}{|f|^2}|dz|$ and $\int_c\frac{|f'|}{1+|f|^2}|dz|$ are infinite. This means that the curve c has infinite length measured by an admissible metric $\rho|dz|=\frac{1}{\sqrt{M(f)}}$ $\frac{|f'|}{1+|f|^2}|dz|$. Therefore, the family of such curves as c has infinite extremal length, that is, $f(z)$ has a limit along each curve except a family of curves with infinite extremal length.

Thirdly, we take a meromorphic function $w=f(z)$ for which $M(f)<\infty$ as above, and a point w_0 in the w-sphere at which $S(t)=\iint_{|w-w_0|<t}\left(\frac{|f'|}{1+|f|^2}\right)^2 dxdy=o\left(t^2\log\frac{1}{t}\right)$. The extremal length of the family Γ_3 of curves in w-sphere which converge to w_0 is infinite. Though extremal length of a family of curves is smaller in general than extremal length of the family of image-curves by an analytic mapping, the inverse image of Γ_3 in R by $w=f(z)$ has infinite extremal length by force of the condition $S(t)=o\left(t^2\log\frac{1}{t}\right)$. (cf. A. Pfluger [8]) This means that the extremal length of the

family of curves along which $\lim f(z)=w_0$ is infinite when $f(z)$ satisfies the above condition at w_0.

The above second and third results are, in a sence, extention of A. Beurling's theorem [2] to the case of a Riemann surface. Thereupon, we are going to study the behavior of the family of curves with infinite extremal length when the curves converge to an ideal boundary of a Riemann surface. That is, we want to know how the family of curves is related to the notion of "boundary" of a Riemann surface. For this purpose we consider, for the present, Kuramochi's compactification (completion) of a Riemann surface and its boundary, and study the continuity of extremal length at the boundary and the relation between extremal length and the capacity of Kuramochi boundary.

2. Kuramochi boundary and extremal length

Let R be an open Riemann surface with positive boundary and $\{R_n\}$ be its regular exhaution, and we suppose $R-R_0$ is connected. We consider a function $N_n(p, q)$ of p in R_n-R_0 which satisfies the following conditions.

(1) $N_n(p, q)$ is harmonic in $R_n-\bar{R}_0$ except a point $q\in R_n-\bar{R}_0$ in a neighborhood of which $N_n(p, q)+\log|p-q|$ is harmonic.[1]

(2) $N_n(p, q)$ is continuous and equals zero on ∂R_0.

(3) The normal derivative $\dfrac{\partial N_n(p, q)}{\partial n}=0$ on ∂R_n.

According to Kuramochi [6] we put

$$D^*_{R_n-R_0-v_r(q)}(N_{n+i}(p, q),\ N_n(p, q))$$
$$=\int_{\partial R_n+\partial R_0} N_{n+i}(p, q)\frac{\partial}{\partial n}N_n(p, q)ds+\int_{\partial v_r(q)}\{N_{n+i}(p, q)+\log|p-q|\}\frac{\partial}{\partial n}N_n(p, q)ds$$
$$=\int_{\partial v_r(q)}\{N_{n+i}(p, q)+\log|p-q|\}\frac{\partial}{\partial n}N_n(p, q)ds$$

1) We use the letter p, q not only to represent the point of R but also the uniformizing parameter at p, q.

where $v_r(q)$ is a disk centered at q and with radius r in a parametric disk of q.[2] We define $D^*_{R_n-R_0}\ (N_{n+i}(p, q), N_n(p, q))$ by

$$\lim_{r\to 0} D_{R_n-R_0-v_r(q)}(N_{n+i}(p, q), N_n(p, q))= \lim_{p\to q} 2\pi(N_{n+i}(p, q)+ \log |p-q|)$$

and $D^*_{R_n-R_0}\ (N_n(p, q))$ by

$$\lim_{r\to 0} D_{R_n-R_0-v_r(q)}(N_n(p, q)) = \lim_{p\to q} 2\pi(N_n(p, q)+ \log |p-q|)$$

Then, for $m\leqq n$

$$\begin{aligned}0\leqq D_{R_m-R_0}(N_n(p, q)-N_{n+i}(p, q))&\leqq D_{R_n-R_0}(N_n(p, q)-N_{n+i}(p, q))\\ &< 2\pi\, \{D^*_{R_n-R_0}(N_n(p, q))-D^*_{R_{n+i}-R_0}(N_{n+i}(p, q)).\end{aligned}$$[3]

And, since $N_n(p, q)\geqq G_m(p, q)$ in R_m-R_0,

$$\begin{aligned}D^*_{R_n-R_0}(N_n(p, q)) &= \lim_{p\to q} 2\pi(N_n(p, q)+ \log |p-q|)\\ &\geqq \lim_{p\to q} \{G_m(p, q)+ \log |p-q|\} = L>-\infty\,,\end{aligned}$$

where $G_m(p, q)$ is Green's function in R_m-R_0 with pole at q. From these inequalities $D^*_{R_n-R_0}(N_n(p, q))$ decreases monotonely when n increases and is lower bounded. So $\{D^*_{R_n-R_0}(N_n(p, q))\}_n$ converges and $\lim_{n\to\infty} D_{R_m-R_0}(N_n(p, q)-N_{n+i}(p, q))=0$, that is, $\{N_n(p, q)\}_n$ converges in mean in R_m-R_0. And since $N_n(p, q)=0$ on ∂R_0. Here, since m is arbitray, $\{N_n(p, q)\}_n$ converges uniformly on every compact set in $R-R_0$. We denote the limit function by $N(p, q)$ and call it N-Green's function of $R-R_0$ with pole at p.

Using this N-Green's function Kuramochi compactified R as following. Let $\{q_i\}$ be a sequence of points in $R-R_0$ which has no points of accumulation in $R-R_0$. When p stays in any compact set, $\{N(p, q_i)\}_i$ is, from some i on, a uniformly bounded sequence of harmonic function of p and it forms a normal family. A sequence $\{q_i\}$ of points of $R-R_0$ having no accumulation point in $R-R_0$, for which the corresponding $\{N(p, q_i)\}_i$ converge to a harmonic function, is called fundamental. Two fundamental sequences are called equivalent if their corresponding $N(p, q)$'s have the same

2) This modified Dirichlet integral $D^*(u, v)$ is equal to ordinary one when u, v are harmonic.

3) About this calculation, see Kuramochi [6] p. 6.

limit. This has the usual properties of an equivalence relation and the class of all fundamental sequences equivalent to a given one determines an ideal boundary point of R. The set of all such ideal boundary points are denoted by B.

We define a distance of $q_1, q_2 \in R-R_0$ as

$$\delta(q_1, q_2) = \sup_{p \in R_1 - R_0} \left| \frac{N(p, q_1)}{1+N(p, q_1)} - \frac{N(p, q_2)}{1+N(p, q_2)} \right|$$

By completion of $R-R_0$ as a metric space with the above metric we reach the above compact space $R-R_0+B$. Thus, $R-R_0+B$ is a complete metric space and the subspace $R-R_0$ is homeomorphic to itself with original topology. B is a closed subset of $R-R_0+B$.

Here, we prove the following proposition.

Proposition 1.[4] *The family* $\mathfrak{N}$ *of curves,* c*, which converge to the ideal boundary of* R *and each of which does not converge to any point of* B *has infinite extremal length.*

Proof. From the definition, we know $N(p, q)=N(q, p)$ for $p, q \in R-R_0$. So, for a fixed $p \in R_1-R_0$, $N(p, q)$ has finite Dirichlet integral over $R-R_1$ as a function of q. From the definition of δ-metric, $c \in \mathfrak{N}$ means that, for a certain $p \in R_1-R_0$, $N(p, q)$ has not a limit along c as a function of q, that is, there are at least two non-equivalent fundamental sequences on c. The family of such curves $\{c_p\}$, along each of which $N(p, q)$ has not limit, has infinite extremal length. And, by continuity (harmonicity) of $N(p, q)$ about p ($q \in R-R_1+B$), if $N(p, q)$ has not limit along c as a function of q for a certain p, then it has not a limit along c for all p' which are sufficiently close to p. Since R_1-R_0 is separable, we can choose countable number of points, $\{p_i\}$, which are dense in R_1-R_0, and corresponding family of curves $\{c_{p_i}\}$. For any $p \in R_1-R_0$, each curve of $\{c_p\}$ belongs to a certain $\{c_{p_i}\}$ from the above continuity. Thus, $\mathfrak{N}$ is the union of countable number of $\{c_{p_i}\}$'s and has infinite extremal length.[5]

4) This proposition was proved by M. Ohtsuka in a more general method [7].

5) $\frac{1}{\lambda \mathfrak{N}} \leq \sum_{i=1}^{\infty} \frac{1}{\lambda_i}$, where λ_i is the extremal length of $\{c_{p_i}\}$ (Ohtsuka [7]).

This proposition says, in other words, that each curve of a family with finite extremal length converges to a point of B except curves of a subfamily with infinite extremal length. We study, in the following, accessible points and capacity of sets of points of B.

3. Extremal length and the capacity of a set of Kuramochi boundary

Let A be a closed set in B and A_m be its neighborhood; $A_m=\left\{p\in R-R_0+B;\ \delta(p, A)<\frac{1}{m}\right\}$. And let $\omega_{m,n}$ be a harmonic function in $R-R_0-\bar{A}_m$[6] such that $\omega_{mn}=0$ on ∂R_0, $\omega_{mn}=1$ on $\partial A_m\cap R_n$ and $\frac{\partial\omega_{mn}}{\partial n}=0$ on ∂R_n-A_m. Then, the Dirichlet integral $D_{mn}(\omega_{mn})$ $=D_{R_n-R_0-A_m}(\omega_{mn})$ equals to the reciprocal $\frac{1}{\lambda_{mn}}$ of extremal length of the family Γ_{mn} of curves which join ∂R_0 and ∂A_m, and increases monotonely with n and has finite limit. And, the inequality $D_{mn}(\omega_{mn}-\omega_{m\,n+i})\leqq D_{m\,n+i}(\omega_{m\,n+i})-D_{mn}(\omega_{mn})$ holds. Thus, ω_{mn} converges uniformly on every compact set in $R-R_0-A_m$. We write $\omega_m=\lim_{n\to\infty}\omega_{mn}$. Then,

$$\lim_{n\to\infty} D_{mn}(\omega_{mn}) = D_{R-R_0-A_m}(\omega_m) = \frac{1}{\lambda_m},$$

where λ_m is extremal length of the family Γ_m of curves which join ∂R_0 and ∂A_m ([3] p. 1) We put $D_m(\omega_m)=D_{R-R_0-A_m}(\omega_m)$. Since λ_m increases monotonely with m, $D_m(\omega_m)$ decreases monotonely. According to Kuramochi, we call $\lim_{m\to\infty} D_m(\omega_m)$ capacity of the set A. If it is positive, ω_m converges to a harmonic function ω_A uniformly on every compact set in $R-R_0$ and $D_{R-R_0}(\omega_A)=\lim_{m\to\infty} D_m(\omega_m)$. When the family Γ_A of curves which start from ∂R_0 and converge with respect to the filter $\{A_m\}$ has finite extremal length λ_A, the capacity of A is positive because $\lim_{m\to\infty} D_m(\omega_m)=\lim_{m\to\infty}\frac{1}{\lambda_m}\geqq\frac{1}{\lambda_A}>0$.

Proposition 2. *Let A be a closed set in B with zero capacity.*

6) We suppose $R_0\cap A_m=\phi$ and $A_m\cap R_n\neq\phi$, and we take as the domain of ω_{mn} the connected component which has common boundary with R_0.

Then, extremal length of the family Γ_A^* *of curves*[7] *which separate* A *and* ∂R_0 *in* $R-R_0$ *is zero.*

Proof. We consider the family $\Gamma_0^*=\{\gamma\}$ of level curves of ω_A, i.e. $\omega_A=c$ on γ for $0\leqq c<1$. This is a subfamily of the family Γ_A^*. By the definition of extremal length λ_0^* of Γ_0^*, there exists, any $\varepsilon>0$, an admissible metric $\rho|dz|$ for which $\lambda_0^*-\varepsilon<\left(\int_\gamma \rho|dz|\right)^2 =\left(\int_\gamma \rho d\omega_A^*\right)^2$, where ω_A^* is a conjugate harmonic function of ω_A. We take $\omega_A+i\omega_A^*$ as a local parameter. Then, by Schwarz inequality

$$\left(\int_\gamma \rho d\omega_A^*\right)^2 \leqq \int_\gamma d\omega_A^* \int_\gamma \rho^2 d\omega_A^* = D_{R-R_0}(\omega_A)\int_\gamma \rho^2 d\omega_A^*,^{8)}$$

and integrating with respect to ω_A, we get

$$\lambda_0^*-\varepsilon = (\lambda_0^*-\varepsilon)\int_0^1 d\omega_A \leqq D_{R-R_0}(\omega_A)\iint_{R-R_0} \rho^2 d\omega_A^* d\omega_A \leqq D_{R-R_0}(\omega_A).$$

ε being arbitrary, we have $\lambda_0^*\leqq D_{R-R_0}(\omega_A)$.
And since $\Gamma_0^*\subset\Gamma_A^*$, extremal length λ_A^* of Γ_A^* is smaller than λ_0^*. Therefore, if $D_{R-R_0}(\omega_A)=0$, then $\lambda_A^*=0$.

Corollary. *If a point* p *of* B *has zero capacity, there exists a family of curves in* $R-R_0$ *which separate* R_0 *and the point* p, *and whose* ρ*-length converges to zero for all admissible metric* $\rho|dz|$.

Proposition 3. *For a closed subset* A *of* B *we have*

$$\lambda_A = \lim_{m\to\infty}\lambda_m = \frac{1}{D_{R-R_0}(\omega_A)}$$

Proof. Since we have already shown that $D_{R-R_0}(\omega_A) = \lim_{m\to\infty}\frac{1}{\lambda_m}\geqq\frac{1}{\lambda_A}$, it is sufficient to prove the opposite inequality. If $D_{R-R_0}(\omega_A)=0$ the proposition is trivial, so we suppose $D_{R-R_0}(\omega_A)$

7) Each curve of Γ_A^* consists of countable number of connected components, and we suppose Γ_A^* contains level curves $\omega_A=c$.

8) $\int_\gamma d\omega_A^*=D_{R-R_0}(\omega_A)$ for almost all c of the interval $(0,1)$ (See Kuramochi [6] p. 14.)

$=\lim_{m\to\infty} D_m(\omega_m)>0$ and denote it by d. First, we consider the extremal length λ_0 of the family Γ_0 of level curves, $\gamma: \omega_A^*=c$, where ω_A^* is a conjugate harmonic function of ω_A. For any $\varepsilon>0$, there exists, from the definition of λ_0, a metric $\rho|dz|$ such that $\iint_{R-R_0}\rho^2 dxdy\leqq 1$ and $\left(\int_\gamma \rho|dz|\right)^2>\lambda_0-\varepsilon$. Taking $\omega_A+i\omega_A^*$ as a local parameter, we have

$$\lambda_0-\varepsilon<\left(\int_\gamma \rho d\omega_A\right)^2\leqq\int_\gamma d\omega_A\int_\gamma \rho^2 d\omega_A\leqq\int_\gamma \rho^2 d\omega_A$$

except a possible subfamily of Γ_0 with infinite extremal length. Integrating both sides of the inequality with respect to $d\omega_A^*$, we have

$$(\lambda_0-\varepsilon)\int_{\partial R_0} d\omega_A^*=d(\lambda_0-\varepsilon)\leqq\iint_{R-R_0}\rho^2 d\omega_A d\omega_A^*\leqq 1$$

Since ε is arbitrary we have $\lambda_0\leqq\frac{1}{d}$. And $\Gamma_0\subset\Gamma_A$ except a set of curves with infinite extremal length, because, ω_A has not value 1 on a closed set of a positive capacity except A ([6] p. 55). So, $\lambda_A\leqq\lambda_0$ and

$$\lambda_A\leqq\frac{1}{d}=\frac{1}{D_{R-R_0}(\omega_A)},$$

and we complete the proof.

By this proposition, we can define the capacity of a closed set A by the reciprocal of extremal length λ_A instead of $D_{R-R_0}(\omega_A)$.

From proposition 1 and 3, we have

Proposition 4. *Let A be a closed subset of B with positive capacity, then A contains at least one accessible point.*

4. An application

We divide the boundary B into two parts. One of them is the set B_a of all accessible points of B, and the other is the set B_0 of all non-accessible points. The proposition 4 shows that the set B_0 is of inner capacity zero, that is, B_0 does not contain any closed set with positive capacity. If a point $p \in B_a$, there exists a curve γ converging to the point p and connected components, $v_n(\gamma)$, of the neighborhoods, $V_n(p)$ of p. And the system $\{v_n(\gamma)\}$ satisfies the conditions such that $v_n(\gamma) \supset v_{n+1}(\gamma)$, $\bigcap \overline{v_n(\gamma)} \supset p$ and each of $\{v_n(\gamma)\}$ contains an end part of γ from a point of γ on.

We consider a Riemann surface R whose boundary B does not contain a point whose capacity is positive. Kuramochi calls such a point "singular". Let $w=f(z)$ be a meromorphic function on R such that $\iint_R \left(\frac{|f'|}{1+|f|^2}\right)^2 dxdy < \infty$. We put $M(p, \gamma)=\bigcap_n \overline{f(v_n(\gamma))}$, then, it is either a continuum or one point. We denote by $\delta M(p, \gamma)$ the diameter of $M(p, \gamma)$ measured in w-sphere by the spherical metric. Let C be a disk on w-sphere and $f^{-1}(C)$ be its inverse image in R consisting of countable number of connected components. From the definition of $\delta M(p, \gamma)$, we have easily

Lemma 1. *If diameter of C, $\delta(C) < \delta_0$, there does not exist a pair of a point p of B and a curve γ converging to p such that $\delta M(p, \gamma) > \delta_0$ and $v_n(\gamma)$ are contained, from a certain number n_0 on, in one component of $f^{-1}(C)$.*

Let $\{C_{n,i}\}$ be a set of spherical disks in w-sphere with radius $\frac{1}{n}$ which cover w-sphere and such that any disks with radius $\frac{1}{3n}$ is contained in a certain one of them. We denote by $T_{n,i}$ the set of points, p, of B_a such that any component of $f^{-1}(C_{n,i})$ does not contain $v_m(\gamma)$ completely for any m and γ converging to p, and by S_n the set of points, p, of B_a such that $\delta M(p, \gamma) \geqq \frac{1}{n}$ for all γ converging to p.

Lemma 2. $S_n \subset \bigcap_i T_{3n,i}$.

Proof. Let p be a point of S_n. If $f^{-1}(C_{3n,i})$ contains a subsequence $\{v_m(\gamma)\}$ $(m > m_0)$ for a path γ converging to p, then we have $\delta M(p, \gamma) \geqq \frac{1}{n}$ by the definition of S_n. But $\delta(C_{3n,i}) = \frac{1}{3n} < \frac{1}{n}$. This contradicts to lemma 1, and p must belong to $T_{3n,i}$. This is valid for all i, so we have $S_n \subset \bigcap_i T_{3n,i}$, q.e.d.

From lemma 2, $\bigcup_n S_n \subset \bigcup_n \bigcap_i T_{3n,i}$. Now we shall prove the following theorem which is, in a sence, an extension of Beurling's theorem to the case of a Riemann surface.

Theorem. *At each accessible point, except the union of countable number of sets of accessible points whose inner capacity is zero, there exists a contracting system of components of the neighborhood for which the cluster set of meromorphic function $f(z)$ such that $\iint_R \left(\frac{|f'|}{1+|f|^2}\right)^2 dxdy < \infty$ is one point.*

Proof. We suppose that there exists a number n_0 such that $S_{n_0/3}$ contains a closed set F of positive capacity. Then $\bigcap_i T_{n_0,i}$ also contains F. Let $\{C_i\}$ be a set of disks with radius $\frac{1}{5n_0}$ and $\bigcup_i$ (interior of C_i) covers w-sphere. We denote by G_{ij} components of $f^{-1}(C_i)$. Then, since $\bigcup_{ij} G_{ij} \supset R-R_0$ $\sum_{ij} \omega_{\bar{G}_{ij} \cap F} \geqq \omega_F$ we can find a $G_{ij}=G$ such that $\omega_{F \cap \bar{G}}>0$. Then, $F \cap \bar{G}$ contains a poiut p of B_a. Let $\{C'_i\}$ be disks, C'_i, with radius $\frac{2}{5n_0}$ and centered at the center of C_i. We choose, in the same way as above, a component G' of $\{f^{-1}(C'_i)\}_i$ such that $\omega_{F \cap \bar{G}'}>0$ and G' contains G. Since the extremal length of the family Γ' of curves which separate ∂C_i and $\partial C'_i$ is finite $\left(=\frac{2\pi}{\log 2}\right)$, the extremal length of the family Γ of curves which separate ∂G and $\partial G'$ is finite, because each curve of Γ' contains an image of a curve of Γ by an analytic mapping $w=f(z)$. Hence, Γ contains either a closed curve γ_0 in $G'-G$, or a curve whose end parts converge to accessible points p_1 and p_2 of $\overline{G'-G} \cap B$. In the first case, we have $\delta(p, \partial G')>\delta(p, \gamma_0)>0$, and $V_m(p) \subset G'$ for $m>1/\delta(p, \partial G')$. In the second case, if $p_i \neq p$ $(i=1, 2)$, then $\delta(p, \partial G')>0$. We suppose $p_1=p$, and consider a family Γ^* of curves which join ∂G and $\partial G'$. Each curve of Γ^* is an inverse image of a curve of a family $\Gamma^{*\prime}$ which joins ∂C_i and $\partial C'_i$ by an analytic function $f(z)$ such that $M(f)=\iint_R \left(\frac{|f'|}{1+|f|^2}\right)^2$

$dxdy<\infty$, and the extremal length of $\Gamma^{*\prime}$ is positive $\left(\geqq\frac{1}{2\pi}\log 2\right)$. So the extremal length of the family Γ^* is also positive.[9] But, since B does not contain a singular point, the extremal length of the family of curves which separate p and ∂R_0 is zero by Proposition 2. Therefore, we can choose, for the point p, a curve γ converging to p and a system $\{v_m(\gamma)\}$ which is contained in G' from some number m_0 on, because if $\partial G'$ always meets $v_m(\gamma)$, the extremal length of Γ^* must be zero. In the case of $p_2=p$, we have the same result. Thus, in all case we can find, for the point p, a curve γ converging to p and a system $\{v_m(\gamma)\}$ contained in G' from some m_0 on.

Therefore, for such $\{v_m(\gamma)\}$, $\delta M(p,\gamma)\leqq\frac{4}{5n_0}$ by lemma 1. But, since $p\in S_{n_0/3}$ $\delta M(p,\gamma)\geqq\frac{3}{n_0}$. This is a contradiction, and we conclude the theorem.

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9) Since $\iint_{f^{-1}(c_i'-c_i)\cap R}\left(\frac{|f'|}{1+|f|^2}\right)^2 dxdy\leqq M(f)<\infty$,

the metric $\rho_0|dz|=\frac{1}{\sqrt{M(f)}}\frac{|f'|}{1+|f|^2}|dz|$ is admissible for Γ^*.

And $\inf_{\gamma\in\Gamma^*}\int_\gamma \rho_0|dz|=\inf_{\gamma'\in\Gamma^{*\prime}}\frac{1}{\sqrt{M(f)}}\int_{\gamma'}\frac{|dw|}{1+|w|^2}\geqq\frac{1}{\sqrt{M(f)}}\left(\mathrm{Tan}^{-1}\frac{2}{5n_0}-\mathrm{Tan}^{-1}\frac{1}{5n_0}\right)>0$.