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On the generalized Dirichlet problem for plurisubharmonic functions[1]

Dedicated to Professor A. Kobori on the occasion of his sixtieth birthday

By

Yukio KUSUNOKI

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Introduction. With the Perron's method in the classical potential theory Bremerman [8] first treated the Dirichlet problem for plurisubharmonic functions. The base domains D considered were mainly bounded domains of holomorphy of the form $\{z;\ V(z)<0\}$ where V is plurisubharmonic on the closure of D. His lower solution in D does not necessarily attain the boundary values even if the boundary and boundary function are nice. In fact he showed that the lower solution attains (in his sense) the continuous boundary value only if it is prescribed on the Silov boundary $S(D)$ of D. A generalization of this result was given by Górski [10] for more general domain D and $S^*(D)$ (Silov boundary with respect to plurisubharmonic functions (see Siciak [16])).

In this paper we shall study further such a generalized Dirichlet problem with various applications to functions of several complex variables and plurisubharmonic functions. First in §1, for given boundary function f we define the plurisubharmonic lower solution $\underline{T}f$ and plurisuperharmonic upper solution $\bar{T}f$ without regard to the Silov boundary of the base domain and introduce the notion

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of pl-barrier (sec. 1. 2) at a boundary point to see the boundary behavior of these solutions. If a pl-barrier exists at a point q, then our solutions actually attain the boundary value $f(q)$ provided that f is continuous at q and bounded on the whole boundary (Corollary 1. 1). Relations between Bremerman's solutions and ours are discussed in sec. 1. 4. The existence of pl-barrier requires a severe restriction for the local shape of boundary and has close connections with the pseudo-convexity in several complex variables. For instance if the boundary is locally flat at q, pl-barrier does not exist at q (Theorem 3. 1). Theorem 3. 4 shows that if every boundary point of D possesses pl-barrier, then D must be a domain of holomorphy. Some criteria for the existence of pl-barriers will be given in sec. 3. 1.

In §2 the maximum principle and removable sets for plurisubharmonic functions are discussed. A set E on the boundary of a bounded domain $D \subset C^n$ is called inner (outer) pl-measure zero if our lower (upper) solution in D for the characteristic function of E vanishes identically. Sets of inner pl-measure zero characterize the maximum principle for plurisubharmonic functions (Theorem 2. 1). Some examples show that the sets of inner pl-measure zero are not "small" as in the case $n=1$, namely sets of real dimension $2n-1$ $(n>1)$ happen to be of inner pl-measure zero. In sec. 2. 2 a characterization for the set of outer pl-measure zero and a theorem of Riesz type will be given. We define in sec. 2. 3 another notion, the pl-removability. According to Grauert-Remmert [11] an analytic set of codimension 1 is of pl-removable. Their proof is also applicable for a non-analytic set with the same dimension and we show that a hypersphere (surface) in $R^{2n-1} \subset C^n$ is of pl-removable. In terms of pl-removability one can state a generalization (Theorem 2. 9') of the well known theorem on removable sets for holomorphic functions.

Sec. 3. 3 contains some applications of above results. We shall give an example showing that the Bochner-Martin's conjecture is false for a domain of inholomorphy. Such an example was first given by Bremerman [5] in a tube domain. He gave another example in a bounded domain (Reinhart region) [6]. Our domain is "nearly" shell and the reasoning is very simple from our point of view. It is noted (Theorem 3. 5) that we cannot take a shell itself as our example, as in the ring domain in $C^n(n>1)$ there does not exist any plurisubharmonic function which attains the boundary value 1 and 0 resp. on the inner and outer boundary.

Finally in § 4 we mention about the generalization of our Dirichlet problem over complex manifolds and discuss on some classes of complex manifolds as a generalization of the classification of open Riemann surfaces (cf. Ahlfors and Sario [1] Chap. IV) As a result different from the case $n=1$ it is noteworthy that in the exterior of a ball (or polydisc) in $C^n(n\geq 2)$ there does not exist any non-constant plurisubharmonic function bounded above (Example III 5)).

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§ 1. Generalized Dirichlet problem

1. 1. Let D be a domain in the space C^n of n complex variables. A real valued function u on D is called *plurisubharmonic* if the following conditions are fulfilled: (a) $-\infty\leq u(z)<\infty$ (b) u is upper semi-continuous (c) the restriction of u to any analytic plane $E=\{z=(z_1,\cdots,z_n);\ z_i=z_i^0+a_i t\ (i=1,\cdots,n)\}$ is subharmonic in $E\cap D$. A function v is called *plurisuperharmonic* if $-v$ is plurisubharmonic. We denote by $\underline{P}(D)$ ($\bar{P}(D)$) the set of plurisubharmonic (plurisuperharmonic) functions in D. For a real valued function f on ∂D (the boundary of D) we denote by $\underline{\mathcal{F}}(f, D)$ ($\overline{\mathcal{F}}(f, D)$) the set of functions $u\in\underline{P}(D)$ ($v\in\bar{P}(D)$) satisfying the boundary conditions:

$$(1)\qquad \overline{\lim_{z\to\zeta}}\, u(z)\leq f(\zeta)\qquad (\text{resp. } \underline{\lim_{z\to\zeta}}\, v(z)\geq f(\zeta))$$

where $z\in D$ and $\zeta\in\partial D$. In the following D is assumed to be bounded, unless otherwise stated. Now we set

$$(2)\qquad \underline{T}f(z)=\underline{T}_D f(z)=\limsup_{z'\to z}\,[\sup_{u\in\underline{\mathcal{F}}(f,D)} u(z')]$$

It is known that $\underline{T}f \in \underline{P}(D)$ if f is bounded above. Replacing $\underline{P}(D)$ by the set of subharmonic functions in $D \subset C^n = R^{2n}$ we get the usual Perron's lower envelope $\underline{H}f$ in place of $\underline{T}f$, where we need not take "limsup" on account of the harmonicity of $\underline{H}f$. Let $\bar{H}f$ be the Perron's upper envelope, then we know

$$\inf_{\partial D} f \leq \underline{T}f(z) \leq \underline{H}f(z) \leq \bar{H}f(z) \leq \sup_{\partial D} f, \quad z \in D \tag{3}$$

We can analogously define the function $\bar{T}f(z)$ by replacing $\underline{\mathfrak{F}}(f, D)$, limsup and sup in (2) by $\bar{\mathfrak{F}}(f, D)$, liminf and inf respectively. $\bar{T}f \in \bar{P}(D)$ if f is bounded below. We have

$$\inf_{\partial D} f \leq \underline{H}f(z) \leq \bar{H}f(z) \leq \bar{T}f(z) \leq \sup_{\partial D} f, \quad z \in D \tag{4}$$

$$\bar{H}f = -\underline{H}(-f), \quad \bar{T}f = -\underline{T}(-f) \tag{5}$$

In connection with the functiontheory of several complex variables our main concern is in the functions $\underline{T}f$ and $\bar{T}f$, but above inequalities will give ous useful estimates.

1. 2, To see the boundary behavior of $\underline{T}f$ and $\bar{T}f$ we introduce the notion of plurisubharmonic barrier as follows. Let ζ_0 be a point of ∂D. Suppose there exists a neighborhood N of ζ_0 and a function $v(z)$ defined in $D \cap N$ satisfying the following conditions: for given positive numbers ε, α and a neighborhood $N'(\subset N)$ of ζ_0 we have

$$\begin{gathered} \lim_{z \to \zeta_0} v(z) = 0 \quad z \in D \cap N \\ v(z) \leq \varepsilon, \ z \in D \cap N' \quad \text{and} \quad v(z) \leq -\alpha, \ z \in D \cap N - N' \end{gathered} \tag{6}$$

Then we call $v(z)$ a *pl-barrier resp. barrier at* ζ_0 *with respect to* D according as v is plurisubharmonic or subharmonic. We note that if a usual (superharmonic) barrier $\omega > 0$ (e.g. Petrovski [15] § 31) exists, $c\omega$ with a suitable negative constant c is a barrier in our sense. Evidently *pl*-barrier is a barrier at the same point, but the existence of barrier does not necessarily imply the existence of *pl*-barrier which will be seen later. Now if *pl*-barrier (or barrier) would exist at ζ_0, we can extend it globally, namely for the *pl*-barrier $v(z)$ there exists a function $\hat{v}(z) \in \underline{P}(D)$ having the properties:

$$
(7)\qquad \begin{gathered} \lim_{z\to\zeta_0} \hat{v}(z) = 0 \quad z\in D \\ \hat{v}(z)\leq \varepsilon,\ z\in D\cap N' \quad \text{and} \quad v(z)\leq -\alpha,\ z\in D-N' \end{gathered}
$$

Indeed, let

$$
\hat{v}_n(z) = \begin{cases} \max(v(z),\ -\alpha+1/n), & z\in D\cap N' \\ -\alpha+1/n, & z\in D-N' \end{cases}
$$

and $\hat{v}(z)=\lim_{n\to\infty}\hat{v}_n(z)$, then it is seen that $\hat{v}(z)$ satisfies our conditions.

1. 3. THEOREM 1. *Let D be a bounded domain in C^n and f be bounded above on ∂D. If there exists a barrier at $\zeta_0\in\partial D$, then*

$$
(8)\qquad \overline{\lim_{z\to\zeta_0}}\ \underline{T}f(z)\leq \overline{\lim_{z\to\zeta_0}}\ \bar{H}f(z)\leq \overline{\lim_{\zeta\to\zeta_0}}\ f(\zeta),\ z\in D,\ \zeta\in\partial D
$$

Further if there exists a pl-barrier at ζ_0, then

$$
(9)\qquad \overline{\lim_{z\to\zeta_0}}\ \bar{T}f(z)\leq \overline{\lim_{\zeta\to\zeta_0}}\ f(\zeta)
$$

PROOF. We shall prove the inequality (9) only. Set $A=\overline{\lim_{\zeta\to\zeta_0}} f(\zeta)$. We may assume $A<M=\sup f$ on ∂D, because if $A=M$, (9) is already valid by (4).

Case 1. $-\infty<A$. Let α be a number $>2(M-A)>0$. For any $\varepsilon>0$ we choose a sufficiently small neighborhood N' of ζ_0 such that $N'\subset N$ and $f(\zeta)<A+\varepsilon/2$, $\zeta\in N'\cap\partial D$ where N is the domain of *pl*-barrier at ζ_0. Then we take a globally defined *pl*-barrier $\hat{v}\in\underline{P}(D)$ at ζ_0 satisfying the condition (7). Now it is easily checked that the plurisuperharmonic function

$$
(10)\qquad w(z) = A+\varepsilon-v(z)(M-A)/\alpha
$$

belongs to the class $\bar{\mathfrak{F}}(f, D)$. Hence $\bar{T}f(z)\leq w(z)$. Letting $z\to\zeta_0$, we have $\hat{v}(z)\to 0$ and $\overline{\lim_{z\to\zeta_0}}\ \bar{T}f(z)\leq A+\varepsilon$, which implies (9).

Case 2. $A=-\infty$. Choose $\alpha>2\varepsilon^2(M'+1/2\varepsilon)$, $M'>\max(M, 0)$ and $N'\subset N$ such that $f(\zeta)<-1/\varepsilon$, $\zeta\in N'\cap\partial D$. Now instead of (10) we have merely to consider the function

$$
-1/2\varepsilon-v(z)(M'+1/2\varepsilon)/\alpha
$$

where $\hat{v}$ is a globally defined *pl*-barrier corresponding those α and N', q.e.d.

A boundary point ζ_0 will be called *pl-regular* (*regular*) if there exists a *pl*-barrier (barrier) at ζ_0. Then we have by (4) (5) (8) and (9) the following

COROLLARY 1.1. *Let f be bounded on ∂D and continuous at $\zeta_0 \in \partial D$. If ζ_0 is pl-regular (resp. regular), then the following* (11), (12) (*resp.* (12)) *hold*:

$$\lim_{z\to\zeta_0} \underline{T}f(z) = \lim_{z\to\zeta_0} \bar{T}f(z) = f(\zeta_0) \qquad z \in D \tag{11}$$

$$\lim_{z\to\zeta_0} \underline{H}f(z) = \lim_{z\to\zeta_0} \bar{H}f(z) = f(\zeta_0) \qquad z \in D \tag{12}$$

COROLLARY 1.2. *Let f be bounded above on ∂D and upper semi-continuous at $\zeta_0 \in \partial D$. If ζ_0 is regular and $f(\zeta_0) = \inf_{\partial D} f(\geq -\infty)$, then $\underline{T}f(z) \to f(\zeta_0)$, $z \to \zeta_0$.*

Thus we know that if f is continuous on ∂D and every boundary point is *pl*-regular (regular), then $\underline{T}f$ and $\bar{T}f$ ($\underline{H}f$ and $\bar{H}f$) attain the boundary value f on ∂D. It is then noted that $\underline{H}f$ and $\bar{H}f$ coincide by maximum principle and they give a unique harmonic solution of the classical Dirichlet problem, however $\underline{T}f$ and $\bar{T}f$ are not necessarily identical each other (see sec. 3.1). $\underline{T}f$ and $\bar{T}f$ are identical, hence pluriharmonic if and only if they take the same value at a point of D.

As other elementary properties of the operator $\underline{T}$ we have

(13)

$\underline{T}(cf) = c\underline{T}f$ for $c \geq 0$, $\underline{T}(cf) = c\bar{T}f$ for $c \leq 0$

$\underline{T}(f+c) = \underline{T}f + c = \underline{T}f + \underline{T}c$ (c: const.)

$\underline{T}f + \underline{T}g \leq \underline{T}(f+g)$ (subadditive)

that is, $\underline{T}$ is not necessarily linear (see Remark (ii), sec. 2.1)

$|\underline{T}f - \underline{T}g| \leq \sup_{\partial D} |f-g|$

1.4. Here we shall compare our upper and lower solutions with those of Bremerman [8]. His procedure is as follows:

(α) D is a bounded domain of holomorphy such that $D = \{z; V(z) < 0\}$ where V is continuous and plurisubharmonic on the closure $\bar{D}$.

(β) Boundary function f is given on the Silov boundary $S(D)$ of D and continuous there.

(γ) Lower solution $\underline{u}f$ is defined by (2) where $\underline{\mathfrak{F}}(f, D)$ should be replaced by the set $\mathfrak{L}(f, D)$ of functions which are plurisubharmonic on $\bar{D}$ and are smaller or equal f on $S(D)$.

Under these circumstances we know ([8]) that for any $v \in \mathfrak{L}(f, D)$ $v(z) \leq \sup\limits_{S(D)} f = M$, hence $\underline{u}f \leq M$, which implies that $\underline{u}f$ can not take on $\partial D - S(D)$ any prescribed value greater than M. Now let $\hat{f}$ be a continuous function on ∂D which is equal f on $S(D)$. If $\hat{f}(\zeta) \geq M$, $\zeta \in \partial D - S(D)$, every element of $\mathfrak{L}(f, D)$ belongs to $\underline{\mathfrak{F}}(f, D)$ hence

$$\underline{u}f(z) \leq \underline{T}\hat{f}(z) \leq \bar{T}\hat{f}(z) \leq \bar{u}f(z), \qquad z \in D.$$

While in case $\hat{f} \leq M$, $\underline{T}\hat{f}$ can be actually smaller than $\underline{u}f$. Indeed, the following example shows that

(i) for the constant function $f = M$ on $S(D)$ $\underline{u}f \equiv M$ while $\underline{T}\hat{f}$ is non-constant $\leq M$ and takes the value M on $S(D)$.

(ii) There exists a boundary point which has a barrier but no *pl*-barriers.

EXAMPLE I. Let P be a polydisc:

$$P = \{|z_1| < 1, \ |z_2| < 1\}.$$

then $S(P) = \{|z_1| = |z_2| = 1\}$. We show first that every boundary point ζ of P has a barrier. Let $\zeta = (\zeta_1, \zeta_2)$, $|\zeta_1| = 1$, $|\zeta_2| \leq 1$. Take a point z_1^* such that $|z_1^*| > 1$, $\arg z_1^* = \arg \zeta_1$, then the ball

$$|z_1 - z_1^*|^2 + |z_2 - \zeta_2|^2 \leq \rho^2 \equiv |z_1^* - \zeta_1|^2$$

lies outside of $\bar{P}$ except ζ. Hence with a suitable constant $c > 0$ the function

$$c[(|z_1 - z_1^*|^2 + |z_2 - \zeta_2|^2)^{-1} - \rho^{-2}]$$

gives a (harmonic) barrier at ζ. Now we take a point $\zeta_0 \in \partial P - S(P)$. Let $f = M$ on $S(P)$ and $\hat{f}$ be a non-negative continuous function $\leq M$ on ∂P such that $\hat{f} = f$ on $S(P)$ and $\hat{f}(\zeta_0) = 0$. Clearly $\underline{u}f \equiv M$, while $\underline{T}\hat{f} (\leq M)$ is non-constant, because by Corollary 1.2

$\underline{T}\hat{f}(z)\to 0$ for $z\to\zeta_0$ and since each point $\zeta\in S(P)$ is *pl*-regular (Bremerman [8]), $\underline{T}\hat{f}(z)\to M$ $(z\to\zeta)$ by Corollary 1. 1. *Pl*-barrier does not exist at $\zeta\in\partial P-S(P)$ (see Remark (i) in sec. 2. 1).

§ 2. Maximum principle and removable sets

2. 1. Let D be a bounded domain in C^n and E be a set on ∂D. Let χ_E be the function which is 1 on E and 0 on $\partial D-E$. We say that E is of *inner resp. outer pl-measure zero with respect to D if*

$$\underline{T}_D\chi_E(z)\equiv 0 \ \textit{resp.}\ \bar{T}_D\chi_E(z)\equiv 0\,, \qquad z\in D^{1)}$$

THEOREM 2. 1. *A set E on ∂D is of inner pl-measure zero if and only if the following maximum principle holds: if $u\in\underline{P}(D)$ is bounded above and*

$$\varlimsup_{z\to\zeta} u(z)\leq 0,\ \zeta\in\partial D-E \tag{14}$$

then we have $u(z)\leq 0$ for any $z\in D$.

PROOF. If the maximum principle holds, every element of $\underline{\mathcal{F}}(\chi_E, D)$ is non positive on D hence $\underline{T}\chi_E\equiv 0$. To prove the converse take $u\in\underline{P}(D)$ which is $\leq K$ in D and satisfies the condition (14). It suffices to consider the case $K>0$. Then $u/K\in\underline{\mathcal{F}}(\chi_E, D)$ hence $u/K\leq\underline{T}\chi_E\equiv 0$ that is, $u\leq 0$ in D.

COROLLARY 2. 1. *Let u be pluriharmonic and bounded in D. If $\lim_{z\to\zeta} u(z)=0$ at each point $\zeta\in\partial D$ except a set of inner pl-measure zero, then $u\equiv 0$.*

The following two theorems show that the sets of inner *pl*-measure zero are not "small" as in the case $n=1$.

THEOREM 2. 2. *Let $D_R=D_1^R\times\cdots\times D_n^R$ be a polydisc in $C^n(n\geq 2)$ where $D_j^R=\{|z_j|<R\}\,(j=1,\cdots,n)$ and $S(D_R)$ be the Silov boundary $(=\{|z_1|=\cdots=|z_n|=R\})$, then $S(D_R)$ and $\partial D_R-S(D_R)$ are of inner pl-measure zero.*

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1) In case of $n=1$, E is then exactly a set of inner resp. outer harmonic measure zero.

PROOF. In case of $\partial D_R - S(D_R)$ it suffices to prove that for any $u \in \underline{P}(D_R)$ which is bounded above and

$$\overline{\lim_{z\to\zeta}}\, u(z) \leq 0,\ \zeta \in S(D_R), \qquad z \in D_R \tag{15}$$

we have $u(z) \leq 0$, $z \in D_R$. Now from (15) and the compactness there is a positive number r_0 such that for $\varepsilon > 0$

$$u(z) \leq \varepsilon,\ z \in S(D_r), \qquad r_0 \leqslant r < R \tag{16}$$

By the approximation theorem (Bremerman [7]), for $\varepsilon > 0$ and such a $\bar{D}_r$ there exist a finite number of holomorphic functions $f_1, \cdots, f_k$ in D_R and positive constants $c_1, \cdots, c_k$ such that

$$\begin{aligned} u(z) - \varepsilon &\leq \max_{j=1,\cdots,k} \{c_j \log |f_j(z)|\} \\ &\leq u(z) + \varepsilon, \qquad z \in \bar{D}_r. \end{aligned} \tag{17}$$

By (16), (17) $c_j \log |f_j(z)| \leq 2\varepsilon$ on the Silov boundary $S(D_r)$, hence $c_j \log |f_j(z)| \leq 2\varepsilon$ in $\bar{D}_r$. We have therefore $u(z) \leq 3\varepsilon$, $z \in \bar{D}_r$ by (17). Since $r \to R$ for $\varepsilon \to 0$, $u(z) \leqq 0$, $z \in D_R$.

Next we prove that $u = \underline{T}\chi_E(E = S(D_R))$ vanishes identically. For fixed $z_n^0 \in D_n^R$, $u(z_1, \cdots, z_{n-1}, z_n^0)$ is plurisubharmonic in $D'_R = D_1^R \times \cdots \times D_{n-1}^R$. Moreover since every point of ∂D_R has a barrier, $u \to 0$ for $(z_1, \cdots, z_{n-1}) \to \partial D'_R$ (Corollary 1.2), hence $u(z_1, \cdots, z_{n-1}, z_n^0) = 0$ in D'_R by maximum principle. Since z_n^0 is arbitrary in D_n^R, $u \equiv 0$, q.e.d.

This example shows that the union of two sets of inner *pl*-measure zero is not necessarily of inner *pl*-measure zero.

REMARK. From above theorems one can see the following facts stated in sec. 1.4.

(i) Every point of $\partial P - S(P)$ is not *pl*-regular.

(ii) The inequality actually occurs in (13).

Indeed, let $\hat{f}$ be the function defined in sec. 1.4, then we have $\underline{T}\hat{f}(z) \leq \sup f$, $z \in P$ by Theorems 2.1 and 2.2, hence (i) follows from Corollary 1.1. As for (ii) we have only to take the functions f and g such that $f = \chi_{S(P)}$, $g = \chi_{\partial P - S(P)}$, then $\underline{T}f = \underline{T}g \equiv 0$ and $\underline{T}(f+g) \equiv 1$.

(iii) There exists a set which is of inner *pl*-measure zero, but not of inner harmonic measure zero. For instance, $D \equiv P$ (Example I) and $E = \partial P - S(P)$.

THEOREM 2. 3. *Let D be a bounded domain in $C^n (n \geq 2)$ and H be a hyperplane of real dimension $2n-1$ and $E = H \cap \partial D$. Suppose every point of $\partial D - E$ has a barrier with respect to D, then E is of inner pl-measure zero. This is not true for $n=1$.*

PROOF. Without loss of generality we may assume $H = \{z = (z_1, \cdots, z_n)\,;\ Re\, z_n = 0\}$. Let D_x be the intersection of D and hyperplane $H_x = \{z\,;\ Re\, z_n = x\}$ parallel to $H = H_0$, then D_x is open (or ϕ) in H_x and

$$D = [z \in D_x,\ x \in \text{some intervals } I]$$

The boundary of each component of D_x is a subset of $\partial D \cap H_x$. Let $U(z) = \underline{T}_D \chi_E(z)$, then $0 \leq U \leq 1$ and $U(z) \to 0$ for $z \to \zeta \in \partial D - E$ by hypothesis and Corollary 1. 2. Now we show

$$U(z) = 0,\ z \in D_x \qquad (0 \neq x \in I) \tag{18}$$

Let $D_{xy} = D_x \cap \{z\,;\ \text{Im}\, z_n = y\}$. It is an open set (or ϕ) in $C^{n-1} = \{z\,;\ z_n = x + iy\}$ and

$$D_x = [z \in D_{xy},\ y \in \text{some intervals } J].$$

Since $U(z)$ restricted on D_{xy} $(x \neq 0)$ is plurisubharmonic with respect to $(z_1, \cdots, z_{n-1})$ and equal zero on $\partial D_{xy} \subset \partial D_x (\partial D_x \cap E = \phi)$, we have $U(z) = 0$, $z \in D_{xy}$ by maximum principle. This holds for any $y \in J$, which prove (18). It remains to prove $U(z) = 0$ on D_0 if $D_0 \neq \phi$. Let $z_0 \in D_0$. Since U is subharmonic in $D \subset R^{2n}$, there is a ball $N (\subset D)$ with center z_0 and radius δ such that

$$0 \leq U(z_0) \leq \omega_0^{-1} \int_N U d\omega_{2n}$$

where $\omega_0 = \int_N d\omega_{2n}$ denotes the volume of N. By Fubini's theorem

$$\int_N U d\omega_{2n} = \int_{-\delta}^{\delta} dx \int_{N \cap D_x} U d\omega_{2n-1}.$$

The right hand side is zero by (18), hence $U(z_0) = 0$.

2. 2. THEOREM 2. 4. *A set E on the boundary of a bounded domain D in C^n is of outer pl-measure zero with respect to D if and only if there exists a positive plurisuperharmonic function v $(\not\equiv \infty)$ in D such that $v(z) \to +\infty$ as z tends to any point of E.*

PROOF. Suppose $\bar{T}_D \chi_E \equiv 0$, then there is a point $z_0 \in D$ such that for a sequence of functions $v_n(z) \in \mathfrak{F}(\chi_E, D)$ $v_n(z_0) < 1/2^n$. Indeed, for any $z^* \in D$ there are sequences $\{z_\nu\}$ and $\{u_\nu^n(z)\}$ such that $u_\nu^n(z_\nu) \to \varepsilon_\nu (n \to \infty)$ $u_\nu^n \in \mathfrak{F}(\chi_E, D)$ and $\varepsilon_\nu \to 0$ with $z_\nu \to z^*$. For large ν the balls $(\subset D)$ with centers z_ν and same volume ω contain a fixed ball N. Since u_ν^n are superharmonic and ≥ 0

$$u_\nu^n(z_\nu) \geq \omega^{-1} \int_N u_\nu^n(z) d\omega^{2n}$$

By Fatou's lemma we have

$$\begin{aligned} \varepsilon_\nu = \varliminf_{n \to \infty} u_\nu^n(z_\nu) &\geq \varliminf_{n \to \infty} \omega^{-1} \int_N u_\nu^n(z) d\omega^{2n} \\ &\geq \omega^{-1} \int_N \varliminf_{n \to \infty} u_\nu^n(z) d\omega^{2n} \geq \omega^{-1} \int_N \inf_{n,\nu} u_\nu^n(z) d\omega^{2n} \geq 0 \,. \end{aligned}$$

For $\varepsilon_\nu \to 0$, we know that $\inf_{n,\nu} u_\nu^n(z)$ is zero almost everywhere in N! Now $v(z) = \sum_{n=1}^{\infty} v_n(z)$ is positive and plurisuperharmonic $(\not\equiv \infty)$ in D, and for any positive integer N

$$\varliminf_{z \to \zeta} v(z) \geq \sum_{n=1}^{N} \varliminf_{z \to \zeta} v_n(z) \geq N, \qquad \zeta \in E$$

which means $v(z) \to +\infty$ for $z \to \zeta$. Conversely suppose there exists such a function $v(z)$, then for any positive integer n $v(z)/n \geq \bar{T}\chi_E(z) \geq 0$. Since there is a point $z_0 \in D$ where $v(z_0) < \infty$, $\bar{T}\chi_E(z_0) = \lim_{n \to \infty} v(z_0)/n = 0$, therefore $\bar{T}\chi_E \equiv 0$ in D by minimum principle.

As an easy consequence of this theorem we get the following

THEOREM 2. 5. *Let D be a bounded domain in C^n. Let*

$$\varphi(z) = (\varphi_1(z), \cdots, \varphi_m(z)) \qquad (m \geq 1)$$

be a holomorphic mapping of D into C^m and $\varphi(D)$ denotes the image of D in C^m. Suppose that (α) F is a set in C^m such that there exists a positive plurisuperharmonic function Ω $(\not\equiv \infty)$ which is *defined in some domain* $G \supset \varphi(D)$ *and becomes* $+\infty$ *on* $G \cap F$. (β) $\varphi(z)$ *has a limit* $\in F$ *when* z *tends to each point of the set* $E(\subset \partial D)$ *whose outer pl-measure is positive. Then* $\varphi(D) \subset F$.

COROLLARY 2.5. *Let* $\varphi_j(z)(j=1,\cdots,m)$ *be bounded in* D *and* F *be the* $(m-\lambda)$*-dimensional analytic set in* C^m *such that*

$$F = \{w \in C^m,\ w_k = a_k(k = 1,\cdots,\lambda)\} \qquad (1 \leq \lambda \leq m).$$

If $\varphi = (\varphi_1,\cdots,\varphi_m)$ *satisfies the condition* (β), *then* $\varphi(D) \subset F$.

Indeed, the set F fulfils the condition (α) with the function

$$\Omega(w) = -\log|(w_1-a_1)\cdots(w_\lambda-a_\lambda)| + C$$

where $C = \lambda \log(\max_j \sup_{z\in D} |\varphi_j(z)| + \max_k |a_k|)$.

COROLLARY 2.5′. *Let* f *be a non-constant holomorphic function on* $\bar{D}$ *and* $E = \{z\,;\, f(z)=0\}$ *be an analytic set, then* $E \cap \bar{D}$ *is of outer pl-measure zero with respect to* $D-E$.

REMARK. (i) A finite union of sets $E_j \subset \partial D$ is of outer *pl*-measure zero if each E_j is so (cf. (13)).

(ii) Theorem 2.5 is strictly sharper than Theorem 11 (Kusunoki [13]) in the sense that there exists a set E such that the harmonic measure of E is zero, but E is not of outer *pl*-measure zero. For example,

$$D = \{|Re\, z_i| < 1,\ |Im\, z_i| < 1,\ (i = 1, 2)\}$$
$$E = \{Im\, z_1 = -1,\ Im\, z_2 = 0\} \cap \partial D$$

E is of harmonic measure zero. If $\bar{T}_D \chi_E = 0$, there is a function $v \in \bar{P}(D)$ such that $v < \infty$ a.e. and $v(p) \to \infty$ for $p \to E$. We note that there is a point $p_0(z_1^0, z_2^0) \in F \equiv D \cap \{Im\, z_2 = 0\}$ where $v(p_0) < \infty$. Otherwise F would be of outer pl-measure zero with respect to $D \cap \{Im\, z_2 > 0\}$! Since $v(z_1, z_2^0)$ is superharmonic ($\not\equiv \infty$) and tends to ∞ for $z_1 \to L = \{|Re\, z_1| < 1,\ Im\, z_1 = -1\}$, L must be of harmonic measure zero with respect to $D \cap \{z_2 = z_2^0\}$, which is absurd.

2.3. Here we mention about another closely related notion. A closed set F in C^n is called *pl-removable* if F is nowhere dense and for each point $\zeta \in F$ there is a neighborhood N of ζ such that every plurisubharmonic function bounded above on $N \cap (C^n - F)$ possesses a unique plurisubharmonic continuation onto $N \cap F$. A finite union of *pl*-removable sets is *pl*-removable.

THEOREM 2.6. *Let D be a bounded domain bounded by a finite number of surfaces Γ_i. Suppose that $E \subset \partial D$ is pl-removable and each point of $\Gamma_i - E$ $(\neq \phi)$ is regular, then E is of inner pl-measure zero with respect to D.*

In fact, let $U(z) = \underline{T}_D \chi_E(z)$, then $0 \leq U \leq 1$ and under our assumption $U(z) \to 0$ for $z \to \zeta \in \partial D - E$. Define $U(z) = 0$ outside of $D \cup E$, then U is plurisubharmonic outside E. Indeed, U is continuous on $\partial D - E$, and on any analytic plane $L = \{z_i = \zeta_i + ta_i\}$ through $\zeta \in \partial D - E$ there is a disc with center $t = 0$ and disjoint with the closed set $L \cap E$, on which the mean value property holds as $U(0) = 0$. Since E is *pl*-removable, U is plurisubharmonic in C^n hence $U \equiv 0$ by maximum principle.

THEOREM 2.7. (Grauert-Remmert [11]) *Any analytic set of codimension* 1 *is of pl-removable.*

THEOREM 2.8. *A hypersphere (surface) S of real dimension $2n-1$ in C^n is of pl-removable.*

PROOF. We may assume that S is the intersection of sphere $S = \{z\,;\ |z_1|^2 + \cdots + |z_n|^2 = r^2\}$ and a hyperplane $H = \{z\,;\ Re\, z_n = r'\}$, $|r'| < r$. Let $E = \{z_i = z_i^0 + \alpha_i t\}$ $(i = 1, \cdots, n)$ be any analytic plane, then for $S \cap E$ there corresponds at most two points in t-plane, because for $\hat{S} \cap E$ and $H \cap E$ there correspond respectively a circle $C = \{|t|^2 \sum |\alpha_i|^2 + 2\, Re \sum \bar{z}_i^0 \alpha_i t + \sum |z_i^0|^2 = r^2\}$ and a line $L = \{Re\, \alpha_n t = r' - Re\, z_n^0\}$ in t-plane, hence for $S \cap E = (\hat{S} \cap E) \cap (H \cap E)$ there correspond $C \cap L$, i.e. at most two points. Now let $u(z)$ be plurisubharmonic in the neighborhood $N (\subset C^n)$ of a point $\zeta \in S$ and bounded above. Since u is subharmonic in $N \cap E$ except at most two points, u is subharmonically continuable onto $N \cap E$. Thus by definining at each $z \in S \cap N$

$$u(z) = \overline{\lim_{z' \to z}}\, u(z'),\ z' \in N - S$$

and using the same argument as in [11] we know that S is *pl*-removable.

THEOREM 2.9. *Let E be a pl-removable set contained in domain D in C^n. Then any pluriharmonic function bounded in $D-E$ possesses a unique pluriharmonic continuation onto E.*

PROOF. Let z_0 be any point of E. Since the problem is local it is sufficient to take an open ball $B \subset D$ containing z_0 and prove that any pluriharmonic function $\Omega(z)$ bounded in $B - B\cap E$ $(\neq \phi)$ is pluriharmonically continuable onto $B\cap E$. As E is non dense, we can choose B so that $\partial B - E \neq \phi$. Now since E is *pl*-removable $\Omega(z)$ can be continued onto $B\cap E$ as a plurisubharmonic and plurisuperharmonic function, which are denoted by $\Omega_1(z)$ and $\Omega_2(z)$ respectively. Note that $\partial B\cap E$ is of inner *pl*-measure zero with respect to B by Theorem 2.6. Applying Theorem 2.1 to $\Omega_1 - \Omega_2$ we have

$$\Omega_1(z) - \Omega_2(z) \leq 0, \qquad z \in B$$

But since $\Omega_1(z) = \Omega_2(z) = \Omega(z)$ for $z \in B - B\cap E$ it follows that $\Omega_1(z) \equiv \Omega_2(z)$, $z \in B$ by maximum principle. That is, Ω_1 gives a unique pluriharmonic continuation of Ω, q.e.d.

A function u which is, plurisubharmonic and plurisuperharmonic is not only continuous but also infinitely differentiable (as it is harmonic in R^{2n}) and satisfies the partial differential equations

$$\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} = 0, \qquad j, k = 1, \cdots, n$$

that is, u is pluriharmonic. A function u of class C_2 is locally the real part of a holomorphic function if and only if u satisfies above equations. Hence we have the following theorem equivalent with Theorem 2.9, which is regarded under Theorem 2.7 as a generalization of the classical theorem on removability, namely

THEOREM 2.9′. *Let E be pl-removable, then every holomorphic function bounded in $D-E$ possesses a unique holomorphic continuation onto E.*

§ 3. Pl-barriers and pseudo-convexity

3.1. We shall state some criteria for the existence of *pl*-barriers. Let D be a domain in C^n and ζ be a boundary point of D.

THEOREM 3.1. *If D is locally strictly convex, that is, if there exists a neighborhood N of ζ such that $N\frown\bar{D}-\zeta$ lies entirely in one side of $(2n-1)$-dimensional plane through ζ, then ζ is pl-regular. If $D\subset C^n$ $(n>1)$ is locally flat at ζ, that is, $N\frown D$ is a half sphere, then ζ is not pl-regular.*

In fact, there exists a non-singular linear transformation $\xi_j=\sum a_{jk}z_k+b_j$ $(j=1,\cdots,n)$ for which the supporting plane and ζ are carried respectively into the hyperplane $Re\,\xi_1=0$ and the origin. We may assume that the image of $N\frown\bar{D}-\zeta$ lies in the half space $\{Re\,\xi_1<0\}$. Then for a suitable positive constant c

$$v(z)=c\,Re\,\xi_1=c\,Re\,(\sum_{k=1}^{n}a_{1k}z_k+b_1)$$

is a pluriharmonic barrier at ζ. The last statement follows from Theorem 2.3 and Corollary 1.1, q.e.d.

Here we shall give an example stated in sec. 1.3 such that $\underline{T}f$ and $\bar{T}f$ attain the same boundary value f, but do not coincide each other. Let B be a ball in $C^n(n>1)$. Take a function u which is harmonic but not pluriharmonic in B. For instance, $u=(Re\,z_1)^2-(Re\,z_2)^2$. Let f be the restriction of u on ∂B, then since B is locally strictly convex, $\underline{T}f$ and $\bar{T}f$ (in B) attain the boundary value f on ∂B. Suppose $\underline{T}f\equiv\bar{T}f$, then clearly

$$\underline{T}f=\bar{T}f=\underline{H}f=\bar{H}f=u \qquad \text{in } B$$

hence u must be pluriharmonic, which is a contradiction.

THEOREM 3.2. *If D is locally strongly pseudo-convex at ζ, that is, if there is a neighborhood N of ζ and a strongly plurisubharmonic function ω in N for which $N\frown D$ can be expressed as $\{z\,;\,\omega(z)<0\}$, then ζ is pl-regular.*

Since ω is twice continuously differentiable, the eigenvalues of the hermitian matrix $(\partial^2\omega/\partial z_i\partial\bar{z}_j)$ have a positive lower bound, say m, in a neighborhood $N'(\subset N)$ of ζ. Hence for a suitable positive constant c and $0<\delta<m$

$$v(z) = c[\omega(z)-\delta\sum_{i=1}^{n}|z_i-\zeta_i|^2],\ z\in N'$$

is a pl-barrier at $\zeta=(\zeta_1,\cdots,\zeta_n)$.

COROLLARY 3.2. *Suppose there exists in N a C_∞-function $\Phi(z)$ satisfying Levi-Krzoska condition such that (a) $\frac{\partial\Phi}{\partial z_j}$ $(j=1,\cdots,n)$ do not vanish simultaneously at ζ (b) $\sum\frac{\partial^2\Phi}{\partial z_j\partial\bar{z}_k}\xi_j\bar{\xi}_k>0$ for any complex $(\xi_1,\cdots,\xi_n)\neq(0,\cdots,0)$ satisfying $\sum\frac{\partial\Phi}{\partial z_j}\xi_j=0$ at ζ (c) $D\cap N=\{z;\Phi(z)<0\}$. Then ζ is pl-regular.*

Indeed, it is known (e.g. [12]) that under these conditions there exists a strongly plurisubharmonic function ω such that $\omega=u\Phi\in C_\infty$, where u is a positive function in some neighborhood of ζ.

3.2. THEOREM 3.3. *If every boundary point of a bounded domain D in C^n is pl-regular, then D is a domain of holomorphy. The converse is not true.*

PROOF. Let $z^0=(z_1^0,\cdots,z_n^0)$ be a point of D and set

$$u(z) = \sum_{i=1}^{n}|z_i-z_i^0|^2.$$

Let f be the restriction of u on ∂D. Under our hypothesis and Corollary 1.1. the function $v(z)=\bar{T}_Df(z)$ attains the boundary value f on ∂D. Hence

$$\omega(z) = u(z)-v(z)$$

is plurisubharmonic in D and $\omega(z)\to0$ for $z\to\zeta\in\partial D$, therefore $\omega\leq0$ in D. Moreover $\omega(z)\not\equiv0$. In fact since $\inf f>0$, $v(z)>0$ in D and $\omega(z_0)=-v(z_0)<0$. Thus the domains

$$D_n = \{z;\ \omega(z)+1/n<0\},\qquad n=1,2,\cdots$$

are relatively compact in D and pseudo-convex. Since

$$D_n \subset D_{n+1}, \quad D = \lim_{n\to\infty} D_n$$

D is pseudo-convex, that is, a domain of holomorphy by a celebrated theorem of Oka. The converse statement is false, for instance the Example I in sec. 1.4.

THEOREM 3.4. (I) *$D \subset C^n$ is a domain of holomorphy if and only if there exists an exhaustion $\{D_n\}$ of D, i.e. $D_n \subset D_{n+1}$, $D = \bigcup D_n$ such that every point of ∂D_n is pl-regular (with respect to D_n).* (II) *For any domain $D \subset C^n$ there exists an exhaustion $\{D_n\}$ of D such that every point of ∂D_n is pl-regular except a pl-removable set (inner pl-measure zero).*

PROOF. (I) A domain of holomorphy can be approximated by strongly pseudo-convex domains $\{D_n\}$. Each point of ∂D_n is *pl*-regular (Theorem 3.2). Conversely if there exists such an exhaustion, each D_n is a domain of holomorphy by Theorem 3.3, hence $D = \lim D_n$ is a domain of holomorphy by Behnke-Stein's theorem.

(II) Let $\{D_n\}$ be an exhaustion of D such that each D_n is a finite union of balls in D, then by Theorem 3.1 or 3.2 every point of ∂D_n is *pl*-regular except a set on ∂D_n consisting of intersections of the balls, which are *pl*-removable by Theorem 2.8 and of inner *pl*-measure zero with respect to D_n by Theorem 2.6.

3.3. As a simple application of our Dirichlet problem we shall give an example of domain $D \subset C^n$ (not a domain of holomorphy) for which the following facts hold:

(a) *There exists a plurisubharmonic function in D which is not plurisubharmonically continuable onto the envelope $E(D)$ of holomorphy of D.*

(b) *A plurisubharmonic function does not necessarily attain its supremum on D at the Silov boundary $S(D)$ of D.*

(a) is the disproof to the modified Bochner-Martin's conjecture [4] for which Bremerman [5] gave first an example in a tube domain. He gave another example [6] in a bounded domain (Reinhart region).

EXAMPLE II. Let B_ρ stands for an open ball with center 0 and radius ρ. Let $B_r^R = B_R - \bar{B}_r$ $(0 < r < R)$. We take a finite covering $\{N_j\}$ of $\bar{B}_r^R$ such that each N_j is a ball and the union $N = \bigcup N_j$ is contained in some ring domain $B_{r_1}^{R_1}$ $(0 < r_1 < r,\ R < R_1)$. Now our domain is

$$D(= D_R) = N \cap B_R . \tag{19}$$

Let α and β denote the inner resp. outer boundary of D and f be the function defined as

$$f = 1 \quad \text{on} \quad \alpha,\ f = 0 \quad \text{on} \quad \beta .$$

There exists a *pl*-barrier at every boundary point of D except a *pl*-removable set F on α which consists of intersections of spheres ∂N_j. Therefore

$$U(z) = \underline{T}_D f(z) \in \underline{P}(D)$$

approaches the boundary value $f(\zeta)$ for $z \to \zeta \in \partial D - F$, hence U is non-constant. Clearly U cannot be continued plurisubharmonically onto the envelope $E(D) = B_R$ of holomorphy of D on account of maximum principle, which shows (a). Since the Silov boundary of D is β, (b) has been also shown.

Furthermore we can prove that U *does not have any plurisubharmonic continuation onto* B_ρ^R $(0 < \rho < r_1)$, *a proper subset of* $E(D)$. Indeed, otherwise the function U defined as 0 outside of B_R is then plurisubharmonic and bounded above outside of B_{r_1}, but such a function must be a constant (sec. 4. 2), which is absurd.

We note that one can choose above domain D arbitrarily close to ring domain B_r^R, but cannot take B_r^R itself by the following reason.

THEOREM 3. 5. *Let D be a bounded domain in C^n such that every point of ∂D is pl-regular. Let B be a closed ball contained in D. Then*

(i) *for any non-negative continuous function φ on ∂D there exists a function $\omega(z) \in \underline{P}(D-B)$ which attains φ on ∂D and* 0 *on ∂B.*

(ii) *there does not exist any plurisubharmonic function on $D-B$ which attains* 0 *on ∂D and* 1 *on ∂B, provided that $n > 1$.*

PROOF. (i) It suffices to take $\omega(z)=\underline{T}_{D-B}f(z)\geq 0$, where $f=\varphi$ on ∂D and $f=0$ on ∂B. (Corollaries 1.1 and 1.2)

(ii) Suppose there exists a function $\Omega(z)\in\underline{P}(D-B)$ which is 0 on ∂D and 1 on ∂B. By (i) there is a function $\omega\in\underline{P}(D-B)$ which is 1 on ∂D and 0 on ∂B. Then the function

$$v(z) = \max(\Omega(z), \quad \omega(z))\in\underline{P}(D-B)$$

attains 1 on $\partial(D-B)$. v is non-constant, because if $v\equiv 1$, Ω or ω would take the maximum 1 in $D-B$, hence reduce to a constant. Since

$$D-B=\bigcup_{n=1}^{\infty} D_n, \quad D_n = \{v(z)-1+1/n<0\}$$

and D_n are relatively compact pseudo-convex domains, $D-B$ must be a domain of holomorphy. However this is false if $n>1$, for every holomorphic function in $D-B$ is holomorphically continued into B.

§ 4. Some classes of complex manifolds

4.1. First we mention about the generalization of our Dirichlet problem onto complex manifolds. Since the plurisubharmonicity is invariant under one-to-one holomorphic mapping and pl-barriers are defined locally, almost all our results can be carried over complex manifolds (cf. [8]). Furthermore the base domain need not be relatively compact.

Let X be a non-compact complex manifold and D be an open set on X. We shall consider the compactification of X, for instance, by adding the Alexandroff point A_∞ and define our lower and upper solutions on D. Let f be a function defined on the boundary $\partial D(\subset X)$ of D and at the point A_∞. We denote by $\underline{\mathcal{F}}^*(f, D)$ be the set of $u\in\underline{P}(D)$ which satisfy the boundary condition (1) and the condition : for $\varepsilon>0$ there exists a compact set K on X such that

$$u(p)<f(A_\infty)+\varepsilon, \quad p\in D\cap(X-K)$$

Now $\underline{T}_D f$ is defined by (2) by using $\underline{\mathcal{F}}^*$ instead of $\underline{\mathcal{F}}$. We can then analogously prove the fundamental inequality (9) for the present D, where ∂D should be read the boundary of D in X.

4. 2. As a generalization of the classification of open Riemann surfaces (cf. Ahlfors-Sario [1] chap. IV) we shall consider some classes of complex manifolds which are invariant under one-to-one holomorphic mappings. For each n (≥ 1) we denote by O^n_{pl} the class of non-compact complex manifolds of dimension n on which there does not exist any non-constant plurisubharmonic function bounded above.

EXAMPLE III. The following manifolds belong to classes O^n_{pl} $(n=1, 2,\cdots)$;

1) Complex n-dimensional space C^n
2) Complex manifold removed a pl-removable set (e.g. an analytic set of dimension at most $n-1$) from a compact one.
3) Open Riemann surfaces of parabolic type ($n=1$)
4) Product $X\times Y$ of $X\in O^m_{pl}$ and $Y\in O^n_{pl}$
5) Exterior of a ball (or polydisc) in C^n ($n\geq 2$). (compare with Y^n in Example IV)

We shall prove only the case of polydisc in 5). The following proof is essentially due to T. Nishino. Let $P=\{|z_j|<1, j=1,\cdots,n\}$ be a polydisc. We show that $C^n-\bar{P}\in O^n_{pl}$. Suppose there exists a non-constant plurisubharmonic function V bounded above in $C^n-\bar{P}$ and take two points $a=(a_1,\cdots,a_n)$, $b=(b_1,\cdots,b_n)$ such that $V(a)\neq V(b)$. We may assume $|a_1|>1$. Then the analytic plane

$$\pi_1:\ z_1=a_1,\quad z_j=a_j+\alpha_j t\,(j=2,\cdots,n),\quad \alpha_2\alpha_3\cdots\alpha_n\neq 0$$

contains the point a and $\pi_1\cap\bar{P}=\phi$. By 1) (or 3)) we have $V=V(a)$ on π_1.To show $V(a)=V(b)$ we proceed as follows.

Case (i) $|b_1|\leq 1$. Then some b_j (e.g. b_2) is greater than 1 in absolute value. Now there is an analytic plane π_2 through the point b such that $\pi_2\cap\bar{P}=\phi$ and $\pi_1\cap\pi_2\neq\phi$, for instance,

$$\pi_2:\ z_1=b_1+\beta_1\tau,\quad z_2=b_2,\quad z_k=b_k+\beta_k\tau\qquad(k=3,\cdots,n)$$

where $\beta_1=a_1-b_1$ and $\beta_k=a_k-b_k+\alpha_k t_0$ with $t_0=(b_2-a_2)/\alpha_2$. Then $V=V(b)$ on π_2, it follows that $V(a)=V(b)$, which is absurd.

Case (ii). $|b_1|>1$. Then there is a point $c=(c_1,\cdots,c_n)\in\pi_1$ such that $|c_2|>1$, hence there exists an analytic plane π_3 which contains the points c and $b^*=(b_1, c_2, b_3,\cdots)$ and $\pi_3\cap\bar{P}=\phi$. Finally we take an analytic plane π_4 for which the points b^*, $b\in\pi_4$ and $\pi_4\cap\bar{P}=\phi$. Then we have $V(a)=V(c)=V(b^*)=V(b)$, which contradicts our hypothesis. q.e.d.

We note that since $C^n\in O^n_{pl}$, the following statement gives a generalization of Liouville's theorem:

Every bounded holomorphic function on $X\in O^n_{pl}$ reduces to a constant.

4. 3. Characterization of O^n_{pl}.

THEOREM 4. 1. *A complex manifold X belong to O^n_{pl} if and only if any one of the following conditions is fulfilled.*

α) *(Maximum principle) Let G be any domain on X and u be a plurisubharmonic function bounded above on G and satisfies*

$$\overline{\lim_{p\to q}}\, u(p)\leq m \quad \textit{for any} \quad q\in\partial G(\subset X)$$

then we have $u(p)\leq m$ throughout G.

β) *Let G be any domain on X and g be the function which is 0 on $\partial G(\subset X)$ and $g(A_\infty)=1$, then $\underline{T}_G g$ vanishes identically,*

PROOF. Compare [1] p. 204 and the proof of Theorem 2. 1.

COROLLARY 4. 1. *Let G be any domain in $X\in O^n_{pl}$ and f be holomorphic in G. If $\overline{\lim_{p\to q}}\,|f(p)|\leq m$ for any $q\in\partial G$, then there holds that either f is unbounded in G, or $|f|\leq m$ throughout G.*

For a paracompact manifold X we have further the following characterization. Let $\{X_n\}_{n=0}^{\infty}$ be an exhaustion $(X_n\Subset X_{n+1})$ of X where X_n are relatively compact domains. Let g_n be functions on $\partial X'_n(X'_n=X_n-\bar{X}_0)$ which is 1 on ∂X_n and 0 on ∂X_0. One sees easily that the sequence of functions

$$U_n(p)=\underline{T}_{X'_n}g_n \qquad (n=1, 2,\cdots)$$

is monoton decreasing, hence converges to a limit function

$$U(p)=\lim_{n\to\infty}U_n(p)\in\underline{P}(X-\bar{X}_0),\qquad 0\leq U\leq 1.$$

THEOREM 4.2. *A paracompact X belongs to O^n_{pl} if and only if $U\equiv 0$.*

PROOF. First we show under the condition $U\equiv 0$ that any plurisubharmonic function u bounded above on X reduces to a constant. Let $m=\sup_{\bar{X}_0}u$. $m<M=\sup_X u$ unless u is a constant. Since the function $(u-m)/(M-m)$ belongs $\mathfrak{F}^*(g_n, X'_n)$ $(n=1,2,\cdots)$ it is dominated by U_n in X'_n. Hence for $n\to\infty$ we know that $u\leq m$ in $X-\bar{X}_0$, hence $u\leq m$ in X, which implies that u must be a constant.

To prove the converse we first show that if $U\not\equiv 0$, we have

$$M=\sup_{X-\bar{X}_0}U(p)=1 \tag{20}$$

For fixed n (>1) consider the compact set $\bar{X}_n-X_{n-1}$. From the upper semi-continuity of U's there exists a number ν_0 $(>n)$ such that for given $\varepsilon>0$

$$U(p)\leq U_\nu(p)\leq M+\varepsilon\quad(\nu\geq\nu_0),\quad p\in\bar{X}_n-X_{n-1}.$$

For any $u\in\mathfrak{F}^*(g_\nu, X'_\nu)$ we have $u/(M+\varepsilon)\leq U_\nu/(M+\varepsilon)\leq 1$ in $\bar{X}_n-X_{n-1}$, hence $u/(M+\varepsilon)$ belongs $\mathfrak{F}^*(g_n, X'_n)$, which implies that

$$\frac{U_\nu(p)}{M+\varepsilon}\leq U_n(p),\quad p\in X'_n.$$

Hence $U(p)/(M+\varepsilon)\leq U_n(p)$, $p\in X'_n$. Letting $n\to\infty$, we know $1\leq M$, consequently $M=1$.

Now we show that $X\notin O^n_{pl}$ if $U\not\equiv 0$. Take a domain X'_0 $(\Subset X_0)$ which is the image of the domain B_R-D_R (B_R, D_R: domains in Example II) lying in some local coordinate. Then

$$w_n(p)=T_{X_n-\bar{X}'_0}g'_n(p)\qquad(n>1)$$

is non-constant, where g'_n is 1 on ∂X_n and 0 on $\partial X'_0$. Let $u\in\mathfrak{F}^*(g_n, X'_n)$, then $v(p)=\max(u(p),0)$ extended as 0 on $X_0-X'_0$ belongs $\mathfrak{F}^*(g'_n, X_n-\bar{X}'_0)$, hence $u(p)\leq v(p)\leq w_n(p)$, $p\in X'_n$. Consequently $U_n(p)\leq w_n(p)$ $p\in X'_n$. Since w_n is non-constant ≤ 1

$$U(p) \leq U_n(p) \leq m = \sup_{\bar{X}_1 - X_0} w_n < 1, \ p \in \bar{X}_1 - X_0$$

Therefore if $X \in O^n_{pl}$, then by Theorem 4.1 we would have

$$U(p) \leq m < 1, \quad p \in X - \bar{X}_0$$

which contradicts with (20).

COROLLARY 4.2. *If there exists a continuous plurisuperharmonic function $\omega(p)$ defined outside of a compact set on X such that $\omega(p) \to +\infty$ only if $p \to A_\infty$, then $X \in O^n_{pl}$.*

PROOF. With a suitable constant $c > 0$ we may take the sets $\{p ; \omega(p) - c < n\}$ as above X_n. Since for any $u \in \underline{\mathfrak{F}}^*(g_n, X'_n)$ $u(p) \leq (\omega(p) - c)/n$, $p \in X'_n = X_n - \bar{X}_0$.

$$0 \leq U(p) \leq U_n(p) \leq (\omega(p) - c)/n, \quad p \in X'_n$$

Hence $U(p) \equiv 0$ for $n \to \infty$, which implies $X \in O^n_{pl}$.

We note that the converse of this statement is true for $n = 1$.

4.4. Finally we refer to classes of complex manifolds defined in terms of holomorphic functions. We denote by O^n_A (resp. O^n_{AB}) the class of complex manifolds of dimension n on which there does not exist any non-constant holomorphic (resp. bounded holomorphic) functions. Clearly

$$O^n_{pl} \subset O^n_{AB}, \quad O^n_A \subsetneqq O^n_{AB} \qquad (n = 1, 2, \cdots).$$

However there is generally no inclusion relation (Example V) between O^n_A and O^n_{pl} except the case $n = 1$ where $O^1_A = \phi$ (Behnke-Stein [2]). We shall first show that the inclusion $O^n_{pl} \subset O^n_{AB}$ is *strict*.

EXAMPLE IV. Under the same notation as in Example II,

$$Y^n \equiv C^n - (B_R - \bar{D}_R) \in O^n_{AB} \quad \text{and} \quad \notin O^n_{pl} \qquad (n \geq 2)$$

Indeed, let g be a non-constant continuous function on ∂Y^n ($\subset C^n$) and $g(A_\infty) < \infty$, then by the generalization of Theorem 1 mentioned in sec. 4.1, the function $\underline{T}_{Y^n} g \in \underline{P}(Y^n)$ attains g on ∂Y^n except a pl-removable set. Hence it is non-constant, moreover bounded above ($\leq \sup_{\partial Y^n} g$) which means $Y^n \notin O^n_{pl}$. Note that we can choose Y^n arbitrarily close to the exterior of a ball which belongs to O^n_{pl} (5), Example III), but these two are not equivalent under one-to-one holomorphic mappings. As for $n=1$

$$Y^1 = C^1 - S$$

belongs to O^1_{AB} but not to O^1_{pl}, where S is the generalized Cantor set which has linear measure zero, but positive capacity ([14] p. 145–149; [1] p. 252–253). We note that

$$Z^n = C^{n-1} \times (C^1 - Y^1)$$

gives another example such that $Z^n \in O^n_{AB}$ and $\notin O^n_{pl}$. Z^n is a domain of holomorphy while Y^n ($n \geq 2$) is not. $C^n - Z^n$ is non-compact while $C^n - Y^n$ is compact in C^n.

EXAMPLE V. From a complex manifold $W_0 \in O^n_A$ ($n \geq 2$) remove a set which is the image of a ball B_R in some local coordinate and insert the domain D_R (of (19)), then we get a complex manifold W such that

$$W \in O^n_A \quad \text{and} \quad W \notin O^n_{pl}$$

On the other hand, the space C^n belongs to O^n_{pl} but $\notin O^n_A$.

As for class O^n_{AB} few results are known even in case of $n=1$. Here we just note that Corollary 4.1 is not valid generally on $X \in O^n_{AB}$.

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