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Remarks on matric groups

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The present note consists of two chapters. In Chapter I, we treat a connected algebraic group G and its Borel subgroup B and prove that if V is a rational G-module of finite dimension and if V_0 is its direct summand as a B-module, then V_0 is a direct summand of V as a G-module. In Chapter II, we generalize main results of a recent paper[1] of the senior author to the case of a matric group over a commutative ring.

Chapter I

Let G be a connected linear algebraic group with universal domain K and let B be a Borel subgroup of G. Then

Theorem 1. *Let V be a finite K-module on which G acts rationally. A submodule V_0 of V is a direct summand of V as a G-module if (and only if) V_0 is a direct summand of V as a B-module.*

Proof. Assume that V_0 is a direct summand of V as a B-module. Then it is well known that the property is characterized by the existence of an endomorphism p of V satisfying the following three conditions:[2] 1) $pV=V_0$, 2) $p^2=p$, and 3) $gp=pg$ for every $g\in B$. Now, the vector space Hom(V, V) becomes a ratoinal G-module by the operation $f^g=g^{-1}fg$ ($f\in$Hom(V, V), $g\in G$). In this G-module, the said endomorphism p is B-invariant because it satisfies the condition 3). From this, it follows that p is G-invari-

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1) M. Nagata, *Invariants of a group in an affine ring*, this Journal Vol. 3, No. 3 (1964) 369-377.

2) This p is called Reynolds operator by D. Mumford, *Geometric invariant theory*, forthcoming.

ant[3], because G/B is projective[4] and the map $g \to g^{-1}pg$ induces a morphism from the projective variety G/B into the affine space Hom (V, V) and therefore the image must be a single point. Thus, in the above conditions, "$g \in B$" can be replaced by "$g \in G$" and we see that V_0 is a direct summand of V as a G-module.

Corollary. *$H^1(G, V)$ is a submodule of $H^1(B, V)$.*

Chapter II

Let R be a commutative algebra over a commutative ring A with unit element. When a group G acts on R as A-isomorhisms, it is an interessting question to ask under what condition the ring of invariants in R is of finite type over A. The case where A is a field has been studies by many authors. In this chapter we show that the answer is affirmative when A is Noetherian, R is of finite type over A and the action of G is semi-reductive (the definition will be given below) under some small conditions.

1. When we say rings or algebras, we always mean in this note those which are commutative and which have units. We suppose from now on through out this chapter that R is an algebra over a ring A and that G is a group which acts on R so that 1) every element of G acts on R as an A-automorphism, 2) $\sum_{g \in G} x^g A$ is a finite A-module for every element $x \in R$.

Remark.[5] If R is finitely generated over A and if A is Noetherian, then the conditions 1) and 2) imply that there exists a finite set of elements $g_1, \cdots, g_n$ of G such that $x \in R$ is G-invariant if and only if $x^{g_i} = x$ for $i = 1, \cdots, n$.

2. We say that the action of G is *semi-reductive* if it satisfies the following condition, too:
If $xA+N$ and N are finife G-stable A-submodules of R and if x modulo N is G-invariant, then there is a G-invariant elment y in R

3) Lemma 5 in K. Otsuka, *On oıbit spaces by torus groups*, this Journal Vol. 3 No. 3 (1964) 287-294.

4) See A. Borel, *Groupes linéaires algébriques*, Ann. of Math. Vol. 64 (1956) 20-82.

5) This remark was orally communicated to the junior author by H. Matsumura.

such that y is expressed in a homogeneous form in $x, n_1, \cdots, n_r$ of positive degree with coefficients in A and monic in x, where $n_1, \cdots, n_r$ are suitable elements of N.

We denote the set of G-invariants in R by $I_G(R)$. Then we obtain the following lemma similarly as in Lemma 5.1.B in the paper referred in foot-note 1):

Lemma 2.1. *Let R' be a G-homomorphic image of R. If the action of G on R is semi-reductive, then for each $x \in I_G(R')$, there is a natural number t such that x^t is in the image of $I_G(R)$. Consequently, $I_G(R')$ is integral over the image of $I_G(R)$.*

If we apply this to the case where R' is an integral domain, we obtain the following corollary quite easily.

Corollary 2.2. Under the same assumption as above, if $\mathfrak{p}$ is a G-stable prime ideal of R, then $I_G(R/\mathfrak{p})$ is purely inseparable over $I_G(R)/(\mathfrak{p} \cap I_G(R))$.

We can adapt also Lemma 5.2.B in the referred paper above, and we get:

Lemma 2.3. *Under the same assumption as above, if $h_1, \cdots, h_s$ are elements of $I_G(R)$, then every element of $(\sum_i h_i R) \cap I_G(R)$ is nilpotent modulo $\sum_i h_i I_G(R)$.*

In order to adapt the lemma, one should note that if the action of G on R is semi-reductive, then the action of G on any G-homomorphic image of R is also semi-reductive.

3. Assume in this section that R is a graded ring; $R = \sum_{i=0} R_i$, R_i being the module of homogeneous elements of degree i. Suppose also that R_0 is a finite A-module and that both A and R are Noetherian. Suppose furthermore that each R_i is G-stable, and that the action of G on R is semi-reductive.

Then, first of all, the argument in §6 of the paper in foot-note 1) can be adapted and we obtain

Theorem 3.1. *In the above situation, if $\mathfrak{a}$ is a G-stable graded ideal of R, then $I_G(R/\mathfrak{a})$ is an A-algebra of finite type.*

Then the argument in § 7 of the same paper can be adapted and we obtain

Theorem 3.2. *In the above situation, if A is a pseudo-geometric ring,*[6] *then for an arbitrary G-stable ideal $\mathfrak{a}$ of R, $I_G(R/\mathfrak{a})$ is an A-algebra of finite type.*

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6) As for the definition of a pseudo-geometric ring, see M.Nagata, *Local rings*, John Wiley, New York, 1962.