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On the generalized Hopf homomorphism and the higher composition.

Part II. $\pi_{n+i}(S^n)$ for $i=21$ and 22.

By

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Introduction

The present paper is the continuation of the previous one [2] and is devoted to the computation of $\pi_{n+i}(S^n)$, the $(n+i)$-th homotopy group of the n-sphere for $i=21$ and 22.

The 2-primary components of $\pi_{n+i}(S^n)$, which we denote by π^n_{n+i}, are determined in [4] for $i\leqq 19$ and in [1] for $i=20$.

The main results of this paper are stated as follows by making use of *generators* given in [4] and [1].

Theorem A.

$$\pi^2_{23}=\{\eta_2\circ\nu'\circ\bar{\mu}_6,\ \eta_2\circ\nu'\circ\eta_6\circ\mu_7\circ\sigma_{16}\}\cong Z_2\oplus Z_2,$$

$$\pi^3_{24}=\{\nu'\circ\eta_6\circ\bar{\mu}_7\}\cong Z_2,$$

$$\pi^4_{25}=\{E\nu'\circ\eta_7\circ\bar{\mu}_8,\ \nu_4\circ\zeta_7\circ\sigma_{18},\ \nu_4\circ\eta_7\circ\bar{\mu}_8\}\cong Z_2\oplus Z_8\oplus Z_2,$$

$$\pi^5_{26}=\{\alpha,\ \nu_5\circ\eta_8\circ\bar{\mu}_9\}\cong Z_2\oplus Z_2,\quad \alpha\in E^{-1}(\eta_6\circ\bar{\kappa}_7),$$

$$\pi^6_{27}=\{\eta_6\circ\bar{\kappa}_7\}\cong Z_2$$

$$\pi^7_{28}=\{\eta_7\circ\bar{\kappa}_8,\ \sigma'\circ\kappa_{14}\}\cong Z_2\oplus Z_2,$$

$$\pi^8_{29}=\{\eta_8\circ\bar{\kappa}_9,\ E\sigma'\circ\kappa_{15},\ \sigma^3_8,\ \sigma_8\circ\kappa_{15}\}\cong Z_2\oplus Z_2\oplus Z_4\oplus Z_2,$$

$$\pi^9_{30}=\{\eta_9\circ\bar{\kappa}_{10},\ \sigma_9\circ\kappa_{16},\ \sigma^3_9\}\cong Z_2\oplus Z_2\oplus Z_2,$$

$$\pi^{10}_{31}=\{\eta_{10}\circ\bar{\kappa}_{11},\ \sigma_{10}\circ\kappa_{17},\ \sigma^3_{10}\}\cong Z_2\oplus Z_2\oplus Z_2,$$

$$\pi^{11}_{32}=\{\eta_{11}\circ\bar{\kappa}_{12},\ \sigma_{11}\circ\kappa_{18},\ \sigma^3_{11},\ \theta'\circ\mu_{23}\}\cong Z_2\oplus Z_2\oplus Z_2\oplus Z_2,$$

$$\pi^{12}_{33}=\{\eta_{12}\circ\bar{\kappa}_{13},\ \sigma_{12}\circ\kappa_{19},\ \sigma^3_{12},\ E\theta'\circ\mu_{24},\ \theta\circ\mu_{24}\}$$

$$\cong Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2,$$

$$\pi_{34}^{13} = \{\eta_{13}\circ\bar{\kappa}_{14},\ \sigma_{13}^3,\ E\theta\circ\mu_{25},\ \lambda\circ\nu_{31}\} \cong Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_4,$$

$$\pi_{35}^{14} = \{\eta_{14}\circ\bar{\kappa}_{15},\ \sigma_{14}^3,\ E\lambda\circ\nu_{32}\} \cong Z_2 \oplus Z_2 \oplus Z_4,$$

$$\pi_{36}^{15} = \{\eta_{15}\circ\bar{\kappa}_{16},\ \sigma_{15}^3,\ E^2\lambda\circ\nu_{33}\} \cong Z_2 \oplus Z_2 \oplus Z_2,$$

$$\pi_{37}^{16} = \{\eta_{16}\circ\bar{\kappa}_{17},\ \sigma_{16}^3,\ E^3\lambda\circ\nu_{34},\ \nu_{16}^*\circ\nu_{34}\} \cong Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2,$$

$$\pi_{38}^{17} = \{\eta_{17}\circ\bar{\kappa}_{18},\ \sigma_{17}^3,\ \nu_{17}^*\circ\nu_{35}\} \cong Z_2 \oplus Z_2 \oplus Z_2,$$

$$\pi_{39}^{18} = \{\eta_{18}\circ\bar{\kappa}_{19},\ \sigma_{18}^3,\ \nu_{18}^*\circ\nu_{36}\} \cong Z_2 \oplus Z_2 \oplus Z_2,$$

$$\pi_{40}^{19} = \{\eta_{19}\circ\bar{\kappa}_{20},\ \sigma_{19}^3,\ \nu_{19}^*\circ\nu_{37},\ \bar{\beta}\circ\eta_{39}\} \cong Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2,$$

$$\pi_{41}^{20} = \{\eta_{20}\circ\bar{\kappa}_{21},\ \sigma_{20}^3,\ E\bar{\beta}\circ\eta_{40},\ \bar{\bar{\beta}}\circ\eta_{40}\} \cong Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2,$$

$$\pi_{42}^{21} = \{\eta_{21}\circ\bar{\kappa}_{22},\ \sigma_{21}^3,\ E\bar{\bar{\beta}}\circ\eta_{41}\} \cong Z_2 \oplus Z_2 \oplus Z_2,$$

$$\pi_{43}^{22} = \{\Delta\iota_{45},\ \eta_{22}\circ\bar{\kappa}_{23},\ \sigma_{22}^3\} \cong Z \oplus Z_2 \oplus Z_2,$$

$$\pi_{44}^{23} = \{\eta_{23}\circ\bar{\kappa}_{24},\ \sigma_{23}^3\} \cong Z_2 \oplus Z_2,$$

$$(G_{21}:2) = \{\eta\circ\bar{\kappa},\ \sigma^3\} \cong Z_2 \oplus Z_2.$$

Theorem B.

$$\pi_{24}^2 = \{\eta_2\circ\nu'\circ\eta_6\circ\bar{\mu}_7\} \cong Z_2,$$

$$\pi_{25}^3 = \{\varepsilon_3\circ\kappa_{11}\} \cong Z_2,$$

$$\pi_{26}^4 = \{\varepsilon_4\circ\kappa_{12},\ \nu_4\circ\bar{\zeta}_7,\ \nu_4\circ\bar{\sigma}_7\} \cong Z_2 \oplus Z_8 \oplus Z_2,$$

$$\pi_{27}^5 = \{\varepsilon_5\circ\kappa_{13},\ \nu_5\circ\bar{\zeta}_8,\ \nu_5\circ\bar{\sigma}_8\} \cong Z_2 \oplus Z_2 \oplus Z_2,$$

$$\pi_{28}^6 = \{\rho'''\circ\sigma_{21},\ \bar{\nu}_6\circ\kappa_{14},\ \varepsilon_6\circ\kappa_{14},\ \nu_6\circ\bar{\sigma}_9\} \cong Z_4 \oplus Z_2 \oplus Z_2 \oplus Z_2,$$

$$\pi_{29}^7 = \{\sigma'\circ\rho_{14},\ \bar{\nu}_7\circ\kappa_{15},\ \varepsilon_7\circ\kappa_{15},\ \nu_7\circ\bar{\sigma}_{10}\} \cong Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2,$$

$$\pi_{30}^8 = \{E\sigma'\circ\rho_{15},\ \bar{\nu}_8\circ\kappa_{16},\ \varepsilon_8\circ\kappa_{16},\ \nu_8\circ\bar{\sigma}_{11},\ \sigma_8\circ\rho_{15},\ \sigma_8\circ\bar{\varepsilon}_{15}\}$$

$$\cong Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_{32} \oplus Z_2,$$

$$\pi_{31}^9 = \{\sigma_9\circ\rho_{16},\ \varepsilon_9\circ\kappa_{17},\ \nu_9\circ\bar{\sigma}_{12},\ \sigma_9\circ\bar{\varepsilon}_{16}\} \cong Z_{16} \oplus Z_2 \oplus Z_2 \oplus Z_2,$$

$$\pi_{32}^{10} = \{\sigma_{10}\circ\rho_{17},\ \varepsilon_{10}\circ\kappa_{18},\ \nu_{10}\circ\bar{\sigma}_{13}\} \cong Z_{16} \oplus Z_2 \oplus Z_2,$$

$$\pi_{33}^{11} = \{\sigma_{11}\circ\rho_{18},\ \varepsilon_{11}\circ\kappa_{19},\ \nu_{11}\circ\bar{\sigma}_{14}\} \cong Z_{16} \oplus Z_2 \oplus Z_2,$$

$$\pi_{34}^{12}=\{\sigma^{*'''},\ \sigma_{12}\circ\rho_{19}\pm 2\sigma^{*'''},\ \varepsilon_{12}\circ\kappa_{20},\ \nu_{12}\circ\bar{\sigma}_{15}\}$$
$$\cong Z_{32}\oplus Z_4\oplus Z_2\oplus Z_2,$$
$$\pi_{35}^{13}=\{\rho_{13}\circ\sigma_{28},\ \varepsilon_{13}\circ\kappa_{21},\ \nu_{13}\circ\bar{\sigma}_{16}\}\cong Z_{16}\oplus Z_2\oplus Z_2,$$
$$\pi_{36}^{14}=\{\sigma^{*''},\ \omega_{14}\circ\nu_{30}^2,\ \varepsilon_{14}\circ\kappa_{22},\ \nu_{14}\circ\bar{\sigma}_{17}\}\cong Z_{16}\oplus Z_2\oplus Z_2\oplus Z_2,$$
$$\pi_{37}^{15}=\{\sigma^{*'},\ \omega_{15}\circ\nu_{31}^2,\ \varepsilon_{15}\circ\kappa_{23},\ \nu_{15}\circ\bar{\sigma}_{18}\}\cong Z_{16}\oplus Z_2\oplus Z_2\oplus Z_2,$$
$$\pi_{38}^{16}=\{\sigma_{16}^*,\ E\sigma^{*'},\ \omega_{16}\circ\nu_{32}^2,\ \varepsilon_{16}\circ\kappa_{24},\ \nu_{16}\circ\bar{\sigma}_{19}\}$$
$$\cong Z_{16}\oplus Z_{16}\oplus Z_2\oplus Z_2\oplus Z_2,$$
$$\pi_{39}^{17}=\{\sigma_{17}^*,\ \omega_{17}\circ\nu_{33}^2,\ \varepsilon_{17}\circ\kappa_{25},\ \nu_{17}\circ\bar{\sigma}_{20}\}\cong Z_{16}\oplus Z_2\oplus Z_2\oplus Z_2,$$
$$\pi_{40}^{18}=\{\sigma_{18}^*,\ \varepsilon_{18}\circ\kappa_{26},\ \nu_{18}\circ\bar{\sigma}_{21}\}\cong Z_{16}\oplus Z_2\oplus Z_2,$$
$$\pi_{41}^{19}=\{\sigma_{19}^*,\ \varepsilon_{19}\circ\kappa_{27},\ \nu_{19}\circ\bar{\sigma}_{22}\}\cong Z_{16}\oplus Z_2\oplus Z_2,$$
$$\pi_{42}^{20}=\{\sigma_{20}^*,\ \varepsilon_{20}\circ\kappa_{28},\ \nu_{20}\circ\bar{\sigma}_{23},\ \Delta\nu_{41}+2\sigma_{20}^*\}\cong Z_{16}\oplus Z_2\oplus Z_2\oplus Z_4,$$
$$\pi_{43}^{21}=\{\sigma_{21}^*,\ \varepsilon_{21}\circ\kappa_{29},\ \nu_{21}\circ\bar{\sigma}_{24}\}\cong Z_8\oplus Z_2\oplus Z_2,$$
$$\pi_{44}^{22}=\{\sigma_{22}^*,\ \varepsilon_{22}\circ\kappa_{30},\ \nu_{22}\circ\bar{\sigma}_{25}\}\cong Z_4\oplus Z_2\oplus Z_2,$$
$$\pi_{45}^{23}=\{\sigma_{23}^*,\ \varepsilon_{23}\circ\kappa_{31},\ \nu_{23}\circ\bar{\sigma}_{26}\}\cong Z_2\oplus Z_2\oplus Z_2,$$
$$\pi_{46}^{24}=\{\varepsilon_{24}\circ\kappa_{32},\ \nu_{24}\circ\bar{\sigma}_{27}\}\cong Z_2\oplus Z_2,$$
$$(G_{22}:2)=\{\varepsilon\circ\kappa,\ \nu\circ\bar{\sigma}\}\cong Z_2\oplus Z_2.$$

The main tool of the computation is the following exact sequence: (Proposition 4.2 of [4])

$$(T)\qquad \cdots\longrightarrow\pi_i^n\xrightarrow{E}\pi_{i+1}^{n+1}\xrightarrow{H}\pi_{i+1}^{2n+1}\xrightarrow{\Delta}\pi_{i-1}^n\xrightarrow{E}\pi_i^{n+1}\xrightarrow{H}\cdots$$

In appendix, we shall see a table of the groups $\pi_{n+21}(S^n)$ and $\pi_{n+22}(S^n)$ containing the odd components.

§ 7. Computations of the 2-primary components of $\pi_{n+21}(S^n)$.

First we have

$$\pi_{23}^2=\{\eta_2\circ\nu'\circ\bar{\mu}_6,\ \eta_2\circ\nu'\circ\eta_6\circ\mu_7\circ\sigma_{16}\}\cong Z_2\oplus Z_2$$

by (5.2) of [4] and [1]. By Lemma 5.7 and Proposition 2.5

of [4], we have

$$\Delta(\nu_5\circ\bar{\mu}_8) = \eta_2\circ\nu'\circ\bar{\mu}_6,$$

(7.1) $$\Delta(\nu_5^2\circ\kappa_{11}) = \eta_2\circ\nu'\circ\nu_6\circ\kappa_9 = 0,$$

$$\Delta(\nu_5\circ\eta_8\circ\mu_9\circ\sigma_{18}) = \eta_2\circ\nu'\circ\eta_6\circ\mu_7\circ\sigma_{16}$$

and $E\pi^2_{23} = 0.$

It follows from the exactness of (T) that $H : \pi^3_{24} \to \pi^5_{24}$ maps π^3_{24} isomorphically onto the kernel of $\Delta : \pi^5_{24} \to \pi^2_{22}$ which is onto (see p.45 of [1]). By Theorem 12.9 of [4] and [1], π^5_{24} and π^2_{22} have 16 and 8 elements respectively. Thus the kernel of Δ is isomorphic to Z_2. By Proposition 2.2 and (5.7) of [4] we have

$$H(\nu'\circ\eta_6\circ\bar{\mu}_7) = \eta_5^2\circ\bar{\mu}_7$$

$$= 4\bar{\zeta}_5 \qquad \text{by (7.14) of [4],}$$

and $4\bar{\zeta}_5 \neq 0$ by Theorem 12.9 of [4]. Consequently we have obtained that

(7.2) $$\pi^3_{24} = \{\nu'\circ\eta_6\circ\bar{\mu}_7\} \cong Z_2.$$

By (5.6) and Theorem 12.8 of [4],

$$\pi^4_{25} = \{E\nu'\circ\eta_7\circ\bar{\mu}_8,\ \nu_4\circ\zeta_7\circ\sigma_{18},\ \nu_4\circ\eta_7\circ\bar{\mu}_8\} \cong Z_2 \oplus Z_8 \oplus Z_2.$$

Consider the exact sequence

$$\cdots \longrightarrow \pi^9_{27} \xrightarrow{\Delta} \pi^4_{25} \xrightarrow{E} \pi^5_{26} \xrightarrow{H} \pi^9_{26} \xrightarrow{\Delta} \pi^4_{24} \longrightarrow \cdots,$$

where $\pi^9_{27} = \{\sigma_9\circ\zeta_{16},\ \eta_9\circ\bar{\mu}_{10}\} \cong Z_8 \oplus Z_2$ and $\pi^9_{26} = \{\sigma_9\circ\eta_{16}\circ\mu_{17},\ \nu_9\circ\kappa_{12},\ \bar{\mu}_9,\ \eta_9\circ\mu_{10}\circ\sigma_{19}\} \cong Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$ by Theorems 12.8 and 12.7 of [4]. By (7.16) and (5.8) of [4]

$$\Delta(\sigma_9\circ\zeta_{16}) = x(\nu_4\circ\sigma'\circ\zeta_{14}) \pm E\varepsilon'\circ\zeta_{14}$$

$$= x'(\nu_4\circ\zeta_7\circ\sigma_{18})$$

for odd integers x, x', since we have

$$\begin{aligned}
\varepsilon'\circ\zeta_{13} &\in \varepsilon'\circ\{\nu_{13},\ 8\iota_{16},\ 2\sigma_{16}\} && \text{by the definition of } \zeta_{13}\\
&= -\{\varepsilon',\ \nu_{13},\ 8\iota_{16}\}\circ 2\sigma_{17} && \text{by Proposition 1.4 of [4]}\\
&\supset \pi^3_{17}\circ 2\sigma_{17}\\
&= \{2\varepsilon_3\circ\nu^2_{11}\circ\sigma_{17}\}\\
&= 0.
\end{aligned}$$

$$\Delta(\eta_9\circ\bar{\mu}_{10}) = E\nu'\circ\eta_7\circ\bar{\mu}_8 \qquad \text{by (5.8) of [4].}$$

Thus $E\pi^4_{25} = \{\nu_5\circ\eta_8\circ\bar{\mu}_9\} \cong Z_2$ and

(7.3) $H : \pi^5_{27} \to \pi^9_{27}$ *is trivial.*

By (7.16), (5.13) and (5.8) of [4],

$$\Delta(\sigma_9\circ\eta_{16}\circ\mu_{17}) = E\nu'\circ\varepsilon_7\circ\mu_{15}+\nu_4\circ\sigma'\circ\eta_{14}\circ\mu_{15} \notin \text{Im } E,$$

$$\Delta(\bar{\mu}_9) = E\nu'\circ\bar{\mu}_7,$$

$$\Delta(\eta_9\circ\mu_{10}\circ\sigma_{19}) = E\nu'\circ\eta_7\circ\mu_8\circ\sigma_{17}$$

and these Δ-images are independent [1]. So, we have an exact sequence : $0\to Z_2 \to \pi^5_{27} \to Z_2$. By Lemma 6.1 of [2], there exists an element α in π^5_{26} such that $H(\alpha)=\nu_9\circ\kappa_{12}$ and $2\alpha=0$, whence we have

$$\pi^5_{26} = \{\alpha,\ \nu_5\circ\eta_8\circ\bar{\mu}_9\} \cong Z_2 \oplus Z_2.$$

It is seen in (16.4) of [1] that $\Delta : \pi^{11}_{27} \to \pi^5_{25}$ is a monomorphism. So, E induces an isomorphism of $\pi^5_{26}/\Delta(\pi^{11}_{28})$ onto π^6_{27}. For generators of π^{11}_{28} we have, by use of (5.10) of [4],

$$\begin{aligned}
(7.4)\quad \Delta(\sigma_{11}\circ\eta_{18}\circ\mu_{19}) &= \nu_5\circ\eta_8\circ\sigma_9\circ\eta_{16}\circ\mu_{17}\\
&= \nu_5\circ(E\sigma'\circ\eta_{15}+\bar{\nu}_8+\varepsilon_8)\circ\eta_{16}\circ\mu_{17} && \text{by (7.4) of [4]}\\
&= 4\nu_5\circ E\sigma'\circ\zeta_{15}+\nu^4_5\circ\mu_{17}+\nu_5\circ\varepsilon_8\circ\eta_{16}\circ\mu_{17}\\
& && \text{by (7.14) and Lemma 6.3 of [4]}\\
&= \nu_5\circ\mu_8\circ\eta^2_{17}\circ\sigma_{19}\\
&= 4(\nu_5\circ\zeta_8\circ\sigma_{19}) = 0 && \text{by (7.14) of [4],}
\end{aligned}$$

$$\Delta(\bar{\mu}_{11}) = \nu_5\circ\eta_8\circ\bar{\mu}_9,$$

$$\Delta(\nu_{11}\circ\kappa_{14}) = \nu_5\circ\eta_8\circ\nu_9\circ\kappa_{12} = 0 \qquad \text{by (5.7) of [4].}$$

It follows that $\pi_{27}^6 = \{E\alpha\} \cong Z_2$ where $E\alpha = \eta_6\circ\bar{\kappa}_7$ since $E^3\alpha \equiv E^2(\eta_6\circ\bar{\kappa}_7)$ mod $E^2(\nu_5^2\circ\pi_{27}^{12} + \pi_{13}^6\circ\kappa_{13}) = 0$ by Lemma 6.1 of [2] and $E^2 : \pi_{27}^6 \to \pi_{29}^8$ is a monomorphism as is seen in the following.

Consider the exact sequence

$$\cdots \to \pi_{29}^{13} \xrightarrow{\Delta} \pi_{27}^6 \xrightarrow{E} \pi_{28}^7 \xrightarrow{H} \pi_{28}^{13} \xrightarrow{\Delta} \cdots,$$

where $\pi_{29}^{13} = \{\sigma_{13}\circ\mu_{20}\} \cong Z_2$ and $\pi_{28}^{13} = \{\rho_{13}, \bar{\varepsilon}_{13}\} \cong Z_2 \oplus Z_2$.

We have seen in the computation of π_{27}^7 [1] that the last homomorphism Δ has the kernel generated by $\bar{\varepsilon}_{13}$, which is the H-image of $\sigma'\circ\kappa_{14}$. The order of $\sigma'\circ\kappa_{14}$ is 2. By Lemma 5.4 and Proposition 12.20 of [4]

$$\Delta(\sigma_{13}\circ\mu_{20}) = \Delta H(\sigma'\circ\rho_{14}) = 0.$$

This implies that the above E is a monomorphism and

(7.5) $H : \pi_{29}^7 \to \pi_{29}^{13}$ *is an epimorphism.*

The following result has been proved :

$$\pi_{28}^7 = \{\eta_7\circ\bar{\kappa}_8, \sigma'\circ\kappa_{14}\} \cong Z_2 \oplus Z_2.$$

It follows immediately from (5.15) and Theorem 9.1 of [4] that

$$\pi_{29}^8 = \{\eta_8\circ\bar{\kappa}_9, E\sigma'\circ\kappa_{15}, \sigma_8^3, \sigma_8\circ\kappa_{15}\} \cong Z_2 \oplus Z_2 \oplus Z_4 \oplus Z_2.$$

Consider the exact sequence :

$$\cdots \to \pi_{31}^9 \xrightarrow{H} \pi_{31}^{17} \xrightarrow{\Delta} \pi_{29}^8 \xrightarrow{E} \pi_{30}^9 \xrightarrow{H} \pi_{30}^{17} = 0,$$

where $\pi_{31}^{17} = \{\sigma_{17}^2, \kappa_{17}\} = Z_2 \oplus Z_2$. By (5.16) of [4],

$$\Delta\kappa_{17} = (2\sigma_8 - E\sigma')\circ\kappa_{15} = E\sigma'\circ\kappa_{15}$$

and $\quad \Delta\sigma_{17}^2 = \pm(2\sigma_8^3 - E\sigma'\circ\sigma_{15}^2).$

Here we remark $E\sigma'\circ\sigma_{15}^2 = 0$. For, the relation $H(\sigma'\circ\sigma_{14}^2) = \eta_{13}\circ\sigma_{14}^2$

$=(\bar{\iota}_{13}+\varepsilon_{13})\circ\sigma_{21}=0$ implies that $\sigma'\circ\sigma_{14}^2$ belongs to $E\pi_{27}^6=\{\eta_7\circ\bar{\kappa}_8\}$. If $E\sigma'\circ\sigma_{15}^2=\eta_8\circ\bar{\kappa}_9$, by the exactness of the above sequence, we have that $2\sigma_9^3=E^2\sigma'\circ\sigma_{16}^2=\eta_9\circ\kappa_{10}\neq 0$. But $2\sigma_9^3=0$ since $2\sigma_{16}^2=0$. This is a contradiction and we have proved the required relation.

It follows from the exactness of the above sequence that

$$\pi_{30}^9=\{\eta_9\circ\bar{\kappa}_{10},\ \sigma_9\circ\kappa_{16},\ \sigma_9^3\}\cong Z_2\oplus Z_2\oplus Z_2$$

and

(7.6) $H:\pi_{31}^9\to\pi_{31}^{17}$ *is trivial.*

$\pi_{32}^{19}=\pi_{31}^{19}=0$ by Theorems 7.12 and 7.13 of [4]. Thus it is clear from the exactness of (T) that

$$\pi_{31}^{10}=\{\eta_{10}\circ\bar{\kappa}_{11},\ \sigma_{10}\circ\kappa_{17},\ \sigma_{10}^3\}\cong Z_2\oplus Z_2\oplus Z_2.$$

We have seen in *p.* 50 of [1] that the kernel of the homomorphism $\Delta:\pi_{32}^{21}\to\pi_{30}^{10}$ is isomorphic to Z_2 and generated by $4\zeta_{21}=\eta_{21}^2\circ\mu_{23}$. By Lemma 7.5 of [4], $H(\theta'\circ\mu_{23})=\eta_{21}^2\circ\mu_{23}$. From the exactness of the sequence :

$$0=\pi_{33}^{21}\to\pi_{31}^{10}\xrightarrow{E}\pi_{32}^{11}\xrightarrow{H}Z_2\to 0,$$

we have

$$\pi_{32}^{11}=\{\sigma_{11}^3,\ \eta_{11}\circ\bar{\kappa}_{12},\ \sigma_{11}\circ\kappa_{18},\ \theta'\circ\mu_{23}\}\cong Z_2\oplus Z_2\oplus Z_2\oplus Z_2.$$

By Lemma 2.4,

$$\Delta\zeta_{23}=\Delta H(\sigma^{*\prime\prime\prime})=0\quad\text{for}\quad \Delta:\pi_{34}^{23}\to\pi_{32}^{11}.$$

Whence

(7.7) $E:\pi_{32}^{11}\to\pi_{33}^{12}$ *is a monomorphism.*

Also we know that $\Delta:\pi_{33}^{23}\to\pi_{31}^{11}$ is trivial by (16.8) of [1]. Therefore we have an exact sequence

$$0\to\pi_{32}^{11}\xrightarrow{E}\pi_{33}^{12}\xrightarrow{H}\pi_{33}^{23}\to 0.$$

where $\pi_{33}^{23}=\{\eta_{23}\circ\mu_{24}\}\cong Z_2$.

In π_{33}^{12}, we have an element $\theta\circ\mu_{24}$ of order 2 which is mapped to $\eta_{23}\circ\mu_{24}$ under H, by Lemma 7.5 of [4]. Thus

$$\pi_{33}^{12}=\{\sigma_{12}^3,\ \eta_{12}\circ\bar{\kappa}_{13},\ \sigma_{12}\circ\kappa_{19},\ E\theta'\circ\mu_{24},\ \theta\circ\mu_{24}\}$$
$$\cong Z_2\oplus Z_2\oplus Z_2\oplus Z_2\oplus Z_2.$$

Consider the following exact sequence :

$$\cdots\longrightarrow\pi_{35}^{25}\overset{\Delta}{\longrightarrow}\pi_{33}^{12}\overset{E}{\longrightarrow}\pi_{34}^{13}\overset{H}{\longrightarrow}\pi_{34}^{25}\overset{\Delta}{\longrightarrow}\pi_{32}^{12}\longrightarrow\cdots,$$

where $\pi_{35}^{25}=\{\eta_{25}\circ\mu_{26}\}\cong Z_2$. By the relation (7.30) of [4].

(7.8) $\Delta(\eta_{25}\circ\mu_{26})=E\theta'\circ\mu_{24}$.

For the last homomorphism Δ in the above sequence, by (16.11) of [1], we know that its kernel is of order 2 and generated by ν_{25}^3. Thus we have an exact sequence :

$$0\longrightarrow E\pi_{33}^{12}\longrightarrow\pi_{34}^{13}\overset{H}{\longrightarrow}\{\nu_{25}^3\}\longrightarrow 0,$$

where $E\pi_{33}^{12}=\{\sigma_{13}^3,\ \eta_{13}\circ\bar{\kappa}_{14},\ \sigma_{13}\circ\kappa_{20},\ E\theta\circ\mu_{24}\}\cong Z_2\oplus Z_2\oplus Z_2\oplus Z_2$.

By (6.1) of [2], we have that

$$H(\lambda\circ\nu_{31})=H(\lambda_0\circ\nu_{31})=\nu_{25}^3.$$

By Proposition 2.12 of [2], we have

$$2(\lambda_0\circ\nu_{31})\in\{\sigma_{13},\ \nu_{20},\ 2\nu_{23},\ \nu_{26}\}\circ 2\nu_{31}=\sigma_{13}\circ\{\nu_{20},\ 2\nu_{23},\ \nu_{26},\ 2\nu_{29}\}.$$

Lemma 7.1. *For sufficiently large n, i.e., for* $n\geqq 16$

$$\{\nu_n,\ 2\nu_{n+3},\ \nu_{n+6},\ 2\nu_{n+9}\}\equiv\kappa_n \mod \sigma_n^2 .$$

The proof is quite similar to the discussions in *p.* 40 of [1] and we omit it.

Corollary. $2(\lambda\circ\nu_{31})\equiv\sigma_{13}\circ\kappa_{20}$ mod σ_{13}^3, *and the order of* $\lambda\circ\nu_{31}$ *is four.*

For, we have $2(\lambda_0\circ\nu_{31})\equiv\sigma_{13}\circ\kappa_{20}$ mod σ_{13}^3 by Lemma 7.1. Since $H(\lambda\circ\nu_{31}-\lambda_0\circ\nu_{31})=0$, $2(\lambda\circ\nu_{31})-2(\lambda_0\circ\nu_{31})\in 2E\pi_{33}^{12}=0$. Corollary follows.

These discussions show

$$\pi_{34}^{13}=\{\sigma_{13}^3,\ \eta_{13}\circ\bar{\kappa}_{14},\ E\theta\circ\mu_{25},\ \lambda\circ\nu_{31}\}\cong Z_2\oplus Z_2\oplus Z_2\oplus Z_4.$$

It follows from (16.10) of [1] that the homomorphism H in the following exact sequence is trivial :

$$\cdots \longrightarrow \pi_{36}^{27} \xrightarrow{\Delta} \pi_{34}^{13} \xrightarrow{E} \pi_{35}^{14} \xrightarrow{H} \pi_{35}^{27} \longrightarrow \cdots,$$

where $\pi_{36}^{27} = \{\nu_{27}^3, \mu_{27}, \eta_{27}\circ\varepsilon_{28}\} \cong Z_2 \oplus Z_2 \oplus Z_2$.

We have

$$\Delta(\nu_{27}^3) = \Delta(\nu_{27})\circ\nu_{28}^2 = 0 \qquad \text{by (10.21) of [4]},$$

$$\Delta(\mu_{27}) = E\theta\circ\mu_{25} \qquad \text{by (7.30) of [4]}$$

and
$$\begin{aligned}\Delta(\eta_{27}\circ\varepsilon_{28}) &= \Delta(\eta_{27}\circ\varepsilon_{28}+\nu_{27}^3)\\ &= \Delta(H(\sigma^{*\prime\prime})) \qquad \text{by (2) of Lemma 6.2 of [2]}\\ &= 0.\end{aligned}$$

It follows that

$$\pi_{35}^{14} = \{\sigma_{14}^3, \eta_{14}\circ\bar{\kappa}_{15}, E\lambda\circ\nu_{32}\} \cong Z_2 \oplus Z_2 \oplus Z_4$$

and

(7.9) *the kernel of* $\Delta : \pi_{36}^{27} \longrightarrow \pi_{34}^{13}$ *is generated by* ν_{27}^3 *and* $\eta_{27}\circ\varepsilon_{28}$.

Next consider the exact sequence

$$\cdots \longrightarrow \pi_{37}^{29} \xrightarrow{\Delta} \pi_{35}^{14} \xrightarrow{E} \pi_{36}^{15} \xrightarrow{H} \pi_{36}^{29} \xrightarrow{\Delta} \pi_{34}^{14} \longrightarrow \cdots,$$

where $\pi_{37}^{29} = \{\bar{\nu}_{29}, \varepsilon_{29}\} \cong Z_2 \oplus Z_2$ and $\pi_{36}^{29} = \{\sigma_{29}\} \cong Z_{16}$.

Since $\Delta\sigma_{29}$ is of order 16 [1], the above H is trivial.

Now consider $E^3(\lambda\circ\nu_{31})$. It follows from Lemma 12.18 of [4] that

$$E^3(\lambda\circ\nu_{31}) - 2\nu_{16}^*\circ\nu_{34} = \Delta\nu_{33}^2.$$

We have

$$\begin{aligned}2\nu_{16}^*\circ\nu_{34} &\in \{\sigma_{16}, 2\sigma_{23}, \nu_{30}\}\circ 2\nu_{34}\\ &= -\sigma_{16}\circ\{2\sigma_{23}, \nu_{30}, 2\nu_{33}\}\\ &\supset -2\sigma_{16}\circ\{\sigma_{23}, \nu_{30}, 2\nu_{33}\}\\ &\subset -2\sigma_{16}\circ\pi_{37}^{23} = \sigma_{16}\circ 2\pi_{37}^{23}\\ &= 0,\end{aligned}$$

where the indeterminacy is $2\sigma_{16}^2\circ\pi_{37}^{30} + \sigma_{16}\circ\pi_{34}^{23}\circ\nu_{34} = 0$.

Thus we have

(7.10). $2\nu_{16}^*\circ\nu_{34}=0$, $E^3(\lambda\circ\nu_{31})=\Delta\nu_{33}^2$ *and* $2E^3(\lambda\circ\nu_{31})=\Delta(2\nu_{33}^2)=0$.

By (4) of Lemma 6.2 of [2], $\Delta\sigma_{31}=\Delta(H\sigma_{16}^*)=0$. Since σ_{31} generates π_{38}^{31} it follows from the exactness of (T)

(7.11). $E: \pi_{36}^{15}\rightarrow \pi_{37}^{16}$ *is a monomorphism and* $2E^2(\lambda\circ\nu_{31})=0$.

On the other hand $2E\lambda\circ\nu_{32}\neq 0$, and this is a non-trivial kernel of $E: \pi_{35}^{14}\rightarrow \pi_{36}^{15}$. By (3) of Lemma 6.2 of [2]

$$\Delta(\bar{\nu}_{29}+\varepsilon_{29}) = \Delta H(\sigma^{*\prime}) = 0.$$

Since $\bar{\nu}_{29}$ and ε_{29} generate $\pi_{37}^{29}\cong Z_2\oplus Z_2$, we have

(7.12) $$2E\lambda\circ\nu_{32} = \Delta\bar{\nu}_{29} = \Delta\varepsilon_{29}$$

and

(7.13) *the kernel of* $\Delta: \pi_{37}^{29}\rightarrow \pi_{35}^{14}$ *is generated by* $\bar{\nu}_{29}+\varepsilon_{29}=H(\sigma^{*\prime})$.

Consequently, we have obtained

$$\pi_{36}^{15} = E\pi_{34}^{14} = \{\sigma_{15}^3,\ \eta_{15}\circ\bar{\kappa}_{16},\ E^2\lambda\circ\nu_{33}\}\cong Z_2\oplus Z_2\oplus Z_2.$$

From (7.11), we have an exact sequence

$$0\longrightarrow \pi_{36}^{15}\xrightarrow{E}\pi_{37}^{16}\xrightarrow{H}\pi_{37}^{31},$$

where $\pi_{37}^{31}=\{\nu_{31}^2\}\cong Z_2$. By Lemma (12.14) of [4], we have
$H(\nu_{16}^*\circ\nu_{34})\equiv\nu_{31}^2 \mod 2\nu_{31}^2=0$.
It follows from the first relation of (7.10) the above sequence splits and

$$\pi_{37}^{16} = \{\sigma_{16}^3,\ \eta_{16}\circ\bar{\kappa}_{17},\ E^3\lambda\circ\nu_{34},\ \nu_{16}^*\circ\nu_{34}\}\cong Z_2\oplus Z_2\oplus Z_2\oplus Z_2.$$

Consider the exact sequence

$$\cdots\longrightarrow \pi_{39}^{17}\xrightarrow{H}\pi_{39}^{33}\xrightarrow{\Delta}\pi_{37}^{16}\xrightarrow{E}\pi_{38}^{17}\longrightarrow\pi_{38}^{33}=0,$$

where $\pi_{39}^{33}=\{\nu_{33}^2\}\cong Z_2$. By the second relation of (7.10) we have

$$\pi_{38}^{17} = \{\sigma_{17}^3,\ \eta_{17}\circ\bar{\kappa}_{18},\ \nu_{17}^*\circ\nu_{35}\}\cong Z_2\oplus Z_2\oplus Z_2$$

and

(7.14) $H: \pi_{39}^{17}\rightarrow\pi_{39}^{33}$ *is trivial.*

From the exactness of $0=\pi_{40}^{35}\rightarrow\pi_{38}^{17}\overset{E}{\rightarrow}\pi_{39}^{18}\rightarrow\pi_{39}^{35}=0$,

it follows immediately

$$\pi_{39}^{18}=\{\sigma_{18}^{3},\ \eta_{18}\circ\kappa_{19},\ \nu_{18}^{*}\circ\nu_{36}\}\cong Z_2\oplus Z_2\oplus Z_2.$$

As is shown in p.53 of [1], the kernel of $\Delta:\pi_{40}^{37}\rightarrow\pi_{38}^{18}$ is generated by $4\nu_{37}=\eta_{37}^{3}$. By Lemma 16.4 of [1] we have that

$$2(\bar{\beta}\circ\eta_{39})=0 \quad\text{and}\quad H(\bar{\beta}\circ\eta_{39})=\eta_{37}^{3} \quad\text{for } \bar{\beta}\in\pi_{39}^{19}.$$

Therefore the sequence

$$0=\pi_{41}^{37}\rightarrow\pi_{39}^{18}\overset{E}{\rightarrow}\pi_{40}^{19}\overset{H}{\rightarrow}Z_2\rightarrow 0$$

splits and

$$\pi_{40}^{19}=\{\sigma_{19}^{3},\ \eta_{19}\circ\bar{\kappa}_{20},\ \nu_{19}^{*}\circ\nu_{37},\ \bar{\beta}\circ\eta_{39}\}\cong Z_2\oplus Z_2\oplus Z_2\oplus Z_2.$$

Next we prove

(7.15) $$\xi_{13}\circ\nu_{31}\equiv\sigma_{13}^{3}\mod\sigma_{13}\circ\kappa_{20}.$$

By the definition of ξ_{13} stated in p.153 of [4],

$$\begin{aligned}\xi_{13}\circ\nu_{31}&\in\{\sigma_{13},\ \nu_{20},\ \sigma_{23}\}\circ\nu_{31}\\&=-\sigma_{13}\circ\{\nu_{20},\ \sigma_{23},\ \nu_{30}\}\\&\equiv\ \sigma_{13}^{3}\quad\mod\sigma_{13}\circ\kappa_{20}\end{aligned}$$

since $\{\nu_{20},\ \sigma_{23},\ \nu_{30}\}\equiv\sigma_{20}^{2}\mod\kappa_{20}$ by Example 4 in p.85 of [4]. Hence by Corollary 12.25 of [4],

(7.16) $$\Delta\nu_{39}=(\nu_{19}^{*}+\xi_{19})\circ\nu_{37}\equiv\nu_{19}^{*}\circ\nu_{37}+\sigma_{19}^{3}\mod\sigma_{19}\circ\kappa_{26}.$$

In the exact sequence

$$0\rightarrow\Delta\pi_{42}^{39}\rightarrow\pi_{40}^{19}\overset{E}{\rightarrow}\pi_{41}^{20}\overset{H}{\rightarrow}\text{Kernel }\Delta\rightarrow 0,$$

$\Delta\pi_{42}^{39}$ is generated by the element of (7.16) and Kernel Δ is already known to be trivial as in p.54 of [1]. By Lemma 16.5 of [1] we have

$$H(\bar{\bar{\beta}}\circ\eta_{40})=\eta_{39}^{2}\quad\text{and}\quad 2\,\bar{\bar{\beta}}\circ\eta_{40}=0.$$

Therefore we have

$$\pi_{41}^{20} = \{\sigma_{20}^3,\ \eta_{20}\circ\bar{\kappa}_{21},\ E\bar{\beta}\circ\eta_{40},\ \bar{\bar{\beta}}\circ\eta_{40}\} \cong Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2.$$

By Lemma 16.4 of [1], $\Delta\eta_{41}=E\bar{\beta}$ and $\Delta\eta_{41}^2=E\bar{\beta}\circ\eta_{40}$. Since η_{41} and η_{41}^2 generate π_{42}^{41} and π_{43}^{41} respectively and since $E\bar{\beta}\neq 0$,

$$\pi_{42}^{21} = E\pi_{41}^{20} = \{\sigma_{21}^3,\ \eta_{21}\circ\bar{\kappa}_{22},\ E\bar{\bar{\beta}}\circ\eta_{41}\} \cong Z_2 \oplus Z_2 \oplus Z_2.$$

In the exact sequence

$$\pi_{44}^{22} \xrightarrow{H} \pi_{44}^{43} \xrightarrow{\Delta} \pi_{42}^{21} \xrightarrow{E} \pi_{43}^{22} \xrightarrow{H} \pi_{43}^{43} \xrightarrow{\Delta} \pi_{41}^{21},$$

$\pi_{44}^{43}=\{\eta_{43}\}\cong Z_2$ and $\pi_{43}^{43}=\{\iota_{43}\}\cong Z$. As is seen in [1], $\Delta\iota_{43}$ is of order 2, whence $H\pi_{43}^{22}$ is generated by $2\iota_{43}$. By Proposition 2.7 of [4], $2\iota_{43}=\pm H\Delta(\iota_{45})$. We have

$$\Delta\eta_{43} = E\bar{\beta}\circ\eta_{41}$$

by Lemma 16.5 of [1]. Thus

$$\pi_{43}^{22} = \{\sigma_{22}^3,\ \eta_{22}\circ\bar{\kappa}_{23},\ \Delta\iota_{45}\} \cong Z_2 \oplus Z_2 \oplus Z.$$

It follws easily from the exactness of (T)

$$\pi_{44}^{23} = \{\sigma_{23}^3,\ \eta_{23}\circ\bar{\kappa}_{24}\} \cong Z_2 \oplus Z_2,$$

and $(G_{21} : 2) = \{\sigma^3,\ \eta\circ\bar{\kappa}\} \cong Z_2 \oplus Z_2$ in stable range.

Finally we prove

Proposition 7.2. $2\lambda\circ\nu_{31} = \sigma_{13}\circ\kappa_{20}$ *and* $\sigma_n\circ\kappa_{n+7} = 0$ *for* $n\geqq 15$.

proof. Let n be sufficiently large. As is well known there exists an element β_1 of $\pi_7(SO(n))$ such that $J(\beta_1)=\sigma_n \in \pi_{n+7}(S^7)$ for Hopf-Whitehead J-homomorphism. We have $\sigma_n\circ\eta_{n+7}=J(\beta_1)\circ E^n\kappa_7=J(\beta_1\circ\kappa_7)$, while $\beta_1\circ\kappa_7 \in \pi_{21}(SO(n))=0$ since $21\equiv 5 \pmod 8$.

Assume that $2\lambda\circ\nu_{31}\neq\sigma_{13}\circ\kappa_{20}$, then $2\lambda\circ\nu_{31}=\sigma_{13}\circ\kappa_{20}+\sigma_{13}^3$, by Corollary to Lemma 7.1. By (7.10), $\sigma\circ\kappa+\sigma^3=0$ in stable range. Thus $\sigma^3=\sigma\circ\kappa=0$, but this contradicts to the above result. Consequently $2\lambda\circ\nu_{31}=\sigma_{13}\circ\kappa_{20}$ and $\sigma_{15}\circ\kappa_{22}=0$ by (7.11). q. e. d.

§8. Computations of the 2-primary components of $\pi_{n+22}(S^n)$.

By (5.2) of [4] and by (7.2), we obtain immediately

$$\pi_{24}^2 = \{\eta_2\circ\nu'\circ\mu_6\circ\bar{\mu}_7\} \cong Z_2.$$

We have $E(\eta_2\circ\nu'\circ\eta_6\circ\bar{\mu}_7)=0$ by Lemma 5.7 of [4]. By the exactness of (T), π_{25}^3 is isomorphic to $H\pi_{25}^3$ which is $\{\nu_5^2\circ\kappa_{11}\}\cong Z_2$ by (7.1).

By Lemma 6.1 of [4], $H(\varepsilon_3)=\nu_5^2$, so $H(\varepsilon_3\circ\kappa_{11})=\nu_5^2\circ\kappa_{11}$ and

$$\pi_{25}^3 = \{\varepsilon_3\circ\kappa_{11}\} \cong Z_2.$$

By (5.6) and Theorem 12.9 of [4], we obtain immediately

$$\pi_{26}^4 = \{\varepsilon_4\circ\kappa_{12},\ \nu_4\circ\bar{\zeta}_7,\ \nu_4\circ\bar{\sigma}_7\} \cong Z_2 \oplus Z_8 \oplus Z_2.$$

Consider the exact sequence

$$\cdots \longrightarrow \pi_{28}^9 \xrightarrow{\Delta} \pi_{26}^4 \xrightarrow{E} \pi_{27}^5 \xrightarrow{H} \pi_{27}^9 \xrightarrow{\Delta} \pi_{25}^4 \longrightarrow \cdots$$

where $\pi_{28}^9 = \{\bar{\zeta}_9,\ \bar{\sigma}_9\} \cong Z_2 \oplus Z_2$.

By the definition of $\bar{\zeta}_6$ and the relation $\nu'\circ\zeta_6=0$,

$$\begin{aligned}
\nu'\circ\bar{\zeta}_6 &\in \nu'\circ\{\zeta_6,\ 8\iota_{17},\ 2\sigma_{17}\} \\
&= \{\nu',\ \zeta_6,\ 8\iota_{17}\}\circ 2\sigma_{18} && \text{by Proposition 1.4 of [4]} \\
&\subset \pi_{18}^3\circ 2\sigma_{18} \\
&= 0 && \text{by Theorem 10.5 of [4].}
\end{aligned}$$

Therefore we have, by use of (5.8) of [4],

$$\begin{aligned}
\Delta\bar{\zeta}_9 &= \pm(2\nu_4-E\nu')\circ\bar{\zeta}_7 \\
&= \pm 2\nu_4\circ\bar{\zeta}_7.
\end{aligned}$$

By the definition of $\bar{\sigma}_6$ and the relation $\nu'\circ\nu_6=0$,

$$\begin{aligned}
\nu'\circ\bar{\sigma}_6 &\in \nu'\circ\{\nu_6,\ \varepsilon_9+\bar{\nu}_9,\ \sigma_{17}\} \\
&= \{\nu',\ \nu_6,\ \varepsilon_9+\bar{\nu}_9\}\circ\sigma_{18} && \text{by Proposition 1.4 of [4]} \\
&= \{\nu',\ \nu_6,\ \eta_9\circ\sigma_{10}\}\circ\sigma_{18} \\
&\supset \{\nu',\ \nu_6,\ \eta_9\}\circ\sigma_{11}^2 && \text{by Poposition 1.3 of [4]} \\
&\subset \pi_{11}^2\circ\sigma_{11}^2 = \{\varepsilon_3\circ\sigma_{11}^2\} = 0 && \text{by Lemma 10.7 of [4].}
\end{aligned}$$

Therefore we have

$$\nu'\circ\bar{\sigma}_6 \equiv 0 \mod \nu'\circ\pi^6_{18}\circ\sigma_{18}+\pi^3_{10}\circ\eta_{10}\circ\sigma^2_{11} = 0,$$

and $\Delta\bar{\sigma}_9 = \pm(2\nu_4-E\nu')\circ\bar{\sigma}_7 = 0.$

By (7.3), the homomorphism H in the above exact sequence is trivial. It follows that

$$\pi^5_{27} = \{\varepsilon_5\circ\kappa_{13},\ \nu_5\circ\bar{\zeta}_8,\ \nu_5\circ\bar{\sigma}_8\} \cong Z_2 \oplus Z_2 \oplus Z_2.$$

Consider the exact sequence

$$\cdots \longrightarrow \pi^{11}_{29} \xrightarrow{\Delta} \pi^5_{27} \xrightarrow{E} \pi^6_{28} \xrightarrow{H} \pi^{11}_{28} \xrightarrow{\Delta} \pi^5_{26} \longrightarrow \cdots,$$

where π^{11}_{29} is generated by $\eta_{11}\circ\bar{\mu}_{12}$, ξ' and λ'. By (5.10) of [4]

$$\begin{aligned}\Delta(\eta_{11}\circ\bar{\mu}_{12}) &= \nu_5\circ\eta^2_8\circ\bar{\mu}_{10}\\ &= 4\nu_5\circ\bar{\zeta}_8 \qquad \text{by Lemma 12.4 of [4]}\\ &= 0.\end{aligned}$$

By Proposition 2.7 of [4] we have

$$H\Delta(2\lambda) = H\Delta(E^2\lambda') = 2\lambda',$$

whence $H\Delta(\lambda) \equiv \lambda' \mod \{\eta_{11}\circ\mu_{12},\ 2\xi',\ 4\lambda'\}.$

So, $\Delta\lambda' \equiv 0 \mod \{\Delta 2\xi'\}=0.$

Similarly $\Delta\xi' \equiv 0 \mod \{\Delta 2\xi'\}=0.$

Therefore $\Delta : \pi^{11}_{29} \longrightarrow \pi^5_{27}$ is trivial. By (7.4) the kernel of Δ: $\pi^{11}_{28} \longrightarrow \pi^5_{26}$ is isomorphic to $Z_2 \oplus Z_2$ and generated by $\sigma_{11}\circ\eta_{18}\circ\mu_{19}$ and $\nu_{11}\circ\kappa_{14}$. We have

$$\begin{aligned}\nu_{11}\circ\kappa_{14} &= H(\bar{\iota}_6)\circ\kappa_{14} &&\text{by Lemma 6.2 of [4]}\\ &= H(\bar{\iota}_6\circ\kappa_{14}) &&\text{by Proposition 2.2 of [4],}\\ \sigma_{11}\circ\eta_{18}\circ\mu_{19} &= \eta_{11}\circ\sigma_{12}\circ\mu_{19} &&\text{by Lemma 6.4 of [4]}\\ &= H(\rho''')\circ\mu_{19} &&\text{by (10.12) of [4]}\\ &= H(\rho'''\circ\mu_{19}) &&\text{by Proposition 2.2 of [4].}\end{aligned}$$

Moreover

$$2(\nu_{11}\circ\kappa_{14}) = \nu_{11}\circ(2\kappa_{14}) = 0.$$

By the definition of $\bar{\zeta}_9$, we have

$$\begin{aligned}
\nu_6\circ\bar{\zeta}_9 &\in \nu_6\circ\{\zeta_9,\ 8\iota_{20},\ 2\sigma_{20}\}_5 \\
&\subset \{\nu_6\circ\zeta_9,\ 8\iota_{20},\ 2\sigma_{20}\}_5 && \text{by Proposition 1.2 of [4]} \\
&= \{2\sigma''\circ\sigma_{13},\ 8\iota_{20},\ 2\sigma_{20}\}_5 && \text{by (10.7) of [4]} \\
&\subset \{\sigma''\circ\sigma_{13},\ 16\iota_{20},\ 2\sigma_{20}\}_5 && \text{by Proposition 1.2 of [4]} \\
&\supset 2\sigma''\circ\{\sigma_{13},\ 16\iota_{20},\ \sigma_{20}\}_5 && \text{by Proposition 1.2 of [4]} \\
&\ni 4\sigma''\circ\rho_{13} && \text{by Lemma 10.9 of [4]} \\
&= 0,
\end{aligned}$$

where the indeteminacy is $\rho'''\circ 2\sigma_{21} = 2\rho'''\circ\sigma_{21}$. Thus

$$\nu_6\circ\bar{\zeta}_9 = 2n(\rho'''\circ\sigma_{21}) \qquad \text{for } n = 0 \text{ or } 1.$$

But we know that $E : \pi^5_{27} \rightarrow \pi^6_{28}$ is a monomorphism and $\nu_6\circ\bar{\zeta}_9 \neq 0$, since $\Delta : \pi^{11}_{29} \rightarrow \pi^5_{27}$ is trivial. This implies that $n=1$, i.e.,

(8.1) $$\nu_6\circ\bar{\zeta}_9 = 2\rho'''\circ\sigma_{21}.$$

We have obtained

$$\pi^6_{28} = \{\rho'''\circ\sigma_{21},\ \bar{\nu}_6\circ\kappa_{14},\ \varepsilon_6\circ\kappa_{14},\ \nu_6\circ\bar{\sigma}_9\} \cong Z_4 \oplus Z_2 \oplus Z_2 \oplus Z_2.$$

Next consider the homomorphism

$$\Delta : \pi^{13}_{30} \rightarrow \pi^6_{28},$$

where $\pi^{13}_{30} = \{\varepsilon^*_{13},\ \sigma_{13}\circ\eta_{20}\circ\mu_{21},\ \nu_{13}\circ\kappa_{16},\ \bar{\mu}_{13}\} \cong Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2.$

By Example 5.12 of [2], we have

$$\Delta(\varepsilon^*_{13}) = \Delta H(\sigma'\circ\omega_{14}) = 0.$$

We choose an element τ' from the secondary composition $\{\rho'', 8\iota_{22}, 2\sigma_{22}\}$, then

$$H(\tau') \in H\{\rho'',\ 8\iota_{22},\ 2\sigma_{22}\}$$

$$\begin{aligned} &\subset \{H(\rho''), 8\iota_{22}, 2\sigma_{22}\} && \text{by Proposition 2.3 of [4]} \\ &= \{\mu_{13}, 8\iota_{22}, 2\sigma_{22}\} \\ &\supset \{\mu_{13}, 2\iota_{22}, 8\sigma_{22}\} \\ &\ni \bar{\mu}_{13} && \text{by the definition of } \bar{\mu}_{13}. \end{aligned}$$

Thus $\bar{\mu}_{13} \equiv H(\tau') \mod \mu_{13}\circ\pi_{30}^{22}+\pi_{22}^{13}\circ 2\sigma_{22}=\{\varepsilon_{13}\circ\mu_{21}, \bar{\nu}_{13}\circ\mu_{21}\}$
whence $\Delta\bar{\mu}_{13} \equiv \Delta H(\tau') \mod \{\Delta(\varepsilon_{13}\circ\mu_{21}), \Delta(\bar{\nu}_{13}\circ\mu_{21})\}=0$
and $\Delta\bar{\mu}_{13} = 0.$

Furthermore

$$\Delta(\sigma_{13}\circ\eta_{20}\circ\mu_{21}) = (\Delta\eta_{13})\circ\sigma_{18}\circ\mu_{19} = 0$$

and

$$\Delta(\nu_{13}\circ\kappa_{16}) = (\Delta\nu_{13})\circ\kappa_{14} = \pm 2\bar{\nu}_6\circ\kappa_{14} = 0.$$

Thus the homomorphism Δ is trivial, and we have an exact sequence

$$0 \longrightarrow \pi_{28}^{6} \xrightarrow{E} \pi_{29}^{7} \xrightarrow{H} \pi_{29}^{13} \longrightarrow 0,$$

by use of (7.5), where $\pi_{29}^{13} = \{\sigma_{13}\circ\mu_{20}\} \cong Z_2$. Consider the elements $\rho''\circ\sigma_{22}$ and $\sigma'\circ\rho_{14}$ in π_{29}^{7}.

$$\begin{aligned} H(\rho''\circ\sigma_{22}) &= \mu_{13}\circ\sigma_{22} && \text{by (10.12) of [4]} \\ &= \sigma_{13}\circ\mu_{20} \\ &= \eta_{13}\circ\rho_{14} && \text{by Proposition 12.20 of [4]} \\ &= H(\sigma')\circ\rho_{14} && \text{by Lemma 5.14 of [4]} \\ &= H(\sigma'\circ\rho_{14}). \end{aligned}$$

It follows

$$\rho''\circ\sigma_{22} \equiv \sigma'\circ\rho_{14} \mod E\pi_{29}^{6} = \{E\rho'''\circ\sigma_{22}, \bar{\varepsilon}_7\circ\kappa_{15}, \varepsilon_7\circ\kappa_{15}, \nu_7\circ\bar{\sigma}_{10}\},$$

and

$$\begin{aligned} 2\sigma'\circ\rho_{14} &\equiv 2\rho''\circ\sigma_{22} \mod 2E\rho'''\circ\sigma_{22} \\ &= E(\rho'''\circ\sigma_{21}). \end{aligned}$$

So, $2\sigma'\circ\rho_{14} = (2n+1)E(\rho'''\circ\sigma_{21})$ for some integer n.

In any way the order of $\sigma' \circ \rho_{14}$ is 8, and we have

$$\pi_{29}^{7} = \{\sigma' \circ \rho_{14},\ \bar{\iota}_7 \circ \kappa_{15},\ \varepsilon_7 \circ \kappa_{15},\ \nu_7 \circ \bar{\sigma}_{10}\} \cong Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2.$$

It is easily verified by use of (5.15) of [4] that

$$\pi_{30}^{8} = \{E\sigma' \circ \rho_{14},\ \bar{\iota}_8 \circ \kappa_{16},\ \varepsilon_8 \circ \kappa_{16},\ \nu_8 \circ \bar{\sigma}_{11},\ \sigma_8 \circ \rho_{15},\ \sigma_8 \circ \bar{\varepsilon}_{15}\}$$

$$\cong Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_{32} \oplus Z_2.$$

Lemma 8.1. $\eta_7 \circ \sigma_8 \circ \kappa_{15} = 0$.

Proof. $\sigma_8 \# \bar{\varepsilon}_3 = \sigma_{11} \circ \bar{\varepsilon}_{18} = \bar{\varepsilon}_{11} \circ \sigma_{26}$ by Proposition 3.1 of [4]

$\in \{\varepsilon_{11},\ 2\iota_{19},\ \nu_{19}^2\} \circ \sigma_{26}$

$= \varepsilon_{11} \circ \{2\iota_{19},\ \nu_{19},\ \sigma_{25}\}$ by Proposition 1.4 of [4].

In the stable range, we have

$\eta \circ \langle 2\iota,\ \nu^2,\ \sigma\rangle = \langle \eta,\ 2\iota, \nu^2\rangle \circ \sigma$ by (3.5) of [4]

$= \varepsilon \circ \sigma$ by (6.1) of [4]

$= 0$ by Theorem 14.1 of [4].

Since $\eta \circ \kappa = \bar{\varepsilon} \neq 0$ and $\eta \circ \sigma^2 = 0$, we obtain

$$\langle 2\iota,\ \nu^2,\ \sigma\rangle \equiv 0 \mod \sigma^2.$$

It follows from this relation that

$$\bar{\varepsilon}_{11} \circ \sigma_{26} \equiv 0 \mod \varepsilon_{11} \circ \sigma_{19}^2,$$

where $\varepsilon_{11} \circ \sigma_{19}^2 = 0$ by Lemma 10.7 of [4]. That is

$$\bar{\varepsilon}_{11} \circ \sigma_{26} = \sigma_{11} \circ \bar{\varepsilon}_{18} = 0.$$

The exactness of $0 = \pi_{34}^{21} \rightarrow \pi_{32}^{10} \xrightarrow{E} \pi_{33}^{11}$ implies that

$$\bar{\varepsilon}_{10} \circ \sigma_{25} = \sigma_{10} \circ \bar{\varepsilon}_{17} = 0.$$

By Lemma 10.7 of [4],

$$\Delta \sigma_{19}^2 = (\sigma_9 \circ \eta_{16} + \bar{\iota}_9 + \varepsilon_9) \circ \sigma_{17}^2 = 0.$$

Since π_{33}^{19} is generated by σ_{19}^2 and κ_{19}, we have

(8.2) $$\sigma_9 \circ \bar{\varepsilon}_{16} \in \mathrm{Ker}(E : \pi_{31}^{9} \rightarrow \pi_{32}^{10}) = \Delta \pi_{33}^{19} = \{\Delta \kappa_{19}\}.$$

Here we have $\sigma_9\circ\bar{\varepsilon}_{16}\neq 0$, as the kernel of $E:\pi_{30}^8\to\pi_{31}^9$ is generated by

(8.3) $\Delta\rho_{17} = \pm(2\sigma_8 - E\sigma')\circ\rho_{15}$ by (5.16) of [4]

and $\Delta\varepsilon_{17} = \pm(2\sigma_8 - E\sigma')\circ\bar{\varepsilon}_{15}$

$= E\sigma'\circ\bar{\varepsilon}_{15} = E\sigma'\circ\eta_{15}\circ\kappa_{16}$ by (10.3) of [4]

$= (\eta_8\circ\sigma_9 + \bar{\nu}_8 + \varepsilon_8)\circ\kappa_{16}$ by (7.5) of [4].

Hence

(8.4) $$\Delta\kappa_{19} = \sigma_9\circ\bar{\varepsilon}_{16}.$$

On the other hand

$\Delta\kappa_{19} = (\sigma_9\circ\eta_{16} + \bar{\nu}_9 + \varepsilon_9)\circ\kappa_{17}$ by (7.1) of [4]

$= \sigma_9\circ\bar{\varepsilon}_{16} + \bar{\nu}_9\circ\kappa_{17} + \varepsilon_9\circ\kappa_{17}$ by (10.23) of [4].

So, $\eta_9\circ\sigma_{10}\circ\kappa_{17} = \bar{\nu}_9\circ\kappa_{17} + \varepsilon_9\circ\kappa_{17} = 0$ by Lemma 6.4 of [4],

whence $(\bar{\nu}_8 + \varepsilon_8)\circ\kappa_{16} \in \mathrm{Ker}\,(E:\pi_{30}^8\to\pi_{31}^9) = \Delta\pi_{32}^{17}$,

i.e., $(\bar{\nu}_8+\varepsilon_8)\circ\kappa_{16} = \pm\Delta(a\ \rho_{17} + b\ \bar{\varepsilon}_{17})$ for some integers a, b

$= 2a\ \sigma_8\circ\rho_{15} - E(a\ \sigma'\circ\rho_{14} + b(\eta_7\circ\sigma_8\circ\kappa_{15} + \bar{\nu}_7\circ\kappa_{15} + \varepsilon_7\circ\kappa_{15}))$.

So, $2a \equiv 0 \mod 32$

or $a = 16\ a'$ for some integer a'.

Since $16(\Delta\rho_{17}) = 0$ and $(\bar{\nu}_8+\varepsilon_8)\circ\kappa_{16} \neq 0$, we have

(8.5) $$\Delta\bar{\varepsilon}_{17} = (\bar{\nu}_8+\varepsilon_8)\circ\kappa_{16}.$$

On the other hand $\Delta\bar{\varepsilon}_{17} = \eta_8\circ\sigma_9\circ\kappa_{16} + (\varepsilon_8+\bar{\nu}_8)\circ\kappa_{16}$, we have then

$$\eta_8\circ\sigma_9\circ\kappa_{16} = 0,$$

and by use of the monomorphism $E:\pi_{29}^7\to\pi_{30}^8$, we obtain the required relation $\eta_7\circ\sigma_8\circ\kappa_{15} = 0$. q. e. d.

By (7.6) we have the following exact sequence :

$$\cdots\longrightarrow \pi_{32}^{17} \xrightarrow{\Delta} \pi_{30}^8 \xrightarrow{E} \pi_{31}^9 \longrightarrow 0,$$

where $$\pi_{32}^{17} = \{\rho_{17},\ \bar{\varepsilon}_{17}\}.$$

It follows from (8.3) and (8.5) that

$$\pi_{31}^{9}=\{\sigma_9\circ\rho_{16},\ \varepsilon_9\circ\kappa_{17},\ \nu_9\circ\bar{\sigma}_{12},\ \sigma_9\circ\bar{\varepsilon}_{16}\}\cong Z_{16}\oplus Z_2\oplus Z_2\oplus Z_2.$$

Consider the following exact sequence :

$$\pi_{33}^{19}\xrightarrow{\Delta}\pi_{31}^{9}\xrightarrow{E}\pi_{32}^{10}\longrightarrow\pi_{32}^{19}=0,.$$

where $\pi_{33}^{19}=\{\sigma_{19}^2,\ \kappa_{19}\}\cong Z_2\oplus Z_2.$

By (8.2) and (8.4) we have

$$\pi_{32}^{10}=\{\sigma_{10}\circ\rho_{17},\ \varepsilon_{10}\circ\kappa_{18},\ \nu_{10}\circ\bar{\sigma}_{13}\}\cong Z_{16}\oplus Z_2\oplus Z_2.$$

It is verified from the results $\pi_{34}^{21}=\pi_{33}^{21}=0$ that

$$\pi_{33}^{11}=\{\sigma_{11}\circ\rho_{18},\ \varepsilon_{11}\circ\kappa_{19},\ \nu_{11}\circ\bar{\sigma}_{14}\}\cong Z_{16}\oplus Z_2\oplus Z_2.$$

By (7.7), we have an exact sequence

$$0=\pi_{35}^{23}\longrightarrow\pi_{33}^{11}\xrightarrow{E}\pi_{34}^{12}\xrightarrow{H}\pi_{34}^{23}\longrightarrow 0,$$

where $\pi_{34}^{23}=\{\zeta_{23}\}\cong Z_8.$

The element $\sigma^{*\prime\prime\prime}$ in Lemma 6.2 of [2] has the properties $H(\sigma^{*\prime\prime\prime})=\zeta_{23}$ and $8\sigma^{*\prime\prime\prime}\equiv 4\sigma_{12}\circ\rho_{19}$ mod $\{8\sigma_{12}\circ\rho_{19}\}$, whence the order of $\sigma^{*\prime\prime\prime}$ is 32.

By an easy calculation we have

$$\pi_{34}^{12}=\{\sigma^{*\prime\prime\prime},\ \sigma_{12}\circ\rho_{19}\pm 2\sigma^{*\prime\prime\prime},\ \varepsilon_{12}\circ\kappa_{20},\ \nu_{12}\circ\bar{\sigma}_{15}\}\cong Z_{32}\oplus Z_4\oplus Z_2\oplus Z_2.$$

We have

$$2\rho'\circ\sigma_{24}\in\{\sigma_9,\ 16\iota_{16},\ \sigma_{16}\}\circ 2\sigma_{24}\quad\text{by the definition of }\rho'$$

$$=-\sigma_9\circ\{16\iota_{16},\ \sigma_{16},\ 2\sigma_{23}\}\quad\text{by Proposition 1.4 of [4]}$$

$$\equiv -2\sigma_9\circ\rho_{16}\quad\text{mod}\quad\sigma_9\circ\pi_{24}^{16}\circ 2\sigma_{24}+\{2\sigma_9\circ\Delta\iota_{33}\},$$

since it is verified that $<16\iota,\ \sigma,\ 2\sigma>$ contains 2ρ in [4]. $2\pi_{24}^{16}=0$ and $2\sigma_9\circ\Delta\iota_{33}=2[\sigma_9,\ \sigma_9]=2\Delta\iota_9\circ\sigma_{17}^2=0$. Thus

(8.6) $$2\rho'\circ\sigma_{24}=-2\sigma_9\circ\rho_{16}.$$

Applying E^4

(8.7) $4\rho_{12}\circ\sigma_{28}=-2\sigma_{13}\circ\rho_{20}$ *and* $8\rho_{13}\circ\sigma_{28}=\pm 8E\sigma^{*'''}$.

Consider the exact sequence

$$\cdots\longrightarrow \pi_{36}^{25}\xrightarrow{\Delta}\pi_{34}^{12}\xrightarrow{E}\pi_{35}^{13}\xrightarrow{H}\pi_{35}^{25}\longrightarrow\cdots,$$

where $\pi_{36}^{25}=\{\zeta_{25}\}\cong Z_8$, $\pi_{35}^{25}=\{\eta_{25}\circ\mu_{26}\}\rightarrow\cdots\cong Z_2$ and H is trivial by (7.8).

By Proposition 3.2 of [4] we have that

$$\Delta(E^{16}(\eta_9^2\circ\mu_{11})\circ E^{21}\iota_{15})=E^{21}\rho^{\mathrm{IV}}\circ E^{19}\sigma_8+E^4\sigma_8\circ E^{14}\rho^{\mathrm{IV}}$$

i. e.,
$$\begin{aligned}4\Delta\zeta_{25}&=\Delta(\eta_{25}^2\circ\mu_{27})\\&=8E^3\rho'\circ\sigma_{27}+16\sigma_{12}\circ\rho_{19}\\&=8\sigma_{12}\circ\rho_{19}\qquad\text{by (8.7) and } 16\sigma_{12}\circ\rho_{19}=0.\end{aligned}$$

Thus $\Delta\zeta_{25}$ is of order 8.

Also, by Proposition 2.7 of [4],

$$\begin{aligned}H\Delta(\zeta_{25})&=\pm 2\zeta_{23}\\&=H(2\sigma^{*'''}).\end{aligned}$$

So, $$\pm\Delta\zeta_{25}\equiv 2\sigma^{*'''}\mod E\pi_{33}^{11},$$

whence $\pm\Delta\zeta_{25}=2\sigma^{*'''}+x\sigma_{12}\circ\rho_{19}+y\varepsilon_{12}\circ\kappa_{20}+z\nu_{12}\circ\bar{\sigma}_{15}$ for some integers x, y and z.

Since $\Delta\zeta_{25}$ is of order 8, we have

$$\begin{aligned}0=8\Delta\zeta_{25}&=16\sigma^{*'''}+8x\sigma_{12}\circ\rho_{19}\\&=8(x+1)\sigma_{12}\circ\rho_{19}\quad\text{by Lemma 6.2 of [2],}\end{aligned}$$

whence $x\equiv 1\mod 2$.

Thus $\Delta\zeta_{25}\equiv 2\sigma^{*'''}+(2n-1)\sigma_{12}\circ\rho_{19}\mod\{\varepsilon_{12}\circ\kappa_{20},\ \nu_{12}\circ\bar{\sigma}_{15}\}$ for some integer n. An easy computation shows

$$\pi_{35}^{13}=\{\rho_{13}\circ\sigma_{28},\ \varepsilon_{13}\circ\kappa_{21},\ \nu_{13}\circ\bar{\sigma}_{16}\}\cong Z_{16}\oplus Z_2\oplus Z_2.$$

Consider the exact sequence

$$\cdots\longrightarrow\pi_{37}^{27}\xrightarrow{\Delta}\pi_{35}^{13}\xrightarrow{E}\pi_{36}^{14}\xrightarrow{H}\pi_{36}^{27}\xrightarrow{\Delta}\pi_{34}^{13}\longrightarrow\cdots,$$

where $\pi_{37}^{27}=\{\eta_{27}\circ\mu_{28}\}\cong Z_2$ and the image of H is $\{\nu_{27}^3, \eta_{27}\circ\varepsilon_{28}\}\cong Z_2 \oplus Z_2$ by (7.9). We have

$$H(\omega_{14}\circ\nu_{30}^2)=\nu_{27}^3 \qquad \text{by Lemma 12.15 of [4],}$$

$$H(\sigma^{*'''})=\nu_{27}^3+\eta_{27}\circ\varepsilon_{28} \qquad \text{by Lemma 6.2 of [2].}$$

By Proposition 3.2. of [4],

$$\begin{aligned}\Delta(\eta_{27}\circ\varepsilon_{28}) &= 8\rho_{13}\circ\sigma_{28}+8\sigma_{13}\circ\rho_{20}\\ &= 24\rho_{13}\circ\sigma_{28} \qquad \text{by (8.7)}\\ &= 8\rho_{13}\circ\sigma_{28}\neq 0.\end{aligned}$$

Then we have obtained an exact sequence

$$0\longrightarrow E\pi_{35}^{13}\longrightarrow \pi_{36}^{14}\longrightarrow H\pi_{36}^{13}\longrightarrow 0,$$

where $\quad E\pi_{35}^{13}=\{\rho_{14}\circ\sigma_{29},\ \varepsilon_{14}\circ\kappa_{22},\ \nu_{14}\circ\bar{\jmath}_{17}\}\cong Z_8\oplus Z_2\oplus Z_2.$

Obviously

$$2\omega_{14}\circ\nu_{30}^2=\omega_{14}\circ(2\nu_{30}^2)=0.$$

As we have the relation

$$\rho_{14}\circ\sigma_{29}\equiv 2\sigma^{*''} \mod \sigma_{14}\circ E\pi_{35}^{20}+2\pi_{29}^{14}\circ\sigma_{29}=\{2\rho_{14}\circ\sigma_{29}\}$$

by Lemma 6.2 of [2]. we see that the order of $\sigma^{*''}$ is 16. Thus

$$\pi_{36}^{14}=\{\sigma^{*''},\ \omega_{14}\circ\nu_{30}^2,\ \varepsilon_{14}\circ\kappa_{22},\ \nu_{14}\circ\bar{\jmath}_{17}\}\cong Z_{16}\oplus Z_2\oplus Z_2\oplus Z_2.$$

By (7.13), we have an exact sequence

$$\cdots\longrightarrow \pi_{38}^{29}\xrightarrow{\Delta}\pi_{36}^{14}\xrightarrow{E}\pi_{37}^{15}\xrightarrow{H} Z_2\longrightarrow 0,$$

where $\pi_{38}^{29}=\{\nu_{29}^3,\ \mu_{29},\ \eta_{29}\circ\varepsilon_{30}\}\cong Z_2\oplus Z_2\oplus Z_2$ and the last Z_2 is generated by $\bar{\nu}_{29}+\varepsilon_{29}=H(\sigma^{*'})$. By Lemme 6.2 of [2],

$$E\sigma^{*''}\equiv 2\sigma^{*'} \mod \rho_{15}\circ\sigma_{30}\equiv 2E\sigma^{*''}.$$

By Proposition 3.2 of [4],

$$\Delta(E^{16}H(\rho'')\circ E^{23}\iota_{15})=E^7\rho''\circ E^{21}\sigma_8+E^6\sigma_8\circ E^{14}\rho'',$$

whence $\quad \Delta(\mu_{29})=4\rho_{14}\circ\sigma_{29}+4\sigma_{14}\circ\rho_{21}$

$$= 12\rho_{14}\circ\sigma_{29} \qquad \text{by} \quad (8.7)$$

$$= 4\rho_{14}\circ\sigma_{29}$$

$$= 8\sigma^{*\prime\prime} \neq 0 \qquad \text{by Lemma 6.2 of [2].}$$

Therefore

(8.8) $\quad 8E\sigma^{*\prime\prime} = 4\rho_{15}\circ\sigma_{30} = 0,$ *and the order of* $\sigma^{*\prime}$ *is* 16.

By (10.10) of [4], we have

$$\Delta(\eta_{29}\circ\varepsilon_{30}) = 4\sigma_{14}^2\circ\varepsilon_{28} = 0$$

and
$$\Delta(\nu_{29}^3) = 4\sigma_{14}^2\circ\bar{\nu}_{28} = 0.$$

These discussions imply

$$\pi_{37}^{15} = \{\sigma^{*\prime},\ \omega_{15}\circ\nu_{31}^2,\ \varepsilon_{15}\circ\kappa_{23},\ \nu_{15}\circ\bar{\zeta}_{18}\} \cong Z_{16} \oplus Z_2 \oplus Z_2 \oplus Z_2.$$

Next, π_{39}^{31} is isomorphic to $Z_2 \oplus Z_2$ and generated by $\bar{\nu}_{31}$ and ε_{31}. By Proposition 3.2 of [4]

$$\Delta(\bar{\nu}_{31}+\varepsilon_{31}) = E^3(\xi_{12}\circ\eta_{30})\circ E^{30}\nu_4 + E^{11}\nu_4\circ E^6(\xi_{12}\circ\eta_{30})$$

$$= \xi_{15}\circ\eta_{33}\circ\nu_{34} + \nu_{15}\circ\xi_{18}\circ\eta_{36}$$

$$= 2\xi_{15}\circ\eta_{33}\circ\nu_{34}$$

$$= 0,$$

whenceΔ $\quad \Delta\bar{\nu}_{31} = \Delta\varepsilon_{31}$

Choose an element $\bar{\nu}_{31}^*$ from the secondary composition $\{\sigma_{16},\ 2\sigma_{23},\ \bar{\nu}_{30}\}_1$, then

$$H(\bar{\nu}_{31}^*) \in H\{\sigma_{16},\ 2\sigma_{23},\ \bar{\nu}_{30}\}_1$$

$$= -\Delta^{-1}(2\sigma_{15}^2)\circ E^2\bar{\nu}_{29} \qquad \text{by Proposition 2.6 of [4]}$$

$$= \bar{\nu}_{31}.$$

Therefore $\Delta\varepsilon_{31} = \Delta\bar{\nu}_{31} = \Delta H(\bar{\nu}_{31}^*) = 0.$
So, we have an exact sequence

$$0 \longrightarrow \pi_{37}^{15} \xrightarrow{E} \pi_{38}^{16} \xrightarrow{H} \pi_{38}^{31}.$$

By Lemma 6.2 of [2], $H(\sigma_{16}^*) \equiv \sigma_{31} \bmod 2\sigma_{31}$. σ_{31} generates $\pi_{38}^{31} \cong Z_{16}$.

The order of σ_{16}^{*} is 16 since $16\sigma_{16}^{*}\equiv\{\sigma_{16}.\ 2\sigma_{23},\ \sigma_{30}\}\circ 16\iota_{38}\equiv-\sigma_{16}\circ\{2\sigma_{23},\ \sigma_{30},\ 16\iota_{37}\}\equiv-\sigma_{16}\circ(2\rho_{23})=0$ mod $16\pi_{38}^{16}=0$ by (8.7) and (8.8). Thus the above sequence splits:

$$\pi_{38}^{16}=\{\sigma_{16}^{*},\ E\sigma^{*\prime},\ \omega_{16}\circ\nu_{32}^{2},\ \varepsilon_{16}\circ\kappa_{24},\ \nu_{16}\circ\bar{\jmath}_{19}\}\cong Z_{16}\oplus Z_{16}\oplus Z_{2}\oplus Z_{2}\oplus Z_{2}.$$

Consider the exact sequence

$$\pi_{40}^{33}\xrightarrow{\Delta}\pi_{38}^{16}\xrightarrow{E}\pi_{39}^{17}\xrightarrow{H}\pi_{39}^{33}\longrightarrow\pi_{37}^{16}$$

where $\pi_{40}^{33}=\{\sigma_{33}\}\cong Z_{16}$ and $\pi_{39}^{33}=\{\nu_{33}^{2}\}\cong Z_{2}$. By (7.10), $\Delta\nu_{33}^{2}=E^{3}(\lambda\circ\nu_{31})\neq 0$. Thus the above E is an epimorphism.

By Lemma 6.2 of [2],

$$2\sigma_{16}^{*}=E\sigma^{*\prime}\quad\text{mod}\quad\{\rho_{16}\circ\sigma_{31},\ \Delta\sigma_{33}\}.$$

However we have, by Proposition 2.4 of [4],

$$H\Delta(\sigma_{33})=\pm 2\sigma_{31}=H(\pm 2\sigma_{16}^{*}),$$

whence $\Delta\sigma_{33}\equiv\pm 2\sigma_{16}^{*}$ mod $E\pi_{37}^{15}$.

Thus $\pm\Delta\sigma_{33}\equiv 2\sigma_{16}^{*}-E\sigma^{*\prime}$ mod $\{\rho_{16}\circ\sigma_{31}\}$.

It follows

$$\pi_{39}^{17}=E\pi_{38}^{16}=\{\sigma_{17}^{*},\ \omega_{17}\circ\nu_{33}^{2},\ \varepsilon_{17}\circ\kappa_{25},\ \nu_{17}\circ\bar{\jmath}_{20}\}\cong Z_{16}\oplus Z_{2}\oplus Z_{2}\oplus Z_{2}.$$

In the exact sequence

$$\pi_{41}^{35}\xrightarrow{\Delta}\pi_{39}^{17}\xrightarrow{E}\pi_{40}^{18}\xrightarrow{H}\pi_{40}^{35}=0,$$

$\pi_{41}^{35}=\{\nu_{35}^{2}\}\cong Z_{2}$. By the relation $\Delta\nu_{35}=\omega_{17}\circ\nu_{33}$ in *p*.170 of [4], $\Delta\nu_{35}^{2}=\omega_{17}\circ\nu_{33}^{2}$.

Thus

$$\pi_{40}^{18}=\{\sigma_{18}^{*},\ \varepsilon_{18}\circ\kappa_{26},\ \nu_{18}\circ\bar{\jmath}_{21}\}\cong Z_{16}\oplus Z_{2}\oplus Z_{2}.$$

Lemma 8.2. *In the stable range* $\sigma\circ\rho=\rho\circ\sigma=0$ *and* $\sigma^{*}=0$.

Proof. There exists an element $\beta_{2}\in\pi_{15}(SO(n))$ such that $J(\beta_{2})=\rho_{n}$, where n is sufficiently large. $\beta_{2}\circ\sigma_{15}\in\pi_{22}(SO(n))=0$ since $22\equiv 6$ (mod 8). Thus $\rho\circ\sigma=\sigma\circ\rho=E^{\infty}J(\beta_{2}\circ\sigma_{15})=0$.

Next consider $\sigma^{*}\in\langle\sigma,\ 2\sigma,\ \sigma\rangle$. By (3.10) of [4],

$$<\sigma, 2\sigma, \sigma> = <\sigma, 2\sigma, 2\sigma> = 2<\sigma, 2\sigma, \sigma>$$

mod $\{\sigma\circ\eta\circ\kappa, \sigma\circ\rho\}=\{\sigma\circ\eta\circ\kappa\}$. $\sigma\circ\kappa=0$ by Lemma 8.1. Thus $<\sigma, 2\sigma, \sigma> = 0$, and $\sigma^* = 0$. q. e. d.

Lemma 8.3. $8\sigma^*_{20} = 4\Delta\nu_{41}$,

$$4\sigma^*_{21} = \Delta\eta^2_{43},$$

$$2\sigma^*_{22} = \Delta\eta_{45}$$

and $\sigma^*_{23} = \Delta\iota_{47}$.

Proof. Obviously $2\Delta\eta_{45}=2\Delta\eta^2_{43}=0$. $2\Delta\iota_{47}=\pm\Delta H(\iota_{49})=0$. Im $E\cap\{\Delta\nu_{41}\}=\{4\Delta\nu_{41}\}$ since $H(\Delta\nu_{41})=\pm2\nu_{37}$ is of order 4. These elements $\Delta\iota_{47}$, $\Delta\eta_{45}$, $\Delta\eta^2_{43}$ and $4\Delta\nu_{41}$ generate the kernel of E^∞ : $\pi^{18}_{40}\to (G_{22}\,; 2)$, which is at most 16 elements. On the other hand σ^*_{18} is of order 16 and vanishes in the stable range by Lemma 8.2. Then Lemma 8.3 follows immediatelely. q. e. d.
Consider the exact sequences

$$\pi^n_{n+22} \xrightarrow{E} \pi^{n+1}_{n+23} \xrightarrow{H} \pi^{2n+1}_{n+23} \xrightarrow{\Delta} \pi^n_{n+21}$$

for $n \geqq 18$. The above discussions clarify the kernels of E. We also see in §7 that Δ are monomorphisms except the case $n=19$. For $n=19$ the kernel of Δ is generated by $2\nu_{37}=\pm H(\Delta\nu_{41})$. Therefore the following results are verified easily :

$$\pi^{19}_{41} = \{\sigma^*_{19}, \varepsilon_{19}\circ\kappa_{27}, \nu_{19}\circ\bar{\sigma}_{22}\} \cong Z_{16}\oplus Z_2\oplus Z_2,$$

$$\pi^{20}_{42} = \{\sigma^*_{20}, \Delta\nu_{41}+2\sigma^*_{20}, \varepsilon_{20}\circ\kappa_{28}, \nu_{20}\circ\bar{\sigma}_{23}\} \cong Z_{16}\oplus Z_4\oplus Z_2\oplus Z_2,$$

$$\pi^{21}_{43} = \{\sigma^*_{21}, \varepsilon_{21}\circ\kappa_{29}, \nu_{21}\circ\bar{\sigma}_{24}\} \cong Z_8\oplus Z_2\oplus Z_2,$$

$$\pi^{22}_{44} = \{\sigma^*_{22}, \varepsilon_{22}\circ\kappa_{30}, \nu_{22}\circ\bar{\sigma}_{25}\} \cong Z_4\oplus Z_2\oplus Z_2,$$

$$\pi^{23}_{45} = \{\sigma^*_{23}, \varepsilon_{23}\circ\kappa_{31}, \nu_{23}\circ\bar{\sigma}_{26}\} \cong Z_2\oplus Z_2\oplus Z_2,$$

$$\pi^{24}_{46} = \{\varepsilon_{24}\circ\kappa_{32}, \nu_{24}\circ\bar{\sigma}_{27}\} \cong Z_2\oplus Z_2$$

and $(G_{22}: 2) = \{\varepsilon\circ\kappa, \nu\circ\bar{\sigma}\} \cong Z_2\oplus Z_2$.

Appendix. Table of $\pi_{n+21}(S^n)$ and $\pi_{n+22}(S^n)$.

For the completeness of this paper we quote from [5] the odd primary components of $\pi_{n+21}(S^n)$ and $\pi_{n+22}(S^n)$.

The results are the followings:

$$\text{odd component of } \pi_{2m+22}(S^{2m+1}) \cong \begin{cases} 0 & m=1 \text{ and } m \geqq 5, \\ Z_3 & m=2, 3, 4, \end{cases}$$

$$\text{odd component of } \pi_{2m+23}(S^{2m+1}) \cong \begin{cases} Z_{105} & m=1, \\ Z_{45} & m=2, \\ Z_9 & m=3, 4, \\ Z_3 & m=5, \\ 0 & m \geqq 6. \end{cases}$$

By Serre's isomorphism

$\pi_{i-1}(S^{2m-1}: p) \oplus \pi_i(S^{4m-1}: p) \cong \pi_i(S^{2m}: p)$, p : *odd prime*,

we can compute easily $\pi_{n+21}(S^n: p)$ and $\pi_{n+22}(S^n: p)$ for even n.

In the following table, an integer n indicates a cyclic group Z_n of order n, the symbol "∞" an infinite cyclic group Z, the symbol "+" the direct sum of groups and $(2)^k$ indicates the direct sum of k-copies of Z_2.

Table of $\pi_{n+21}(S^n)$ and $\pi_{n+22}(S^n)$.

$n=$	1	2	3	4	5	6	7
$\pi_{n+21}(S^n) \cong$	0	$(2)^2$	2	$24+(2)^2$	$6+2$	6	$6+2$
$\pi_{n+22}(S^n) \cong$	0	2	210	$9240+6+2$	$90+(2)^2$	$180+(2)^3$	$72+(2)^3$

$n=$	8	9	10	11
$\pi_{n+21}(S^n) \cong$	$12+(2)^3$	$6+(2)^2$	$6+(2)^2$	$(2)^4$
$\pi_{n+22}(S^n) \cong$	$1440+24+(2)^4$	$144+(2)^3$	$144+6+2$	$48+(2)^2$

$n=$	12	13	14	15
$\pi_{n+21}(S^n)\cong$	$6+(2)^4$	$4+(2)^3$	$4+(2)^2$	$(2)^3$
$\pi_{n+22}(S^n)\cong$	$2016+12+(2)^2$	$16+(2)^2$	$16+(2)^2$	$16+(2)^3$

$n=$	16	17	18	19
$\pi_{n+21}(S^n)\cong$	$(2)^4$	$(2)^3$	$(2)^3$	$(2)^4$
$\pi_{n+22}(S^n)\cong$	$240+16+(2)^3$	$16+(2)^3$	$16+(2)^2$	$16+(2)^2$

$n=$	20	21	22	23	$n\geqq 24$
$\pi_{n+21}(S^n)\cong$	$(2)^4$	$(2)^3$	$\infty+(2)^2$	$(2)^2$	$(2)^2$
$\pi_{n+22}(S^n)\cong$	$48+4+(2)^2$	$8+(2)^2$	$4+(2)^2$	$(2)^3$	$(2)^2$

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