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Remark on coefficient fields in complete local rings

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Let $R' \supseteq R$ be complete local rings (not necessarily Noetherian) of prime characteristic p such that R' dominates R. Denote the residue class fields of R' and R by K' and K, respectively. Let g be the natural homomorphism of R' onto K'.

In [2, p. 92], it is proved that under the assumption $R'^{p^n} \subseteq R$ for some positve integer n, there exists a coefficient field of R which is extendable to one of R' if (1) $g(R'^{p^i} \cap R) = K'^{p^i} \cap K$ $(i = 1, 2, \cdots)$ and (2) there exists a p-basis B of K' such that $K'^{p^n}(C) = K$ where $C = \{b^{p^e} \mid b \in B,\ e = e(b)$ is the exponent of b over $K\}$.

Our purpose is to note the following two extensions of this result for K' pure inseparable and of arbitrary exponent over K. Neither depends on R' being integral over R.

Theorem 1. Suppose $\bigcap_i^{\infty} R'^{p^i}[R] = R$. If (1) $g(R'^{p^i} \cap R) = K'^{p^i} \cup K$ $(i = 1, 2, \cdots)$ and (2) there exists a p-basis B of K' such that $K'^{p^i}(C) \supseteq K$ $(i = 1, 2, \cdots)$, then there exists a coefficient field of R which is extendable to one of R'.

The assumption $\bigcap_i^{\infty} R'^{p^i}[R] = R$ implies the maximal perfect subfield $P = \bigcap_i^{\infty} K'^{p^i}$ of K' is contained in K. The following considers the case $P \supseteq P \cap K$. Let $P' = \bigcap_i^{\infty} R'^{p^i}$.

Theorem 2. Suppose $g(P' \cap R) = P \cap K$. If (1) of Theorem 1 holds and (2) there exists a set $D \subseteq P \cap K$ and a p-basis B of K'

such that $K^p(D, C)=K$, then there exists a coefficient field of R which is extendable to one of R'.

That these two theorems extend the result in [2, p. 92] is a consequence of the following: $K'^{p^n}(C)=K$ implies $K'^{p^{n+1}}(C)=K^p(C)$ and since $K'^{p^n}=K'^{p^{n+1}}(B^{p^n})\subseteq K'^{p^{n+1}}(C)$, we have $K^p(C)=K$. For Theorem 1, $K^p(C)=K$ implies $K^{p^i}(C)=K$ whence $K'^{p^i}(C)\supseteq K$ $(i=1, 2,\cdots)$. Clearly, $R'^{p^n}\subseteq R$ implies $\bigcap_i^\infty R'^{p^i}[R]=R$. For Theorem 2, we can take D to be the empty set and clearly $K'^{p^n}\subseteq K$ implies $g(P'\cap R)=P\cap K$ since $g\bigcap_i^\infty R^{p^i}=P\subseteq K$ here. Thus, by Theorem 2, the assumption $R'^{p^n}\subseteq R$ in [2] can be replaced by $K'^{p^n}\subseteq K$ whence R' need not be integral over R.

Proof of Theorem 1. By (1) there exists a set of representatives B_0 of B in R' such that for all $b\in B$, $b^{p^e}\in K$ implies $b_0^{p^e}\in R$ where $b_0\in B_0$ and $gb_0=b$. By the existence lemma in [2, p. 91], there exists a coefficient field k' of R' containing B_0. Clearly, k' contains a set of representatives $C_0\subseteq R$ of C by the choice of B_0. By (2), $gk'^{p^i}(C_0)\supseteq K$. Thus, there exists a field $k\subseteq\bigcap_i^\infty k'^{p^i}(C_0)$ such that $gk=K$. Since $C_0\subseteq R$ and $\bigcap_i^\infty R'^{p^i}[R]=R$, $k\subseteq R$. Q.E.D.

Proof of Theorem 2. By hypothesis and (1), there exists a set of representatives B_0 of B in R' such that $R'^{p^i}[B_0]$ $(i=1, 2, \cdots)$ contain sets of represenatives D_0, $C_0\subseteq R$ of D, C respectively. By (2), there exists a subset $G\subseteq D\cup C$ which is a p-basis of K. Hence, $R'^{p^i}[B_0]$ contains a set of representatives $G_0\subseteq R$ of G. The remainder of the proof follows as in the proof of Theorem 27, [3, p, 306], from the inclusion $R'^{p^i}[B_0]\supseteq R^{p^i}[G_0]$ and from the completeness of R' and R. Q.E.D.

If K is separable algebraic over $(P\cap K)(C)$, then (2) of Theorem 2 holds. It follows that B is an algebraically independent set over P and thus over $P\cap K$. Hence C is an algebraically independent set over $P\cap K$. By footnote 14, [1, p. 378], $D\cup C$ is a p-basis of $(P\cap K)(C)$ where D is any p-basis of $P\cap K$. Since K is separable algebraic over $(P \cap K)(C)$, $D \cup C$ is a p-basis of K by footnote 14 or Theorem 8, [1].

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