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Some remarks on the 14th problem of Hilbert

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As for the 14th problem of Hilbert, we know that the answer is negative (see [2]). But there are several known sufficient conditions for the affirmative answer (those in [6] and [8] are important). In the present paper, we are to give some sufficient conditions of a new type but related to the one given in [4]. We shall treat the problem in a generalized form. Namely, we consider a pseudo-geometric integral domain K, satisfying two conditions $(K, 1)$ and $(K, 2)$ below, as a ground ring.[1)]

$(K, 1)$ The altitude formula holds for K.

$(K, 2)$ Normal spots over K are analytically irreducible.

Let $A_1, \cdots, A_n$ be normal affine rings over K and let A be the direct sum of them. Let R be an integral domain containing K and contained in A such that (i) the field of quotients $Q(R)$ of R is a subring of the total quotient ring of A (i.e., the natural homomorphism from R into A_i (given by multiplying the identity of A_i) is injective (=isomorphism into)) and (ii) $R=A\cap Q(R)$. Our aim is to give some sufficient conditions for this ring R to be an affine ring over K. In our treatment, we assume one more condition for

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1) As for the words pseudo-geometric, altitude formula, affine rings, height, depth and so on, they are understood as those in our book [5]. As for these conditions $(K, 1)$ and $(K, 2)$, we may start with the derived normal ring of K instead of K itself. Then $(K, 1)$ follows from $(K, 2)$ (See [3]). Therefore it is really enough to assume $(K, 2)$ only. Spot = locality.

R. Namely, we assume that for every prime ideal $\mathfrak{p}$ of R, there is a prime ideal $\mathfrak{p}'$ of A which lies over $\mathfrak{p}$. Then our main results can be stated as follows:

Theorem 1. *If* $\mathfrak{m}R_{\mathfrak{m}}$ *has a finite basis for every maximal ideal* $\mathfrak{m}$ *of* R, *then* R *is finitely generated over* K.

Theorem 2. *Assume that for every pair of maximal ideal* $\mathfrak{m}$ *and a prime ideal* $\mathfrak{p}$ *of height* 1 *in* R *such that* $\mathfrak{p}\subseteq\mathfrak{m}$, *the maximal ideal* $\mathfrak{m}R_{\mathfrak{m}}/\mathfrak{p}R_{\mathfrak{m}}$ *has a finite basis and that every normal spot over* K *contains a prime element unless the spot is a field,*[2] *then* R *is finitely generated over* K.

We shall show also in this paper that the main theorem in [4] can be simplified if the ground rings are restricted to such one in the present paper.

§1. The main theorems.

We shall make use of the following lemma of Zariski [7]:[3]

Lemma 1. *If a normal spot* P *over* K *is dominated by another spot* Q *over* K, *then* P *is a subspace of* Q (*under their natural topologies*).

Now we shall prove Theorem 1. So, we assume that $\mathfrak{m}R_{\mathfrak{m}}$ has a finite basis for every maximal ideal $\mathfrak{m}$ of R. Since A_i are normal rings and since $R=A\cap Q(R)$, there is a normal affine ring B over K with an ideal $\mathfrak{a}$ such that R is the $\mathfrak{a}$-transform $T(\mathfrak{a})$ of B (see [4]). Assume for a moment that R is not finitely gederated over K. Then, as was shown in the proof of Theorem 4 in [1] or in that of Lemma 2.7 in [4], there is a sequence of normal affine rigns $B=B_0\subset B_1\subset\cdots\subset B_r\subset\cdots$ with ideals $\mathfrak{a}\subset\mathfrak{a}_1\subset\cdots\subset\mathfrak{a}_r\subset\cdots$ such that (1) $R=\bigcup_i B_i$ and (2) each $\mathfrak{a}_i$ is different from B_i and is the intersection of prime ideals of height 1 in B_i which contains $\mathfrak{a}$. Let $\mathfrak{a}^*$ be

2) This second condition is satisfied by fields.

3) Though Zariski stated this result for spots over a field, his Theorem 1 in [7] is good for the general case and we have no difficulty in proving this lemma.

the union of all $\mathfrak{a}_i$ and let $\mathfrak{m}$ be a miximal ideal of R containing $\mathfrak{a}^*$. Since $\mathfrak{m}R_{\mathfrak{m}}$ has a finite basis, B may be replaced by one B_i which contains a basis for $\mathfrak{m}R_{\mathfrak{m}}$. Thus we may assume that $\mathfrak{m}''=\mathfrak{m}\cap B$ generates $\mathfrak{m}R_{\mathfrak{m}}$ (in $R_{\mathfrak{m}}$). Since there is a prime ideal $\mathfrak{m}'$ of A which lies over $\mathfrak{m}$, we see that $R/\mathfrak{m}$ is finite algebraic over $B/\mathfrak{m}''$. Now we consider the rings $B_{\mathfrak{m}''}$, $R_{\mathfrak{m}}$ and $A_{\mathfrak{m}'}$. $B_{\mathfrak{m}''}$ and $A_{\mathfrak{m}'}$ are normal spots over K, and therefore $B_{\mathfrak{m}''}$ is a subspace of $A_{\mathfrak{m}'}$ Since $R_{\mathfrak{m}}$ is in between of them, we see that $R_{\mathfrak{m}}$ is a local ring which may not be Noetherian and $B_{\mathfrak{m}''}$ is a subspace of $R_{\mathfrak{m}}$. Namely, the completion $(B_{\mathfrak{m}}'')^*$ of $B_{\mathfrak{m}''}$ can be regarded as a subring of the completion $(R_{\mathfrak{m}})^*$ of $R_{\mathfrak{m}}$. Since $\mathfrak{m}''$ generates $\mathfrak{m}R_{\mathfrak{m}}$ and since $R/\mathfrak{m}$ is finite algebraic over $B/\mathfrak{m}''$, we see that $(R_{\mathfrak{m}})^*$ is integral over $(B_{\mathfrak{m}''})^*$. Since $R_{\mathfrak{m}}$ is contained in the field of quotients of B, it follows now that $R_{\mathfrak{m}}$ is integral over $B_{\mathfrak{m}''}$ and therefore $R_{\mathfrak{m}}=B_{\mathfrak{m}''}$ (see [5] (37.4)). This contradicts to the infiniteness of the sequence $B\subset B_1\subset\cdots\subset B_r\subset\cdots$ and the proof of Theorem 1 is completed.

Now we shall prove Theorem 2. By virtue of Theorem 1, we have only to show that $\mathfrak{m}R_{\mathfrak{m}}$ has a finite basis for every maximal ideal $\mathfrak{m}$ of R. Take a normal affine ring B with an ideal $\mathfrak{a}$ such that R is the $\mathfrak{a}$-transform $T(\mathfrak{a})$ of B and set $\mathfrak{m}''=\mathfrak{m}\cap B$. Then $B_{\mathfrak{m}''}$ is a normal spot, hence it has a prime element, say p. Then p is a prime element in $R_{\mathfrak{m}}$. Therefore our assumption say that $\mathfrak{m}R_{\mathfrak{m}}/pR_{\mathfrak{m}}$ has a finite basis, which implies that $\mathfrak{m}R_{\mathfrak{m}}$ had a finite basis. This completes the proof of Theorem 2.

§2. Supplementary remarks.

(1) On the proof of the main theorem of [4]. The main theorem of [4] is as follows:

Let K be a pseudo-geometric ring and let A be a finitely generated ring over K. If a ring R which is in between K and A is strongly submersive in A, then $R_{red}=R/$(the radical of R)

is a finite K-algebra.

What we want to remark here is that if K is such one as in §1, and if furthermore K satisfies the condition that every normal spot over K has a semi-prime element unless the spot is a field, then the assertion can be proved in much simpler way.

In deed, it is easy to reduce the assertion to the case as in the begining of this paper. Take B and $\mathfrak{a}$ as in the proof of Theorem 2, and we take a semi-prime element s of $B_{\mathfrak{m}''}$. Then s is semi-prime in $R_{\mathfrak{m}}$ and $R_{\mathfrak{m}}/sR_{\mathfrak{m}}$. is a subdirect sum of a finite number of Noetherian rings (using induction argument on height $\mathfrak{m}$), hence is Noetherian. Therefore $\mathfrak{m}R_{\mathfrak{m}}$ has a finite basis, and R is a finite K-algebra. One should note that in this case, among the preliminary results in §1 of [4], we need only Lemma 1.2 which asserts that if R is strongly submersive in A and if $\mathfrak{a}$ is an ideal of R, then $R/\mathfrak{a}$ is strongly submersive in $A/\mathfrak{a}A$.

By the way, we like to note here that the above mentioned proof really yields

Theorem 2*. *Let K and R be as in the beginning of this paper. If, for every pair of maximal ideal $\mathfrak{m}$ of R and a prime ideal $\mathfrak{p}$ of height 1 in R contained in $\mathfrak{m}$, the ring $R_{\mathfrak{m}}/\mathfrak{p}R_{\mathfrak{m}}$ is Noetherian and if every normal spot over K has a semi-prime element unless the spot is a field, then it follows that R is finitely generated over K.*

(2) On the height of prime ideals of R.

For an integral domain I, we consider the following chain condition:[4)]

(C) If $\mathfrak{p}$ is a prime ideal of I, then every descending chain of prime ideals in I which begins with $\mathfrak{p}$ and ends with 0 can be refined so that its length is equal to the height of $\mathfrak{p}$.

We begin with an easy

4) One can show easily that the chain condition (C) is satisfied by K in the beginning of the present paper.

Lemma 2. *Let I be a Noetherian integral domain for which the altitude formula and the chain condition (C) hold good. Then these conditions hold for every affine ring over I.*

Since the proof is easy, we omit it.

Now we come to the main remark:

Theorem 3. *Let I be a Noetherian integral domain for which the altitude formula and the chain condition (C) hold. Let A be an affine ring over I and let R be a subring of A which contains K. If $\mathfrak{p}$ is a prime ideal of R for which there is a prime ideal $\mathfrak{p}'$ of A which lies over $\mathfrak{p}$, then*

$$\textit{height }\mathfrak{p}+\textit{trans. deg}_{I/(\mathfrak{p}\cap I)}R/\mathfrak{p}=\textit{height}(\mathfrak{p}\cap I)+\textit{trans. deg}_I R.$$

Proof. If $\mathfrak{p}=0$, then the aesertion is obvious, and we use induction argument on height $\mathfrak{p}$. Let I^* be an affine ring over I which is contained in R. Then, by the altitude formula applied to $\mathfrak{p}^*=\mathfrak{p}\cap I^*$, we have

$$\text{height }\mathfrak{p}^*+\text{trans. deg}_{I/(\mathfrak{p}\cap I)}I^*/\mathfrak{p}^*=\text{height }(\mathfrak{p}\cap I)+\text{trans. deg}_I I^*.$$

Therefore the assertion is equivalent to the formula for I^* instead of I. Therefore I may be replaced by any of such I^* by virtue of Lemma 2. In particular, we may assume that $R/\mathfrak{p}$ is algebraic over $I/(\mathfrak{p}\cap I)$. Considering $I_{(\mathfrak{p}\cap I)}$ instead of I, we may assume that $\mathfrak{p}\cap I$ is the unique maximal ideal of I. Now, let $\mathfrak{P}$ be the set of prime ideals of A which lie over $\mathfrak{p}$. We may assume that $\mathfrak{p}'$ is maximal in $\mathfrak{P}$. Since $R/\mathfrak{p}$ is a field by our assumption and since $A/\mathfrak{p}'$ is an affine ring over the field $R/\mathfrak{p}$, we see that $\mathfrak{p}'$ is a maximal ideal of A and $A/\mathfrak{p}'$ is algebraic over the field $R/\mathfrak{p}$. Let $\mathfrak{q}'$ be a prime ideal of A such that (i) $\mathfrak{q}'\subset\mathfrak{p}'$. (ii) depth $\mathfrak{q}'=1$ and (iii) $\mathfrak{p}A\not\subseteq\mathfrak{q}'$. Set $\mathfrak{q}=\mathfrak{q}'\cap R$.

Case 1. Assume that $\mathfrak{q}\cap I=\mathfrak{p}\cap I$. Then, since $A/\mathfrak{q}'$ is an affine ring over the field $I/(\mathfrak{p}\cap I)$, we see that trans. $\deg_{I(\mathfrak{p}\cap I)}A/\mathfrak{q}'=1$.

Since $R/\mathfrak{q}$ is a subring of $A/\mathfrak{q}'$ and since $\mathfrak{q}\neq\mathfrak{p}$, we see that trans. $\deg_{I/(\mathfrak{p}\cap I)}R/\mathfrak{q}=1$. Now, by induction assumption, we have: height $\mathfrak{q}+$ trans. $\deg_{I/(\mathfrak{p}\cap I)}R/\mathfrak{q}=$ height $(\mathfrak{p}\cap I)+$ trans. $\deg_I R$. Since height $\mathfrak{p}\geq$(height $\mathfrak{q})+1$, we see that height $\mathfrak{p}+$ trans. $\deg_{I/(\mathfrak{p}\cap I)}R/\mathfrak{p}\geq$ height $(\mathfrak{p}\cap I)+$ trans. $\deg_I R$. The converse inequality holds obviously because I is Noetherian and we settle this case.

Case 2. Assume that $\mathfrak{q}\cap I\neq\mathfrak{p}\cap I$. Since $A/\mathfrak{q}'$ is an affine ring over the ring $I/(\mathfrak{q}\cap I)$ and since $\mathfrak{p}'/\mathfrak{q}'$ is a maximal ideal of height 1 which lies over the maximal ideal $(\mathfrak{p}\cap I)/(\mathfrak{q}\cap I)$, we see that $A/\mathfrak{q}'$ is algebraic over $I/(\mathfrak{q}\cap I)$ and that $(\mathfrak{p}\cap I)/(\mathfrak{q}\cap I)$ is of height 1. By the chain condition (C), we see that height $(\mathfrak{p}\cap I)=1+$ +height $(\mathfrak{q}\cap I)$, and therefore we settle this case similarly as in Case 1 above. Thus the proof of Theorem 3 is completed.

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