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Finitely generated rings over a valuation ring

By

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We shall try to generalize certain results on finitely generated rings over noetherian rings to the case of finitely generated rings over general (commutative) rings.[1] To begin with, the writer likes to express his thanks to David Mumford, for that some ideas in this note, especially that of Theorem 3, arose in conversation with him.

One of our main results is the following:

Theorem 1. *Let an integral domain A be finitely generated over its subring R. Let* $\mathfrak{p}$ *be a prime ideal of R. Assume that* $\mathfrak{q}$ *is a prime ideal of A which lies over* $\mathfrak{p}$ *such that* $\mathfrak{q}$ *is a minimal prime divisor of* $\mathfrak{p}A$*. Then we have*

$$\text{trans. deg}_{R/\mathfrak{p}}\, A/\mathfrak{p} \geq \text{trans. deg}_R A.$$

Then we come to the following theorems on finitely generated rings over a valuation ring:

Theorem 2. *Let V be a valuation ring of rank (=Krull dimension) r, and let* $A = V[a_1, \cdots, a_n]$ *be a finitely generated integral domain over V. If* $\mathfrak{p}$ *is a prime ideal of A, then every maximal chain of prime ideals in A which begins with* $\mathfrak{p}$ *and ends with* 0 *has length equal to*

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1) As for terminology, we shall use mainly the one in our book, M. Nagata, *Local rings*, John Wiley, New York (1962).

$$\text{height}(\mathfrak{p}\cap V)+\text{trans. deg}_V A-\text{trans. deg}_{V/(\mathfrak{p}\cap V)} A/\mathfrak{p}.$$

Theorem 3. *Let V be a valuation ring and let $A=V[a_1, \cdots, a_n]$ be a finitely generated ring over V. Let $\mathfrak{a}$ be the kernal of the homomorphism ϕ from the polynomial ring $V[X_1, \cdots, X_n]$ onto A such that $\phi(X_i)=a_i$. If no non-zeroelement of V is a zero-divisor in A, or equivalently, if A is a flat V-module, then $\mathfrak{a}$ has a finite basis.*

On the other hand, our proof of Theorem 1 can be applied to prove the following:

Let R be a ring and let $A_1, \cdots, A_n$ be finitely generated rings over R such that for each pair (i, j) $(i, j\leq n)$, there is a non-zero-divisor $a_{ij}\in A_i$ such that $A_i[A_j]=A_i[a_{ij}^{-1}]$ (hence these A_i have the same total ring of quotients). Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ be prime ideals of R and let R_0 be a subring of R. Then:

Theorem 4. *There are finitely generated rings R' and R'_i $(i=1, \cdots, n)$ over R_0 such that* (1) $R\supseteq R'\subseteq A'_i\subseteq A_i=R[A'_i]$, (2) *$A'_i$ contains a_{ij} and $A'_i[A'_j]=A'_i[a_{ij}^{-1}]$ for every pair (i, j) $(i, j\leq n)$ and* (3) *for each $\mathfrak{p}_k$, setting $P_k=R_{\mathfrak{p}_k}/\mathfrak{p}_k R_{\mathfrak{p}_k}$, it holds that $A'_i\otimes_{R'}P_k\cong A_i\otimes_R P_k$.*

Theorem 1, in case R is a valuation ring (one can reduce it to that case), is implicit in the geometric statement: a specialization of an n-dimensional variety is an n-dimensional cycle i.e., all its component have dimension n (cf. Lemma 2.1 below), which, we presume, has been made by some one. However, all recent authors who have laid down algebraic foundations for algebraic geometry seem oddly (in spite of their generalized treatment in other respect) to restrict themselves to the case R noetherian.

Theorem 3 can be viewed as one explanation of why Theorem 1 is valid. In fact, in the language of schemes,[2] it asserts:

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2) As for schemes, see, for inst., articles of A. Grothendieck (either "*Elements*" or "*S. G. A.*") or lecture notes by Dieudonné on algebraic geometry.

Theorem 3′. *If π: $X \rightarrow \operatorname{Spec} R$ is a flat morphism of schemes of finite type and if R is a valuation ring, then π is finitely presented.*

As for Theorem 4, it asserts the following: Under the assumption on A_i, we have a scheme M of finite type (defined by these A_i) over $\operatorname{Spec} R$. Then, for any given finite number of $\mathfrak{p}_i \in \operatorname{Spec} R$ and for a given subring R_0 of R, there exist an affine scheme S' over R_0 and a scheme M' of finite type over S' such that (1) M is a closed subscheme of $M' \times_{S'} \operatorname{Spec} R$ and (2) the fibre over $\mathfrak{p}_k$ on M is isomorphic to that on $M' \times_{S'} \operatorname{Spec} R$.

1. The proof of Theorem 1.

We know already that if R is noetherian then the assertion is true.[3] But, we shall prove the assertion without using the result but using a more fundamental fact that the altitude formula (=dimension formula) holds for any locality over a prime integral domain.[4]

Let K be the field of quotients of R. Considering $R_{\mathfrak{p}}$, we may assume that $(R, \mathfrak{p})$ is quasi-local. Consider a polynomial ring $R[X_1, \cdots, X_n]$ and a surjective R-homomorphism ϕ: $R[X] \rightarrow A$. Let $\mathfrak{a}$ be the kernel of ϕ. Take a finite number of elements $f_1, \cdots, f_m$ such that (i) $\sum f_i K[X] = \mathfrak{a} K[X]$ and (ii) $\mathfrak{a} + \mathfrak{p} R[X] = \sum f_i R[X] + \mathfrak{p} R[X]$. Let R_0 be the ring generated by the coefficients of these f_i over the prime integral domain of R and set $R_1 = (R_0)_{\mathfrak{p} \cap R_0}$, $\mathfrak{p}_1 = \mathfrak{p} \cap R_1$, K_1 = the field of quotients of R_1 and $A_1 = R_1[a_1, \cdots, a_n]$, here $a_i = \phi(X_i)$. By (i), $K[a_1, \cdots, a_n] \cong K_1[a_1, \cdots, a_n] \otimes_{K_1} K$ whence $\text{trans.}\deg_R A = \text{trans.}\deg_{R_1} A_1$. By (ii), $A/\mathfrak{p}A \cong (A_1/\mathfrak{p}_1 A_1) \otimes_{R_1/\mathfrak{p}_1} R/\mathfrak{p}$. Therefore $\text{trans.}\deg_{R/\mathfrak{p}} A/\mathfrak{q} = \text{trans.}\deg_{R_1/\mathfrak{p}_1} A_1/(\mathfrak{q} \cap A_1)$ and $\mathfrak{q} \cap A_1$

3) See, for inst., (35.6) in the book cited in 1).

4) A prime integral domain is either a finite field or the ring of rational integers, hence is either a field or a pseudo-geometric Dedekind domain. Validity of the altitude formula is known for much more general case (see, for inst., the book cited above). For this special case, see, M. Nagata, *A general theory of algebraic geometry over Dedekind domins, I,* Amer. J. Math. 78 (1956), pp. 78–116.

contains a minimal prime divisor of $\mathfrak{p}_1 A_1$. Thus we may assume that R is a locality over the prime integral domain.[5] Then by the altitude formula, height $\mathfrak{q}$ + trans. $\deg_{R/\mathfrak{p}} A/\mathfrak{q}$ = height $\mathfrak{p}$ + trans. $\deg_R A$. Let $y_1, \cdots, y_r$ be a system of parameters of R. Then r = height $\mathfrak{p}$ and $\mathfrak{q}$ is a minimal prime divisor of the ideal $\sum y_i A$. Thus we have height $\mathfrak{q} \leq r$ = height $\mathfrak{p}$, and therefore trans. $\deg_{R/\mathfrak{p}} A/\mathfrak{q} \geq$ trans. $\deg_R A$.

2. Proof of Theorem 2.

Lemma 2.1. *Under the notation of Theorem* 1, *if R is a valuation ring, then*

$$\text{trans. } \deg_{R/\mathfrak{p}} A/\mathfrak{q} = \text{trans. } \deg_R A.$$

Proof. Let $z_1, \cdots, z_r$ be elements of A such that they are algebraically independent modulo $\mathfrak{p}$ over $R/\mathfrak{p}$. If they algebraically dependent over R, then there is a non-trivial relation $f(z_1, \cdots, z_r) = 0$ over R. Since R is a valuation ring, we may assume that some coefficients are not in $\mathfrak{p}$, then it contradicts to the choice of the z_i. Thus trans. $\deg_{R/\mathfrak{p}} A/\mathfrak{q} \leq$ trans. $\deg_R A$. Hence we have the equality.

Lemma 2.2. *Assume that a valuation ring V is contained in a ring R in which non-zero elements of V are non-zero-divisors. If $\mathfrak{p}R \neq R$ for a prime ideal $\mathfrak{p}$ of V, then for any prime ideal $\mathfrak{q}$ of V contained in $\mathfrak{p}$, we have $\mathfrak{q}R \cap V = \mathfrak{q}$; if $\mathfrak{p}'$ is a prime ideal of R which lies over $\mathfrak{p}$, then there is a prime ideal $\mathfrak{q}'$ of R lying over $\mathfrak{q}$ and such that $q' \subseteq \mathfrak{p}'$.*

Proof. Assume that a is an element of $\mathfrak{q}R \cap V$ which is not in $\mathfrak{q}$. Then there are elements q_i of $\mathfrak{q}$ and elements r_i of R such that $a = \sum q_i r_i$. Then $1 = \sum (q_i a^{-1}) r_i \in \mathfrak{q} V_{\mathfrak{q}} R = \mathfrak{q}R$. Therefore $\mathfrak{q}R = R$, which is a contradiction. If this is applied to $R_{\mathfrak{p}'}$ instead of R, then we have the existence of $\mathfrak{q}'$, and the proof is completed.

Now we shall prove Theorem 2 by induction on r + trans. $\deg_V A$.

5) If we use the result on noetherian case, then this completes the proof.

Let $\mathfrak{p}=\mathfrak{p}_0\supset\mathfrak{p}_1\supset\cdots\supset\mathfrak{p}_s=0$ be a maximal chain. If $s=0$, then $\mathfrak{p}=0$ and the assertion is obvious. Assume that $s\geq 1$. Set $\mathfrak{q}=\mathfrak{p}_{s-1}\cap V$.

(i) Assume that $\mathfrak{q}=0$. Then, denoting by K the field of quotients of V, $\mathfrak{p}_{s-1}K[a_1, \cdots, a_n]$ is of height 1, whence we see that trans. $\deg_V A/\mathfrak{p}_{s-1}=$ trans. $\deg_V A-1$. Therefore the induction assumption gives us the result in this case.

(ii) Assume that $\mathfrak{q}\neq 0$. Then $\mathfrak{p}_{s-1}$ is a minimal prime divisor of $\mathfrak{q}A$. Therefore Lemma 2.1 shows that trans. $\deg_{V/\mathfrak{q}} A/\mathfrak{p}_{s-1}=$ trans. $\deg_V A$. Therefore, by our induction assumption, $s-1=$ height$(\mathfrak{p}\cap V/\mathfrak{q})+$trans. $\deg_{V/\mathfrak{q}} A/\mathfrak{p}_{s-1}-$trans. $\deg_{V/(\mathfrak{p}\cap V)} A/\mathfrak{p}=$height $\mathfrak{p}-$height$\mathfrak{q}+$trans. $\deg_V A-$trans. $\deg_{V/(\mathfrak{p}\cap V)} A/\mathfrak{p}$. By Lemma 2.2, we see that height$\mathfrak{q}=1$, and we complete the proof of Theorem 2.

The above proof of Theorem 2 can be applied easily to prove the following:

Theorem 2′. *Let V be a valuation ring and let A be a finitely generated integral domain over V. Let $\mathfrak{p}$ be a prime ideal of A. Set $s=$trans. $\deg_V A-$trans. $\deg_{V/(\mathfrak{p}\cap V)} A/\mathfrak{p}$. Then $s\geq 0$. Let $\mathfrak{p}\cap V=\mathfrak{q}_0\supset\mathfrak{q}_1\supset\cdots\supset\mathfrak{q}_t\supseteq 0$ be a chain of prime ideals in V and let $s_0, \cdots, s_t$ be non-negative integers. Then there is a descending chain of prime ideals $\mathfrak{p}=\mathfrak{p}_0\supset\mathfrak{p}_1\supset\cdots\supset\mathfrak{p}_n$ in A such that there are s_α of these $\mathfrak{p}_i$ which lie over $\mathfrak{q}_\alpha$ for each $\alpha=0, 1, \cdots, t$ if and only if $\sum s_i\leq s$.*

3. Proof of Theorem 3.

We begin with the following proposition which is related to the statement of Theorem 3.

Proposition 3.1. *A module M over a valuation ring V is a flat module if and only if no non-zero element of V is a zero-divisor with respect to M.*

Proof. Considering $V\oplus M$, we may assume that M is a ring containing V (principle of idealization). (i) Only if part: For $0\neq x\in V$, we have $0=[0:x]_V\otimes_V M\cong[0:x]_M$ and therefore x is not a zero-divisor with respect to M. (ii) If part: A finitely generated ideal $\mathfrak{a}$ of V is principal: $\mathfrak{a}=xV$. Therefore $\mathfrak{a}M=xM\cong xV\otimes_V M$. Therefore M is flat. This completes the proof.

Corollary 3.2. *Let V be a valuation ring of a field K and let $A=V[a_1, \cdots, a_n]$ be a finitely generated ring over V. Consider a polynomial ring and surjective homomorphism $\phi: V[X_1, \cdots, X_n] \to A$ $(\phi(X_i)=a_i)$. Let $\mathfrak{a}$ be the kernel of ϕ, and let $f_1, \cdots, f_r$ be elements of $\mathfrak{a}$ which generate $\mathfrak{a}K[X_1, \cdots, X_n]$. Let K_1 be a subfield of K containing all coefficients of these f_i and set $V_1=V\cap K_1$. If no non-zero element of V_1 is a zero-divisor in A (or, rather, in $V_1[a_1, \cdots, a_n]$), then $A\cong V_1[a_1, \cdots, a_n]\otimes_{V_1} V$.*

Proof. $V_1[a_1, \cdots, a_n]$ is a flat module over V_1, hence $V_1[a_1, \cdots, a_n]\otimes_{V_1} V\subseteq V_1[a_1, \cdots, a_n]\otimes_{V_1} K\cong K[a_1, \cdots, a_n]$, which proves the assertion.

Now we shall prove Theorem 3. We denote by $\mathfrak{m}$ the maximal ideal of V.

(1) Homogeneous case. First we assume that $\mathfrak{a}$ is a homogeneous ideal of $V[X]$. Let F_d be the module of homogeneous forms of degree d in $V[X]$ and set $\mathfrak{a}_d=F_d\cap\mathfrak{a}$. Because of the assumption on A, $\mathfrak{a}_d$ has the property that $cf\in\mathfrak{a}_d$, $0\neq c\in V$ and $f\in F_d$ imply $f\in\mathfrak{a}_d$. We claim that $\mathfrak{a}_d$ is a finite free V-module. Namely,

Lemma 3.3.[6] *If a submodule M of F_d has the property that $cf\in M$, $0\neq c\in V$, $f\in F_d$ imply $f\in M$, then M is a finite free V-module and $\mathfrak{m}M=M\cap\mathfrak{m}F_d$.*

Proof. The last equality is obvious. Take elements $m_1, \cdots, m_t$ of M such that residue classes of these elements form a linearly independent basis for $M/(M\cap\mathfrak{m}F_d)$ over the field $V/\mathfrak{m}$. Let $b_1, \cdots, b_u$ be a free base of F_d such that $m_i=b_i$ for $i\leq t$. Let M' be the module generated by these m_i. Assume now that there is an element $m\in M$ which is not in M'. Then $m=\sum c_i b_i$ with $c_i\in V$. Subtracting

6) Note that F_d can be replaced by an arbitrary finite free V-module.

$\sum_{t\geq i} c_i b_i$, we may assume that $c_i=0$ for $i\leq t$. We may assume that $c_n V=\sum_i c_i V$. Then we see that $c_n^{-1}m\in M$ and $(c_n^{-1}m)$ modulo $\mathfrak{m}$ is not in $M/(M\cap\mathfrak{m}F_d)$, which is a contradiction.

Now we go back to the proof of Theorem 3. Let $f_1, \cdots, f_t$ be homogeneous elements of $\mathfrak{a}$ such that their residue classes modulo $\mathfrak{m}$ generate $\mathfrak{a}/(\mathfrak{a}\cap\mathfrak{m}V[X])$ in $(V/\mathfrak{m})[X]$ and let $\mathfrak{a}'$ be the homogeneous ideal generated by these f_i. Then $\mathfrak{a}'\leq\mathfrak{a}$. Set $\mathfrak{a}'_d=\mathfrak{a}'\cap F_d$. Since $\mathfrak{a}'+\mathfrak{m}V[X]=\mathfrak{a}+\mathfrak{m}V[X]$, we have $\mathfrak{a}'_d+\mathfrak{m}F_d=\mathfrak{a}_d+\mathfrak{m}F_d$. Since $\mathfrak{a}_d\cap\mathfrak{m}F_d=\mathfrak{m}\mathfrak{a}_d$, we see that $\mathfrak{a}'_d+\mathfrak{m}\mathfrak{a}_d=\mathfrak{a}_d$, which shows that $\mathfrak{a}'_d=\mathfrak{a}_d$ by virtue of the lemma of Krull-Azumaya. Thus $\mathfrak{a}=\mathfrak{a}'$, and $\mathfrak{a}$ has a finite basis.

(2) General case. Take a transcendental element y_0 and set $y_i=a_i y_0$ $(i=1, 2, \cdots, n)$, $A'=V[y_0, \cdots, y_n]$. Then the kernel $\mathfrak{a}'$ of $\phi\colon V[X_0, \cdots, X_n]\to A'$ (defined similarly) has a finite basis, say $f'_1, \cdots, f'_t$. Let f_i be the element of $V[X_1 \cdots, X_n]$ which is obtained from f'_i setting $X_0=1$. We claim that $\mathfrak{a}$ is generated by $f_1, \cdots, f_t$. It is obvious that $f_i\in\mathfrak{a}$. Conversely, let f be an arbitrary element of $\mathfrak{a}$. Let f' be a homogenized polynomial from f introducing X_0. Then $f'\in\mathfrak{a}'$, and we see that f is in the ideal generated by these f_i. This completes the proof.

4. Proof of Theorem 4.

Let $b_{i1}, \cdots, b_{it}$ be a set of generators of A_i over R and let $\mathfrak{q}_i$ be the kernel of the R-homomorphism $\phi_i\colon R[X_1, \cdots, X_t]\to A_i$ $(\phi_i(X_j)=b_{ij})$. For each $\mathfrak{p}_k$, there is a finite subset $\mathfrak{c}_{ik}$ of $\mathfrak{q}_i$ such that $A_i\otimes_R P_k\cong P_k[X_1, \cdots, X_t]/\mathfrak{c}_{ik}P_k[X]$. Let R' be a finitely generated ring over R_0 such that (1) $R'\subseteq R$, (2) R' contains all coefficients of members of $\mathfrak{c}_{ik}$, (3) $a_{ij}\in R'[b_{i1}, \cdots, b_{it}]$, (4) $b_{jk}\in R'[b_{i1}, \cdots, b_{it}, a_{ij}^{-1}]$. Then $A'_i=R'[b_{i1}, \cdots, b_{it}]$ are the required rings.

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