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On some analytic families of polarized algebraic varieties

By

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Let a complex manifold X parametrize a complex analytic family of non-singular polarized varieties in a projective space. When we identify isomorphic members of the family, this identification defines an equivalence relation $\mathfrak{R}$ on X. On the other hand, when F is the Zariski-closure of the set of Chow-coordinates of members of the family in the smallest algebraic variety containing it, $\mathfrak{R}$ almost always induces on F a closed equivalence relation in the sense of algebraic geometry, which we shall denote by $\mathfrak{R}'$. Under certain conditions, we are going to show, among other things, that analytically defined quotient space, the analytic variety of moduli, and algebro-geometrically defined quotient space, the algebro-geometric variety of moduli, are the same thing. Conditions are stated as axioms, which are satisfied when X is a bounded domain, R is defined by a proper discontinuous group G acting on X, the field of G-invariant meromorphic functions form an algebraic function field of dimension $m = \dim X$ and when F has a Zariski-open subset F_0 satisfying the following conditions: (i) F_0 carries an equivalence relation $\mathfrak{R}'$, compatible with $\mathfrak{R}$ defined on X by G; (ii) Every member of the analytic family is contained in the family defined by F_0, and every member of the latter family is equivalent, with respect to R', to a member of the former family; (iii) R' is a closed equivalence relation in the sense of

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algebraic geometry, and orbits of points of F_0 with respect to R' are irreducible and have the constant dimension. Actually, under slightly milder assumptions than those described above, we are going to show the existence of the analytic and algebro-geometric varieties of moduli at the same time and see that they are the same. This is sometimes rather convenient, since we may be able to tie up algebraically defined concepts with analytic concepts.

Before we actually discuss our main topics, we shall discuss a few preliminary results, many of which are well-known. In particular, in §2 and §3, we shall discuss the case when X is the Siegel space of degree n and G is the paramodular group. A purpose of this is to show actually that the axioms we are going to consider in §4 are satisfied in this particular case. As for more general cases than the Siegel space and the paramodular group, the readers are referred to Satake (II), Shimura (II, III, IV, V, VI, VII), Siegel (II) and Kuga (I). At the moment, the author does not know if in some of these general cases all the axioms in §4 are satisfied, but hopes that most of them would do so. In §5, rough materials are constructed which will be refined in §6 and pasted together to get analytic as well as algebro-geometric variety of moduli. In §7 we shall discuss the possibility of projective embeddings of the varieties of moduli and the fields of moduli.

For general properties of Abelian and theta-functions, the readers are referred to Siegel (I), Weil (II) and Conforto (I). For discontinuous groups on bounded domains, Poincaré-series and automorphic functions, readers are referred to Siegel (I) and Pyatecki-Shapiro (I). The author would like to point out that the case when X is the Siegel space and when G is the modular group was treated by Baily (II).

Conventions. Since we are going to deal with objects with mixed structure, we shall agree to use the following conventions. Whenever open sets, closed sets, closure are mentioned, these are relative to the ordinary complex topology, unless the contrary is specifically mentioned. Corresponding notions relative to Zariski-topology will be indicated by "Z-open sets", "Z-closed sets" and by "Z-closure". As far as algebro-geometric terminology are concerned, they will be based on Weil (V). Therefore, for instance, an "algebraic variety" will mean an absolutely irreducible algebraic variety in the sense of Zariski. When Y is a cycle or an analytic subset of an algebraic variety or a complex manifold, $|Y|$ will denote always the support of Y.

§1. Let X^m be a connected complex manifold, P^N a projective space and Z^{m+n} a connected complex submanifold of $X\times P$ (the topology on Z may be stronger than the one induced on it by the topology of $X\times P$). When (X, Z) satisfies the following three conditions, we say that (X, Z) *defines an analytic family* $\{Z(x); x\in X\}$ in P.

(a) When π is the projection of Z on X, there is a non-singular algebraic subvariety $Z(x)$ in P corresponding to each point x of X such that $|Z(x)|$ is the set of points in Z with the projection x;

(b) When we regard $Z(x)$ as a closed complex submanifold of P, $x\times Z(x)$ is a closed complex submanifold of Z;

(c) When z is a point of Z, there exist an open subset W_z of Z containing z such that $\pi(W_z)$ is open on X, an open subset U of C^m (the complex Eucledean space), an isomorphism ψ between $\pi(W_z)$ and U, an open subset V of C^n and an isomorphism φ between W_z and $U\times V$ such that the following diagram is commutative:

$$\begin{array}{ccc} W_z & \xrightarrow{\ \pi\ } & \pi(W_z) \\ \varphi\downarrow & & \uparrow\psi \\ U\times V & \xrightarrow{\ \pi'\ } & U \end{array}$$

(π' denotes the projection of $U\times V$ on U).

Let (X, Z) and (X', Z') define analytic families in projective spaces. When there is an isomorphism g between Z and Z' and an isomorphism h between X and X' such that $\pi\circ g=h\circ\pi$, we say that the analytic families defined by (X, Z), (X', Z') are *isomorphic.*

Let $z=(x, y)$ be a point of Z and S an affine Z-open subset of P containing y. When we set $W_0=W\cap\pi(W)\times S$, W_0 is open in W and in Z since the topology on Z is finer than the induced topology. Hence $\varphi(W_0)$ is open in $U\times V$. Set $\varphi(z)=\varphi(x, y)=(u, v)$ with $u=\psi(x)$ (cf. (*c*)). There is an open subset U^* of U containing u and an open subset V^* of V containing v such that $U^*\times V^* \subset\varphi(W_0)$. When W^* is the set of points in W_0 which are mapped into $U^*\times V^*$ by φ, it is open on Z, contains z and $\pi'\circ\varphi=\psi\circ\pi$ is satisfied on W^*. Let $\tilde{f}_1, \cdots, \tilde{f}_N$ be the affine coordinate functions on S. They induce holomorphic functions $f_1, \cdots, f_N$ on $U^*\times V^*$. Rearranging indices if necessary, and then setting $f_0=1$, we have found the following two consequences of (a), (b) and (c).

(d) To each point $z=(x, y)$ of Z, contained in $x\times Z(x)$, there is an open subset U^* of x on X, an open subset V^* of C^n, a point v of V^* and a set $(f_0, \cdots, f_N)$ of holomorphic functions on $U^*\times V^*$ such that $y=(f_0(x, v) : \cdots : f_N(x, v))$ and that the map determined by $f_0 : \cdots : f_N$ maps $x'\times V^*$, $x'\in U^*$, to an open subset of $x'\times Z(x')$;

(e) The map determined by $f_0 : \cdots : f_N$ has rank n everywhere on $x'\times V^*$ when $x'\in U^*$.

Conversely, let X^m be a connected complex manifold. Assume that a non-singular algebraic variety $Z(x)$ in P^N correspond to each x on X and that the conditions (d), (e) are satisfied by X and by the set $\{x\times Z(x)\}$. Then, defining $|Z|$ to be the set of points (x, y) in $X\times P$ such that y is contained in $Z(x)$, it is not difficult to show that Z becomes a connected complex submanifold of $X\times P$ when we define a suitable topology which is finer than the induced topology. By doing so, it is not hard to show that (X, Z) satisfy (a), (b), (c). Therefore, the conditions (a), (b), (c) and the conditions (d), (e) are equivalent.

In the following, we shall give a proof of this for the sake of convenience.

LEMMA 1. *Let X^m be a complex manifold and $\{A(x)^n ; x\in X\}$ a set of non-singular varieties in a projective space P^N, satisfying* (d) *and* (e). *Let Z be the set of points (x, y) in $X\times P$ such that $y\in|A(x)|$. Then Z has a structure of a connected complex submanifold of $X^m\times P$ satisfying* (a), (b) *and* (c).

Proof. Let $(x_0, y_0)\in Z$. Let U be an open subset of X, containing x_0, V an open subset of C^n, v_0 a point of V and $f_0, \cdots, f_N$

holomorphic functions on $U\times V$ such that $f_i=1$ for some i, $(f_0(x_0, y_0), \cdots, f_N(x_0, y_0))$ is a set of affine coordinates of y_0 and that $\partial(f_0, \cdots, f_N)/\partial(v)$ is of rank n everywhere on $U\times V$. Then the map F of $U\times V$ into $X\times P$ defined by $(x, v)\rightarrow(x, y)$ is holomorphic and of rank n everywhere on $U\times V$. Moreover, $F(x, v)$ is a point of $x\times A(x)$. Since the rank of the map F is maximal, we can find an open subset W_0 of $U\times V$, containing (x_0, v_0) such that $F_0=F/W_0$ is 1–1 on W_0. Set $D_0=F_0(W_0)$. D_0 is a subset of Z. Thus, we have associated to (x_0, y_0) an open subset W_0 of $C^m\times C^n$ (we have identified U with an open subset of C^m), a subset D_0 of Z containing (x_0, y_0) and a 1–1 holomorphic map $F_0: W_0\rightarrow D_0$. When (x_α, y_α) and (x_β, y_β) are two points of Z and $(W_\alpha, D_\alpha, F_\alpha)$, $(W_\beta, D_\beta, F_\beta)$ are corresponding sets, $F_\alpha^{-1}\circ F_\beta$ and $F_\beta^{-1}\circ F_\alpha$ are clearly holomorphic maps. We define topology τ on Z as follows. When W'_α is an open subset of W_α, we define $F_\alpha(W'_\alpha)$ to be open. This generates topology on Z which we call τ. Then F_α is a homeomorphism of W_α into (Z, τ). Then, (Z, τ) has a structure of a connected complex manifold having the $(D_\alpha, F_\alpha^{-1})$ as coordinate neighbourhoods of the (x_α, y_α). It is clear that the canonical injection of our complex manifold into $X\times P$ is holomorphic. Our lemma is thereby proved.

Example 1. Let X^m be a non-singular algebraic variety, Z^{m+n} a subvariety (algebraic) of $X\times P^N$ and assume that the following conditions are satisfied: (i) $x\times P$ and Z intersect properly on $X\times P$ for every point x of X; (ii) When we define $Z(x)$ by $(x\times P)\cdot Z=x\times Z(x)$, $Z(x)$ is a non-singular subvariety of P. Then (X, Z) defines an analytic family of non-singular varieties in P.

Example 2. Let H_n be the Siegel space of degree n, i.e. H_n consists of n-rowed complex symmetric matrices W such that the imaginary parts are positive definite. Let $E=\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$ be the canonical form of a non-degenerate Riemann-form which is carried by a complex torus of dimension n. Then $(I_n, D^{-1}W)=R_W$, $W\in H_n$, is a Riemann-matrix, having E as a principal matrix (I_n denotes the n-rowed identity matrix). Denoting by the same letter R_W the discrete subgroup of C^n generated by column vectors of

R_W over the ring of integers, C^n/R_W is a complex torus T_W of dimension n. E defines on T_W a homology class $\gamma_{E,W}$ of complex analytic divisors, and every complex analytic divisor on T_W which is in $\gamma_{E,W}$ is non-degenerate (i.e. defined as the zero of a non-degenerate theta-function). Conversely, when a complex torus T^n and a homology class γ on T containing a non-degenerate complex analytic divisor are given, then, there is a suitable E and a point W in H_n such that T and T_W are isomorphic (with respect to the complex structures) and that the isomorphism transforms γ to $\gamma_{E,W}$ (cf. Siegel (I), Weil (II)).

Let m be a fixed positive integer satisfying $m\geqslant 3$. Then there is a set of holomorphic functions $\Theta_0, \cdots, \Theta_N$ on $H_n\times C^n$ satisfying the following conditions: (i) When we regard $\Theta_\alpha(W, z)$ as a function on C^n, it is a theta-function relative to R_W and $\Theta_\alpha=0$ define on T_W a complex analytic divisor belongings to $m\gamma_{E,W}$; (ii) Regarding the $\Theta_\alpha(W, z)$ as functions on C^n, the map $z\rightarrow(\Theta_0(W, z):\cdots:\Theta_N(W, z))$ determines a projective embedding Θ_W of T_W into a projective space; (iii) $\Theta_W(T_W)=A_W$ is the underlying variety of an Abelian variety, $\deg(A_W)=m^n\cdot n!\det(D)$, $N=m^n\det(D)-1$, A_W is not contained in a hyperplane in P^N and the set of hyperplane sections of A_W forms a complete linear system; (iv) Set $O_W=(\Theta_0(W, 0):\cdots:\Theta_N(W, 0))$; then there is one and only one Abelian variety such that A_W is the underlying variety and O_W is the neutral element; When we denote this by (A_W, O_W), Θ_W is an isomorphism of T_W and (A_W, O_W) as complex Lie groups (cf. Conforto (I), Siegel (I), Weil (II)).

Take $X=H_n$, $Z(W)=A_W$ (resp. $Z(W)=A_W\times O_W$), then it can be verified easily that the conditions (d) and (e) are satisfied.

Remark. From now on, *a non-singular subvariety of a projective space will be regarded always as a polarized variety such that a hyperplane section is a polar divisor*, unless the contrary is specifically mentioned. Hence, A_W is a polarized variety such that $\Theta_W(\gamma_{E,W})$ is a polar divisor class.

When Y is a positive cycle in a projective space. We denote by $c(Y)$ the Chow-point of Y. We shall end this paragraph with the following proposition which is more or less well-known but the only place where we can find a proof is in Shimura (II) (in a slightly stronger condition).

PROPOSITION 1. *Let (X, Z) define an analytic family of non-singular varieties in a projective space. Then the map $x \rightarrow c(Z(x))$ is a holomorphic map of X into a projective space.*

Proof. It is enough to prove that our map is locally holomorphic. Let $\dim Z(x)=n$, k a field of definition of $Z(x)$, P^N the ambient projective space of $Z(x)$ and L_t a generic linear variety in P^N of codimension n over k (where we denote by t the set of defining linear equations for L_t, identified with a point in an affine space). Express $Z(x)\cdot L_t$ as $\Sigma_\alpha(y^{(\alpha)})$ in a suitable affine Z-open subset of P. Without loss of generality, we may assume that $y_0^{(\alpha)}=1$. By our condition (d), there is an open subset U_α of X, containing x, an open subset V_α of C^n, a point $v^{(\alpha)}$ of V_α, and a set $(f^{(\alpha)})=(f^{(\alpha)}{}_0, \cdots, f^{(\alpha)}{}_N)$ of holomorphic functions on $U_\alpha \times V_\alpha$ such that $(f^{(\alpha)}(x, v^{(\alpha)}))=(y^{(\alpha)})$. By (e), the map $f^{(\alpha)}$ defined by the $f^{(\alpha)}{}_i$, mapping $x \times V_\alpha$ into P^N has rank n everywhere. We have $\Sigma_0{}^N t_{ji} f^{(\alpha)}{}_i(x, v^{(\alpha)})=0$. Let $(x^*, v^*) \in U_\alpha \times V_\alpha$ and regard $F^{(\alpha)}{}_j = \Sigma_0{}^N t_{ji} f^{(\alpha)}{}_i(x^*, v^*)$ as a function of (t, x^*, v^*). Then the $F^{(\alpha)}{}_j$ are holomorphic in a neighbourhood of $(t, x, v^{(\alpha)})$ and has non-vanishing jacobian there with respect to v^*. Therefore, the coordinates of $v^{(\alpha)}$ are locally holomorphic functions of (t, x). Let $t'=(t'_1, \cdots, t'_N)$ be a set of independent variables over $k(t, y^{(\alpha)})$, and set $t''_\alpha = \Sigma_1{}^N t'_i f^{(\alpha)}{}_i(x, v^{(\alpha)})$. Then t''_α is locally a holomorphic function of (t', t, x).

Let $G(T)$, $(T)=(T_{ij})$, $0 \leqslant i \leqslant n$, $0 \leqslant j \leqslant N$, be the Chow-form of $Z(x)$. By a fundamental property of Chow-forms (cf. v.d. Waerden (I)), we have $G(t''_\alpha, t', t)=0$. Hence (t''_α, t', t) can be identified with a point on the hypersurface G defined by $G(T)=0$. G is a variety defined over k and (t''_α, t', t) is a generic point of G over k. When we normalize $G(T)$ so that some coefficient is 1, then the coefficients of $G(T)$ can be obtained rationally from the coordinates of a finite set of independent generic points of G over k, by applying the Cramer's rule. When we denote by S the space of (t', t), the above arguments show that there is an open subset W of the product of S, in suitable number, and an open subset U' of X containing x, such that the coefficients of $G(T)$ are holomorphic on $W\times U'$. But clearly these coefficients are independent of points of W. Hence they are locally holomorphic functions of x.

§ **2.** In this paragraph, we summarize known results about (H_n, Z) (cf. Example 2 of § 1) from analytic stand point. Here Z is either $Z(W)=A_W$ or $A_W\times O_W$. $Z(W)$ and $Z(W')$ are isomorphic if and only if there is a non-singular n-rowed complex matrix Y and a $2n$-rowed integral unimodular matrix M such that $R_{W'}=YR_WM$. Denote by G_E the set of transformations $W\to W'$ defined by the relation $R_{W'}=YR_WM$. Then G_E is a group of complex analytic automorphisms of H_n and its action on H_n is properly discontinuous. G_E is called a paramodular group, and it is commensurable with the modular group G_{E_0}, where $E_0=\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Moreover, there is a suitable subgroup G'_E of G_E of finite index which acts fixed point free on H_n.

H_n/G_E has a structure of normal analytic space (cf. H. Cartan), which can be compactified, as a topological space, by adding a finite set of spaces of dimensions at most $n(n+1)/2-2$ (cf. Satake (II)). Furthermore, this compactified space has a structure of a normal analytic space which is a prolongation of that of H_n/G_E (cf. Baily (I)). Therefore, when $n\geqq 2$, a meromorphic function on H_n/G_E can be extended uniquely to a meromorphic function on the compactified normal analytic space and the set of such functions forms a finitely generated field of dimension at most $n(n+1)/2$ (cf. Remmert (I)). By mapping H_n into the generalized unit circle by the well-known map, and using there Poincaré series, we can show the existence of a G_E-invariant meromorphic function on H_n which separates two points on H_n which are not congruent modulo G_E. Furthermore, when W is a point of H_n such that only the identity of G_E (res. G'_E) leaves it invariant, then there is a set of $n(n+1)/2$ G_E-(resp. G'_E-) invariant meromorphic functions on H_n such that the functional determinant does not vanish at W. This can be proved by the same technique as above, using Poincaré series of sufficiently high weight. From these two remarks, it follows that the condition (C') of §4 is satisfied by (H_n, Z).

§ **3.** We summarize here well-known facts and immediate consequences of them in algebraic geometry which will be needed later.

LEMMA 2. *Let (A, O_A), (B, O_B) be two Abelian varieties in one and the same projective space, where O_A, O_B are neutral elements of the groups. Assume that the set of hyperplane sections of A (resp. B) forms a complete linear system. Then (A, O_A) and (B, O_B) are isomorphic if and only if A and B are projectively equivalent.*

LEMMA 3. *Let (A, O_A) be a polarized Abelian variety. Then the group of automorphisms of (A, O_A) is a finite group. When A is a subvariety of a projective space, the set of automorphisms of A which are induced by projective transformations is a finite set.*

Let (A, O_A) be a polarized Abelian variety and X an ample polar divisor on A (i.e. a polar divisor such that the complete linear system determined by X defines a projective embedding of A). Denote by $\mathfrak{A}(X)$ the set of positive divisors on A which are algebraically equivalent to X and by $\Lambda(X)$ the set of positive divisors which are linearly equivalent to X. By definition, X, i.e. $\Lambda(X)$ defines a non-degenerate projective embedding f_X of A. f_X is uniquely determined up to a projective transformation. f_X can be extended uniquely to an isomorphism of polarized Abelian varieties, mapping (A, O_A) on $(f_X(A), f_X(O_A))$, which we shall denote by the same letter. When U is a set of polar divisor on A, we denote by $\mathfrak{P}(A, U)$ the set of the $f_X(A)$, where the X are ample divisors contained in U. We definine $\mathfrak{P}((A, O_A), U)$ similarly. Then we get the following corollary.

COROLLARY. $\mathfrak{P}(A, \mathfrak{A}(X)) = \mathfrak{P}(A, \Lambda(X))$.

LEMMA 4. *Let (A, O_A) be a polarized Abelian variety and X an ample polar divisor on A. Let $\mathfrak{p}(A, \mathfrak{A}(X))$ be the set of Chow-points of members of $\mathfrak{P}(A, \mathfrak{A}(X))$. We define $\mathfrak{p}((A, O_A), \mathfrak{A}(X))$*

similarly[1]. $\mathfrak{p}(A, \mathfrak{A}(X))$, $\mathfrak{p}((A, O_A), \mathfrak{A}(X))$ *are both algebraic varieties and have the smallest fields of definitions. When we denote by* $\bar{\mathfrak{p}}(A, \mathfrak{A}(X))$, $\bar{\mathfrak{p}}((A, O_A), \mathfrak{A}(X))$ *the Z-closures of* $\mathfrak{p}(A, \mathfrak{A}(X))$, $\mathfrak{p}((A, O_A), \mathfrak{A}(X))$, $\mathfrak{p}(A, \mathfrak{A}(X))$ *and* $\bar{\mathfrak{p}}(A, \mathfrak{A}(X))$ *and also* $\mathfrak{p}((A, O_A), \mathfrak{A}(X))$ *and* $\bar{\mathfrak{p}}((A, O_A), \mathfrak{A}(X))$ *have the same smallest fields of definitions.*

All these results are discussed in Matsusaka (I) in slightly more general form. Moreover, following two lemmas can be deduced as easy exercise from the results contained in the paper quoted above.

LEMMA 5. *Let* (A, O_A) *be a polarized Abelian variety and* X, Y *two ample polar divisors on* A. *Then* $\mathfrak{p}(A, \mathfrak{A}(X))$ *and* $\mathfrak{p}(A, \mathfrak{A}(Y))$ *have the same smallest field of definition. The same is true for* $\mathfrak{p}((A, O_A), \mathfrak{A}(X))$ *and* $\mathfrak{p}((A, O_A), \mathfrak{A}(Y))$.

Because of this lemma, it is possible to use A for (A, O_A) in many problems concerning moduli. Incidentally, the definition for the field of moduli given in Matsusaka (I) and Shimura (II) are different in appearance but they are actually the same, which can be shown easily. Hence, the field of moduli of a polarized Abelian variety (A, O_A) over a fixed field k_0 is given by the smallest field of definition of $\mathfrak{p}(A, \mathfrak{A}(X))=\mathfrak{p}(A, \Lambda(X))$ containg k_0, when X is an ample polar divisor.

LEMMA 6. *Let* (A_i, O_i), (B_i, O'_i), $i=1, 2$, *be four polarized Abelian varieties and assume that* (A_1, O_1) *and* (A_2, O_2) *are isomorphic. Assume further that there is a discrete valution ring* $\mathfrak{o}$ *such that* $((B_1, O'_1), (B_2, O'_2))$ *is a specialization of* $((A_1, O_1), (A_2, O_2))$ *over* $\mathfrak{o}$. *Let* Γ *be the graph of the isomorphism between* (A_1, O_1) *and* (A_2, O_2) *and extend the above specialization to a specialization* Γ' *of* Γ *over* $\mathfrak{o}$. *Then* Γ' *is the graph of an isomorphism between* (B_1, O'_1) *and* (B_2, O'_2).

This is an immediate consequence of the principle of degeneration of Zariski (cf. Zariski (1)), the compatibility of specializations with algebraic projection (cf. Shimura (1)) and of the fact that $|\Gamma'|$ is an algebraic subgroup of $B_1\times B_2$. This is also a special case

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1) When (B, O_B) is a member $\mathfrak{P}((A, O_A), \mathfrak{A}(X))$, we consider the Chow-point of $B\times O_B$.

of a more general result for non-ruled varieties (cf. Matsusaka-Mumford (1)).

Before we state and prove the next lemma, we summarize the results about the dual Abelian variety of a given Abelian variety (cf. Weil (II) and Igusa (I)). Let $\tilde{V}$ be the underlying real vector space of a complex vector space V of dimension n and J the complex structure such that $(\tilde{V}, J)=V$. Let D be a discrete subgroup of V of the maximum rank over the field of real numbers R and set $T=V/D$ which is a complex torus of dimension n. When $\tilde{x}\in\tilde{V}$, denote by x the corresponding point of V and by x' the corresponding point on T. Assume that (V, D) admits a non-degenerate Riemann-form E, i.e. a skew-symmetric bilinear form on V, integral valued on $D\times D$, such that $E(\tilde{x}, J\tilde{x})$ is symmetric and positive definite. Let X be a complex analytic divisor on T, belonging to the divisor class determined by E. Then $X\sim X_{x'}$ on T if and only if $E(\tilde{x}, \tilde{d})\equiv 0 \bmod Z$ for all $d\in D$. Let $\tilde{V}^*$ be the dual of $\tilde{V}$ and $\tilde{g}$ the map of $\tilde{V}$ into $\tilde{V}^*$ mapping $\tilde{x}$ to $E(\tilde{x},\ )$. $\tilde{g}$ is surjective and R-linear. Let $\tilde{D}^*$ be the set of those $E(\tilde{x},\ )$ such that $E(\tilde{x}, \tilde{d})\equiv 0 \bmod Z$ for all $d\in D$. Then $\tilde{g}(\tilde{d})\in\tilde{D}^*$ and the latter is a discrete subgroup of $\tilde{V}^*$ of rank $2n$ over R. J is an automorphism of $\tilde{V}$ such that $J^2=-1$. Hence J has the dual automorphism J^* of $\tilde{V}^*$, and J^{*-1} is a complex structure on $\tilde{V}^*$ such that $\tilde{g}(J\tilde{x})=J^{*-1}\tilde{g}(\tilde{x})$. Hence $\tilde{g}$ defines a complex linear map g of V onto $V^*=(\tilde{V}^*, J^{*-1})$ such that $g(D)\subset D^*$, where D^* corresponds to $\tilde{D}^*$ in V^*. Therefore, g induces a complex homomorphism α of T on the complex torus $T^*=V^*/D^*$, the dual torus. When $d_1, \cdots, d_{2n}$ is a set of generators of D, then we can introduce a coordinate system on $\tilde{V}$ such that d_i becomes the standard unit vector e_i. When $\{E(\tilde{a}_1,\ ), \cdots, E(\tilde{a}_{2n},\ )\}$ is the dual basis of $\tilde{V}^*$, then this is a set of generators of $\tilde{D}^*$. Hence, we can introduce the coordinate system on $\tilde{V}^*$ such that $E(\tilde{a}_i,\ )$ is the standard unit vector e_i (for the above description of dual torus, see Weil (II) and Igusa (I)). When we denote by $\boldsymbol{E}$ and $M(\alpha)$ the representations of E and $\tilde{g}$ with respect to these coordinate systems, we see at once that $\boldsymbol{E}=M(\alpha)$.

(V^*, D^*) also admits a non-degenerate Riemann-form and hence T and T^* can be identified with Abelian varieties. (α, T^*) is then the Picard variety of T and α is the canonical homomorphism determined by the divisor class of X. Moreover, when suitable l-adic coordinate systems are introduced on Abelian varieties T, T^* and $M(\alpha)$ is viewed as a matrix with l-adic integral entries, we have $M(\alpha)=M_l(\alpha)$ (cf. Weil (I)).

In general, let A and A' be Abelian varieties such that A' is a specialization of A over Q. Let C be a divisor on A and (β, B) the Picard variety of A such that $\beta(u)=\mathrm{Cl}(C_u-C)$. When Γ is the graph of β, we can find a suitable model of B such that (A', B', Γ', C') is a specialization of (A, B, Γ, C) over Q, where B' is an Abelian variety, Γ' is the graph of a homomorphism β', (β', B') is the Picard variety of A' and $\beta'(u')=\mathrm{Cl}(C'_{u'}-C')$. This is well-known and easy to prove in our case. As for the general case of specialization over a discrete valuation-ring, see Koizumi (I).

LEMMA 7. *Let (A, O), (A', O') be two Abelian varieties in a projective space and assume that the latter is a specialization of the former over the field Q of rational numbers. Let C, C' be hyperplane sections of A, A' and E, E' Riemann-forms associated with the divisor classes on A, A' determined by C, C'. Then E and E' have the same normal form.*

Proof. Let (β, B) be the Picard variety of A such that $\beta(u)=\mathrm{Cl}(C_u-C)$, (β', B') the Picard variety of A' such that $\beta'(u')=\mathrm{Cl}(C'_{u'}-C')$ and assume that B has been chosen so that (A', B', Γ') is a specialization of (A, B, Γ) over Q, where Γ (resp. Γ') is the graph of β (resp. β'). Denote by $\mathfrak{g}(l,\ )$ the group of points of orders which are powers of a prime l on Abelian varieties. Since the graph of an endomorphism $l^m\delta$ on A specializes to such on A' over $A\rightarrow A'$ ref. Q, this specialization defines an isomorphism σ between $\mathfrak{g}(l, A)$ and $\mathfrak{g}(l, A')$. The same is true for B, B' and $\mathfrak{g}(l, B)$, $\mathfrak{g}(l, B')$. We denote by σ again such an isomorphism. Then it is possible to introduce an l-adic coordinate system on A' (resp. B') such that x in $\mathfrak{g}(l, A)$ (resp. $\mathfrak{g}(l, B)$) and $\sigma(x)$ have the same l-adic coordinates. When this is done, β and β' have the same l-adic representation, i.e. $M_l(\beta)=M_l(\beta')$. Since this is true for all l, it follows that $M(\beta)$ and $M(\beta')$ have the same elementary divisors. Consequently, E and E' have the same normal form.

PROPOSITION 2. *Let (A^n, O) be a polarized Abelian variety, Y a basic polar divisor of A and E the non-degenerate Riemann-form associated with the divisor class determined by Y. Let m be a positive integer such that $m \geqslant 3$ and set $N = m^n \det(D)/n! - 1$, where D is such that $\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$ is the normal form of an integral representation of E. Denote by $\mathfrak{F}_m$ the set of non-singular subvarieties B in the projective space P^N, which are underlying varieties of Abelian varieties, satisfying the following conditions*: (i) *The set of hyperplane sections of B forms a complete linear system*; (ii) *The Riemann-form associated with the divisor class of hyperplane sections of B is given by mE. Then the set of Chow-points F_m of members of $\mathfrak{F}_m$ is an algebraic variety defined over Q. When we denote by $\mathfrak{F}'_m$ the set consisting of $B \times u$, where B is in $\mathfrak{F}_m$ and u is a point of B, the set of Chow-points of members of $\mathfrak{F}'_m$ is an algebraic variety defined over Q.*

Proof. Let B be a member of $\mathfrak{F}_m$. Using the same notations as in Example 2, there is a point W in H_n such that A_W and B are isomorphic (as polarized varieties). When φ is the map: $W \to c(A_W)$, it is a holomorphic map of H_n into a projective space by Prop. 1. $\varphi(H_n)$ is contained in F_m and an element of $\mathfrak{F}_m$ can be obtained from an element of the form A_W by a projective transformation (cf. Lemma 2). Denote by G the group variety $PGL(N)$ and by g an element of G. Set $X = H_n \times G$ and $Z(W, g) = g(A_W)$. Then it is easy to verify that (X, Z) defines an analytic family of non-singular varieties in P^N. When ψ denotes the map $(W, g) \to c(g(A_W))$, it is a holomorphic map of $H_n \times G$ into a projective space by Prop. 1, and $\psi(H_n \times G) = F_m$. $H_n \times G$ is a connected complex manifold, and hence, ψ is defined by a finite set of meromorphic functions on $H_n \times G$. Taking algebraic relations among them into consideration, it is easy to see that the Z-closure $\bar{F}_m$ of F_m is an algebraic variety. Let k be a field of definition of $\bar{F}_m$. Then a generic point of $\bar{F}_m$ over k is also a point of F_m by Lemma 7. Let V be the set of points in $\bar{F}_m$, which correspond to underlying varieties of Abelian varieties and which are not contained in any hyperplane in P^N, then V is a Z-open subset of $\bar{F}_m$ (cf. Hoyt (I)). F_m is contained in V. But, from the Riemann-Roch theorem $l(X)=X^{(n)}/n!$, it follows that the varieties corresponding to points of V also satisfy the condition (i). Then F_m contains V by Lemma 7. Thus $F_m=V$. From the definition of V and Lemma 7, V is invariant under the action of an automorphism of the field of complex numbers. Hence F_m is an algebraic variety defined over Q. The corresponding fact for $\mathfrak{F}'_m$ can be deduced from this easily.

§ **4.** In order to study analytic families of non-singular varieties more closely, we list here some basic properties satisfied by our Example (2), the paramodular family of Abelian varieties. In order to do so, we shall discuss briefly about an equivalence relation on algebraic varieties.

Let V be an algebraic variety and Y a *Z-closed subset* of $V\times V$. Assume that Y defines on V an *equivalence relation* $\mathfrak{R}$, i.e. when we define $v\sim v'$ if and only if (v, v') is in Y, this is an equivalence relation on V. Assume further that every component Y_i of Y has the projection V on V and that $\dim Y_i=\dim Y_j$ for every pair (i, j). *We shall consider equivalence relation of this type only* on algebraic varieties. Set $\dim V=n$ and $\dim Y=n+r$. When we denote by $K(V/\mathfrak{R})$ the set of rational functions on V which are invariant with respect to $\mathfrak{R}$, then it is a finitely generated field of dimension $n-r$ over the universal domain (the field of complex numbers in the case we are interested in). When k is a field of definition of V over which Y is closed, then we say that $\mathfrak{R}$ is defined over k.

The paramodular families we have studied in § 2 and § 3 satisfy the following properties. It is an analytic family (X, Z) such that:

(A) X carries a properly discontinuous group G of automorphisms;
(B) There is a set $\mathfrak{F}$ of non-singular varieties in a projective space P^N, containing every $Z(x)$, satisfying the following properties:

(i) The set of Chow-points of $\mathfrak{F}$ is an algebraic variety F;

(ii) F carries an equivalence relation $\mathfrak{R}$, such that $Z(x) \sim Z(x')$ with respect to $\mathfrak{R}$ if and only if x and x' are congruent modulo G;

(iii) The set of Chow-points of members of $\mathfrak{F}$ which are equivalent to some $Z(x)$, $x \in X$, contains a Z-open subset of F;

(iv) When A is a member of $\mathfrak{F}$, the set of Chow-points of members of $\mathfrak{F}$ which are equivalent to A forms an algebraic subvariety of F of constant dimension.

Before we list some other properties satisfied by the paramodular families, we shall discuss some consequences of (A) and (B).

LEMMA 8. *Let* $(F, \mathfrak{R})$ *be as in* (B) *and* k *a common field of definition of* F *and* $\mathfrak{R}$. *When* $E(a)$ *is the set of points of* F *which are equivalent to* $a \in F$, *then* $E(a)$ *is* $k(a)$*-closed.*

This is easy to prove.

LEMMA 9. *Let the notatons be as in Lemma* 8 *and denote by* $\overline{E(a)}$ *the* Z*-closure of* $E(a)$ *in the ambient projective space. When* a *is a generic point of* F *over* k, *the smallest field of definition* K_a *of* $\overline{E(a)}$ *containing* k *is contained in* $k(a)$ *and* $\dim_k K_a = \dim X$.

Proof. The first part of our lemma follows from Lemma 8. Let $\tilde{v}$ be the Chow-point of $\overline{E(a)}$ and $\tilde{V}$ the locus of $\tilde{v}$ over k. Clearly, $K_a = k(\tilde{v})$. When we set $\tau(a) = \tilde{v}$, τ is a rational map of F into $\tilde{V}$. Replacing F by its normalization, if necessary, we may assume without loss of generality that τ is defined at every point of $E(a)$. By (B)-(iii), there is a point x in X such that $c(Z(x)) \in E(a)$. Since the map $\varphi : x \rightarrow c(Z(x))$ is holomorphic on X, the map $x \rightarrow \tilde{v}$ is a homomorphic map of an open subset U of X containing x into $\tilde{V}$. The image of F by τ contains a Z-open subset of $\tilde{V}$ (cf. Weil (VI)). Consequently, the image of U by the map $x \rightarrow \tilde{v}$ contains a Z-open subset of $\tilde{V}$, which proves $\dim X \geq \dim \tilde{V}$. On the other hand, from (A), (B)-(ii) and from the definition of $\tilde{V}$, it follows easily that $\dim X \leqslant \dim \tilde{V}$.

COROLLARY. *Let* $(F, \mathfrak{R})$ *be as in* (B). *Then the field* $K(F/\mathfrak{R})$

is an algebraic function field of dimension $m=\dim X$, *and is a subfield of the field of G-invariant meromorphic functions on X.*

Besides (A) and (B), the paramodular families further satisfy the following conditions:

(C) When $(x_1, \cdots, x_t)$ is a finite set of points of X, such that no two points are congruent modulo G, there is a G-invariant meromorphic function h, algebraic over $K(F/\mathfrak{R})$, such that h is holomorphic at the x_i and that $h(x_i)\neq h(x_j)$ for $i\neq j$;

(D) G contains a subgroup G_0 of finite index in G which operates fixed point free on X; moreover, when $(x_1, \cdots, x_t)$ is a finite set of points on X, there is a finite set of G_0-invariant meromorphic functions on X, algebraic over $K(F/\mathfrak{R})$, which are holomorphic with non-vanishing Jacobians at the x_i.

In the discussions which follow, we sometimes use the following stronger conditions (B′) and (C′), satisfied by the paramodular families.

(B′) In (B), (i), (ii), (iv) remain as they are, but (iii) is replaced by

(iii)′ The set of Chow-points of members of $\mathfrak{F}$ which are equivalent to some $Z(x)$, $x\in X$, is a Z-open subset of F;

(C′) When $(x_1, \cdots, x_t)$ is a set of points on X such that no two points are congruent modulo G, then there is a finite set of holomorphic functions $h_0, \cdots, h_r$ on X satisfying the following conditions:

(i) When $i\neq j$, there is an index s such that $h_s(x_i)\neq h_s(x_j)$, $h_s(x_i)\cdot h_s(x_j)\neq 0$;

(ii) h_i/h_j is a G-invariant meromorphic function on X, algebraic over $K(F/\mathfrak{R})$;

(iii) When $x_{t+1}\in X$ is not congruent to any x_i modulo G, there is a set $(h'_0, \cdots, h'_{r'})$ of holomorphic functions on X, containing the set of monomials in the h_i of a fixed degree d, such that it satisfies (i) and (ii) with respect to $(x_1, \cdots, x_t, x_{t+1})$.

Remark. In general, when X is isomorphic to a symmetric domain, then (C′) can be verified except for (ii), by using the Poincaré sesies. H_n is one of such domains. In the case when

$n=1$, it is well-known and easy to verify that $K(F/\mathfrak{N})$ exhausts the modular functions in one variable. As we remarked in §2, $K(F/\mathfrak{N})$ exhausts the paramodular functions and is the set of invariant meromorphic functions when $n\geqslant 2$. In general, when X is the product of irreducible domains of dimensions at least 2 and when G is the so-called "normal discrete group", then Pyatecki̇̆-Šapiro showed that G-invariant meromorphic-functions on X form an algebraic function field of dimension $m=\dim X$. The same is true when X/G is compact. Hence, in these cases, again the algebraic nature of meromorphic functions over $K(F/\mathfrak{N})$ in (C), (D) and (C′) are verified because of the corollary of Lemma 9.

§5. In this paragraph, *we assume that* (X, Z) *is an analytic family of non-singular varieties in a projective space, satisfying the conditions* (A), (B), (C) *and* (D), unless the contrary is specifically mentioned. Moreover, we set $\dim X=m$. Furthermore, when $(F, \mathfrak{N})$ is the pair which enters in (B), *we fix a common field* k_0 *of definition for* F *and* $\mathfrak{N}$, *and assume that all fields we shall consider contain* k_0.

In order to simplify matters, let us fix the following notations. φ will denote always a holomorphic map of X into a projective space defined by $x\rightarrow c(Z(x))$. When a is a point of F, $E(a)$ will denote always the set of points on F which are equivalent to a and $\overline{E(a)}$ will denote the Z-closure of $E(a)$ in the ambient projective space. When k is a field of definition of F (containing k_0) and u a generic point of F over k, $\tilde{v}=c(\overline{E(u)})$ has a locus over k by Lemma 9, which will be denoted by $\tilde{V}$. There is a rational map of F into $\tilde{V}$ defined by $u\rightarrow\tilde{v}$, which will be denoted by τ. Finally, $\tau\circ\varphi$ is a holomorphic map of an open subset of X into $\tilde{V}$, which will always be denoted by ψ.

Frst we shall show that meromorphic functions on X, which are algebraic over $K(F/\mathfrak{N})$ and are G-invariant, are elements of $K(F/\mathfrak{N})$, i.e. $\mathfrak{N}$-invariant rational functions on F. This follows from the following lemma.

LEMMA 10. *Let* V^n *be a normal algebraic variety in a projective space and* U *an open subset of* V *such that every point of*

U is simple of V. Let $b_0, \cdots, b_r$ *be rational functions on V and f a meromorphic function defined on U, satisfying* $b_0 f^r + b_1 f^{r-1} + \cdots + b_r = 0$. *If there is a common field k of definition for V and for the* b_i *such that U contains all the generic points of V over k, f is a rational function on V.*

Proof. Let V_0 be the set of simple points of V. Then a point in $V - V_0$ is at least of codimension 2 over k and a point in $V_0 - U$ is at least of codimension 1 over k. Let v' be a point of $V_0 - U$ and W an affine open subset of V_0 containing v'. Let (h) be a set of affine coordinate functions in W. Since the b_i are rational functions of (h), there are polynomials $c_0(h), \cdots, c_r(h)$ in (h) such that f satisfies $c_0(h) f^r + c_1(h) f^{r-1} + \cdots + c_r(h) = 0$ in $U \cap W$. Setting $g = c_0(h) \cdot f$, we get $g^r + e_1(h) g^{r-1} + \cdots + e_r(h)$, where the $e_i(h)$ are also plynomials in (h). When we take a sufficiently small neighbourhood W' of v', contained in W, the roots of $Y^r + e_1(h) Y^{r-1} + \cdots + e_r(h) = 0$ are bounded when $v \in W'$. Since the set of generic point of V over k is dense, $U \cap W' \neq \emptyset$. Clearly g is holomorphic in $U \cap W'$, and points of $W' - U \cap W'$ are at least of codimension 1 over k. Hence g can be continued analytically throughout W' and becomes holomorphic there. Consequently, f is meromorphic on W'. Therefore, f is a meromorphic function on V_0. When that is so, f is a rational function on V, as is well-known (if one uses the reduction of singularities due to Hironaka, one can prove this fact directly from the above proof) (cf. Hironaka (I)).

As we announced before our Lemma 10, we get the following proposition easily from this Lemma.

PROPOSITION 3. *Let f be a G-invariant meromorphic function on X and set* $f(x) = \tilde{f}(\psi(x))$. *Then* $\tilde{f}$ *is a rational function on* $\tilde{V}$ *if f is algebraic over* $K(F/\mathfrak{R})$. *Conversely, when* $\tilde{f}$ *is a rational function on* $\tilde{V}$, $f = \tilde{f} \circ \psi$ *is a G-invariant meromorphic function on X, algebraic over* $K(F/\mathfrak{R})$. *The correspondence* $f \rightarrow \tilde{f}$ *gives an isomorphism between the field of G-invariant meromorphic functions on X which are algebraic over* $K(F/\mathfrak{R})$ *and the field of rational functions on* $\tilde{V}$.

LEMMA 11. *Let V be a non-singular variety in a projective* *space and V^* a topological space satisfying the following conditions*: (i) $|V^*|\subset|V|$; (ii) $|V^*|$ *contains a Z-open subset V_0 of V*; (iii) *The injection i of V^* into V is continuous*; (iv) *When Γ is an algebraic curve on V, $|\Gamma|\cap|V^*|$ with the induced topology of V^* is such that no finite set of points is open. Let g be a rational function on V and D a Z-open subset of V such that g is defined and finite at every point of D. Let g^* be a continuous function on an open subset D^* of V^* such that $|D^*|$ contains $|V^*|\cap|D|$ and that $g^*(v'')=g(v'')$ for every v'' in $|V^*|\cap|D|$. Then g is defined at every point v' of $|D^*|$ and $g(v')=g^*(v')$.*

Proof. First we settle our lemma when V is a non-singular curve. We state this as the following lemma.

LEMMA. *Let Γ be a non-singular curve, Γ^* a topological space such that $|\Gamma|=|\Gamma^*|$ and assume that the canonical injection map i of Γ^* into Γ is continuous. Let g be a rational function on Γ and g^* a continuous map of Γ^* into C^1 such that $g^*=g\circ i$ on a dense subset of Γ^*. Then g is defined and finite everywhere on Γ.*

Proof. When $\mathfrak{a}$ is the set of points where g is not finite, g is defined everywhere on $\Gamma-\mathfrak{a}$. When that is so, $g^*=g\circ i$ everywhere on $\Gamma^*-\mathfrak{a}$. Let x be a point of $\mathfrak{a}$ and c a constant such that $g^*(x)+c\neq 0$. Then, there is a finite set of points $\mathfrak{b}$, containing x, such that $1/(g^*+c)$ is continuous on $\Gamma^*-\mathfrak{b}$, $1/(g+c)$ is continuous on $\Gamma-\mathfrak{b}$ and that $1/(g^*+c)=1/(g+c)\circ i$ on $\Gamma^*-\mathfrak{b}$. Moreover, $1/(g^*+c)$ is continuous on $\Gamma^*-(\mathfrak{b}-x)$ and $1/(g+c)$ is continuous on $\Gamma-(\mathfrak{b}-x)$. Hence $1/(g^*+c)=1/(g+c)\circ i$ on $\Gamma^*-(\mathfrak{b}-x)$. This is against to our assumption that g is not finite at x, and $\mathfrak{a}=\emptyset$.

Proof of our lemma. Let k be a common field of definition for V, V_0, D and g. Let Γ_0 be an algebraic curve on V, going through v' and a generic point of V over k and having v' as a simple point. Set $i^{-1}(\Gamma_0)=\Gamma^*_0$. Γ^*_0 is an algebraic curve on V with finer topology than the induced topology. Let $\mathfrak{a}$ be the set of singular points of Γ_0, contained in $|\Gamma^*_0|$, and the points in $|\Gamma^*_0|$ which are not contained in $|D|$. When we call Γ the set $(|\Gamma^*_0|-\mathfrak{a})+v'$,

Γ is also an algebraic non-singular curve on V. Set $i^{-1}(\Gamma)=\Gamma^*$. Then $|\Gamma|=|\Gamma^*|$. g^* induces a continuous function h^* on Γ^* and g induces a rational function h on Γ which is defined and finite at every point of Γ, except at v'. Moreover, $h^*=h\circ i$ on Γ^*-v'. Moreover, Γ^*-v' is dense in Γ^*. Consequently, h is defined at v' and $h(v')=h^*(v')=g^*(v')$ by our lemma. Since this is true for all algebraic curves Γ_0 with the same properties as Γ_0, it follows that g is defined at v' and $g(v')=g^*(v')$.

For the sake of simplicity, we shall adopt the following definitions. Let W be an algebraic variety defined over a field k, h a rational map of W into an algebraic variety U and Γ the Z-closure of the graph of h on $W\times U$. Let w' be a point of W and assume that there is one and only one point $w'\times u'$ on Γ with the projection w' on W. Furthermore, assume that Γ is complete over w'. Then we shall say that h is *single valued at* w'. We shall say also that u' is the *value* of h at w' and denote it by $h(w')$. This is equivalent to the following. Let w be a generic point of W over k. Then $h(w)$ has a uniquely determined specialization u' over k, over the specialization $w\rightarrow w'$ ref. k.

LEMMA 12. *Let V be an algebraic variety in a projective space and Γ a Z-closed subset of $V\times V$ such that every component of Γ has the same dimension and that every such component has the projection V on either factor of the product $V\times V$. Let k be a field of definition for V such that Γ is k-closed. When v' is a point of V, denote by $\Gamma(v')$ the set defined by $v'\times V\cap\Gamma=v'\times\Gamma(v')$. Assume that Γ defines an equivalence relation on V and that, when v is a generic point of V over k, $\Gamma(v')$ is a uniquely determined specialization of $\Gamma(v)$ over k over the specialization $v\rightarrow v'$ ref. k. Let g be a rational function on V defined over k such that $g(v)=g(v'')$ whenever v'' is a generic point of V over k and contained in $\Gamma(v)$. Under these conditions, if g is single valued at v', then g is single valued at every point w' of $\Gamma(v')$ and $g(v')=g(w')$.*

Proof. Let $(v, g(v))\rightarrow(w', t)$ ref. k. By our assumption, we get $(v, g(v), \Gamma(v))\rightarrow(w', t, \Gamma(w'))$ ref. k. Since Γ defines an equivalence relation, $\Gamma(v')=\Gamma(w')$, and v' is a point of $\Gamma(w')$. When that is so, there is a point v_0 of $\Gamma(v)$, which is a generic point of V over k, such that $(v, g(v), \Gamma(v), v_0) \rightarrow (w', t, \Gamma(w'), v')$ ref. k. By our assumptions, $g(v)=g(v_0)$ and g is single valued at v'. Consequently, $t=g(v')$. Our lemma is thereby proved.

Let V be an algebraic variety and Γ a Z-closed subset of $V \times V$. When Γ defines on V an equivalence relation on V and when Γ satisfies conditions described in Lemma 12, we shall call such an equivalence relation on V *admissible*. As an application of Lemma 11, we get the following lemma.

LEMMA 13. *Let f be a G-invariant meromorphic function on X and U a G-invariant open subset of X on which f is holomorphic. Let μ be a holomorphic map of U into a locally Z-closed algebraic subvariety of a projective space with the following properties*: (i) *$\mu(x)=\mu(x')$ implies $x \equiv x'$ mod G*; (ii) *There is a Z-closed subset Γ of $V \times V$ which defines an admissible relation on V such that $(\mu(x), \mu(x'))$ is in Γ if and only if $x \equiv x'$* mod G; (iii) *The set of points on V which are equivalent to points of $\mu(U)$ contains a Z-open subset V_0 of V. Let k be a common field of definition for V and V_0 over which Γ is closed. Assume that there is a rational function g, defined over k, such that $f(x)=g(v)$ when $x \in U$ and when v is a generic point of V over k such that $\mu(x) \sim v$. Under these conditions, when x_0 is a point of U, g is single valued at $v_0=\mu(x_0)$ and $g(v_0)=f(x_0)$.*

Proof. Denote by $\bar{V}$ the Z-closure of V in a projective space, W a non-singular variety in a projective space and h a surjective morphism of W on $\bar{V}$ such that it is a birational transformation (cf. Hironaka (I)). Then (μ, h) is a holomorphic map of $U \times W$ into $\bar{V} \times \bar{V}$. Let $\bar{\Gamma}$ be the Z-closure of Γ on $\bar{V} \times \bar{V}$ and T the set of points in $U \times W$ such that $(\mu(x), h(w))$ is contained in $\bar{\Gamma}$. Then T is an analytic subset of $U \times W$. Regard T as a topological space with the induced topology from $U \times W$. When we set $f'(x, w)=f(x)$, f' is a continuous function on T. When we set $g'=g \circ h$, g' is a rational function on W. If w' is a point of W such that $h(w')=v'$, and if g is defined at v', then g' is defined at w' and $g'(w')=g(v')$. When w is a generic point of W over k

(take W and h so that they are defined over k), there is a point x in U such that $(\mu(x), h(w))$ is in $\bar{\Gamma}$. Since $f(x)=g(h(w))$, it follows that $f'(x, w)=g'(w)$.

Let w_0 be a point of W such that $h(w_0)=v_0$. Suppose that g' is defined at such w_0 and $g'(w_0)=f(x_0)$. Then it can be shown easily that g is single valued at v_0 and $g(v_0)=f(x_0)$.

Let ν be the projection map of T on W. ν is clearly continuous. $\nu(T)$ also clearly contains a Z-open subset $h^{-1}(V_0)$ of W. Introduce on $\nu(T)$ the quotient topology with respect to the equivalence relation on T defined by the map ν. Denote the resulting topological space by W^*. ν determines the canonical map ν^* of T on W^*. f' determines a continuous function g^* on W^*. Then we can verify that W, W^*, g' and g^* satisfy the requirements of Lemma 11. Hence g' is defined at a point w_0 of $|W^*|$ and $g'(w_0)=g^*(w_0)=f'(x', w_0)$ if (x', w_0) is a point of T. Our lemma is thereby proved.

When we apply this lemma to our situation, we get the following proposition at once (cf. Prop. 3).

PROPOSITION 4. *Let x_0 be a point of X and f a G-invariant meromorphic function on X, algebraic over $K(F/\mathfrak{R})$, such that it is holomorphic at x_0. Let $\tilde{f}$ be a rational function on $\tilde{V}$ such that $f=\tilde{f}\circ\psi$ and set $\hat{f}=\tilde{f}\circ\tau$. Then $\hat{f}$ is single valued at $\varphi(x_0)$ and has the value $f(x_0)$.*

COROLLARY 1. *Let f be a G-invariant meromorphic function on X, algebraic over $K(F/\mathfrak{R})$. Let $\hat{f}$ be a rational function on F determined by $f=\tilde{f}\circ\psi$, $\hat{f}=\tilde{f}\circ\tau$. Let x_0 be a point of X, b a point of F such that $b\sim\varphi(x_0)$ and $E(b)$ the set of points on F which are equivalent to b. If f is holomorphic at x_0, then $\hat{f}$ is single valued at every point of $E(b)$ and takes the constant value $f(x_0)$.*

This corollary follows at once from our proposition and from Lemma 12. Similarly, we get

COROLLARY 2. *Let f be a G-invariant meromorphic function on X, algebraic over $K(F/\mathfrak{R})$, and $\hat{f}$ a rational function on F determined by $f=\tilde{f}\circ\psi$, $\hat{f}=\tilde{f}\circ\tau$. When $\hat{f}$ is single valued at b_0 on F, it is single valued at every point of $E(b_0)$ and has the value $\hat{f}(b_0)$.*

Using the fact that X is a complex manifold, Cor. 1 and Cor. 2 above can be generalized at once as follows:

COROLLARY 3. *Let f and $\hat{f}$ be as in Corollary 2 and b_0 a point of F. When $\hat{f}$ is intergral over the local ring of F at b_0, then it is so over the local ring of F at every point of $E(b_0)$. When there is a point x_0 on X such that $\varphi(x_0)=b_0$, and when $\hat{f}$ is integral over the local ring of F at b_0, f is holomorphic at x_0 and $\hat{f}$ is single valued at every point of $E(b_0)$ with the value $f(x_0)$.*

From (C), Prop. 4 and from Cor. 1, Cor. 2 above, we immediately get the following

COROLLARY 4. *Let $x_1, \cdots, x_t$ be points on X and $\varphi(x_i)=b_i$. There is a finite set of meromorphic functions $f_1, \cdots, f_N$ on X, which are G-invariant and algebraic over $K(F/\mathfrak{R})$, satisfying the following conditions:* (i) *When the $\hat{f}_j$ are rational functions of F determined by $f_j=\tilde{f}_j\circ\psi$, $\hat{f}_j=\tilde{f}_j\circ\tau$, the $\hat{f}_j$ are single valued at every point of $E(b_i)$ and take the values c_{ij};* (ii) *If $x_0\in X$, $\varphi(x_0)=b_0$ is such that every $\hat{f}_j$ is single valued at some point b of $E(b_0)$ and takes the value c_{ij}, then b_0 is a point of $E(b_i)$.*

Using Cor. 3 and the technique of normalization in an algebraic extension, we get the following

COROLLARY 5. *In Cor. 4, we can take the $\hat{f}_i$ so that the following additional conditions are satisfied:* (iii) *The $\hat{f}_i$ generate the rational function field of $\tilde{V}$;* (iv) *The geometric image of F by the map into the projective space P^{N-1} determined by $(\hat{f}_1, \cdots, \hat{f}_N)$ is a normal algebraic variety.*

Combining Cor. 4 and Cor. 5, we further get the following

COROLLARY 6. *When $x_1, \cdots, x_t$ are points of X and $\varphi(x_i)=b_i$, there are a normal algebraic variety W, a rational map λ of F into W which is generically surjective, and a holomorphic map μ of an open subset of X, containing the x_i, into W satisfying the following conditions:* (i) *When x' is a point of X such that μ is holomorphic at x', then λ is single valued at $\varphi(x')$ and $\mu(x')=\lambda(\varphi(x'))$;* (ii) *When $x'\in X$ and λ is integral over the local ring of F at $\varphi(x')$, then λ is single valued at $\varphi(x')$, μ is holomorphic at x'*

and $\mu(x')=\lambda(\varphi(x'))$; (iii) *When* $x'\in X$ *and* λ *is single valued at* $\varphi(x')$, *then it is single valued at every point of* $E(\varphi(x'))$ *and has the constant value* $\mu(x')$; (iv) λ *is single valued at the* b_i; *when* Λ *is the Z-closure of the graph of* λ *on* $F\times W$, *then* $\Lambda\cap F\times\lambda(b_i)$ *contains* $E(b_i)\times\lambda(b_i)$; *moreover, when* $b'\times\lambda(b_i)$ *is a point of* Λ *such that* $E(b')\cap\varphi(X)\neq\emptyset$ *and that* λ *is single valued at* b', *then* b' *is a point of* $E(b_i)$; (v) *the correspondence* $\lambda(u)\to\tau(u)$ *is a birational correspondence between* W *and* $\tilde{V}$.

§ **6.** In this paragraph, *we assume again that* (X, Z) *is an analytic family of non-singular algebraic varieties in a projective space, satisfying* (A), (B), (C) *and* (D), unless the contrary is specifically stated. Furthermore, *we fix a point* x_0 *on* X and consequently $b_0=\varphi(x_0)$. We denote by W_{b_0}, λ_{b_0}, μ_{b_0} the variety, the rational map and the holomorphic map constructed in Cor. 6 of Prop. 4, in the special case when $(x_1, \cdots, x_t)$ reduces to x_0. For the sake of simplicity, we shall omit the index b_0 when there is no danger of confusion. By (D), G contains a subgroup G_0 of finite index with the property described in (D).

LEMMA 14. *Let* $G=\sum_1^a G_0\cdot\gamma_i$ *and* f *a* G_0*-invariant meromorphic function on* X, *algebraic over* $K(F/\mathfrak{R})$. *Let* y *be a point of* X *such that* $\varphi(y)\sim b_0$ *and that* f *is holomorphic at the* $\gamma_i(y)$. *Then* f *is integral over the local ring of* W *at* $\lambda(\varphi(y))$.

Proof. Set $f_i(x)=f(\gamma_i(x))$. It is enough to prove that the coefficients of the polynomial $\Pi_i(T-f_i)$ are integral over the local ring. Denote by h one of the coefficients. Then h is holomorphic at y, G-invariant and algebraic over $K(F/\mathfrak{R})$. Hence, there is a rational function g on W such that $h=g\circ\mu$ which is valid at x on X whenever $\mu(x)$ is in a suitable Z-open subset of W (cf. Prop. 3 and Cor. 6, Prop. 4). Take the identity equivalence relation on W. Then our assertion follows easily by applying Lemma 13 to our situation.

Denote by L the set of G_0-invariant meromorphic functions on X which are algebraic over $K(F/\mathfrak{R})$. Since G_0 is a subgroup of G of finite index, L is a finite algebraic extension of the field

$K(W)$ of rational functions of W (cf. Prop. 3 and Cor. 6, Prop. 4). Throughout this paragraph, we shall denote by (W^*, π) a normalization of W in the function field L. W^* is a normal algebraic variety in a projective space, π a proper morphism of W^* onto W and the inverse image of a point of W by π is a finite set of points. From Lemma 14 and from (D), we get easily the following lemma.

LEMMA 15. *Choose y as in the Lemma* 14. *Then there is an open subset U of X, containing the $\gamma_i(y)$ and a holomorphic map μ^* of U into W^* such that μ^* is of rank $m=\dim X$ everywhere on U and that $\mu=\pi\circ\mu^*$. Hence $\mu^*(\gamma_i(y))$ is a simple point of W^* for every i.*

By (B), (iii), the set of points on F which are equivalent to points of $\varphi(X)$ contains a Z-open subset of F. Hence the set contains the largest Z-open subset of F, which we denote by $F^\sharp$. Let k be a common field of definition for F, $\tilde{V}$, τ, λ, W, W^*, π, $F^\sharp$ over which the equivalence relation on F is defined. We fix k this way untill the end of this §.

Let Λ be the Z-closure of the graph of λ on $F\times W$ and π the morphism (identify, π) of $F\times W^*$ on $F\times W$. Let u be a generic point of F over k. By Cor. 6, Prop. 4, we have $F\times\lambda(u)\cap\Lambda=E(u)\times\lambda(u)$, and $E(b_0)\times\lambda(b_0)$ is a component of $F\times\lambda(b_0)\cap\Lambda$ if b_0 is a point of $F^\sharp$. When $b'\times\lambda(b_0)$ is a point of the intersection, not contained in $E(b_0)\times\lambda(b_0)$, then either λ is not finite at b' or $E(b')\cap\varphi(X)=\emptyset$. Let Γ, Γ' be respectively the graphs of Π, π and set $\Pi^{-1}(\Lambda)=pr_{12}(\Gamma\cdot(F\times W^*\times\Gamma'))$. Let w^*, w_0^* be points on W^* such that $\pi(w^*)=\lambda(u)$, $\pi(w_0^*)=\lambda(b_0)$. Then, $F\times w^*\cap\Pi^{-1}(\Lambda)=E(u)\times w^*$, and $E(b_0)\times w_0^*$ is a component of $F\times w_0^*\cap\Pi^{-1}(\Lambda)$. When y is a point of X such that $\varphi(y)\sim b_0$, $\mu^*(y)$ is mapped to $\lambda(b_0)$ by π. Assuming still that b_0 is in $F^\sharp$, $E(b_0)$ is then a component of $F\times\mu^*(y)\cap\Pi^{-1}(\Lambda)$. We set $\Lambda^*=\Pi^{-1}(\Lambda)$. By (B), (iv) and from basic properties of π, it can be shown easily that Λ^* is an algebraic subvariety of $F\times W^*$. Let $\bar{\Lambda}^*$ be the Z-closure of Λ^* in the ambient space.

Let P be the ambient projective space of F. Then, from what we discussed above, it follows that $\bar{\Lambda}^*$ and $P\times w^*$ intersect properly on $P\times W^*$ and $\bar{\Lambda}^*\cdot(P\times w^*)=\overline{E(u)}\times w^*$, where $\overline{E(u)}$ denotes the Z-closure in P. $\overline{E(b_0)}\times\mu^*(y)$ is a proper component of $\bar{\Lambda}^*\cap(P\times\mu^*(y))$ on $P\times W^*$ if b_0 is a point of $F^\sharp$.

LEMMA 16. *Let $w^{*\prime}$ be a point of W^* such that $\bar{\Lambda}^*\cap(P\times w^{*\prime})$ has a proper component $S\times w^{*\prime}$ on $P\times W^*$ and that $S\cap F\neq\emptyset$. Then, there is a point d in F such that $S=\overline{E(d)}$, the Z-closure of $E(d)$ in P. When that is so, $\Lambda^*\cap(F\times w^{*\prime})=E(d)\times w^{*\prime}$.*

Proof. Let d be a common point of S and F. When u is a generic point of F over k and w^* a generic point of W^* over k such that $\lambda(u)=\pi(w^*)$, $\bar{\Lambda}^*\cdot(P\times w^*)=\overline{E(u)}\times w^*$. Then $S=\overline{E(d)}$ follows from (B), (iv), (v) and from the compatibility of specializations with the operation of intersection-product (cf. Shimura (I)). In fact, let K be the algebraic closure of $k(w^{*\prime})$ and d a generic point of S over K. Then d is necessarily in F by our assumption. $d\times w^{*\prime}$ is a specialization of $u\times w^*$ over k. This specialization extends uniquely to $E(u)\to E(d)$ ref. k on F by (B), (v), (iv). It follows that $\overline{E(d)}\times w^{*\prime}$ is contained in $\bar{\Lambda}^*$, and has a point $d\times w^{*\prime}$ in common with $S\times w^{*\prime}$. Therefore, $S\times w^{*\prime}$ is contained in the Z-closure of $\overline{E(d)}\times w^{*\prime}$. On the other hand, $\dim S=\dim E(d)=\dim E(u)$ by (B), (iv) and S is irreducible. Consequently, $S=\overline{E(d)}$. Let $u'\times w^{*\prime}$ be a point of $\bar{\Lambda}^*\cap(P\times w^{*\prime})$ such that $u'\in F$. Then $u'\times w^{*\prime}$ is a specialization of $u\times w^*$ over k. Extend this to a specialization T of $\overline{E(u)}$ over k. The compatibility of specialization with the operation of intersection-product implies that $\overline{E(d)}$ is a component of T. There is a generic point b of F over k, contained in $E(u)$, such that $(u', w^{*\prime}, d)$ is a specialization of (u, w^*, b) over k. u, b, u' and d are points of F and $u\sim b$. Hence $u'\sim d$ by (B), (v). Our lemma is thereby proved.

LEMMA 17. *Let F_λ be the set of points b on F such that λ is finite at b. Then F_λ is a Z-open subset of F, defined over k and λ is single valued at every point of F_λ.*

Proof. Let F' be the normalization of F. The set of points on F' where a rational map is defined is Z-open. Our lemma then follows from this and from Cor. 3, Prop. 4.

LEMMA 18. *Let Λ_λ^* be the restriction of Λ^* on $F_\lambda \times W^*$. Let D_λ^* be the set of simple points $w^{*\prime}$ on W^* such that $F_\lambda \times w^{*\prime} \cap \Lambda_\lambda^*$ is not empty, is irreducible and that its dimension is given by $r = \dim E(u)$, where u is a generic point of F over k. Then D_λ^* is a Z-open subset of W^*, defined over k.*

Proof. Let w^* be a generic point of W^* over k such that $\lambda(u) = \pi(w^*)$. From $\Lambda^* \cdot (F \times w^*) = E(u) \times w^*$ and from Cor. 3, Prop. 4, it follows that $\Lambda_\lambda^* \cdot (F_\lambda \times w^*) = E(u) \times w^*$. Then, it is an easy exercise to show that $\mathfrak{D}_\lambda^*$ contains a Z-open subset of W^*. Moreover, this Z-open subset can be so chosen that it is defined over k. Let $\mathfrak{D}^*$ be the largest Z-open subset of W^* which is contained in $\mathfrak{D}_\lambda^*$. Then $\mathfrak{D}^*$ is also defined over k. Set $\mathfrak{Y} = W^* - \mathfrak{D}^*$. $\mathfrak{Y}$ is k-closed. Let Y be a component of $\mathfrak{Y}$ containing a point $w^{*\prime}$ of $\mathfrak{D}_\lambda^*$. Let v^* be a generic point of Y over the algebraic closure of k. Since $w^{*\prime}$ is simple on W^*, v^* is also simple on W^*. Then the compatibility of specializations with the operation of intersection-product implies that $\Lambda_\lambda^* \cap (F_\lambda \times v^*)$ also contains a component of dimension r. When that is so, Lemma 16 implies that v^* is in $\mathfrak{D}_\lambda^*$. Denote by $\mathfrak{X}'$ the union of those component of $\Lambda_\lambda^* \cap (F_\lambda \times Y)$, having the projection Y on W^*. Remove from Y the projection of $(\Lambda_\lambda^* \cap (F_\lambda \times Y)) - \mathfrak{X}'$ and the intersection of Y with the other components of $\mathfrak{Y}$, other than Y. Denote by Y_0 the remainder. Y_0 is Z-open on Y. Denote by $\mathfrak{X}_0$ the restriction of $\mathfrak{X}'$ on $F_\lambda \times Y_0$. Using Cor. 3, Prop. 4, we see that there is a point d in F_λ such that $\mathfrak{X}_0 \cap (F_\lambda \times v^*) = E(d) \times v^*$ (cf. Lemma 16). Since v^* is a generic point of Y_0 over k and since every component of $\mathfrak{X}_0$ has the projection Y_0 on Y_0, it follows that $\mathfrak{X}_0$ is the point set attached to a subvariety of $F_\lambda \times Y_0$. There is a non-empty Z-open subset $Y_0 - \mathfrak{a}$ of Y_0 such that $v^{*\prime} \in Y_0 - \mathfrak{a}$ implies the following: (a) $\mathfrak{X}_0 \cap (F_\lambda \times v^{*\prime})$ is not empty; (b) The intersection is irreducible; (c) The intersection has dimension r. When that is so, $\Lambda_\lambda^* \cap (F_\lambda \times v^{*\prime})$ has the same properties by the definition of Y_0. Let $\mathfrak{Y}'$ be the union of the components of $\mathfrak{Y}$, other than Y and set $\mathfrak{Y}_0 =$

$\mathfrak{Y}' \cup (Y - Y_0) \cup \mathfrak{a}$. $\mathfrak{Y}_0$ is Z-closed, $W^* - \mathfrak{Y}_0 \neq \mathfrak{D}^*$ and $\mathfrak{D}_\lambda^* \supset W^* - \mathfrak{Y}_0 \supset \mathfrak{D}^*$. Consequently, $\mathfrak{D}_\lambda^* = \mathfrak{D}^*$. Our lemma is thereby proved.

COROLLARY. *When v^* is a point of $\mathfrak{D}_\lambda^*$, there is a point d in F_λ such that $\Lambda_\lambda^* \cap (F_\lambda \times v^*) = E(d) \times v^*$. Moreover, when b_0 is contained in $F^\#$ and y a point of X such that $\varphi(y) \sim b_0$, $\mathfrak{D}_\lambda^*$ contains $\mu^*(y)$ whenever it is simple on W^*. Moreover such y exists.*

The first part of our corollary is contained in the above proof and follows from Lemma 16 and Cor. 3, Prop. 4. $E(b_0) \times \lambda(b_0)$ is a component of $F \times \lambda(b_0) \cap \Lambda$ by Cor. 6 and Cor. 3 of Prop. 4. If we choose y as in Lemma 15, $\mu^*(y) = w_0^*$ is simple on W^*. Moreover, $E(b_0) \times w_0^*$ is a component of $F \times w_0^* \cap \Lambda^*$. Then the rest of our corollary follows from Lemma 16.

PROPOSITION 5. *Let $\mathfrak{D}_\lambda^*$ be as in Lemma* 18 *and $\mathfrak{D}_\lambda$ the set-theoretic image of it by π. Then $\mathfrak{D}_\lambda$ is a Z-open subset of W, defined over k. Let F_λ be as in Lemma* 17, *Λ the Z-closure of the graph of λ of $F \times W$ and Λ_λ the restriction of Λ on $F_\lambda \times W$. When w is a point of $\mathfrak{D}_\lambda$, there is a point u in F_λ such that $\Lambda_\lambda \cap (F_\lambda \times w) = E(u) \times w$. When b_0 is in $F^\#$, $\lambda(b_0)$ is a point of $\mathfrak{D}_\lambda$.*

Proof. Except for the fact that $\mathfrak{D}_\lambda$ is Z-open on W, the rest follows from the above corollary. Then our proposition follows from the following general lemma.

LEMMA. *Let U^n, V^n be normal algebraic varieties and assume that U is in a projective space. Let h be a morphism of U onto V such that the inverse image of points of V by h consists of finitely many points. Then the set-theoretic image of a Z-open set D of U by h is Z-open on V.*

Proof. Let k' be a common field of definition for U, V, h and for D. Let v be a generic point of V over k and set $h^{-1}(v) = \mathfrak{m}(v)$. When P is the ambient projective space of U, $\mathfrak{m}(v)$ is a cycle on P. When we extend the specialization $v \to v'$ ref. k on V to a specialization of $\mathfrak{m}(v)$ in P, there is a finite set of cycles in P such that it has to be one of the cycles in the set. Since V is normal, this implies that $\mathfrak{m}(v)$ has a uniquely determined specialization in P over k, over the specialization $v \to v'$ ref. k. Let A

be the locus of $\iota(\mathfrak{m}(v))$ over k. Then the above arguments show that there is a birational correspondence ν between V and A, which is set-theoretically one-to-one, and a morphism ρ of U onto A such that $\rho=\nu\circ h$. Therefore, in order to prove our lemma, we can replace V by A.

Set $Y=U-D$. Denote by D' the set-theoretic projection of D on A by ρ. When a' is a point of A, denote by $\mathfrak{n}(a')$ the P-cycle with the Chow-point a'. a' is then contained in D' if and only if $\mathfrak{n}(a')$ contains a component which is contained in D. Or, a' does not belong to D' if and only if the support of $\mathfrak{n}(a')$ is contained in Y. The set of such a' forms a Z-closed subset of A by a well-known property of Chow-forms (cf. v.d. Waerden (I)). Our lemma is thereby proved.

PROPOSITION 6. *Let D_λ be as in Prop. 5. Let $\mathfrak{G}_\lambda$ be the set of points a' on F_λ such that $\lambda(a')$ is contained in $\mathfrak{D}_\lambda$. Then $\mathfrak{G}_\lambda$ is a Z-open subset of F_λ defined over k. When a' is in $\mathfrak{G}_\lambda$, then $E(a')$ is contained in $\mathfrak{G}_\lambda$. Denote further by Λ'_λ the Z-closure on $\mathfrak{G}_\lambda\times\mathfrak{D}_\lambda$ of the graph of the restriction of λ on $\mathfrak{G}_\lambda\times\mathfrak{D}_\lambda$. Then the following conditions are satisfied*: (i) *When a' is in G_λ, then $\Lambda'_\lambda\cap(\mathfrak{G}_\lambda\times\lambda(a'))=E(a')\times\lambda(a')$*; (ii) *$\lambda$ is single valued at every point of $E(a')$ and has the value $\lambda(a')$, whenever $a'\in\mathfrak{G}_\lambda$*; (iii) *When b_0 is in $F^\#$, then b_0 is in $\mathfrak{G}_\lambda$*; (iv) *When b is a point of $\mathfrak{G}_\lambda$, and g a rational function on F such that g is single valued and finite at b and that there is a rational function h on D_λ with $g=h\circ\lambda$, then h is defined at $\lambda(b)$.*

Proof. By the definition, λ is single valued at a'. Hence it is single valued at every point of $E(a')$ with the constant value $\lambda(a')$ by Cor. 3, Prop. 4. Hence $E(a')$ is contained in F_λ and consequently in $\mathfrak{G}_\lambda$. (ii) is thereby proved. (i) and (iii) follow from Prop. 5. $\mathfrak{G}_\lambda$ is Z-open since $\mathfrak{D}_\lambda$ is Z-open on W and λ is single valued everywhere on F_λ. When that is so, it is clear that $\mathfrak{G}_\lambda$ is defined over k.

Let u be a generic point of F over k. Set $w=\lambda(u)$ and $w'=\lambda(b)$. Denote by $\mathfrak{m}(w)$ the W^*-cycle $\pi^{-1}(w)$. Since (W^*, π) is a normalization of W in a finite algebraic extension of the rational function field of W, and since W is also normal, it follows that $\mathfrak{m}(w)$ has a uniquely determined specialization over k in the ambient projective space, over the specialization $w \rightarrow w'$ ref. k. Denote this by $\mathfrak{m}(w')$. The support of this is the set of points on W^* which are mapped to w' by π. Among such points, there is at least one point of $\mathfrak{D}_\lambda^*$ by the definition (cf. Lemma 18, Prop. 5). Therefore, from (i) and from Cor. of Lemma 18, we see that $E(b)$ is the uniquely determined specialization of $E(u)$ over k, over the specialization $w \rightarrow w'$ ref. k, since the specializations and the operation of intersection-product are compatible. Since $(w, E(u)) \rightarrow (w', E(b))$ ref. k can be extended to a specialization $(w, E(u), v) \rightarrow (w', E(b), b)$ ref. k, where v is a point of $E(u)$, and since we can take v to be a generic point of F over k, our assumptions imply $g(u)=g(v)$, $g(u)=h(w)$ and consequently that h is single valued at w'. When that is so, h is defined at w'. Our proposition is proved.

COROLLARY. *Assume that b_0 is in $F^\sharp$. Denote by $\mathfrak{G}_{b_0}, \mathfrak{D}_{b_0}, \lambda_{b_0}$ the Z-open subsets $\mathfrak{G}_\lambda$, $\mathfrak{D}_\lambda$ and the rational map λ. Let b_1 be another point of $F^\sharp$ and $\mathfrak{G}_{b_1}, \mathfrak{D}_{b_1}, \lambda_{b_1}$ the corresponding Z-open sets and rational map. Then, there is a birational correspondence $T_{b_1 b_0}$ between $\mathfrak{D}_{b_0}$ and $\mathfrak{D}_{b_1}$ determined by $\lambda_{b_0}(u) \rightarrow \lambda_{b_1}(u)$. The graph of this birational correpondence is Z-closed on $\mathfrak{D}_{b_0} \times \mathfrak{D}_{b_1}$.*

Proof. This follows from our proposition, from Cor. 6, Prop. 4, and from the fact that $w \rightarrow w'$ ref. k on W determines $E(u) \rightarrow E(b)$ ref. k uniquely on F_λ, as we have seen in the above proof.

When we sum up the results of this §, we get the following theorem.

THEOREM 1. *There are, a normal algebraic variety $\mathfrak{D}$, a rational map λ of F into $\mathfrak{D}$, an open subset U of X and a holomorphic map μ of U onto $\mathfrak{D}$ satisfying the following conditions:* (a) *Let $F^\sharp$ be the largest Z-open subset of F, contained in the set of points of F which are equivalent to some points of $\varphi(X)$; then there is a Z-open subset F_0 of F containing $F^\sharp$ such that λ is single valued at every point of F_0;* (b) *When b is a point of F_0, the set $E(b)$ of points on F which are equivalent to b is contained in F_0;*

λ is single valued at every point of $E(b)$ and has the constant value $\lambda(b)$; moreover, when b' is in F_0 and $\lambda(b)=\lambda(b')$, then b' is in $E(b)$; (c) *When x is a point of U, then $\mu(x)=\lambda(\varphi(x))$;* (d) *$U$ is a G-invariant open subset of X, and x, y of U are such that $\mu(x)=\mu(y)$ if and only if x and y are congruent modulo G;* (e) *There is an isomorphism between the field of G-invariant meromorphic functions f, algebraic over $K(F/\mathfrak{R})$, and the field of rational functions on $\mathfrak{D}$ determined by $f=\tilde{f}\circ\mu$;* (f) *When a G-invariant meromorphic function f, algebraic over $K(F/\mathfrak{R})$, is holomorphic at a point x of U, the rational function $\tilde{f}$ is defined at $\mu(x)$;* (g) *When g is a rational function on F such that there is a rational function h on $\mathfrak{D}$ with $g=h\circ\lambda$ and that g is finite at a point b of F_0, then h is defined at $\lambda(b)$.*

COROLLARY. *If* (B′) *is satisfied by (X, Z) instead of* (B), *then we can take $U=X$ and F_0 to be the set of points on F which are equivalent to some points of $\varphi(X)$.*

§7. In this paragraph, *we assume that (X, Z) is an analytic family of non-singular varieties in a projective space, satisfying* (A), (B′), (C′) *and* (D). Let k_0 be the smallest common field of definition of F and the equivalence relation of F. *We fix k_0 and all fields we shall consider will be assumed to contain k_0.*

Let b_0 be a point of F_0 (cf. Th. 1) and $\mathfrak{D}_0$, $\mathfrak{G}_0$, λ_0 the Z-open sets and the rational map constructed in Prop. 6 with respect to b_0. $\mathfrak{G}_0$ contains b_0 (Prop. 5). Let b_1 be a point of $F_0-\mathfrak{G}_0$. Because of the axiom (C′), Cor. 6 of Prop. 4, Prop. 5, Prop. 6 and it Corollary, it is possible to construct $\mathfrak{D}_1$, $\mathfrak{G}_1$, λ_1 for b_1 such that $\mathfrak{G}_1$ contains $\mathfrak{G}_0$ and that $\mathfrak{D}_0$ can be identified as a subset of $\mathfrak{D}_1$ by an isomorphism. When that is so, there is a positive integer s such that $\mathfrak{G}_s=F_0$. Therefore, we get the following theorem.

THEOREM 2. *When (X, Z) satisfies* (A), (B′), (C′), (D), *then $\mathfrak{D}$ in our Theorem* 1 *can be taken so that it is a locally Z-closed subset of a projective space. Moreover, U can be taken to be X and F_0 can be taken to be the set of points of F which are equivalent to some point of $\varphi(X)$.*

Moreover, when we apply Weil's result on the field of definition of an algebraic variety (cf. Weil (IV)) and Th. 1, Th. 2 to our situation, we immediately get the following theorem.

THEOREM 3. *Under the same conditions as stated in our Theorem* 2, $\mathfrak{D}$ *in Th.* 1 *can be taken to be a locally Z-closed normal algebraic variety defined over* k_0 *in a projective space and* λ *a morphism defined over* k_0.

THEOREM 4. *With the same notations and assumptions as in Theorem* 1, *let* k *be a common field of definition of* F, $\mathfrak{D}$ *and* λ *over which the equivalence relation on* F *is defined. Let* u *be a point of* F. *Then* $k(\lambda(u))$ *is the smallest field, containing* k, *satisfying the following condition: When* L *is a field, containing* k, u' *a point of* $E(u')$ *and* α *an* L*-isomorphism of* $L(u')$, *then* $\alpha(u')\sim u$.

Proof. This follows easily from (B′).

When we consider the deformation of algebraic varieties, then we consider usually all possible projective embeddings of members of the family $\mathfrak{F}$ by ample polar divisors. For instance, in the case of Example 2, we consider all possible embeddings of polarized complex tori. Then usually, the following situation arises. There is a set $\tilde{\mathfrak{F}}$ of non-singular polarized varieties in projective spaces such that $\tilde{\mathfrak{F}}$ contains $\mathfrak{F}$ as a subset. Moreover, and equivalence relation is defined on $\tilde{\mathfrak{F}}$, usually in terms of isomorphisms of members of $\tilde{\mathfrak{F}}$, and this equivalence relation induces the equivalence relation on $\mathfrak{F}$ stated in the axiom (B). In this case, we further have the following theorem.

THEOREM 5. *Assume that* (X, Z) *satisfy* (A), (B′), (C′), (D) *and that* $\tilde{\mathfrak{F}}$ *satisfies the following two conditions*: (i) *There is a subfield* k^* *of* k_0 *such that, when* α *is a* k^**-automorphism of the field of complex numbers*, $\tilde{A}, \tilde{B}\in\tilde{\mathfrak{F}}$, $\tilde{A}^{\alpha}\in\tilde{\mathfrak{F}}$, $\tilde{A}\sim\tilde{B}$ *imply* $\tilde{B}^{\alpha}\in\tilde{\mathfrak{F}}$ *and* $A^{\alpha}\sim B^{\alpha}$; (ii) *When* P *is the ambient projective space of members of* $\tilde{\mathfrak{F}}$ *and* $\mathfrak{F}'$ *an algebraic family* (*absolutely irreducible*) *of non-singular varieties in* P *which contains a member of* $\tilde{\mathfrak{F}}$, *then* $\mathfrak{F}'$ *is contained in* $\tilde{\mathfrak{F}}$. *When we take* $\mathfrak{D}$ *and* λ *as in Theorem* 3 *and* u *a point of* F_0, *then* $k_0(\lambda(u))$ *is the smallest field, containing* k^*, *satisfying the* *following property* (M). (M): *Let* L *be a field, containing* k^*, *and* α *an* L*-automorphism of the field of complex numbers. Then there is a member* $\tilde{A}$ *of* $\tilde{\mathfrak{F}}$ *satisfying* $\tilde{A}\sim A$, $c(A)\in E(u)$, $\tilde{A}^{\alpha}\in\tilde{\mathfrak{F}}$. *Moreover, when* $\tilde{B}$ *in* $\tilde{F}$ *satisfies* $\tilde{A}\sim\tilde{B}$ *with respect to this* $\tilde{A}$, *then* $\tilde{B}\sim\tilde{B}^{\alpha}$.

Proof. From (i) and Th. 4, the field $k_0(\lambda(u))$ satisfies the condition (M). Let L be now a field satisfying the condition (M) and α an L-automorphism of the field of complex numbers. Let $\tilde{A}$ be a member of $\tilde{\mathfrak{F}}$ such that $\tilde{A}\sim A$, $c(A)\in E(n)$, $A^{\alpha}\in\tilde{\mathfrak{F}}$. From (i), it follows that $\tilde{A}^{\alpha}\sim A^{\alpha}$, $A^{\alpha}\in\tilde{\mathfrak{F}}$. When we denote by $\mathfrak{F}^{\alpha}$ the algebraic family determined by $\mathfrak{F}^{\alpha}$, then A^{α} is a member of $\mathfrak{F}^{\alpha}$. Comparing dimensions and using (ii), we get $\mathfrak{F}^{\alpha}=\mathfrak{F}$. Hence from (i), it follows that L contains k_0. From (M), it follows that $A^{\alpha}\sim A$. Consequently, α leaves $E(u)$ invariant. This implies that α leaves every element of $k_0(\lambda(u))$ invariant. Our theorem is thereby proved.

It would be clear how we apply this theorem to the question of the field of moduli. Therefore, we shall not go into the detail of it, except to mention the following. In the case of the paramodular family of polarized Abelian varieties, Theorem 5 shows that the field of moduli of a polarized Abelian varieties over Q is generated by the special values of paramodular functions.

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