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Connections, metrics and almost complex structures of tangent bundles

By

Makoto MATSUMOTO

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In a series of papers [3],⋯,[7], a theory of connections of Finsler spaces has been developed from the standpoint of fibre bundles. Let $P(M, \pi, G)$ be the bundle of frames over a differentiable manifold M, and $B(M, \tau, F, G)$ be the tangent bundle over M. The Finsler connection is defined in the induced bundle Q from P by the projection τ of B, and many concepts about connections of Finsler spaces are generalized and systematically treated.

On the other hand, the differential geometry of tangent bundles has been studied by several authors. S. Sasaki [12], [13] introduced a Riemannian metric into tangent bundles of Riemannian manifolds in order to study the behavior of geodesics. K. Yano and E. T. Davies [16] generalized the notion to the case where the base manifold has a Finsler metric. T. Nagano [9] and P. Dombrowski [1] defined the natural almost complex and product structures on tangent bundles. Recently, K. Yano and A. J. Ledger [17], [18] investigated linear connections on tangent bundles and showed that it is possible to obtain some interesting connections from a connection on the base manifold.

The purpose of the present paper is to develop synthetically the differential geometry of tangent bundles from the viewpoint of Finsler connections. Thus, connections, metrics and other structures of tangent bundles may be regarded as a part of contents of the Finslerian geometry.

The terminology and signs of previous papers will be used in the following without too much comment. The theory of Finsler connections from the standpoint of fibre bundles were published systematically in Seminar Note *4* [8] in Japanese.

§1. Linear Finsler connections

Let $P(M, \pi, G)$ be the bundle of frames over a differentiable n-manifold M, where G is the general linear group $GL(n, R)$, and π is the projection $P\to M$ which maps a frame p at a point $x\in M$ into x. It is well known [2], [10] that a linear connection $\underline{\Gamma}$ in P is a distribution $p\in P\to\underline{\Gamma}_p$ such that

1. $P_p = \underline{\Gamma}_p+P^v_p$ (direct sum),
2. $\underline{R}_g\underline{\Gamma}_p = \underline{\Gamma}_{pg}$,

where P_p is the tangent vector space of P at a point p, P^v_p the vertical subspace of P_p, and $\underline{R}_g$ the right translation of P by an element $g\in G$. If we denote by $\hat{G}$ the Lie algebra of G, the connection form $\underline{\omega}$ of the above connection $\underline{\Gamma}$ is a $\hat{G}$-valued *1*-form on P, which is defined by equations

1. $\underline{\omega}\underline{F}(A) = A$,
2. $\underline{\omega}X = 0 \qquad (X\in\underline{\Gamma})$,

where $\underline{F}(A)$ is the fundamental vector field on P corresponding to $A\in\hat{G}$.

Let $B(M, \tau, F, G)$ be the tangent bundle over M, where F is the real vector n-space, and τ the projection $B\to M$ which maps a tangent vector X at $x\in M$ into x. Throughout the present paper, we take a fixed base (e_a), $a=1,\cdots,n$, of the vector space F. Then, a matrix $g=(g^a_b)\in G$ operates on F by the rule $g\cdot e_a=g^b_ae_b$. On the other hand, a frame $p=(p_a)\in P$ is regarded as an admissible mapping $F\to B$ which maps $f=f^a e_a\in F$ into the tangent vector $pf=(p_af^a)\in\tau^{-1}\pi(p)$. If $f\in F$ is then fixed, we obtain a mapping $\kappa_f: P\to B$ such that $p\in P\to pf$. By means of this mapping, the associated connection H with the linear connection $\underline{\Gamma}$ is naturally defined in B by the equation

$$H_b = \kappa_f\underline{\Gamma}_p \qquad (b=pf). \tag{1.1}$$

Now, we consider the induced bundle $\tau^{-1}P = Q(B, \bar{\pi}, G)$ over the total space B of the tangent bundle, where the total space Q is the set $\{(b, p) \in B \times P \mid \tau(b) = \pi(p)\}$ [3]. The projection $\bar{\pi}$ is such that $\bar{\pi}(b, p) = b$, and further we have the induced mapping $\eta : Q \to P$ which maps $q = (b, p)$ into p.

In previous papers [3], [4], we gave a notion of a Finsler connection (Γ^h, Γ^v) in Q such that

1. $Q_q = \Gamma_q^h + \Gamma_q^v + Q_q^v$ (direct sum),
2. $R_g \Gamma_q^h = \Gamma_{qg}^h$, $R_g \Gamma_q^v = \Gamma_{qg}^h$,
3. $\bar{\pi}\Gamma_q^v = B_b^v$,

where Q_q is the tangent vector space to Q at q, Q_q^v the vertical subspace of Q_q, B_b^v the vertical subspace of the tangent vector space B_b to B at b, and R_g the right translation of Q by $g \in G$, namely, $R_g(b, p) = (b, \underline{R}_g(p)) = (b, pg)$.

We shall show how to obtain a special Finsler connection from a linear connection $\underline{\Gamma}$ in P. First of all, from the connection form $\underline{\omega}$ of $\underline{\Gamma}$ and the induced mapping η, we define a $\hat{G}$-valued *1*-form ω on Q by the equation

$$\omega = \underline{\omega}\eta\,. \tag{1.2}$$

It is easy to verify that ω satisfies conditions of a connection form on Q, that is,

1. $\omega F(A) = A$,
2. $\omega R_g = ad(g^{-1})\omega$,

where $F(A)$ is the fundamental vector field on Q corresponding to $A \in \hat{G}$. Therefore, we obtain a connection Γ in Q, whose connection form is the above ω. We denote by l_q the operation of lift with respect to the connection Γ, and define

$$\Gamma_q^h = l_q H_b\,, \quad \Gamma_q^v = l_q B_b^v\,. \tag{1.3}$$

Then, it is obvious that the pair of distributions (Γ^h, Γ^v) as thus obtained is a Finsler connection in Q, which will be called the *linear Finsler connection* derived from the linear connection $\underline{\Gamma}$.

In the following, we shall find certain special properties of the linear Finsler connection. It follows from (1.3) that

$$\pi\Gamma^h_q = H_b, \tag{1.4}$$

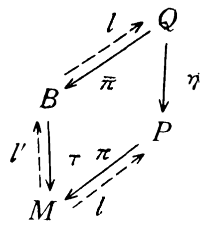
which means that the nonlinear connection [3] induced from (Γ^h, Γ^v) coincides with the associated connection with the original $\underline{\Gamma}$. Next, we shall show the equation

$$B^h(f)_q = \xi_{f_0}B(f)_p \qquad (q=(pf_0, p)), \tag{1.5}$$

where $B(f)$ is the basic vector field on P with respect in $\underline{\Gamma}$, corresponding to $f\in F$ [2, p. 119, the standard horizontal vector field], [10, p. 49], while $B^h(f)$ is the h-basic vector field on Q with respect to the linear Finsler connection [3]. And the mapping $\xi_{f_0}: P\to Q$ is such that $p\in P\to(pf_0, p)$.

Proof of (1.5). Denoting by l'_b the operation of life with respect to the nonlinear connection H, the h-basic vector field $B^h(f)$ is defined by $B^h(f)_q = l_q l'_b pf$ at $q=(b, p)$. From the definition of the linear Finsler connection, it follows that $l'_b = \kappa_{f_0}\underline{l}_p$ at $b=pf_0$, where $\underline{l}_p$ is the operation of lift with respect to the original linear connection $\underline{\Gamma}$. Thus we see first

$$B^h(f)_q = l_q\kappa_{f_0}B(f)_p. \tag{1.6}$$



On the other hand, we get

$$\pi\xi_{f_0}B(f)_p = \kappa_{f_0}B(f)_p,$$

and further, by means of (1.2),

$$\omega\xi_{f_0}B(f)_p = \underline{\omega}B(f)_p = 0.$$

Therefore (1.6) gives (1.5).

It follows from (1.5) that

$$\pi B^h(f)_q = \kappa_{f_0}B(f)_p, \tag{1.7}$$

$$\eta B^h(f)_q = B(f)_p. \tag{1.8}$$

In terms of canonical coordinates (x^i, p^i_a) of P and (x^i, b^i, p^i_a) of Q [3], basic vector fields $B(f)$ and $B^h(f)$ are written in the form

$$B(f) = f^a p_a^i\Big(\frac{\partial}{\partial x^i} - p_b^j \underline{\Gamma}_j{}^k{}_i \frac{\partial}{\partial p_b^k}\Big),$$

(1.9)

$$B^h(f) = f^a p_a^i\Big(\frac{\partial}{\partial x^i} - F^j{}_i \frac{\partial}{\partial b^j} - p_b^j F_j{}^k{}_i \frac{\partial}{\partial p_b^k}\Big),$$

where $\underline{\Gamma}_j{}^k{}_i(x)$ are components of the connection parameter of $\underline{\Gamma}$, while $F^j{}_i(x, b)$, $F_j{}^k{}_i(x, b)$ components of connection parameters of (Γ^h, Γ^v). Hence we have

$$\xi_{f_0} B(f) = f^a p_a^i\Big(\frac{\partial}{\partial x^i} - b^k \underline{\Gamma}_k{}^j{}_i \frac{\partial}{\partial b^j} - p_b^j \underline{\Gamma}_j{}^k{}_i \frac{\partial}{\partial p_b^k}\Big).$$

Thus the equation (1.5) gives the relation between those components of connection parameters as follows:

(1.5′) $\qquad F^i{}_j = b^k \underline{\Gamma}_k{}^i{}_j(x)\,, \qquad F_j{}^i{}_k = \underline{\Gamma}_j{}^i{}_k(x)\,.$

Next, as to a v-basic vector field $B^v(f)$, we shall show the equation

(1.10) $\qquad \eta B^v(f) = 0\,.$

$B^v(f)$ is defined [3] by the equation $B^v(f)_q = l_q p j_\gamma f$ at q, where p is the differential of the admissible mapping p, γ the characteristic field $Q \to F$ such that $q = (b, p) \to p^{-1}b$, and j_f the so-called parallel translation $F \to F_f$ (= the tangent vector space to F at f). The above (1.10) will be easily verified from the fact that $p j_\gamma f$ is vertical and $\eta l_q = \underline{l}_p \tau$.

Let θ be the basic form on P and θ^h be the h-basic form on Q, which are defined by $\theta_p = p^{-1}\pi$ and $\theta_q^h = p^{-1}\tau\bar{\pi}$, $q = (b, p)$. It is obvious that

(1.11) $\qquad \theta^h = \theta\eta\,.$

Now, we consider covariant derivatives. Let $h^h X$ be the Γ^h-component of $X \in Q_q$, and then we have $\eta h^h X = \eta B^h(\theta^h X) = B(\theta\eta X) = \underline{h}\eta X$, where $\underline{h}\eta X$ is the horizontal component of $\eta X \in P_p$. Therefore it follows that

$$D^h\omega = d\omega h^h = d\underline{\omega}\eta h^h = d\underline{\omega}\underline{h}\eta = \underline{\Omega}\eta\,,$$

where $\underline{\Omega}$ is the curvature form of the original connection $\underline{\Gamma}$, while

$D^h\omega=\Omega^h$ is by definition the h-curvature form of the Finsler connection. Hence we obtain $\Omega^h=\underline{\Omega}\eta$. On the other hand, as for the Γ^v-component, it follows from (1.10) that $\eta h^v X=\eta B^v(\theta^v X)=0$, where θ^v is the v-basic form [3]. Referring to this equation, we have $\Omega^v=0$ and $\Omega^{hv}=0$.

Similarly, about h-torsion forms $\Theta^{(h)h}$, $\Theta^{(h)hv}$ and $\Theta^{(h)v}$, we have $\Theta^{(h)h}=\underline{\Theta}\eta$ and $\Theta^{(h)hv}=0$, where $\underline{\Theta}$ is the torsion form of $\underline{\Gamma}$, while $\Theta^{(h)v}$ vanishes from the first.

Finally, in order to obtain v-torsions of the linear Finsler connection, we shall first show the following lemma.

LEMMA [8, p. 105]. *If a Finsler connection satisfies conditions F and C_1, its v-torsion tensors R^1, P^1 and S^1 are given by equations*

$$R^1(f_1, f_2) = -F(R^2(f_1, f_2))\gamma\,, \qquad P^1(f_1, f_2) = -F(P^2(f_1, f_2))\gamma\,,$$
$$S^1(f_1, f_2) = -F(S^2(f_1, f_2))\gamma\,, \qquad (f_1, f_2\in F)\,,$$

where R^2, P^2 and S^2 are h-, hv- and v-curvature tensors respectively, γ is the characteristic field on Q, and F the symbol of constructing a fundamental vector field.

Proof. We shall remember the dual equations of structure [3, (1.4)]:

$$\begin{aligned}
&[B^h(f_1), B^h(f_2)]=F(R^2(f_1, f_2))+B^h(T(f_1, f_2))+B^v(R^1(f_1, f_2))\,,\\
(1.12)\quad &[B^h(f_1), B^v(f_2)]=F(P^2(f_1, f_2))+B^h(C(f_1, f_2))+B^v(P^1(f_1, f_2))\,,\\
&[B^v(f_1), B^v(f_2)]=F(S^2(f_1, f_2)) \qquad\qquad\qquad +B^v(S^1(f_1, f_2))\,.
\end{aligned}$$

According to propositions given in [4], the condition F is that $B^h(f)\gamma=0$, while the condition C_1 is that $B^v(f)\gamma=f$. Hence, Lemma is immediately proved from (1.12).

We now return to consideration of a linear Finsler connection. The definitions of conditions F and C_1 are that $\sigma_f\Gamma^h_q=H_b$ and $\sigma_f\Gamma^v_q=0$ respectively, where σ_f is the mapping $Q\to B$ such that $q=(b, p)\to pf$ for a fixed $f\in F$. Therefore it is clear that the linear Finsler connection satisfies both of conditions F and C_1, and thus its v-torsions are given by Lemma.

Summarizing above results we obtain

PROPOSITION 1. *Curvatures and torsions of the linear Finsler connection derived from a linear connection $\underline{\Gamma}$ are given by equations*

curvatures: $R^2(f_1, f_2) = \underline{R}(f_1, f_2)$, $P^2 = 0$, $S^2 = 0$,
h-torsions: $T(f_1, f_2) = \underline{T}(f_1, f_2)$, $C = 0$,
v-torsions: $R^1(f_1, f_2) = -F(\underline{R}(f_1, f_2))\gamma$, $P^1 = 0$, $S^1 = 0$,

where $\underline{R}$ and $\underline{T}$ are the curvature and the torsion of the original connection $\underline{\Gamma}$ respectively.

In terms of the canonical coordinate (x^i, b^i), the above equation $R^1(f_1, f_2) = -F(\underline{R}(f_1, f_2))\gamma$ are expressed by

$$R_j{}^i{}_k(x, b) = b^h \underline{R}_{h\cdot jk}^{\ i}(x),$$

where $R_j{}^i{}_k$ and $\underline{R}_{h\cdot jk}^{\ i}$ are components of R^1 and $\underline{R}$ respectively.

In the following we shall consider frequently the famous Cartan's connection of Finsler space. In this case, we have shown in a previous paper [7] that the $(h)h$-torsion T vanishes identically, and the $(h)hv$-torsion C coincides with the symbol C used by Cartan, while $(v)h$- and $(v)hv$-torsion R^1, P^1 are given by equations of Lemma, and the $(v)v$-torsions S^1 vanishes identically.

§ 2. Connections of a Finsler type on tangent bundles

We consider the bundle of frames $P'(B, \pi', G')$ over the total space B of the tangent bundle, where G' is of course the general linear group $GL(2n, R)$. Let F' be the real vector $2n$-space, and then F' is identified with $F \times F$. Let us recall the fixed base (e_a) of F in § *1*, and then $e'_a = (e_a, 0)$ and $e'_{(a)} = (0, e_a)$, $a = 1, \cdots, n$, constitute a base of F', where indices with parentheses will run from $n+1$ to $2n$ throughout the remainder of this paper. The above identification $F \times F \to F'$ will be denoted by ρ. Then, a $2n$-matrix $g' = (g'^{\alpha}_{\beta}) \in G'$ operates on F' by the rule $g' \cdot e'_\alpha = g^{\beta}_{\alpha} e'_\beta$, where Greek indices run from 1 to $2n$ from now on.

First of all, we shall establish a homomorphism (Φ, φ) of the bundle Q into P'. Let us define the mapping $\varphi : G \to G'$ by the equation $\varphi(g) = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$, $g \in G$. Next, corresponding to the fixed base (e_a) of F and the Finsler connection under consideration, there are h- and v-basic vector fields $B_a^h = B^h(e_a)$, $B_a^v = B^v(e_a)$, $a=1, \cdots, n$. These $2n$ vectors span the horizontal subspace $\Gamma^h + \Gamma^v$ at each point of Q. Hence

$$p'_a = \pi(B_a^h)_q\,, \quad p'_{(a)} = \pi(B_a^v)\,,$$

form a frame $p' = (p'_a, p'_{(a)}) \in P'$ at the point $b = \pi(q) \in B$. Therefore we obtain the mapping $\Phi : Q \to P'$ such that $\Phi(q) = p'$. The equation $\pi'\Phi = \pi$ obviously holds. According to the fact that $R_g B^h(f) = B^h(g^{-1}f)$ and $R_g B^v(f) = B^v(g^{-1}f)$ [11, (3. 2)], it is seen that $\Phi(q)\varphi(g) = \Phi(qg)$. Consequently, we obtain the bundle homomorphism (Φ, φ).

It is well known [2, p. 76], [10, p. 36] that the homomorphism (Φ, φ) gives a connection Γ' in P' from the Finsler connection $\Gamma = (\Gamma^h, \Gamma^v)$ in Q, which is defined by the equation $\Gamma'_{\Phi(q)} = \Phi(\Gamma_q^h + \Gamma_q^v)$, and, at a point p' which does not belong to $\Phi(Q)$, the horizontal subspace is obtained from $\Gamma'_{\Phi(q)}$ by the suitable right translation from a point in $\Phi(q)$. The Γ' as thus defined is called the *connection of a Finsler type* derived from the Finsler connection $\Gamma = (\Gamma^h, \Gamma^v)$,

A point $p' \in P'$ is regarded as an admissible mapping $F' \to B'$, where B' is the total space of the tangent bundle over B, similar to the case of the bundle of frame P over M. Especially, for $q \in Q$ and $f_1, f_2 \in F$, we have

$$\Phi(q)\rho(f_1, f_2) = \pi\big(B^h(f_1) + B^v(f_2)\big)\,, \tag{2. 1}$$

which is easily verified by the definition of the mapping Φ.

PROPOSITION 2. *Let $F'(A')$ be a fundamental vector field on P' corresponding to $A' \in \hat{G}'$ (the Lie algebra of G'), and θ' be the basic form on P'. Further, let ω' be the connection form of the connection of a Finsler type, and $B'(f')$ be the basic vector field on P'. Then we have*

(2. 2) $\quad \Phi F(A) = F'(\varphi A)\,,$ $\qquad$ (2. 3) $\quad \omega'\Phi = \varphi\omega\,,$

(2. 4) $\quad \theta'\Phi = \rho(\theta^h, \theta^v)\,,$ $\qquad$ (2. 5) $\quad \Phi B^h(f) = B'\big(\rho(f, 0)\big)\,,$

(2. 6) $\quad \Phi B^v(f) = B\big(\rho(0, f)\big)\,.$

Proof. (2. 2) will be easily proved. In order to show (2. 3), consider the $\hat{G}'$-valued *1*-form $\alpha=\omega'\Phi-\varphi\omega$ on Q. According to (2. 2) we have $\alpha F(A)=0$ for any $A\in\hat{G}$, and further, from the definition of Γ', we have $\alpha B^h(f)=0$, $\alpha B^v(f)=0$ for any $f\in F$. Hence we obtain $\alpha=0$, which is (2. 3). Next, we are concerned with (2. 4). For $X\in Q_q$, we have

$$\begin{aligned}\theta'\Phi(X) &= \Phi(q)^{-1}\pi'\Phi(X) = \Phi(q)^{-1}\bar{\pi}(X)\\ &= \Phi(q)^{-1}\bar{\pi}\big(B^h(\theta^h X)+B^v(\theta^v X)\big),\end{aligned}$$

and hence (2. 4) will be obtained in consequence of (2. 1). Finally, it follows from (2. 4) that

$$\theta'\Phi B^h(f) = \rho\big(\theta^h(B^h(f))\big), \quad \theta^v\big(B^h(f)\big) = \rho(f, 0),$$

and it is clear that $\Phi B^h(f)$ is horizontal with respect to the connection Γ'. Therefore we establish (2. 5), and (2. 6) similarly.

In order to obtain the expression of the covariant derivative $\Delta'_X Y$ with respect to the connection Γ' in terms of the one with respect to the Finsler connection (Γ^h, Γ^v), we need following considerations. We first treat a classical vector field X of a Finsler space. It is well known that X depends upon not only a point $X\in M$ but also an element of support x', and hence X is not regarded as a tangent vector field to M. It is also obvious that X is not considered as a tangent vector field to B, because a tangent vector to B has $2n$ components. Thus, it is natural that X should be regarded as a mapping $Q\to F$ such that the equation $XR_g=g^{-1}X$ holds for any $g\in G$ [8, p. 68], (cf. [10, p. 53, Lemma]). In the following, we shall adopt this point of view.

Then, let us construct vector fields $X^=$ and $X^\perp$ of the Finsler space from a tangent vector field X to B, which are mappings $Q\to F$ defined by equations

(2. 7) $\qquad X^=(q) = \theta^h l_q X, \quad X^\perp(q) = \theta^v l_q X,$

at a point $q\in Q$. $X^=$ and $X^\perp$ are called *Finsler h- and v-vector fields* derived from X respectively.

Next, given a tangent vector field X to the base manifold M, we obtain the horizontal lift X^h and the vertical life X^v with respect to the nonlinear connection H [1], [12], [14], [16], [17]. The horizontal life X^h is nothing else but the life $l'_b X$, while the vertical life X^v is naturally defined by

(2.8) $$X_b^v = p j_\gamma p^{-1} X \qquad (\gamma = p^{-1} b),$$

where p is the differential of the admissible mapping $p \in \pi^{-1}\tau(b)$. It is easy to show that the above definition does not depend upon the choice of p. Then, Finsler vector fields derived from the horizontal and the vertical lifts of a tangent vector field X to M are obtained as follows:

(2.9) $$(X^h)^\perp = 0, \quad (X^v)^= = 0,$$

(2.10) $$(X^h)^= = (X^v)^\perp = p^{-1}X \qquad \text{at a point } q = (b, p),$$

which are easily verified from definitions of θ^h and θ^v. Moreover, it follows from (2.1) that

(2.11) $$\big(\Phi(q)\rho(f_1, f_2)\big)^= = f_1, \quad \big(\Phi(q)\rho(f_1, f_2)\big)^\perp = f_2.$$

We now are in a position to be concerned with the covariant derivative $\Delta'_X Y$, where X and Y are tangent vector fields to B. $(\Delta'_X Y)_b$ is by definition equal to $p' \cdot X^*(\theta' Y^*)$, where p' is a point of $\pi'^{-1}(b)$ and X^*, Y^* are lifts of X, Y respectively with respect to the connection Γ'. Since we can take $p' = \Phi(q)$, $q \in \pi^{-1}(b)$, we have

$$(\Delta'_X Y)_b = \Phi(q) \cdot \Phi l_q X(\theta' \Phi l_q Y).$$

It follows from (2.4) and (2.7) that

$$= \Phi(q) \cdot \big(B^h(X^=) + B^v(X^\perp)\big)\big(\rho(Y^=, Y^\perp)\big).$$

Consequently we have from (2.1) that

(2.12) $$\Delta'_X Y = \pi\Big[B^h\big((B^h(X^=) + B^v(X^\perp))Y^=\big) + B^v\big((B^h(X^=) + B^v(X^\perp))Y^\perp\big)\Big].$$

Or, according to (2.11), we have

(2.13) $$(\Delta'_X Y)^= = \big(B^h(X^=) + B^v(X^\perp)\big)Y^=,$$
$$(\Delta'_X Y)^\perp = \big(B^h(X^=) + B^v(X^\perp)\big)Y^\perp.$$

We shall consider the covariant derivative in the special case where the connection Γ' is the *connection of a linear Finsler type*, that is, the one derived from the linear Finsler connection. First of all, putting $X=\underline{X}^h$, $Y=\underline{Y}^h$, where $\underline{X}$ and $\underline{Y}$ are tangent vector fields to M. Then it follows from (2.9) and (2.12) that $\Delta'_X Y =\pi B^h\big(B^h(X^=)Y^=\big)$, and hence from (2.10) that

$$\Delta'_X Y = \pi B^h\big(B^h(p^{-1}\underline{X})p^{-1}\underline{Y}\big) = \pi B^h\big(B^h(\theta \underline{l}\underline{X})(\theta \underline{l}\underline{Y})\eta\big)\,.$$

Then, it follows from (1.8) that

$$= \pi B^h\big(B(\theta \underline{l}\underline{X})\theta \underline{l}\underline{Y}\big) = \pi B^h\big(p^{-1}\underline{\Delta}_{\underline{X}}\underline{Y}\big) = l'\underline{\Delta}_{\underline{X}}\underline{Y}\,,$$

where $\underline{\Delta}$ denotes the covariant derivative with respect to the original linear connection $\underline{\Gamma}$. Thus we obtain $\Delta'_{\underline{X}^h}\underline{Y}^h=(\underline{\Delta}_{\underline{X}}\underline{Y})^h$. In case of $X=X^h$, $Y=Y^v$, and so on, we may proceed in the similar manner, and thus following equations are obtained:

$$\text{(2.14)}\qquad \begin{aligned} &\Delta'_{\underline{X}^h}\underline{Y}^h = (\underline{\Delta}_{\underline{X}}\underline{Y})^h\,, \quad \Delta'_{\underline{X}^v}\underline{Y}^h = 0\,, \\ &\Delta'_{\underline{X}^h}\underline{Y}^v = (\underline{\Delta}_{\underline{X}}\underline{Y})^v\,, \quad \Delta'_{\underline{X}^v}\underline{Y}^v = 0\,. \end{aligned}$$

Finally, we shall be concerned with the torsion T' and the curvature R' of the connection Γ' of a Finsler type. Those tensors will be derived from the dual equation of structure

$$\text{(2.15)}\qquad [B'(f'_1), B'(f'_2)] = F'\big(R'(f'_1, f'_2)\big)+B'\big(T'(f'_1, f'_2)\big)\,.$$

In this equation, taking $f_1, f_2\in F$ and putting $f'_1=\rho(f_1, 0)$, $f'_2=\rho(f_2, 0)$, the left-hand side at a point $p'=\Phi(q)$ is written in the form $[\Phi B^h(f_1), \Phi B^h(f_2)]=\Phi[B^h(f_1), B^h(f_2)]$ in consequence of (2.5). Therefore, according to (1.11), we first obtain $R'\rho\big(f_1, 0), \rho(f_2, 0)\big)\Phi=\varphi R^2(f_1, f_2)$ and $T'\big(\rho(f_1, 0), \rho(f_2, 0)\big)\Phi=\rho\big(T(f_1, f_2), R^1(f_1, f_2)\big)$. By the same way on taking $f'_1=\rho(f_1, 0)$, $f'_2=\rho(0, f_2)$, and so on, we conclude that

$$\text{(2.16)}\qquad \begin{aligned} &R'\big(\rho(f_1, 0), \rho(f_2, 0)\big)\Phi = \varphi R^2(f_1, f_2)\,, \\ &R'\big(\rho(f_1, 0), \rho(0, f_2)\big)\Phi = \varphi P^2(f_1, f_2)\,, \\ &R'\big(\rho(0, f_1), \rho(0, f_2)\big)\Phi = \varphi S^2(f_1, f_2)\,, \end{aligned}$$

and

$$T'\big(\rho(f_1,0),\ \rho(f_2,0)\big)\Phi = \rho\big(T(f_1,f_2),\ R^1(f_1,f_2)\big),$$
$$(2.17)\quad T'\big(\rho(f_1,0),\ \rho(0,f_2)\big)\Phi = \rho\big(C(f_1,f_2),\ P^1(f_1,f_2)\big),$$
$$T'\big(\rho(0,f_1),\ \rho(0,f_2)\big)\Phi = \rho\big(0,\ S^1(f_1,f_2)\big).$$

We shall treat the torsion T' in detail. Equations (2.17) giving the torsion T' are written in the concrete form

$$T'_b{}^a{}_c = T_b{}^a{}_c, \qquad T'_b{}^{(a)}{}_c = R_b{}^a{}_c, \qquad T'_b{}^a{}_{(c)} = C_b{}^a{}_c,$$
$$T'_b{}^{(a)}{}_{(c)} = P_b{}^a{}_c, \qquad T'_{(b)}{}^a{}_{(c)} = 0, \qquad T'_{(b)}{}^{(a)}{}_{(c)} = S_b{}^a{}_c,$$

where it should be remarked that left-hand sides are components of T' with respect to the frame $p'=\Phi(q)$, while right-hand sides are components of torsions of the Finsler connection with respect to the frame $p=\eta(q)$. If (x^i, p^i_a) is the canonical coordinate of p, the canonical coordinate $(x^i, b^i, p'^\lambda_\alpha)$ of p' is as follows:

$$(2.18)\quad p'^i_a = p^i_a, \quad p'^{(i)}_a = -p^j_a F^i{}_j, \quad p'^i_{(a)} = 0, \quad p'^{(i)}_{(a)} = p^i_a.$$

The inverse matrix (p'^{-1}) of (p') is given by

$$(2.19)\quad p'^{-1a}_i = p^{-1a}_i, \quad p'^{-1(a)}_i = p^{-1a}_j F^j{}_i, \quad p'^{-1a}_{(i)} = 0, \quad p'^{-1(a)}_{(i)} = p^{-1a}_i.$$

Therefore, for example, components $T'_j{}^{(i)}{}_{(k)}$ of T' with respect to the canonical coordinate are obtained by the following computation:

$$\begin{aligned}
T'_j{}^{(i)}{}_{(k)} &= T'_\beta{}^\alpha{}_\gamma\, p'^{(i)}_\alpha\, p'^{-1\beta}_j\, p'^{-1\gamma}_{(k)} \\
&= T'_b{}^a{}_{(c)}\, p'^{(i)}_a\, p'^{-1b}_j\, p'^{-1(c)}_{(k)} + T'_b{}^{(a)}{}_{(c)}\, p'^{(i)}_{(a)}\, p'^{-1b}_j\, p'^{-1(c)}_{(k)} \\
&\qquad + T'_{(b)}{}^{(a)}{}_{(c)}\, p'^{(i)}_{(a)}\, p'^{-1(b)}_j\, p'^{-1(c)}_{(k)} \\
&= C_b{}^a{}_c(-p^l_a F^i_l)\, p^{-1b}_j\, p^{-1c}_k + P_b{}^a{}_c\, p^i_a\, p^{-1b}_j\, p^{-1c}_k \\
&\qquad + S_b{}^a{}_c\, p^i_a (p^{-1b}_l F^l{}_j)\, p^{-1c}_k \\
&= -C_j{}^l{}_k F^i{}_l + P_j{}^i{}_k + S_l{}^i{}_k F^l{}_j,
\end{aligned}$$

where $C_j{}^l{}_k$, $P_j{}^i{}_k$ and $S_l{}^i{}_k$ are components of torsions C, P^1 and S^1 respectively with respect to the canonical coordinate. Thus, components $T'_\mu{}^\lambda{}_\nu$ of the torsion T' with respect to the canonical coordinate $(x'^\lambda)=(x^i, b^i)$ are given by equations

$$\begin{aligned}
(2.20)\quad T'_j{}^i{}_k &= T_j{}^i{}_k - C_k{}^i{}_l F^l{}_j + C_j{}^i{}_l F^l{}_k, \\
T'_j{}^{(i)}{}_k &= R_j{}^i{}_k - T_j{}^l{}_k F^i{}_l + S_h{}^i{}_l F^h{}_j F^l{}_k \\
&\qquad + (P_j{}^i{}_l - C_j{}^h{}_l F^i{}_h) F^l{}_k - (P_k{}^i{}_l - C_k{}^h{}_l F^i{}_h) F^l{}_j,
\end{aligned}$$

$$T'_{j}{}^{i}{}_{(k)} = C_{j}{}^{i}{}_{k}, \quad T'_{j}{}^{(i)}{}_{(k)} = P_{j}{}^{i}{}_{k} - C_{j}{}^{l}{}_{k}F^{i}{}_{l} + S_{l}{}^{i}{}_{k}F^{l}{}_{j},$$
$$T'_{(j)}{}^{i}{}_{(k)} = 0, \quad T'_{(j)}{}^{(i)}{}_{(k)} = S_{j}{}^{i}{}_{k}.$$

In case of the connection of a linear Finsler type, it follows from Proposition 1 that components of the torsion tensor T' vanish except $T'_{j}{}^{i}{}_{k}(=\underline{T}_{j}{}^{i}{}_{k})$ and $T'_{j}{}^{(i)}{}_{k}=b^{l}(\underline{R}_{l}{}^{i}{}_{\cdot jk}-\underline{T}_{j}{}^{h}{}_{k}\underline{\Gamma}_{l}{}^{i}{}_{h})$. Therefore the connection of a Finsler type derived from a Finsler connection is not symmetric even if the original connection be linear and symmetric, and it seems undesirable to leaves something as it is. In the next section, we shall consider a symmetrization of the connection.

§ 3. A generalization of the connection of Yano and Ledger

Let $\Gamma_{\mu}{}^{\lambda}{}_{\nu}$ be connection parameters of a connection Γ on a differentiable manifold, and $T_{\mu}{}^{\lambda}{}_{\nu}$ be components of the torsion tensor T of the connection, that is, $T_{\mu}{}^{\lambda}{}_{\nu}=\Gamma_{\mu}{}^{\lambda}{}_{\nu}-\Gamma_{\nu}{}^{\lambda}{}_{\mu}$. Then we obtain the symmetric connection Γ^{*}, whose connection parameters $\Gamma^{*}{}_{\mu}{}^{\lambda}{}_{\nu}$ are given by the equation $\Gamma^{*}{}_{\mu}{}^{\lambda}{}_{\nu}=\Gamma_{\mu}{}^{\lambda}{}_{\nu}-\frac{1}{2}T_{\mu}{}^{\lambda}{}_{\nu}$. This symmetric connection Γ^{*} will be called the one obtained from Γ by *canonical symmetrization.*

In this section, we shall be concerned with the connection Γ^{*} in P', obtained from the connection Γ' of a Finsler type by canonical symmetrization. Therefore connection parameters $\Gamma^{*}{}_{\mu}{}^{\lambda}{}_{\nu}$ of the connection Γ^{*} are defined by the equation

$$\Gamma^{*}{}_{\mu}{}^{\lambda}{}_{\nu} = \Gamma'{}_{\mu}{}^{\lambda}{}_{\nu} - \frac{1}{2}T'{}_{\mu}{}^{\lambda}{}_{\nu}, \tag{3.1}$$

where T' is given by (2.17). Then, it is well known that any symmetric connection in P' may be obtained by adding to $\Gamma^{*}{}_{\mu}{}^{\lambda}{}_{\nu}$ a symmetric tensor of (1, 2)-type.

In order to study the property of the connection Γ^{*}, we shall be concerned with the covariant derivative $\Delta^{*}_{X}Y$ with respect to Γ^{*}. The equation

$$\Delta^{*}_{X}Y = \Delta'_{X}Y + \frac{1}{2}T'(X, Y) \tag{3.2}$$

will be obtained immediately, where $T'(X, Y)$ as usual is the tangent vector to B which is given by the equation $T'(X, Y)_b = p' \cdot T'(p'^{-1}X, p'^{-1}Y)$ at $b \in B$, $p' \in \pi'^{-1}(b)$.

Let us treat (3.2) in detail. If we take tangent vectors $\underline{X}$ and $\underline{Y}$ to the base manifold M, and put $X = \underline{X}^h$, $Y = \underline{Y}^h$, and if we consider the point $p' = \Phi(q)$, $q \in \bar{\pi}^{-1}(b)$, we have from (2.1)

$$\Phi(q)^{-1}\underline{X}^h = \Phi(q)^{-1}\bar{\pi} l_q \underline{X}^h = \rho(\theta^h l_q \underline{X}^h, 0) = \rho(p^{-1}\underline{X}, 0).$$

Therefore, as a consequence of (2.17), we obtain

$$\begin{aligned} T'(\underline{X}^h, \underline{Y}^h) &= \Phi(q) T'\big(\rho(p^{-1}\underline{X}, 0), \rho(p^{-1}\underline{Y}, 0)\big) \\ &= \Phi(q)\rho\big(T(p^{-1}\underline{X}, p^{-1}\underline{Y}), R^1(p^{-1}\underline{X}, p^{-1}\underline{Y})\big), \end{aligned}$$

and it follows from (2.1) that

$$\begin{aligned} &= \bar{\pi}\big(B^h(T(p^{-1}\underline{X}, p^{-1}\underline{Y})) + B^v(R^1(p^{-1}\underline{X}, p^{-1}\underline{Y}))\big) \\ &= l'_b p \cdot T(p^{-1}\underline{X}, p^{-1}\underline{Y}) + p \cdot j_y R^1(p^{-1}\underline{X}, p^{-1}\underline{Y}). \end{aligned}$$

Similarly to the case of T', let us write $T(\underline{X}, \underline{Y}) = p \cdot T(p^{-1}\underline{X}, p^{-1}\underline{Y})$, and $R^1(\underline{X}, \underline{Y}) = p \cdot R^1(p^{-1}\underline{X}, p^{-1}\underline{Y})$, and thus the equation $\Delta^*_{\underline{X}^h}\underline{Y}^h = \Delta'_{\underline{X}^h}\underline{Y}^h + \frac{1}{2}\big((T(\underline{X}, \underline{Y}))^h + (R^1(\underline{X}, \underline{Y}))^v\big)$ will be obtained, where superscripts h and v were introduced in §2. Thus, and by the same way, we shall obtain following equations:

$$(3.3)\quad \begin{aligned} \Delta^*_{\underline{X}^h}\underline{Y}^h &= \Delta'_{\underline{X}^h}\underline{Y}^h + \frac{1}{2}\big((T(\underline{X}, \underline{Y}))^h + (R^1(\underline{X}, \underline{Y}))^v\big), \\ \Delta^*_{\underline{X}^h}\underline{Y}^v &= \Delta'_{\underline{X}^h}\underline{Y}^v + \frac{1}{2}\big((C(\underline{X}, \underline{Y}))^h + (P^1(\underline{X}, \underline{Y}))^v\big), \\ \Delta^*_{\underline{X}^v}\underline{Y}^h &= \Delta'_{\underline{X}^v}\underline{Y}^h + \frac{1}{2}\big(-(C(\underline{Y}, \underline{X}))^h - (P^1(\underline{Y}, \underline{X}))^v\big), \\ \Delta^*_{\underline{X}^v}\underline{Y}^v &= \Delta'_{\underline{X}^v}\underline{Y}^v + \frac{1}{2}\big(S^1(\underline{X}, \underline{Y})\big)^v. \end{aligned}$$

We shall treat the special case where the connection of a Finsler type is linear. According to (2.14) and Proposition 1, above equations are reduced to

$$
\begin{aligned}
(3.4)\qquad & \Delta^*_{\underline{X}^h}\underline{Y}^h = \left(\underline{\Delta}_{\underline{X}}\underline{Y}+\frac{1}{2}\underline{T}(\underline{X},\ \underline{Y})\right)^h+\left(\underline{R}(\underline{X},\ \underline{Y})\gamma\right)^v. \\
& \Delta^*_{\underline{X}^h}\underline{Y}^v = (\underline{\Delta}_{\underline{X}}\underline{Y})^v, \\
& \Delta^*_{\underline{X}^v}\underline{Y}^h = 0, \quad \Delta^*_{\underline{X}^v}\underline{Y}^v = 0.
\end{aligned}
$$

Thus we know that *the canonical symmetrization* Γ^* *of the connection of a linear Finsler type coincides with that of Yano and Ledger* [17], and hence we shall call Γ^* the *generalized Yano-Ledger connection.*

Remark. We should pay attention to the difference in algebraic signs between the first of (3.4) and (14) of the paper [17]. If we denote by $\underset{(2)}{\Gamma}{}_j{}^i{}_k$, $\underset{(2)}{T}{}_j{}^i{}_k$ and $\underset{(2)}{R}{}^i{}_{jkl}$ symbols used in the reference book [2], instead of our $\underline{\Gamma}_j{}^i{}_k$, $\underline{T}_j{}^i{}_k$ and $\underline{R}_j{}^i{}_{\cdot kl}$, we obtain $\underline{\Gamma}_j{}^i{}_k = \underset{(2)}{\Gamma}{}_k{}^i{}_j$, $\underline{T}_j{}^i{}_k = -\underset{(2)}{T}{}_j{}^i{}_k$, $\underline{R}_j{}^i{}_{\cdot kl} = -\underset{(2)}{R}{}^i{}_{jkl}$, according to [2, pp. 143–145], (cf. (1.9) of the present paper). Therefore we have $\underline{T}(\underline{X},\ \underline{Y}) = -\underset{(2)}{T}(\underline{X},\ \underline{Y})$, $\underline{R}(\underline{X},\ \underline{Y}) = -\underset{(2)}{R}(\underline{X},\ \underline{Y})$, Moreover, according to symbols used in [17], we have $\overline{R(X,\ Y)} = -\left(\underline{R}(X,\ Y)\gamma\right)^v$ by means of the equation (7) of [17]. Consequently we find that the above (3.4) coincides with (12), (13), (14) and (18) of [17] entirely.

§4. The lifted Riemannian metric

We have studied, in preceding sections, connections on the tangent bundle arising from Finsler connections, and a Finsler metric, however, was not under discussion. Now we suppose that a Finsler metric function L be given. Let G be the usual Finsler metric tensor defined by L. Then, G is a tensor field of (0, 2)-type, which is regarded as the mapping $Q\to F^*\otimes F^*$ (tensorial product of the dual space F^* of the real vector n-space F) [7], [8, p. 106]. If we take Finsler vector fields $X,\ Y: Q\to F$, then the value $G\left(X(q),\ Y(q)\right)$ is called the *scalar product* of X and Y at a point $q\in Q$, or, more precisely, the one with respect to the *element of support* $b=\pi q$. Thus the value $[G(X(q),\ X(q))]^{\frac{1}{2}}$ is called the *Finslerian length.*

Let X and Y be tangent vector fields to B, and then Finsler h- and v-vector fields $X^{=}$, $X^{\perp}$, $Y^{=}$, $Y^{\perp}$ are given by the rule

(2.7). Then the equation

$$(4.1)\quad \bar{G}(X_b,\ Y_b) = G\big(X^{=}(q),\ Y^{=}(q)\big)+G\big(X^{\perp}(q),\ Y^{\perp}(q)\big), \qquad q=(b,\ p),$$

gives a tensor field $\bar{G}$ of (0, 2)-type on B. It is easy to show that $\bar{G}$ is well defined by (4.1) and does not depend upon the choice of $q\in\bar{\pi}^{-1}b$. From the property of G, it follows that $\bar{G}$ is symmetric and positive-definite. Therefore, if $\bar{G}(X_b,\ X_b)$ is defined as the scalar product of X and Y, and further $[\bar{G}(X_b,\ X_b)]^{\frac{1}{2}}$ as the length of X, we have a Riemannian metric $\bar{G}$ on B, and thus the tangent bundle B over the manifold M is a Riemannian manifold of $2n$ dimensions [12], [16]. The tensor $\bar{G}$ as thus defined will be called the *lifted Riemannian metric.*

Let g_{ab} and $\bar{g}_{\alpha\beta}$ be components of tensors G and $\bar{G}$ with respect to frames $p=\eta(q)$ and $p'=\Phi(q)$ respectively, and then we have from (4.1)

$$(4.2)\qquad \bar{g}_{ab} = g_{ab}\,,\quad \bar{g}_{a(b)} = 0\,,\quad \bar{g}_{(a)(b)} = g_{ab}\,.$$

Moreover, let g_{ij} and $\bar{g}_{\lambda\mu}$ be components of G and $\bar{G}$ with respect to canonical coordinates respectively, and then we obtain

$$(4.2')\qquad \begin{aligned} &\bar{g}_{ij} = g_{ij}+g_{kl}F^k{}_iF^l{}_j\,, \\ &\bar{g}_{i(j)} = F^k{}_ig_{kj}\,.\quad \bar{g}_{(i)(j)} = g_{ij}\,. \end{aligned}$$

Returning to the connection Γ' of a Finsler type, we shall find the covariant differential of the lifted Riemannian metric $\bar{G}$ with respect to this connection Γ'. For this purpose, we first define mappings $\rho_1,\ \rho_2 : F' \to F$ as follows:

$$\rho_1(f'^{\alpha}e'_{\alpha}) = f'^{a}e_a\,,\quad \rho_2(f'^{\alpha}e'_{\alpha}) = f'^{(a)}e_a\,,$$

and denote by $\rho_1^*,\ \rho_2^*$ their dual mappings. The lifted Riemannian metric $\bar{G}$ is considered as the mapping $F'\to F'^*\otimes F'^*$ such that $\bar{G}(f'_1,\ f'_2)=\bar{G}(p'f'_1,\ p'f'_2)_{\pi'p'}$ for $p'\in P'$ and $f'_1,\ f'_2\in F'$. Let us take specially $p'=\Phi(q)$. Since $\Phi(q)f'=\Phi(q)\rho(\rho_1(f'),\ \rho_2(f'))$, we have from (2.11) and (4.1)

$$\bar{G}\big(\Phi(q)\big)(f'_1,\ f'_2) = G\big(\rho_1(f'_1),\ \rho_1(f'_1)\big)+G\big(\rho_2(f'_1),\ \rho_2(f'_2)\big)\,.$$

Hence, by using tensorial product of mappings, we obtain the equation

(4.3) $\quad \bar{G}\Phi = \rho_1^*\otimes\rho_1^*G+\rho_2^*\otimes\rho_2^*G\,.$

Now, it follows from (2.5) and (2.6) that

$$\Delta'\bar{G} = d\bar{G}\big(B'(e'_\alpha)\big)\otimes e'^\alpha = B_a^h(\bar{G}\Phi)\otimes e'^a+B_a^v(\bar{G}\Phi)\otimes e'^{(a)}\,,$$

and from (4.3) that

$$= B_a^h(\rho_1^*\otimes\rho_1^*G+\rho_2^*\otimes\rho_2^*G)\otimes e'^a+B_a^v(\rho_1^*\otimes\rho_1^*G+\rho_2^*\otimes\rho_2^*G)\otimes e'^{(a)}\,.$$

According to definitions of h- and v-differentials Δ^h, Δ^v [3], we thus establish

(4.4)
$$\begin{aligned}\Delta'\bar{G} &= (\rho_1^*\otimes\rho_1^*\otimes\rho_2^*+\rho_2^*\otimes\rho_2^*\otimes\rho_1^*)\Delta^h G\\ &\quad+(\rho_1^*\otimes\rho_1^*\otimes\rho_2^*+\rho_2^*\otimes\rho_2^*\otimes\rho_2^*)\Delta^v G\,.\end{aligned}$$

In terms of components with respect to frames $p=\eta(q)$ and $p'=\Phi(q)$, (4.4) is expressed in the form

(4.4′)
$$\begin{aligned}\bar{g}_{ab;c} &= \bar{g}_{(a)(b);c} = g_{ab|c}\,,\\ \bar{g}_{ab;(c)} &= \bar{g}_{(a)(b);(c)} = g_{ab}|_c\,,\\ \bar{g}_{a(b);c} &= \bar{g}_{a(b);(c)} = 0\,,\end{aligned}$$

where semicolons mean the covariant differentiation with respect to Γ', and $g_{ab|c}$, $g_{ab}|_c$ are Finslerian convariant derivatives, that is,

$$g_{ab|c} = g_{ij|k}p_a^i p_b^j p_c^k\,,\quad g_{ab}|_c = g_{ij}|_k p_a^i p_b^j p_c^k\,,$$

$$g_{ij|k} = \frac{\partial g_{ij}}{\partial x^k}-\frac{\partial g_{ij}}{\partial b^l}F^l{}_k-g_{lj}F_i{}^l{}_k-g_{il}F_j{}^l{}_k\,,$$

$$g_{ij}|_k = \frac{\partial g_{ij}}{\partial b^k}-g_{lj}C_i{}^l{}_k-g_{il}C_j{}^l{}_k\,.$$

Next, we shall investigate the covariant differential $\Delta^*\bar{G}$ with respect to the generalized Yano-Ledger connection Γ^*. First, for any $f'_1, f'_2, f'_3\in F'$, we have from (3.1)

$$\begin{aligned}\Delta^*\bar{G}(f'_1, f'_2, f'_3) &= \Delta'\bar{G}(f'_1, f'_2, f'_3)\\ &\quad+\frac{1}{2}\big(T'_*(f'_1, f'_2, f'_3)+T'_*(f'_2, f'_1, f'_3)\big)\,,\end{aligned}$$

where we put $T'_*(f'_1, f'_2, f'_3)=\bar{G}\big(p'\cdot T'(f'_1, f'_3), p'\cdot f'_2\big)$, that is, $T'_{*\lambda\mu\nu}=\bar{g}_{\mu\kappa}T'_\lambda{}^\kappa{}_\nu$ in terms of canonical coordinate. If follows from (4.1) that

$$T'_*(f'_1, f'_2, f'_3) = G\big((p'\cdot T'(f'_1, f'_3))^=, (p'\cdot f'_2)^=\big) + G\big((p'\cdot T'(f'_1, f'_3))^\perp, (p'\cdot f'_2)^\perp\big).$$

Putting $p'=\Phi(q)$, and paying attention to (2.11), we have $\big(p'\cdot T'(f'_1, f'_3)\big)^= = \rho_1 T'(f'_1, f'_3)$, and $\big(p'\cdot T'(f'_1, f'_3)\big)^\perp = \rho_2 T'(f'_1, f'_3)$. Therefore we obtain from (4.4)

$$\begin{aligned}
&\Delta^*\bar{G}(f'_1, f'_2, f'_3)\\
&\quad = \Delta^h G\big(\rho_1(f'_1), \rho_1(f'_2), \rho_1(f'_3)\big) + \Delta^h G\big(\rho_2(f'_1), \rho_2(f'_2), \rho_1(f'_3)\big)\\
&\qquad + \Delta^v G\big(\rho_1(f'_1), \rho_1(f'_2), \rho_2(f'_3)\big) + \Delta^v G\big(\rho_2(f'_1), \rho_2(f'_2), \rho_2(f'_3)\big)\\
&\qquad + \frac{1}{2}\big(G(\rho_1 T'(f'_1, f'_3), \rho_1 f'_2) + G(\rho_1 T'(f'_2, f'_3), \rho_1 f'_1)\\
&\qquad\qquad + G(\rho_2 T'(f'_1, f'_3), \rho_2 f'_2) + G(\rho_2 T'(f'_2, f'_3), \rho_2 f'_1)\big).
\end{aligned} \tag{4.5}$$

Especially, if we take $\rho(f, 0)$, $\rho(0, f)$, instead of $f'\in F'$, it follows from (2.17) that

$$\begin{aligned}
&\Delta^*\bar{G}\big(\rho(f_1, 0), \rho(f_2, 0), \rho(f_3, 0)\big)\\
&\quad = \Delta^h G(f_1, f_2, f_3) + \frac{1}{2}\big(T_*(f_1, f_2, f_3) + T_*(f_2, f_1, f_3)\big),\\
&\Delta^*\bar{G}\big(\rho(f_1, 0), \rho(f_2, 0), \rho(0, f_3)\big)\\
&\quad = \Delta^v G(f_1, f_2, f_3) + \frac{1}{2}\big(C_*(f_1, f_2, f_3) + C_*(f_2, f_1, f_3)\big),\\
&\Delta^*\bar{G}\big(\rho(f_1, 0), \rho(0, f_2), \rho(f_3, 0)\big)\\
&\quad = \frac{1}{2}\big(R^1_*(f_2, f_1, f_3) - C_*(f_3, f_1, f_2)\big),\\
&\Delta^*\bar{G}\big(\rho(f_1, 0), \rho(0, f_2), \rho(0, f_3)\big)\\
&\quad = \frac{1}{2}P^1_*(f_2, f_1, f_3),\\
&\Delta^*\bar{G}\big(\rho(0, f_1), \rho(0, f_2), \rho(f_3, 0)\big)\\
&\quad = \Delta^h G(f_1, f_2, f_3) - \frac{1}{2}\big(P^1_*(f_1, f_3, f_2) + P^1_*(f_2, f_3, f_1)\big),\\
&\Delta^*\bar{G}\big(\rho(0, f_1), \rho(0, f_2), \rho(0, f_3)\big)\\
&\quad = \Delta^v G(f_1, f_2, f_3) + \frac{1}{2}\big(S^1_*(f_1, f_2, f_3) + S^1_*(f_2, f_1, f_3)\big).
\end{aligned} \tag{4.6}$$

where we used covariant torsion tensors

$$C_{*jik} = g_{il}C_j{}^l{}_k, \quad T_{*jik} = g_{il}T_j{}^l{}_k,$$
$$R_{*ijk} = g_{il}R_j{}^l{}_k, \quad P_{*ijk} = g_{il}P_j{}^l{}_k, \quad S_{*ijk} = g_{il}S_j{}^l{}_k.$$

We now consider the Riemannian connection $\underline{\Gamma}$ in P derived from a Riemannian metric G on the base manifold M. From the connection $\underline{\Gamma}$, we have the linear Finsler connection (Γ^h, Γ^v) by the method as shown in § 1. Therefore, we obtain the connection of a linear Finsler type Γ' in P' derived from (Γ^h, Γ^v), which will be called the *connection of a Riemannian type*. As have already seen in § 2, this connection is not always symmetric, and, however, we obtain the generalized Yano-Ledger connection Γ^* by the canonical symmetrization. The symmetric connection Γ^* will be called the *Riemannian Yano-ledger connection.* On the other hand, since we have the lifted Riemannian metric $\bar{G}$ from the Riemannian metric G, the Riemannian connection $\bar{\Gamma}$ is defined on B. Thus, we have two symmetric connections Γ^* and $\bar{\Gamma}$. It is obvious that, with respect to $\bar{\Gamma}$, the covariant derivative of $\bar{G}$ vanishes identically. Besides, it is easy to see from (4.6) that there are non-zero components of the covariant derivative of $\bar{G}$ with respect to the Riemannian Yano-Ledger connection Γ^*, that is, $\bar{g}_{a(b)*c}=\frac{1}{2}\gamma^e R_{ebac}$, where $*$ means the covariant differentiation with respect to Γ^*. Consequently we obtain

PROPOSITION 3. *The Riemannian Yano-Ledger connection Γ^* derived from the Riemannian metric G coincides with the Riemannian connection with respect to the lifted Riemannian metric $\bar{G}$ if and only if, the base Riemannian manifold M admits an absolute parallelism.*

§ 5. Almost complex structures on tangent bundles

We now return to consideration of a Finsler connection (Γ^h, Γ^v) in the induced bundle Q over the tangent bundle B. On the total space Q, there are three kinds of essential vector fields, that is, the fundamental vector field $F(A)$, corresponding to $A\in\hat{G}$, which is vertical, and two kinds of basic vector fields $B^h(f)$ and $B^v(f)$, corresponding to $f\in F$, which are h- and v-horizontal respectively.

Let us define a tensor field $\bar{J}$ of (1. 1)-type on Q such that

(5. 1) $\bar{J}F(A)=F(A)\,,\quad \bar{J}B^h(f)=B^v(f)\,,\quad \bar{J}B^v(f)=-B^h(f)\,,$

If we take a tangent vector $X\in Q_q$, X is decomposed, with respect to the Finsler connection, as follows:

$$X=F(\omega X)+B^h(\theta^h X)+B^v(\theta^v X)\,,$$

where ω is the connection form, and θ^h, θ^v are h-, v-basic forms respectively. It follows from (5. 1) that the action of $\bar{J}$ is given by

$$\bar{J}X=G(\omega X)-B^h(\theta^v X)+B^v(\theta^h X)\,.$$

As for those forms ω, θ^h and θ^v, it is easily proved that

(5. 2) $\omega\bar{J}=\omega\,,\quad \theta^h\bar{J}=-\theta^v\,,\quad \theta^v\bar{J}=\theta^h\,.$

Further, as for the right translation R_g of Q by $g\in G$, we have

(5. 3) $\bar{J}R_g=R_g\bar{J}\,.$

By a direct computation from (5. 1), we obtain the important equation

(5. 4) $\bar{J}^2X=-X\,,\qquad$ for the horizontal X.

Now, by making use of the above tensor $\bar{J}$, we obtain an almost complex structure J of the tangent bundle B such that

(5. 5) $J=\bar{\pi}\bar{J}l\,,$

where l is the operation of lift with respect to the Finsler connection under consideration, and $\bar{\pi}$ the projection $Q\to B$. The tensor J as thus defined is really an almost complex structure, because, given $X\in B_b$, we take any point $q\in\bar{\pi}^{-1}b$, and then $\bar{J}l_qX$ horizotal by means of (5. 1), and hence $J^2X=-X$ is obtained in consequeace of (5. 4). This J will be called the *natural almost complex structure* derived from the Finsler connection.

We shall next show that

(5. 6) $J\bar{\pi}=\bar{\pi}\bar{J}\,,$

(5. 7) $\bar{J}l=lJ\,.$

Proof of (5.6). Denoting by vX the vertical part of $X\in Q_q$, we have

$$J\pi X = \pi\bar{J}l\pi X = \pi\bar{J}(X-vX) = \pi\bar{J}X-\pi\bar{J}vX\,.$$

The vector $\bar{J}vX$ is vertical and hence $\pi\bar{J}vX=0$. This prove (5.6).

Proof of (5.7). If we take $X\in B_b$, $\bar{J}lX$ is horizontal and hence we have $lJX=l\pi\bar{J}lX=\bar{J}lX$, which completes the proof.

We shall recall Finsler h- and v-vector fields $X^{=}$ and $X^{\perp}$ derived from a tangent vector field X to B. It follows from (2.7) that $(JX)^{=}=\theta^h lJX$. It follows from (5.7) and (5.2) that $(JX)^{=}=\theta^h\bar{J}lX=-\theta^v lX=-X^{\perp}$. By the similar way we obtain two equations

$$(5.8)\qquad (JX)^{=} = -X^{\perp}\,,\quad (JX)^{\perp} = X^{=}\,.$$

Next, we shall be concerned with horizontal and vertical lifts $\underline{X}^h$, $\underline{X}^v$ of a tangent vector $\underline{X}\in M_x$. Then, we shall prove

$$(5.9)\qquad J\underline{X}^h = \underline{X}^v\,,\quad J\underline{X}^v = -\underline{X}^h\,,$$

which are equivalent to the equation (15) of [1].

Proof of (5.9). Taking a point $q=(b,p)\in Q$, we have an element $p^{-1}\underline{X}=f\in F$, and then

$$J\underline{X}^h = \pi\bar{J}l_q l'_b\underline{X} = \pi\bar{J}l_q l'_b pf\,,$$

which is written, according to definition of $B^h(f)$ and $B^v(f)$, as follows:

$$= \pi\bar{J}B^h(f)_q = \pi B^v(f)_q = pj_\gamma f = pj_\gamma p^{-1}\underline{X}\,,$$

which is the vertical lift $\underline{X}^v$ in consequence of (2.8). Similarly, the second of (5.9) will be verified.

It is concluded from (5.9) that, though the natural almost complex structure J is defined by means of the Finsler connection (Γ^h, Γ^v), *J really depends only upon the nonlinear connection* $H=\pi\Gamma^v$, because any $X\in B_b$ is decomposed into the form

$$X = (\pi X)^h+(pj_\gamma^{-1}p^{-1}v'X)^v\,,\qquad (p\in\pi^{-1}\tau b)\,,$$

where $v'X$ is the vertical part with respect to the nonlinear connection H, and therefore the above decomposition is determined by H only.

PROPOSITION 4. [14] *Let $\bar{G}$ be the lifted Riemannian metric derived from a Finsler metric G, and J be the natural almost complex structure. Then, $(\bar{G}, J)$ is the almost Hermitian structure.*

Proof. From the definition (4.1) of $\bar{G}$, it follows that

$$\bar{G}(JX, JY) = G\big((JX)^{=}, (JY)^{=}\big) + G\big((JX)^{\perp}, (JY)^{\perp}\big),$$

and from (5.8) that

$$= G(-X^{\perp}, -Y^{\perp}) + G(X^{=}, Y^{=}) = \bar{G}(X, Y),$$

which completes the proof.

It is well known that there exists a symmetric connection with respect to which the structure tensor J of the almost complex space is covariant constant, if and only if, the structure is integrable. We shall show, in the next section, that the natural almost complex structure J is not necessarily integrable, and hence there is generally no possibility of finding a symmetric connection with respect to which J is covariant constant. We, however, obtain such a nonsymmetric connection by the natural process as follows:

THEOREM 1. *The natural almost complex structure J is covariant constant with respect to the connection of a Finsler type Γ'.*

Proof. For tangent vector fields X and Y to B, the formula

$$\Delta'_X(JY) = (\Delta'_X J)Y + J(\Delta'_X Y)$$

is derived. From (2.12) and (5.8) it follows the $\Delta'_X(JY) = J(\Delta'_X Y)$, and hence we get $(\Delta'_X J)Y = 0$ for any X and Y. Thus the proof is complete.

Finally, we shall find the covariant derivative $\Delta^*_X J$ of J with respect to the generalized Yano-Ledger symmetric connection Γ^*. From (3.1) and Theorem 1 it follows that

$$(5.10)\quad (\Delta^*_X J)Y = \frac{1}{2}(T'(X, JY) - JT'(X, Y)).$$

According to (2.1), (5.1) and (5.6), we see that

$$T'(X, JY) = \Phi(q)T'\big(\rho(X^{=}, X^{\perp}), \rho(-Y^{\perp}, Y^{=})\big),$$
$$J\Phi(q)\rho(f_1, f_2) = \Phi(q)\rho(-f_2, f_1).$$

Therefore, taking tangent vector fields $\underline{X}$, $\underline{Y}$ to the base manifold M, and referring to notations used in (3.4), we have

$$\begin{aligned}
&(\Delta^*_{\underline{X}^h}J)\underline{Y}^h = \big(C(\underline{X}, \underline{Y})+R^1(\underline{X}, \underline{Y})\big)^h+\big(P^1(\underline{X}, \underline{Y})-T(\underline{X}, \underline{Y})\big)^v,\\
&(\Delta^*_{\underline{X}^h}J)\underline{Y}^v = \big(P^1(\underline{X}, \underline{Y})-T(\underline{X}, \underline{Y})\big)^h-\big(C(\underline{X}, \underline{Y})+R^1(\underline{X}, \underline{Y})\big)^v,\\
&(\Delta^*_{\underline{X}^v}J)\underline{Y}^h = -\big(P^1(\underline{Y}, \underline{X})\big)^h+\big(C(\underline{X}, \underline{Y})\big)^v,\\
&(\Delta^*_{\underline{X}^v}J)\underline{Y}^v = \big(C(\underline{X}, \underline{Y})\big)^h+\big(P^1(\underline{Y}, \underline{X})\big)^v.
\end{aligned} \tag{5.11}$$

In case of a linear Finsler connection, we see that both of $(\Delta^*_{\underline{X}^v}J)\underline{Y}^h$ and $(\Delta^*_{\underline{X}^v}J)\underline{Y}^v$ vanish.

§ 6. The condition of integrability of the natural almost complex structure

Following other authors [1], [14], [16], we shall find the condition of integrability of the natural almost complex structure J on B. Let E be the torsion tensor of the structure J, that is,

(6.1) $E(X, Y) = [X, Y]+J[JX, Y]-J[JY, X]-[JX, JY]$,

for tangent vectors X, Y to B. The condition of integrability is of course that E vanishes identically. P. Dombrowski [1] evaluated the value of E for horizontal and vertical lifts of tangent vectors to the base manifold M. In the following, we shall, however, make use of the dual equation of structure (1.12) by introducing the tensor $\bar{E}$ such that, for tangent vectors X, Y to Q,

(6.2) $\bar{E}(X, Y) = [X, Y]+\bar{J}[\bar{J}X, Y]-\bar{J}[\bar{J}Y, X]-[\bar{J}X, \bar{J}Y]$,

which is the torsion tensor of $\bar{J}$, as it were.

First of all, we shall show that

(6.3) $\pi\bar{E}(l_qX, l_qY) = E(X, Y)$,

for X, $Y \in B_b$, $b=\pi q$.

Proof of (6.3). From (5.6) and (5.7) it follows that

$$\pi\bar{E}(l_qX, l_qY) = \pi[l_qX, l_qY]+J\pi[l_qJX, l_qY] -J\pi[l_qJY, l_qX]-\pi[l_qJX, l_qJY].$$

It is well known [10] that the horizontal part of $[l_qX, l_qY]$ is equal to the lift of $[X, Y]$, and hence $\pi[l_qX, l_qY]=[X, Y]$. Therefore we obtain (6.3).

Now, referring to the dual equation of structure (1.12) and paying attention to the relations $S^1(f_1, f_2)=C(f_1, f_2)-C(f_2, f_1)$, we obtain easily values of $\bar{E}$ for basic vectors $B^h(f)$ and $B^v(f)$ as follows:

$$\begin{aligned} &\bar{E}(B^h(f_1), B^h(f_2))=F(U(f_1, f_2))+B^h(V(f_1, f_2))+B^v(R^1(f_1, f_2)), \\ (6.4)\quad &\bar{E}(B^h(f_1), B^v(f_2))=-F(U(f_1, f_2))+B^h(R^1(f_1, f_2))-B^v(V(f_1, f_2)), \\ &\bar{E}(B^v(f_1), B^v(f_2))=-F(U(f_1, f_2))-B^h(V(f_1, f_2))-B^v(R^1(f_1, f_2)), \end{aligned}$$

where we used tensors U and V defined by

$$(6.5)\quad \begin{aligned} U(f_1, f_2) &= R^2(f_1, f_2)+P^2(f_1, f_2)-P^2(f_2, f_1), \\ V(f_1, f_2) &= T(f_1, f_2)-P^1(f_1, f_2)+P^1(f_2, f_1). \end{aligned}$$

In consequence of (6.3) and (6.4), we now find values of E as follows:

$$\begin{aligned} &E(\pi B^h(f_1), \pi B^h(f_2)) = \pi B^h(V(f_1, f_2))+\pi B^v(R^1(f_1, f_2)), \\ (6.6)\quad &E(\pi B^h(f_1), \pi B^v(f_2)) = \pi B^h(R^1(f_1, f_2))-\pi B^v(V(f_1, f_2)), \\ &E(\pi B^v(f_1), \pi B^v(f_2)) = -\pi B^h(V(f_1, f_2))-\pi B^v(R^1(f_1, f_2)). \end{aligned}$$

Since $\pi B^h(e_a)=p'_a$ and $\pi B^v(e_a)=p'_{(a)}$ constitute a base of the tangent space B_b (cf. §2), equations (6.6) give the value of the torsion tensor E for any $X, Y\in B_b$. Therefore, $E=0$ is equivalent to vanishing of tensors V and R^1. Thus we conclude that

THEOREM 2. [1], [16] *The condition of integrability of the natural almost complex structure J is that tensors R^1 and V vanish identically.*

The tensor R^1 is the $(v)h$-torsion tensor of the Finsler connection under consideration, which, in terms of the canonical coordinate, has components

$$(6.7)\quad R_j{}^i{}_k = \frac{\partial F^i{}_j}{\partial x^k} - \frac{\partial F^i{}_k}{\partial x^j} - \frac{\partial F^i{}_j}{\partial b^l} F^l{}_k + \frac{\partial F^i{}_k}{\partial b^l} F^l{}_j ,$$

And we observe from (6. 5) that components of the tensor V are

$$(6.8)\quad V_j{}^i{}_k = -\frac{\partial F^i{}_j}{\partial b^k} + \frac{\partial F^i{}_k}{\partial b^j} .$$

We shall treat the Finsler connection of Cartan. As already shown in the end of §1, P^1 is equal to $P^i_{o\cdot jk} = A_j{}^i{}_{k|o}$ (Cartan's symbol), which is symmetric with respect to subscripts, and hence $V=0$ in consequence of (6. 5). On the other hand, R^1 is equal to $b^l R_{l\cdot jk}^i$. Thus, Theorem 4. 1 of [16] is obtained as a special case of Theorem 2.

§7. The condition of the almost Kähler structure

S. Tachibana and M. Okumura showed [14] that, in Riemannian case, the structure $(\bar{G}, J)$ is always almost Kähler. In the present section, we shall be concerned with Finsler case. The structure $(\bar{G}, J)$ is by definition almost Kähler if and only if, the 2-form J_* on B be closed, where J_* is given by the equation

$$(7.1)\quad J_*(X, Y) = \frac{1}{2}\big(\bar{G}(X, JY) - \bar{G}(Y, JX)\big), \qquad X, Y \in B_b .$$

It follows from (4. 1) and (5. 8) that J_* is written in the form

$$(7.2)\quad J_*(X, Y) = G(X^\perp, Y^=) - G(Y^\perp, X^=) .$$

In the following, instead of the form J_* on B, we shall first consider the 2-form $\bar{J}_* = J_* \bar{\pi}$ on Q, Since, for $X, Y \in Q_q$, we have $(\bar{\pi}X)^= = \theta^h X$ and $(\bar{\pi}X)^\perp = \theta^v X$ according to (2. 7), it is seen that

$$(7.3)\quad \bar{J}_*(X, Y) = G(\theta^v X, \theta^h Y) - G(\theta^v Y, \theta^h X) .$$

It is clear that the projection $\bar{\pi} : Q \to B$ is onto-mapping, and hence the condition of the almost Kähler structure, i. e., $dJ_* = 0$ coincides with $d\bar{J}_* = 0$. Therefore, in order to find the condition, we shall evaluate the value of $d\bar{J}_*$ in the following.

According to the well known formula of exterior differential of forms, we have

$$3d\bar{J}_*(X, Y, Z) = X\big(\bar{J}_*(Y, Z)\big) + Y\big(\bar{J}_*(Z, X)\big) + Z\big(\bar{J}_*(X, Y)\big) - \bar{J}_*([X, Y], Z) - \bar{J}_*([Y, Z], X) - \bar{J}_*([Z, X], Y),$$

for $X, Y, Z \in Q_q$. Referring to (7.3), the equation is rewritten in the form

$$\begin{aligned} &= X\big(G(\theta^v Y, \theta^h Z) - G(\theta^v Z, \theta^h Y)\big) + Y\big(G(\theta^v Z, \theta^h X) - G(\theta^v X, \theta^h Z)\big) \\ &\quad + Z\big(G(\theta^v X, \theta^h Y) - G(\theta^v Y, \theta^h X)\big) - G(\theta^v[X, Y], \theta^h Z) \\ &\quad + G(\theta^v Z, \theta^h[X, Y]) - G(\theta^v[Y, Z], \theta^h X) + G(\theta^v X, \theta^h[Y, Z]) \\ &\quad - G(\theta^v[Z, X], \theta^h Y) + G(\theta^v Y, \theta^h[Z, X]). \end{aligned}$$

Then, if we take basic vector fields $B^h(f)$, $B^v(f)$ in place of X, Y, Z in the above, and refer to the dual equation of structure (1.12), following equations are obtained:

$$(7.4)\quad \begin{aligned} &3d\bar{J}_*\big(B^h(f_1), B^h(f_2), B^h(f_3)\big) = -S_{123}[R_*(f_1, f_2, f_3)], \\ &3d\bar{J}_*\big(B^h(f_1), B^h(f_2), B^v(f_3)\big) = \Delta^h G(f_1, f_3, f_2) - \Delta^h G(f_2, f_3, f_1) \\ &\qquad + T_*(f_1, f_3, f_2) - P^1_*(f_1, f_2, f_3) + P^1_*(f_2, f_1, f_3), \\ &3d\bar{J}_*\big(B^h(f_1), B^v(f_2), B^v(f_3)\big) = \Delta^v G(f_1, f_3, f_2) - \Delta^v G(f_1, f_2, f_3) \\ &\qquad + C_*(f_1, f_3, f_2) - C_*(f_1, f_2, f_3) - S_*(f_1, f_2, f_3), \\ &d\bar{J}_*\big(B^v(f_1), B^v(f_2), B^v(f_3)\big) = 0, \end{aligned}$$

where $S_{123}[\cdots\cdots]$ is the symbol of summation of terms obtained by cyclic permutation of subscripts. It will, however, be easy to show that the right-hand side of the third equation of (7.4) is identically equal to zero. Hence, the condition is given by equations

$$(7.5)\quad \begin{aligned} &S_{123}[R_*(f_1, f_2, f_3)] = 0, \\ &\Delta^h G(f_1, f_3, f_2) - \Delta^h G(f_2, f_3, f_1) + T_*(f_1, f_3, f_2) \\ &\qquad - P^1_*(f_1, f_2, f_3) + P^1_*(f_2, f_1, f_3) = 0. \end{aligned}$$

THEOREM 3. *The structure* $(\bar{G}, J)$ *is almost Kähler if and only if, equations* (7.5) *are satisfied.*

In terms of the canonical coordinate, above equations are expressed as follows:

$$(7.5')\quad \begin{aligned} &g_{il}R_j{}^l{}_k + g_{jl}R_k{}^l{}_i + g_{kl}R_i{}^l{}_j = 0, \\ &g_{ij|k} - g_{ik|j} + g_{il}T_j{}^l{}_k - g_{jl}P_k{}^l{}_i + g_{kl}P_j{}^l{}_i = 0. \end{aligned}$$

We shall consider the Riemannian case, that is, $\bar{G}$ is the lifted metric derived from the Riemannian metric G on the base manifold M, and J is the natural almost complex structure defined by the associated linear connection H with the Riemannian connection, In this case, the first of (7.5′) holds by the well known identities $S_{jkl}[R_{ijkl}]=0$ with respect to components of the curvature tensor. Further we observe that every terms of the second of (7.5′) always vanish. Consequently we obtain the theorem due to Tachibana and Okumura.

Finally, we shall treat the Finsler connection of E. Cartan. Then, the first of (7.5′) is reduced to $S_{jkl}[R_{0jkl}]=0$ (Cartan's symbol), which has been shown by Cartan. Besides, the second of (7.5′) is also satisfied, because Cartan's connection is metrical, symmetric ($T_j{}^i{}_k=0$) and $g_{jl}P_k{}^l{}_i=A_{jki|0}$ (Cartan's symbol). Therefore we have

COROLLARY. *In case of the Finsler connection of E. Cartan, the structure* $(\bar{G}, J)$ *is almost Kähler.*

From this point of view, the Finsler connection of E. Cartan seems to be very reasonable.

Institute of Mathematics, Yoshida College,
Kyoto University.

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REMARK. In §2, the bundle homomorphism $\Phi: Q \to P'$ was introduced in order to derive the connection of a Finsler type. That is, for a point $q=(b, p)\in Q$, the image $\Phi(q)$ is the frame $p'=(p'_a, p'_{(a)})$, where $p'_a=\pi B^h_a$ and $p'_{(a)}=\pi B^v_a$. On the other hand, we defined the horizontal lift X^h and the vertical lift X^v of a tangent vector $X\in M_x$. It will be easily seen that, if $p=(p_a)$, $a=1, \cdots, n$, then $p'_a=(p_a)^h$ and $p'_{(a)}=(p_a)^v$. From this point of view, the bundle homomorphism (Φ, φ) seems to be natural. Moreover, it can be seen that the homomorphism depends only upon the nonlinear connection H.