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On iterated suspensions II.

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Introduction.

The present paper is the continuation of the previous work [12] with the same title. The sections of this paper are numbered from Section 8 which follows from the last section of the previous work. The notations and the results of the psevious work will be referred such as (1.7), Proposition 3.6, etc.

In Section 8, we shall have *a periodicity* of the following type:

$$\pi_i(Q_{2k}^{2m-1}:p)\approx\pi_{i+2\nu p}(Q_{2k}^{2m+2\nu-1}:p), \qquad \nu=p^{k-1},$$

for $i<4mp-6$, $1\leq k\leq m$ and $k\leq p^2-2p$. It is an open question whether this periodicity holds for meta-stable cases or not. Our method of the proof is a mod p analogy of relative J-homomorphism in [11] and a stunted lens space will be used in place of a stunted real projective space.

Section 9 is a discussion on the homomorphism $\Delta:\pi_{i+4}(S^{2mp+1}:p)\rightarrow\pi_{i+2}(S^{2mp-1}:p)$ in the exact sequence (2.5). The results will be applied, in Section 11, to the computation of $\pi_i(Q_2^{2m-1}:p)$ for unstable cases. We shall see that many of unstable elements are cancelled by Δ.

In Section 10, the existence of unstable elements γ of the third type ($\gamma\notin\operatorname{Im}S^2$, $S^{2p-4}\gamma\neq0$, $S^{2p-2}\gamma=0$) and the fourth type ($\gamma\notin\operatorname{Im}S^2$, $S^{2p}\gamma\neq0$, $S^{2p+2}\gamma=0$) will be proved.

The homotopy groups $\pi_{2m+1+k}(S^{2m+1}:p)$ will be determined for $k<2p^2(p-1)-3$ in Section 11. The result is stated briefly as follows:

$$\pi_{2m+1+k}(S^{2m+1}:p) = A(m, k)+B(m, k)+\sum_{t=1}^{4} U_t(m, k) \qquad \text{(direct sum)}$$

where the subgroup $A(m, k)+B(m, k)$ is a maximal subgroup which is mapped under S^∞ isomorphically into the stable group $(\pi_k^S : p)$, $U_t(m, k)$ are subgroups generated by unstable elements of the t-th types respectively.

The structure of the groups $\pi_{2m+1+k}(S^{2m+1}:p)$ of meta-stable cases will be discussed in Section 12. We shall have an existence theorem of unstable elements of the second type in the groups $\pi_{2m+1+2sp(p-1)-2}(S^{2m+1}:p)$ for $s\not\equiv 0 \pmod p$.

8. Periodicity of $\pi_i(Q_{2k}^{2m-1}:p)$.

In Chapter XI of [11] we have a map

$$f_n^{n+k} : S^{n-1}(P^{n+k-1}/P^{n-1}) \rightarrow Q_k^n = \Omega(\Omega^k S^{n+k}, S^n)$$

which induces $\mathcal{C}_2$-isomorphisms f_{n*}^{n+k} of homotopy groups π_i for $i<4n-3$ [11, Theorem 11.7], where P^r denotes the r-dimensional real projective space. Let $\xi\in\tilde{K}(P^{k-1})$ be the stable class of the canonical line bundle over P^{k-1}, then the order of $J(\xi)$ in $\tilde{J}(P^{k-1})$ is $\nu=2^{\phi(k-1)}$ [1, Example (6.3)], where $\phi(k-1)$ is the number of integers j such that $0<j\leq k-1$ and $j\equiv 0, 1, 2, 4 \pmod 8$. By Proposition 2.6 of [3] P^{n+k-1}/P^{n-1} and $P^{n+k+\nu-1}/P^{n+\nu-1}$ have the same stable homotopy type. Since $S^{n-1}(P^{n+k-1}/P^{n-1})$ is $(2n-2)$-connected we have an isomorphism $\pi_i(S^{n-1}(P^{n+k-1}/P^{n-1}))\approx\pi_{i+2\nu}(S^{n+\nu-1}(P^{n+p+\nu-1}/P^{n+\nu-1}))$ for $i<4n-3$. Therefore we have obtained the following (probably well-known) periodicity of $\pi_i(Q_k^n:2)$.

Theorem 8.0. *Let $\nu=2^{\phi(k-1)}$. If $i<4n-3$, then the groups $\pi_i(Q_k^n)$ and $\pi_{i+2\nu}(Q_k^{n+\nu})$ are $\mathcal{C}_2$-isomorphic.*

In the following we shall try to prove a periodicity of $\pi_i(Q_{2k}^{2m-1}:p)$ for odd prime p and to make some applications. The periodicity of the following type is obtained.

Theorem 8.1. *Let $\nu=p^{k-1}$. If $i<4mp-6$, $1\leq k\leq m$ and $k\leq p^2-2p$ then $\pi_i(Q_{2k}^{2m-1})$ is $\mathcal{C}_p$-isomorphic to $\pi_{i+2\nu p}(Q_{2k}^{2m+2\nu-1})$, i.e.,*

$$\pi_i(Q_{2k}^{2m-1}:p) \approx \pi_{i+2\nu p}(Q_{2k}^{2m+2\nu-1}:p).$$

PROBLEM. *Does the above periodicity hold for meta-stable cases* ($i<2mp^2-5$) *and for general* k? *This is true for* $k=1$.

Denote by $L_p^{2s+1}=S^{2s+1}/Z_p$ the usual $(2s+1)$-dimensional lens space given as in [5]. L_p^r, $r\leq 2s+1$, will be the r-skeleton of L_p^{2s+1} with the usual cellular decomposition $L_p^{2s+1}=S^1\cup e^2\cup\cdots\cup e^{2s}\cup e^{2s+1}$. In the notation of [5], $L_p^{2s}=L_0^s(p)$ and $L_p^{2s+1}=L^s(p)$. It is proved in Theorem 3 of [5]

(8.1). *Let* $\nu=p^{[k/(p-1)]}$, *then* $L_p^{2m}/L_p^{2(m-k)}$ *has the same stable homotopy type of* $L_p^{2m+2\nu}/L_p^{2(m-k)+2\nu}$.

Lemma 8.2. *There exists a map* ($m\geq 1$)

$$h=h_m:\ S^{2m+1}L_p^{(2m+1)(p-1)-2}\rightarrow S^{2m+1}$$

such that in the mapping cone $K=S^{2m+1}\cup_h CS^{2m+1}L_p^{(2m+1)(p-1)-2}$ *of* h *the Steenrod operations* $\mathcal{P}^j:\ H^{2m+1}(K;Z_p)\rightarrow H^{2m+2j(p-1)+1}(K;Z_p)$ *and hence* $\Delta\mathcal{P}^j$ *are isomorphisms for* $1\leq j\leq m$.

Proof. Let $S_p=S^{2m+1}\times\cdots\times S^{2m+1}$ be the product of p copies of S^{2m+1} and let θ_p be the subspace of S_p which consists of the points having the base point $*$ as one of the p coordinates. As the permutation of the factors, the symmetric group $S(p)$ of p letters acts on S_p and θ_p. Let ψ: $(E^{2m+1}, S^{2m})\rightarrow(S^{2m+1}, *)$ be a characteristic map of the $(2m+1)$-cell $S^{2m+1}-*$. From the p-product of ψ, we have a characterictic map ψ^p: $(E^{(2m+1)p}, S^{(2m+1)p-1})\rightarrow(S_p, \theta_p)$ of the $(2m+1)p$-cell $S_p-\theta_p$ such that $S(p)$ acts on $(E^{(2m+1)p}, S^{(2m+1)p-1})$, compatible with ψ^p and for each permutation $\zeta\in S(p)\subset O(p)$ the action of ζ on $E^{(2m+1)p}$ is given by a matrix $\zeta\otimes E$ where E stands for the unit $(2m+1)$-matrix. Let ζ be a cyclic permutation which generates a cyclic subgroup Z_p of $S(p)$. Then the characteristic equation of the matrix $\zeta\otimes E$ is $(x^p-1)^{2m+1}=0$. Thus, by suitable orthogonal transformation of the coodinates in $E^{(2m+1)p}$ we may identify $E^{(2m+1)p}$ with a join $S^{(2m+1)(p-1)-1}*E^{2m+1}$; Z_p acts freely on $S^{(2m+1)(p-1)-1}$, trivially on E^{2m+1} and linearly with respect to the parameter of the join. It follows that the cyclic product S_p/Z_p of S^{2m+1} is obtained from θ_p/Z_p by attaching $(S^{(2m+1)(p-1)-1}/Z_p)*E^{2m+1}$ by a map $h_0:(S^{2m+1)(p-1)-1}/Z_p)*S^{2m}\rightarrow\theta_p/Z_p$. Up to homotopy equivalence we may change the joins $*S^{2m}$ and

$*E^{2m+1}$ by the $(2m+1)$-suspension S^{2m+1} and its cone CS^{2m+1}, then we have

$$S_p/Z_p = \theta_p/Z_p \cup_{h_0} CS^{2m+1}(S^{(2m+1)(p-1)-1}/Z_p).$$

This lens space $S^{(2m+1)(p-1)-1}/Z_p$ is slightly different with the usual $L_p^{(2m+1)(p-1)-1}$. A representative: $S^1 \to S^{(2m+1)(p-1)-1}/Z_p$ of a generator of $\pi_1(S^{(2m+1)(p-1)-1}/Z_p)$ can be extended over a map $f: L_p^{(2m+1)(p-1)-1} \to S^{(2m+1)(p-1)-1}/Z_p$. From the cohomological structure of the lens spaces it follows that f induces isomorphisms of mod p cohomology groups and so does $S^{2m+1}f$. The map $S^{2m+1}f$ defines a map from the mapping cone $\theta_p/Z_p \cup_{h_1} CS^{2m+1}L_p^{(2m+1)(p-1)-1}$ of $h_1 = h_0 \circ S^{2m+1}f$ into S_p/Z_p which induces isomorphisms of mod p cohomology groups.

It was proved in [6] that for a generator u of $H^{2m+1}(S_p/Z_p; Z_p)$, $\mathcal{P}^j u$ and $\Delta\mathcal{P}^j u$, $1 \leq j \leq m$, are non-zero elements which lie in the image of the injection homomorphism $j^*: H^*(S_p/Z_p, \theta_p/Z_p; Z_p) \to H^*(S_p/Z_p; Z_p)$. By the naturality a similar assertion holds for the mapping cone of h_1.

Let $\pi: \theta_p/Z_p \to \theta_p/S(p)$ be the natural projection. The space $\theta_p/S(p)$ coincides with the $(p-1)$-symmetric product $S_{p-1}/S(p-1)$ of S^{2m+1}. It is known (see [7]) that the canonical inclusion $i: S^{2m+1} = S^{2m+1}/S(1) \to \theta_p/S(p) = S_{p-1}/S(p-1)$ induces an isomorphism $i^*: H^*(\theta_p/S(p); Z_p) \approx H^*(S^{2m+1}; Z_p)$. Remark that $\pi^*: H^{2m+1}(\theta_p/S(p); Z_p) \to H^{2m+1}(\theta_p/Z_p; Z_p)$ is an isomorphism. Put $h_2 = \pi \circ (h_1 | S^{2m+1}L_p^{(2m+1)(p-1)-2})$, then the above non-triviality of $\mathcal{P}^j$ and $\Delta\mathcal{P}^j$ holds for the mapping cone of h_2. Apply Theorem 1.1 to $S^{2m+1}L_p^{(2m+1)(p-1)-2}$ and then apply Theorem 1.2 to the maps h_2 and i, then we have the existence of a map $h: S^{2m+1}L_p^{(2m+1)(p-1)-2} \to S^{2m+1}$ such that $i \circ h$ is homotopic to h_2. Consider the mapping cone K of h and compare with that of h_2, then the non-triviality of $\mathcal{P}^j$ and $\Delta\mathcal{P}^j$ in K is obtained and the lemma follows. q.e.d.

Theorem 8.3. *Assume that* $m \geq k \geq 1$. *Let* $K(m, k)$ *and* $G: K(m, k) \to Q_{2k}^{2m-1}$ *be a CW-complex and a map satisfying the assertion of Proposition* 3.6, *thus* $G^*: H^i(Q_{2k}^{2m-1}; Z_p) \approx H^i(K(m, k); Z_p)$ *for* $i < 4mp-5$*). *Then there exists a map*

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*) In Proposition 3.6, (ii), $4mp-3$ should be read $4mp-5$.

$$f: S^{2m-2}(L_p^{2(m+k-1)(p-1)}/L_p^{2m(p-1)-2}) \to K(m, k)$$

such that the induced homomorphism $f^*: H^*(K(m, k); Z_p) \to H^*(S^{2m-2}(L_p^{2(m+k-1)(p-1)}/L_p^{2m(p-1)-2}); Z_p)$ *is a monomorphism.*

Proof. Put $n=m+k-1$ and $L_p=L_p^{2n(p-1)}$. Let $h'': S^{2n+1}L_p \to S^{2n+1}$ be the restriction of the map h_n of Lemma 8.2 and let $K=S^{2n+1}\cup CS^{2n+1}L_p$ be a mapping cone of h''. We use the notation of Section 3. Extend the canonical inclusion $S^{2n+1}\subset K_{2n+1}$ to a map $k: K\to K_{2n+1}$ and consider the induced map $\Omega k: \Omega(K, S^{2n+1}) \to X_{2n+1} =\Omega(K_{2n+1}, S^{2n+1})$. The cone-construction of K defines naturally a map $h': S^{2n+1}L_p \to \Omega(K, S^{2n+1})$. Then it is easily verified that

$$(\Omega k\circ h')^*(\sigma \mathcal{P}^n w) \neq 0 \quad \text{in} \quad H^{2np}(S^{2n+1}L_p; Z_p)$$

for the fundamental class w of $H^{2n+1}(K_{2n+1}; Z_p)=H^{2n+1}(Z, 2n+1; Z_p)$. As in the proof of Lemma 3.5, we may identify $Q_{2t}^{2n-2t+1}$ with $\Omega(\Omega^{2t}X_{2n+1}, X_{2n-2t+1})$ and we have also that $\sigma^4\mathcal{P}^n w$ is defined and generates $H^{2np-3}(Q_2^{2n-1}; Z_p)$. Let $i: \Omega^3 X_{2n+1} \to Q_2^{2n-1}$ be the map equivalent to the inclusion, then by the naturality of σ we have

$$(\Omega^3(\Omega k\circ h'))^* i^*(\sigma^4\mathcal{P}^n w) \neq 0 \quad \text{in} \quad H^{2np-3}(S^{2n-2}L_p; Z_p).$$

By Theorem 3.1 and by the assumption $1\leq m\leq k$, we have an isomorphism $(\sigma^{2k-2})^{-1}\circ j^*: H^{2np-2k-1}(Q_{2k}^{2m-1}; Z_p) \approx H^{2np-3}(Q_2^{2n-1}; Z_p)$, $j: Q_2^{2n-1}\to Q_{2k}^{2m-1}$. From this we conclude

$$f'^*(\sigma^{2k+2}\mathcal{P}^{m+k-1}w)\neq 0 \quad \text{in} \quad H^{2mp+2(k-1)(p-1)-3}(S^{2m-2}L_p^{2(m+k-1)(p-1)}; Z_p),$$

where $f'=i'\circ\Omega^{2k+1}(\Omega k\circ h'): S^{2m-2}L_p \to \Omega^{2k+1}X_{2n+1} \subset \Omega(\Omega^{2k}X_{2n+1}, X_{2m-1}) =Q_{2k}^{2m-1}$.

If $m=1$, then $m=k=1$ and $K(1, 1)=Y_p^{2(p-1)}$ is homotopy equivalent to $L_p^{2(p-1)}/L_p^{2(p-1)-2}$. Thus the theorem is obvious if $m=1$. So, we may assume that $m\geq 2$. Then $S^{2m-2}L_p^{2(m+k-1)(p-1)}$ is homotopy equivalent to a complex as in Theorem 1.1. The map G induces isomorphisms of $H^i(\ ; Z_p)$ for $i<2m+2(m+k-1)(p-1) \leq 4mp-6$ $(1\leq k\leq m)$. Applying Theorem 1.2 to the maps G and f' we have the existence of a map $f_0: S^{2m-2}L_p^{2(m+k-1)(p-1)} \to K(m, k)$ such that $G\circ f_0$ is homotoptic to f'. Since $K(m, k)$ is $(2mp-4)$-connected, we can choose f_0 such that $f_0=f\circ S^{2m-2}\pi$ for a map

$$f: S^{2m-2}(L_p^{2(m+k-1)(p-1)}/L_p^{2m(p-1)-2}) \to K(m, k)$$

and for a map π of $L_p^{2(m+1-k)(p-1)}$ shrinking $L_p^{2m(p-1)-2}$.

We have seen that f'^* is an epimorphism of $H^{2mp-3+2(k-1)(p-1)}(; Z_p)$, then so are f_0^* and f^*. Thus the following statement $(8.2)_{i,\varepsilon}$ is true for $(i, \varepsilon)=(1, 0)$.

$(8.2)_{i,\varepsilon}$. $f^*: H^t(K(m, k); Z_p) \approx H^t(S^{2m-2}(L_p^{2(m+k-1)(p-1)}/L_p^{2m(p-1)-2}); Z_p)$ *for* $t=t(i, \varepsilon)=2mp-3+2(k-i)(p-1)+\varepsilon$ $(i=1, 2, \cdots, k; \varepsilon=0, 1)$.

We shall prove (8.2) by induction on i. For $t=t(i, 0)$ the Bockstein homomorphisms Δ are isomorphisms of the both sides of $(8.2)_{i,0}$. By the naturality of Δ, $(8.2)_{i,0}$ and $(8.2)_{i,1}$ are equivalent. By use of the relations in Theorem 3.1, we see that for each i, $1<i\leq k$, there exists $\varepsilon=0$ or $=1$ such that $\mathcal{P}^1 H^t(K(m, k); Z_p)\neq 0$ for $t=t(i, \varepsilon)$. If the same non-triviality of $\mathcal{P}^1$ holds in $S^{2m-2}(L_p^{2(m+k-1)(p-1)})/L_p^{2m(p-1)-2})$, then $(8.2)_{i,\varepsilon}$ and $(8.2)_{i-1,\varepsilon}$ are equivalent, hence (8.2) is proved by induction on i.

Let u be a generator of $H^1(L_p^{2(m+k-1)(p-1)}; Z_p)$ and choose generators a_i' of $H^{2mp-3+2i(p-1)}(S^{2m-2}L_p^{2(m+k-1)(p-1)}; Z_p)$ such that $\sigma^{2m-2}(a_i')=u\cdot(\Delta u)^{(m+i)(p-1)-1}$. By use of $\mathcal{P}^1 u=0$, $\mathcal{P}^1(\Delta u)=(\Delta u)^p$ and Cartan's formula, we have

$$\mathcal{P}^t(u\cdot(\Delta u)^{s(p-1)-1}) = \binom{s(p-1)-1}{t} u\cdot(\Delta u)^{(s+t)(p-1)-1}$$

and $$\mathcal{P}^t((\Delta u)^{s(p-1)}) = \binom{s(p-1)}{t} (\Delta u)^{(s+t)(p-1)}.$$

Since $\sigma^{2m-2}(\Delta a_i')=\Delta\sigma^{2m-2}(a_i')=\Delta(u\cdot(\Delta u)^{(m+i)(p-1)-1})=(\Delta u)^{(m+i)(p-1)}$ we have

$$(8.3) \qquad \mathcal{P}^t a_i' = \binom{(m+i)(p-1)-1}{t} a_{i+t}' \quad \textit{and}$$

$$\mathcal{P}^t \Delta a_i' = \binom{(m+i)(p-1)}{t} \Delta a_{i+t}'.$$

Here we may consider that a_i' is a generator of $H^{2mp-3+2i(p-1)}(S^{2m-2}(L_p^{2(m+k-1)(p-1)}/L_p^{2m(p-1)-2}); Z_p)$. In particular the relations

$$\mathcal{P}^1 a_i' = -(m+i+1)a_{i+1}' \quad \text{and} \quad \mathcal{P}^1\Delta a_i' = -(m+i)\Delta a_{i+1}'$$

show the required non-triviality of $\mathcal{P}^1$. (8.2) has been proved and the theorem follows.

Corollary 8.4. *Under the assumption of Theorem* 8.3, *the following relations hold for suitable generators a_i of Theorem* 3.1.

$$\mathcal{P}^t a_i = (-1)^t \binom{(m+i)(p-1)-1}{t} a_{i+t} \quad \textit{and}$$

$$\mathcal{P}^t \Delta a_i = (-1)^t \binom{(m+i)(p-1)}{t} \Delta a_{i+t},$$

$0 \leq i \leq i+t < k$.

Next we shall discuss on some homotopical properties of a sort of complexes containing $K(m, k)$ and stunted lens spaces. First we have

(8.4). *Let k and b be integers such that $k \equiv -b$ (mod $p-1$), $0 \leq b < p-1$. Then we have*

(i) $\pi^S_{2k-1}(Y_p ; Y_p)=0$ *if* $k<b(p^2-p-1)$,

(ii) $\pi^S_{2k}(Y_p ; Y_p)=0$ *if* $k<(b-1)(p^2-p-1)+p-2$.

This follows from (6.1) and (4.1), or more precisely from the list of $\pi^S_i(Y_p ; Y_p)$ in [13].

Lemma 8.5. *Let L be a CW-complex having a sequence of subcomplexes $*=L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_r = L$ such that L_i is a mapping cone $L_{i-1} \cup_{f_i} CY_p^{2n_i-1}$ of a map $f_i: Y_p^{2n_i-1} \to L_{i-1}$, where $n_1 < n_2 < \cdots < n_r$ and $n_r < 2n_1 - 1$. Then, up to homotopy equivalence, L satisfies the following condition. For each i, $1 \leq i \leq r$, let $J(i)$ be the set of integers j such that $n_j \leq n_i - b(p^2-p-1)$ if $n_i - n_j \equiv -b$ (mod $p-1$) and $0 \leq b < p-1$. Then the union $M(n_i) = * + \cup_{j \in J(i)} CY_p^{2n_j-1}$ is a subcomplex of K.*

Proof. Remark that the assumption $n_r < 2n_1 - 1$ means that K is in a stable range. The lemma is proved by changing inductively the attaching map f_i in its homotopy class. Assuming that L_{i-1} has been already modified to satisfy the condition, it is sufficient to prove the injection homomorphism

$$i_* : \pi(Y_p^{2n_i-1} ; M_0(n_i)) \to \pi(Y_p^{2n_i-1} ; L_{i-1})$$

is an epimorphism, where $M_0(n_i) = M(n_i) - CY_p^{2n_i-1}$ and it is a subcomplex of L_{i-1} since $M_0(n_i)$ is the union of the subcomplexes

$M(n_j)$ of L_{i-1} for $j<i$ and $j\in J(i)$. L_{i-1} is obtained from $M_0(n_i)$ by attaching some $CY_p^{2n_k-1}$, $k\notin J(i)$. If $X=X'\cup_f CY_p^{2n_k-1}$ and X' is $(2n_1-2)$-connected, we have an exact sequence

$$\pi(Y_p^{2n_i-1}; X') \xrightarrow{i'_*} \pi(Y_p^{2n_i-1}; X) \xrightarrow{\pi'_*} \pi(Y_p^{2n_i-1}; Y_p^{2n_k}).$$

The last group vanishes by (8. 4), (i). Thus i'_* is an epimorphism. Using this fact we have easily that i_* is an epimorphism. q.e.d.

For example, if $L=S^{2m}(L_p^{2n+2k}/L_p^{2n-2})$, $m+k<n-1$ and $k<p^2-p-1$, then up to homotopy equivalence L is one point union of $p-1$ subcomplexes $M(m+n+k-i)$, $i=0, 1, \cdots, p-2$.

Lemma 8. 6. *Let* $K=Y_p^{2m_1}\cup CY_p^{2m_2-1}\cup\cdots\cup CY_p^{2m_s-1}$ *and* $M_1=Y_p^{2n_1}\cup CY_p^{2n_2-1}\cup\cdots\cup CY_p^{2n_r-1}$ *be CW-complexes satisfying the condition of Lemma* 8. 5; $m_1<m_2<\cdots<m_s$, $n_1<n_2<\cdots<n_r$. *Assume that* $m_1\equiv m_2\equiv\cdots\equiv m_s$ *(mod* $p-1$*),* $n_r\equiv m_1+1$ *(mod* $p-1$*),* $n_r<m_1+(p-3)(p^2-p-1)+p-2$ *and that* $n_j\leq n_r-b(p^2-p-1)$ *if* $n_r-n_j\equiv -b$ *(mod* $p-1$*),* $0\leq b<p-1$. *Then we have* $\pi^S(M_1; K)=0$.

Proof. Let $n_j-m_j\equiv -b$ (mod $p-1$). By the assumption, $n_r-n_j\equiv m_1+1-n_j\equiv b+1\equiv -(p-2-b)$, $0\leq p-2-b<p-1$ and $n_j-m_j\leq n_r-(p-2-b)(p^2-p-1)-m_1<(b-1)(p^2-p-1)+p-2$. It follows from (ii) of (8. 4) $\pi^S(Y_p^{2n_j}; Y_p^{2m_i})=0$ for $1\leq i\leq s$ and $1\leq j\leq r$. By use of homotopy exact sequences we have easily $\pi^S(M_1; K)=0$.

Proof of Theorem 8. 1. Since $G: K(m, k)\rightarrow Q_{2k}^{2m-1}$ induces an isomorphism of $H^i(\ ; Z_p)$ for $i<4mp-5$, $G_*: \pi_i(K(m, k))\rightarrow\pi_i(Q_{2k}^{2m-1})$ is a $\mathcal{C}_p$-isomorphism for $i<4mp-6$. Similarly $\pi_{i+2\nu p}(Q_{2k}^{2m+2\nu-1})$ is $\mathcal{C}_p$-isomorphic to $\pi_{i+2\nu p}(K(m+\nu, k))$ for $i<4mp-6<4(m+\nu)p-6-2\nu p$. Since $K(m, k)$ is $(2mp-4)$-connected, $S^{2\nu p}: \pi_i(K(m, k))\rightarrow\pi_{i+2\nu p}(S^{\nu p}K(m, k))$ is an isomorphism for $i<2(2mp-4)+1=4mp-7$. For $i=4pm-7$, $S^{2\nu p}$ is an epimorphism and its kernel is at most of order 2, hence it is a $\mathcal{C}_p$-isomorphism. So, it is sufficient to prove

(8. 5). *If* $\nu=p^{k-1}$, $1\leq k\leq m$ *and* $k\leq p^2-p$, *then* $K(m+\nu, k)$ *is homotopy equivalent to* $S^{2\nu p}K(m, k)$.

By Theorem 8. 3, there are maps

$$f: S^{2m-2}(L_p^{2(m+k-1)(p-1)}/L_p^{2m(p-1)-2})\rightarrow K(m, k)$$

and $$F: S^{2(m+\nu)-2}(L_p^{2(m+\nu+k-1)(p-1)}/L_p^{2(m+\nu)(p-1)-2})\rightarrow K(m+\nu, k)$$

which induce monomorphisms of $H^*(\ ;Z_p)$. Since $[((k-1)(p-1)+1)/(p-1)]=k-1$, we have by (8.1) that $L_p^{2(m+k-1)(p-1)}/L_p^{2m(p-1)-2}$ and $L_p^{2(m+\nu+k-1)(p-1)}/L_p^{2(m+\nu)(p-1)-2}$ have the same stable homotopy type. For the simplicity we put

$$L = S^{2(m+\nu)-2}(L_p^{2(m+\nu+k-1)(p-1)}/(L_p^{2(m+\nu)(p-1)-2}).$$

L has a sufficiently large connectedness and is in a stable range. Thus L is homotopy equivalent to $S^{2m-2+2\nu p}(L_p^{2(m+k-1)(p-1)}/L_p^{2m(p-1)-2})$. We have obtained

(8.6). *There are maps $F: L \to K(m+\nu, k)$ and $F': L \to S^{2\nu p}K(m, k)$ which induce monomorphisms of $H^*(\ ;Z_p)$.*

Now apply Lemma 8.5 to this complex L and consider the the subcomplexes

$$M = M((m+\nu)p+(k-1)(p-1)-1)$$

and

$$M_1 = M((m+\nu)p+(k-p-1)(p-1)).$$

The complexes M_1 and $K=K(m+\nu, k)$ (or $=S^{2\nu p}K(m, k)$) satisfy the assumption of Lemma 8.6, where $n_r=(m+\nu)p+(k-p-1)(p-1)$, $m_1=(m+\nu)p-1$ and the assumptions $k \leq p^2-2p$ of Theorem 8.1 implies $n_r < m_1+(p-3)(p^2-p-1)+p-2$. It follows from Lemma 8.6 that the restrictions $F|M_1$ and $F'|M_1$ are homotopic to zero. Thus there exist maps h and h' such that the following diagram is homotopy commutative:

$$\begin{array}{ccccc} & & M & & \\ & \swarrow^{F|M} & & \searrow^{F'|M} & \\ K(m+\nu, k) & & \downarrow \pi & & S^{2\nu p}K(m, k) \\ & \nwarrow_{h} & & \nearrow_{h'} & \\ & & M/M_1. & & \end{array}$$

From the definition of M and M_1 we see that the dimensions of the cells of $L-M$ and M_1-* differ from those of $K(m+\nu, k)$ and $S^{2\nu p}K(m, k)$ and that the numbers of the cells of M/M_1, $K(m+\nu, k)$ and $S^{2\nu p}K(m, k)$ are equal. Then it follows from (8.6) that h^* and h'^* are isomorphisms of $H^*(\ ;Z_p)$ hence of $H^*(\ ;Z)$. Thus

h and h' are homotopy equivalences. (8.5) has been proved and we conclude Theorem 8.1. q.e.d.

Observing (8.5) and Proposition 3.6, we have

Proposition 8.7. *Under the assumption of Theorem* 8.1, *the periodic isomorphisms commute with the exact sequence* (3.3), *i.e., we have the following commutative diagram.*

$$\begin{array}{ccccc} \pi_i(Q_{2h}^{2m-1}:p) & \xrightarrow{i_*} & \pi_i(Q_{2k}^{2m-1}:p) & \xrightarrow{\Omega^{-2h}\circ j_*} & \\ \approx\downarrow & & \approx\downarrow & & \\ \pi_{i+2\nu p}(Q_{2h}^{2m+2\nu-1}:p) & \xrightarrow{i_*} & \pi_{i+2\nu p}(Q_{2k}^{2m+2\nu-1}:p) & \rightarrow & \end{array}$$

$$\begin{array}{ccc} \pi_{i+2h}(Q_{2k-2h}^{2m+2h-1}:p) & \xrightarrow{d_*\circ\Omega^{2h+1}} & \pi_{i-1}(Q_{2h}^{2m-1}:p) \\ \approx\downarrow & & \approx\downarrow \\ \pi_{i+2h+2\nu p}(Q_{2k-2h}^{2m+2h+2\nu-1}:p) & \rightarrow & \pi_{i+2\nu p-1}(Q_{2h}^{2m+2\nu-1}:p). \end{array}$$

As an application of the periodicity theorem, we have the following sequences of unstable elements of the second type.

Proposition 8.8. *For each positive integer h such that $h\equiv p$ or $\equiv 2p$ (mod p^2), there exists a sequence $\{\gamma^{(t)}; t=1, 2, \cdots, [(hp-p-2)/(p+1)]\}$ of elements $\gamma^{(t)}\in\pi_{2hp-2t-3}(S^{2h-2t-1}:p)$ satisfying the following relations*:

$$S^2\gamma^{(1)} = p_*I'(\iota_{2hp-1}),$$
$$S^2\gamma^{(t)} = p\cdot\gamma^{(t-1)} \quad \textit{for } t\geq 2$$
and
$$H^{(2)}\gamma^{(t)} = x_t\cdot I'\alpha'_{t+1}(2(h-t-1)p-1)\neq 0 \quad \textit{for some } x_t\not\equiv 0 \ (\textit{mod } p).$$

If $t<\mathrm{Min}([(hp-p-2)/(p+1)], p^2)$ then $p\cdot\gamma^{(t)}=S^2\gamma^{(t+1)}\neq 0$ hence the order of $\gamma^{(t)}$ is a multiple of p^2, moreover the order of $H^{(4)}\gamma^{(t)}$ is p^2.

Proof. For $h=p$ and $h=2p$, we have seen in Theorems 7.1 and 7.4 the existence of $\gamma^{(t)}$, in particular, of $\gamma^{(1)}\in\pi_{2hp-5}(S^{2h-3}:p)$ such that $S^2\gamma^{(1)}=p_*I'(\iota_{2hp-1})$ and $H^{(2)}\gamma^{(1)}=x_1\cdot I'\alpha'_2(2(h-2)p-1)$ for some $x_1\not\equiv 0$ (mod p).

From the commutativity of the diagram

$$\begin{array}{ccccc}\pi_{2hp-3}(Q_2^{2h-1}:p) & \xrightarrow{p_*} & \pi_{2hp-3}(S^{2h-1}:p) & \xleftarrow{S^2} & \pi_{2hp-5}(S^{2h-3}:p) \\ \approx\downarrow\Omega^5 & & \downarrow H^{(4)} & & \downarrow H^{(2)} \\ \pi_{2hp-8}(\Omega^5Q_2^{2h-1}:p) & \xrightarrow{d_*} & \pi_{2hp-8}(Q_4^{2h-5}:p) & \xleftarrow{i_*} & \pi_{2hp-8}(Q_2^{2h-5}:p),\end{array}$$

we have

$$H^{(4)}p_*I'(\iota_{2hp-1})=d_*\Omega^5I'(\iota_{2hp-1})=x_1\cdot i_*I'\alpha_2'(2(h-2)p-1), \quad h=p \text{ or } 2p.$$

Apply Proposition 8.7 to the lower sequence, then we see that this relation holds for each positive integer h such that $h\equiv p, 2p \pmod{p^2}$.

Consider the following exact and commutative diagram:

$$\begin{array}{ccccc}\pi_{2hp-5}(S^{2h-3}:p) & \xrightarrow{S^2} & \pi_{2hp-3}(S^{2h-1}:p) & \xrightarrow{H^{(2)}} & \pi_{2hp-6}(Q_2^{2h-3}:p) \\ \downarrow H^{(2)} & & \downarrow H^{(4)} & & \approx\downarrow\Omega^2 \\ \pi_{2hp-8}(Q_2^{2h-5}:p) & \xrightarrow{i_*} & \pi_{2hp-8}(Q_4^{2h-5}:p) & \xrightarrow{j_*} & \pi_{2hp-8}(\Omega^2Q_2^{2h-3}:p).\end{array}$$

We have $H^{(2)}p_*I'(\iota_{2hp-1})=\Omega^{-2}j_*H^{(4)}p_*I'(\iota_{2hp-1})=\Omega^{-2}j_*i_*I'(\alpha_2'(2(h-2)p-1))=0$, and this implies the existence of an element $\gamma^{(1)}$ such that $S^2\gamma^{(1)}=p_*I'(\iota_{2hp-1})$. The element $\gamma^{(1)}$ satisfies $i_*(H^{(2)}\gamma^{(1)}-x_1\cdot I'\alpha_2'(2(h-2)p-1))=0$. Since $\pi_{2hp-7}(\Omega^2Q_2^{2h-3}:p)=\pi_{2hp-5}(Q_2^{2h-3}:p)=0$ by (6.4), i_* is a monomorphism by the exactness of (3.3), and the relation $H^{(2)}\gamma^{(1)}=x_1\cdot I'\alpha_2'(2(h-2)p-1)$ follows.

Now, applying Theorem 5.4, (i) to the element $\gamma^{(1)}$, we have the existence of an element $\gamma^{(2)}$ satisfying $p\cdot\gamma^{(1)}=S^2\gamma^{(2)}$ and $H^{(2)}\gamma^{(2)}=x_2\cdot I'\alpha_3'(2(h-3)p-1)$ for some integer $x_2\not\equiv 0 \pmod p$. This process can be continued in meta-stable range, i.e., for $t\leq (hp-p-2)/(p+1)$. If $t\leq(hp-p-2)/(p+1)$ we have $p\cdot\pi_{2hp-2t-6}(Q_2^{2h-2t-3}:p)=0$ by Theorem 2.2, $\alpha_t'(2(h-t)p-5)$ and $\alpha_{t+1}'(2(h-t-1)p-3)$ exist by Propoition 4.4, (iii) and Lemma 2.1. Then by Theorem 5.4, (i), there exists an elemen $\gamma^{(t)}$ satisfying $p\cdot\gamma^{(t-1)}=S^2\gamma^{(t)}$ and $H^{(2)}\gamma^{(t)}=x_t\cdot I'a_{t+1}'(2(h-t-1)p-1)$ for some ieteger $x_t\not\equiv 0 \pmod p$ provided the existence and the similar relations for $\gamma^{(t-1)}$. Since $\pi^*i_*\alpha_{t+1}'=\pi^*i^*\alpha^t\delta\alpha=\alpha^t\delta\alpha\delta\neq 0$ by (4.5), we have that $i_*\alpha_{t+1}'\neq 0$, α_{t+1}' is not divisible by p and that $I'\alpha_{t+1}'(2(h-t-1)p-1)=Q^{h-t-1}(\alpha_{t+1}')\neq 0$.

Let $t<\mathrm{Min}([(hp-p-2)/(p+1)], p^2)$ and consider the following commutative and exact diagram:

$$\begin{array}{ccccc} & & \pi_{2hp-2t-5}(S^{2h-2t-3}:p) & \xrightarrow{S^2} & \pi_{2hp-2t-3}(S^{2h-2t-1}:p) \\ & & \downarrow H^{(2)} & & \downarrow H^{(4)} \\ \pi_{2hp-2t-5}(Q_2^{2h-2t-3}:p) & \xrightarrow{d_*\circ\Omega^{-3}} & \pi_{2hp-2t-8}(Q_2^{2h-2t-5}:p) & \xrightarrow{i_*} & \pi_{2hp-2t-8}(Q_4^{2h-2t-5}:p), \end{array}$$

where $d_*\circ\Omega^{-3}=H^{(2)}\circ p_*$ by (5.2). First we have

$$p\cdot H^{(4)}\gamma^{(t)} = H^{(4)}S^2\gamma^{(t+1)} = i_*H^{(2)}\gamma^{(t+1)}$$
$$= x_{t+1}\cdot i_*I'\alpha'_{t+1}(2(h-t-1)p-1)$$

and $$I'\alpha'_{t+1}(2(h-t-1)p-1) = Q^{h-t-1}(\alpha'_{t+1}) \neq 0\,.$$

We see also in (6.4) the group $\pi_{2hp-2t-5}(Q_2^{2h-2t-3}:p)$ is generated by $\bar{Q}^{h-t-1}(\alpha_t)$ and additionally by $Q^{h-t-1}(\beta_1^{p-1})$ if $t+1=p^2-p-1$ and by $\bar{Q}^{h-t-1}(\alpha_1\beta_1^{p-1})$ if $t+1=p^2-p$. By (6.5)

$$d_*\Omega^{-3}\bar{Q}^{h-t-1}(\alpha_t) = H^{(2)}p_*\bar{Q}^{h-t-1}(\alpha_t) = 0\,.$$

In the case $t+1=p^2-p-1$, we have $h-t-2\equiv-(t+2)\equiv 0\not\equiv-1$ (mod p) and

$$d_*\Omega^{-3}Q^{h-t-1}(\beta_1^{p-1}) = H^{(2)}p_*Q^{h-t-1}(\beta_1^{p-1}) = x\cdot Q^{h-t-2}(\alpha_1\beta_1^{p-1})$$

for some $x\not\equiv 0$ (mod p), by Lemma 6.1, (ii). By (4.6), the elements $Q^{h-t-2}(\alpha_1\beta_1^{p-1})$ and $Q^{h-t-2}(\alpha'_{(p-1)p})$ are independent generators. In the case $t+1=p^2-p$, we have $\bar{Q}^{h-t-1}(\alpha_1\beta_1^{p-1})=\bar{Q}^{h-t-1}(\alpha_1)\circ\beta_1^{p-1}(2(h-t)p-5)$ by use of (1.3), (ii). Then

$$d_*\Omega^{-3}\bar{Q}^{h-t-1}(\alpha_1\beta_1^{p-1}) = H^{(2)}p_*\bar{Q}^{h-t-1}(\alpha_1)\circ\beta_1^{p-1}(2(h-t)p-8) = 0$$

by (6.5). We have seen that in all case $I'\alpha'_{t+1}(2(h-t-2)p-1)$ is not in the image of $d_*\circ\Omega^{-3}$. Thus $p\cdot H^{(4)}\gamma^{(t)}=x_{t+1}\cdot i_*I'\alpha'_{t+1}(2(h-t-2)p-1)\neq 0$ and $p\cdot\gamma^{(t)}\neq 0$. By Theorem 2.2, $p^2H^{(4)}\gamma^{(t)}=i_*(p\cdot H^{(2)}\gamma^{(t+1)})=0$. Thus the order of $H^{(4)}\gamma^{(t)}$ is p^2.

9. The homomorphism $\Delta:\pi_{i+4}(S^{2mp+1}:p)\to\pi_{i+2}(S^{2mp-1}:p)$.

The homomorphism Δ in the exact sequence (2.5) is determined for the image of S^2 by the formula

$$\Delta S^2(\alpha) = p\cdot\alpha$$

of (2.7). We shall consider the behaviour of Δ for elements not in the S^2-image. According to Section 2, we understand the homomorphism Δ as follows.

(9. 1). *For the spaces ΩQ_m and Q'_m in Section 2, there are maps $h: \Omega Q_m \to \Omega^4 S^{2mp+1}$, $h': Q'_m \to \Omega Y$ and $i: Q^2 S^{2mp-1} \to \Omega Y$ which induce $\mathcal{C}_p$-isomorphisms of the homotopy groups and the cohomology groups. By these isomorphisms of the p-primary components: $\pi_{i+4}(S^{2mp+1}:p) \approx \pi_i(\Omega Q_m:p)$, $\pi_{i+2}(S^{2mp-1}:p) \approx \pi_i(Q'_m:p)$, the homomorphism*

$$\Delta: \pi_{i+4}(S^{2mp+1}:p) \to \pi_{i+2}(S^{2mp-1}:p)$$

is equivalent to a homomorphism

$$d_*: \pi_i(\Omega Q_m:p) \to \pi_i(Q'_m:p)$$

induced by a map $d: \Omega Q_m = \Omega(\Omega^3 S^{2m+1}, \Omega^2 S^{2m}_{p-1}) \to Q'_m = \Omega(\Omega S^{2m}_{p-1}, S^{2m-1})$.

Let $\mathcal{E} \in \pi_{2mp-3}(\Omega Q_m)$ and $\mathcal{E}' \in \pi_{2mp-3}(Q'_m)$ be elements which correspond to generators of $\pi_{2mp+1}(S^{2mp+1}) \approx \pi_{2mp-1}(S^{2mp-1}) \approx Z$. Then, by (2. 3), we have

$$d_* \mathcal{E} = p \cdot \mathcal{E}'.$$

By use of mapping-cylinder arguments, we may assume that $\mathcal{E}$ and $\mathcal{E}'$ are represented by inclusions of S^{2mp-3} into ΩQ_m and Q'_m respectively, and d maps S^{2mp-3} into S^{2mp-3} by degree p. Furthermore, we may assume that S^{2mp-3} is imbedded in $\Omega^4 S^{2mp+1}$ and $\Omega^2 S^{2mp-1}$ canonnically and in ΩY such that h, h' and i are identical on S^{2mp-3}.

Consider the following commutative and exact diagram:

$$\begin{array}{ccccccc}
\cdots \longrightarrow \pi_i(S^{2mp-3}:p) & \xrightarrow{i_*} & \pi_i(\Omega Q_m:p) & \xrightarrow{j_*} & \pi_i(\Omega Q_m, S^{2mp-3}:p) & \xrightarrow{\partial} & \cdots \\
\downarrow d_* & & \downarrow d_* & & \downarrow d_* & & \\
\cdots \longrightarrow \pi_i(S^{2mp-3}:p) & \xrightarrow{i_*} & \pi_i(Q'_m:p) & \xrightarrow{j_*} & \pi_i(Q'_m, S^{2mp-3}:p) & \xrightarrow{\partial} & \cdots
\end{array}$$

The d_* of the left side satisfied $d_*(\alpha) = p \cdot \alpha$ by (1. 10). The middle one is equivalent to Δ. The d_* of the right side is equivalent to a homomorphism

$$\bar{\Delta}: \pi_{i-1}(Q_4^{2mp-3}:p) \to \pi_{i-1}(Q_2^{2mp-3}:p)$$

by the following isomorphisms (9. 2) obtained from (9. 1):

$$(9.2) \quad \begin{aligned} &\pi_i(\Omega Q_m, S^{2mp-3}:p) \approx \pi_i(\Omega^4 S^{2mp+1}, S^{2mp-3}:p) \approx \pi_{i-1}(Q_4^{2mp-3}:p), \\ &\pi_i(Q'_m, S^{2mp-3}:p) \approx \pi_i(\Omega^2 S^{2mp-1}, S^{2mp-3}:p) \approx \pi_{i-1}(Q_2^{2mp-3}:p). \end{aligned}$$

Then we have the following commutative and exact diagram

$$
(9.3)\qquad
\begin{array}{ccccccc}
\cdots \xrightarrow{p_*} & \pi_i(S^{2mp-3}:p) & \xrightarrow{S^4} & \pi_{i+4}(S^{2mp+1}:p) & \xrightarrow{H^{(4)}} & \pi_{i-1}(Q_4^{2mp-3}:p) & \xrightarrow{p_*}\cdots \\
 & \downarrow \cdot p & & \downarrow \Delta & & \downarrow \bar{\Delta} & \\
\cdots \xrightarrow{p_*} & \pi_i(S^{2mp-3}:p) & \xrightarrow{S^2} & \pi_{i+2}(S^{2mp-1}:p) & \xrightarrow{H^{(2)}} & \pi_{i-1}(Q_2^{2mp-3}:p) & \xrightarrow{p_*}\cdots,
\end{array}
$$

where the four groups of the left side square are considered to be Z if $i=2mp-3$.

Lemma 9.1. *According to Proposition* 3.6, *choose maps* $g: Y_p^{2(mp-1)p-2}\to Q_2^{2mp-3}$ *and* $G: K(mp-1, 2)=Y_p^{2(mp-1)p-2}\cup_h CY_p^{2mp^2-5}\to Q_4^{2mp-3}$ *which induce isomorphisms of* $H^j(\ :Z_p)$ *for* $j<4(mp-1)p-5$. *Let* $\pi: K(mp-1, 2)\to Y_p^{2mp^2-4}$ *be the shrinking map of* $Y_p^{2(mp-1)p-2}$. *Then there exists a map* $D: Y_p^{2mp^2-4}\to Y_p^{2(mp-1)p-2}$ *such that the following diagram is commutative*:

$$
\begin{array}{ccccc}
\pi_{i-1}(K(mp-1, 2):p) & \xrightarrow{\pi_*} & \pi_{i-1}(Y_p^{2mp^2-4}:p) & \xrightarrow{D_*} & \pi_{i-1}(Y_p^{2(mp-1)p-2}:p) \\
\downarrow G_* & & \bar{\Delta} & & \downarrow g_* \\
\pi_{i-1}(Q_4^{2mp-3}:p) & & \longrightarrow & & \pi_{i-1}(Q_2^{2mp-3}:p).
\end{array}
$$

Proof. The isomorphisms of (9.2) are induced by the maps $\Omega h:\Omega(\Omega Q_m, S^{2mp-3})\to\Omega(\Omega^4S^{2mp+1}, S^{2mp-3})=Q_4^{2mp-3}$, $\Omega h':\Omega(Q_m', S^{2mp-3})\to\Omega(\Omega Y, S^{2mp-3})$ and $\Omega i: Q_2^{2mp-3}=\Omega(\Omega^2S^{2mp-1}, S^{2mp-3})\to\Omega(\Omega Y, S^{2mp-3})$. Since these maps induce $\mathcal{C}_p$-isomorphisms of the homotopy groups, they induce isomorphisms of $\pi(K(mp-1, 2);\ )$ by Theorem 1.2. Thus there are maps G_1, G_2 and G_3 such that the following diagram is homotopy commutative:

$$
\begin{array}{c}
K(mp-1, 2)\ (\xrightarrow{D'})\ Y_p^{2(mp-1)p-2} \\
G:\ K(mp-1,2)\to Q_4^{2mp-3},\quad G_1:\ K(mp-1,2)\to\Omega(\Omega Q_m, S^{2mp-3}),\quad G_2:\ K(mp-1,2)\to\Omega(\Omega Y, S^{2mp-3}),\quad G_3:\ K(mp-1,2)\to Q_2^{2mp-3},\quad g:\ Y_p^{2(mp-1)p-2}\to Q_2^{2mp-3} \\
Q_4^{2mp-3}\xleftarrow{\Omega h}\Omega(\Omega Q_m, S^{2mp-3})\xrightarrow{\Omega h'\circ\Omega d}\Omega(\Omega Y, S^{2mp-3})\xleftarrow{\Omega i} Q_2^{2mp-3}.
\end{array}
$$

We have also that $g_*:\pi(K(mp-1, 2); Y_p^{2(mp-1)p-2})\to\pi(K(mp-1, 2); Q_4^{2mp-3})$ is an isomorphism onto by Theorem 1.2. Thus there exists a map D' such that G_3 is homotopic to $g\circ D'$. Then by the definition of $\bar{\Delta}$ the following diagram is commutative:

$$\begin{array}{ccc} \pi_{i-1}(K(mp-1,2):p) & \xrightarrow{D'_*} & \pi_{i-1}(Y_p^{2(mp-1)p-2}:p) \\ \downarrow G_* & \bar{\Delta} & \downarrow g_* \\ \pi_{i-1}(Q_4^{2mp-3}:p) & \longrightarrow & \pi_{i-1}(Q_2^{2mp-3}:p). \end{array}$$

Consider the case $i=2(mp-1)p-2$. Then the above four groups are isomorphic to Z_p, and from (9. 3) we have the following commutative and exact diagram:

$$\begin{array}{ccccc} \pi_{2(mp-1)p+2}(S^{2mp+1}:p) & \xrightarrow{H^{(4)}} & Z_p & \xrightarrow{p_*} & \pi_{2(mp-1)p-3}(S^{2mp-3}:p) \\ \downarrow \Delta & & \downarrow \bar{\Delta} & & \downarrow \cdot p \\ \pi_{2(mp-1)p}(S^{2mp-1}:p) & \xrightarrow{H^{(2)}} & Z_p & \xrightarrow{p_*} & \pi_{2(mp-1)p-3}(S^{2mp-3}:p). \end{array}$$

Here, $H^{(2)}=0$ by the triviality of p Hopf homomorphism. Thus $p_*\bar{\Delta}(Z_p)=p\cdot p_*(Z_p)=0$ implies $\bar{\Delta}(Z_p)=0$. This shows that D' is homotopic to a map D'' such that $D''(S^{2(mp-1)p-3})=*$. Since $\pi_{2(mp-1)p-2}(Y_p^{2(mp-1)p-2})=0$, D'' is homotopic to a map D_0 such that $D_0(Y_p^{2(mp-1)p-2})=*$. Thus $D_0=D\circ\pi$ for a map D: $Y_p^{2mp^2-4}\to Y_p^{2(mp-1)p-2}$.

q.e.d.

Lemma 9. 2. *The map D of Lemma* 9. 1 *represents a generator* $x\cdot\alpha(2(mp-1)p-2)$ *of* $\pi(Y_p^{2mp^2-4}; Y_p^{2(mp-1)p-2})\approx Z_p$ *for some integer* $x\not\equiv 0$ *(mod p).*

Proof. The group $\pi(Y_p^{2mp^2-4}; Y_p^{2(mp-1)p-2})$ is stable, hence isomorphic to $\pi_{2p-2}^S(Y_p; Y_p)\approx Z_p$ and generated by $\alpha(2(mp-1)p-2)$. Thus D represents $x\cdot\alpha(2(mp-1)p-2)$ for some integer x. We assume $x\equiv 0$ (mod p) and lead to a contradiction. To do this it is sufficient to give an element γ of $\pi_{i+4}(S^{2mp+1}:p)$ such that $H^{(4)}\gamma\in \mathrm{Im}\, G_*$ and $\Delta\gamma\notin \mathrm{Im}\, S^2$. Then $\bar{\Delta}H^{(4)}\gamma=H^{(2)}\Delta\gamma\neq 0$ but, since D is homotopic to zero by the assumption $x\equiv 0$, $\bar{\Delta}H^{(4)}\gamma\in \mathrm{Im}\,(g_*\circ D_*\circ\pi_*)=0$ which is a contradiction.

First consider the case $m\not\equiv 1$ (mod p) and let $m=ap-b$, $a\geq 1$, $0\leq b<p-1$. In Proposition 8. 8, let $h=(ap+1)p$ and $t=(b+1)p$. Then there exists $\gamma^{(t-1)}\in\pi_{i+4}(S^{2mp+1}:p)$ and $\gamma^{(t)}\in\pi_{i+2}(S^{2mp-1}:p)$, $i=2hp-2t-5=2mp+2(ap+1)p(p-1)-5$, such that $S^2\gamma^{(t)}=p\cdot\gamma^{(t-1)}$ and the orders of $H^{(4)}\gamma^{(t-1)}$ and $H^{(4)}\gamma^{(t)}$ are p^2. Consider the following commutative and exact diagram:

$$\begin{array}{ccccc}\pi_i(S^{2mp-3}:p) & \xrightarrow{S^2} & \pi_{i+2}(S^{2mp-1}:p) & \xrightarrow{H^{(2)}} & \pi_{i-1}(Q_2^{2mp-3}:p) \\ \downarrow H^{(2)} & & \downarrow H^{(4)} & & \| \\ \pi_{i-3}(Q_2^{2mp-5}:p) & \xrightarrow{i_*} & \pi_{i-3}(Q_4^{2mp-5}:p) & \longrightarrow & \pi_{i-1}(Q_2^{2mp-3}:p)\,.\end{array}$$

By (2.7), $p\cdot H^{(4)}\Delta\gamma^{(t-1)}=H^{(4)}\Delta S^2\gamma^{(t)}=p\cdot H^{(4)}\gamma^{(t)}$. Thus the order of $H^{(4)}\Delta\gamma^{(t-1)}$ is p^2. By Theorem 2.2, $p\cdot\pi_{i-3}(Q_2^{2mp-5}:p)=0$. It follows that $H^{(2)}\Delta\gamma^{(t-1)}\not\equiv 0$ and $\Delta\gamma^{(t-1)}\notin \mathrm{Im}\, S^2$. The fact that $H^{(4)}\gamma^{(t-1)}\in \mathrm{Im}\, G_*$ is essentially proved in the proof of Theorem 5.4, (i) and the details are left to the readers.

For the case $m\equiv 1 \pmod p$ and $m>1$, the proof is quite similar to the above. We use Proposition 8.8 for $h=(ap+2)p$ and $t=(p-1)p$ where $m=(a+1)p+1$, $a\geq 0$.

Finally consider the case $m=1$. Let $i=2(p+1)p-7$ and consider the groups $\pi_{i+4}(S^{2p+1}:p)$ and $\pi_{i+2}(S^{2p-1}:p)$. By Theorem 7.2 and Lemma 6.1, (iii), these groups are isomorphic to Z_p and generated by unstable elements of the first type: $p_*I'(\iota_{2(p+1)p-1})$ and $p_*\bar{Q}^p(\alpha_1)$ respectively. Put $\gamma=p_*I'(\iota_{2(p+1)p-1})$. By Lemma 6.1, (iii), $H^{(2)}p_*(\bar{Q}^p(\alpha_1))=x'\cdot Q^{p-1}(\alpha_2)\not\equiv 0$, $x'\not\equiv 0 \pmod p$. By the exactness of (1.7), $S^2p_*(\bar{Q}^p(\alpha_1))=0$. Then, by Lemma 2.6, $p_*(\bar{Q}^p(\alpha_1))=z\cdot\Delta\gamma$ for some $z\not\equiv 0 \pmod p$. Thus we have $\Delta\gamma=(1/z)p_*(\bar{Q}^p(\alpha_1))\notin \mathrm{Im}\, S^2$. Since $G_*:\pi_{i-1}(K(p-1,2))\to\pi_{i-1}(Q_4^{2p-3})$ is a $\mathcal{C}_p$-isomorphism onto if $i-1<4(p-1)p-5$, we have $H^{(4)}\gamma\in \mathrm{Im}\, G_*$.

Consequently, in all cases we have a contradiction from the assumption $x\equiv 0 \pmod p$. Thus $x\not\equiv 0 \pmod p$. q.e.d.

The following theorem is the main result of this sectin.

Theorem 9.3. *Let* $g: Y_p^{2mp^2-2}\to Q_2^{2mp-1}$ *and* $g': Y_p^{2(mp-1)p-2}\to Q_2^{2mp-3}$ *be maps of Lemma 2.5. For an element* α *of* $\pi_{i+4}(S^{2mp+1}:p)$, *assume that there exists an element* β *of* $\pi_{i-2}(Y_p^{2mp^2-5}:p)$ *such that*

$$H^{(2)}\alpha = g_*(S^3\beta) \quad \text{and} \quad (\alpha\delta(2(mp-1)p-2))\circ\beta = 0\,.$$

We assume further that $g'_*:\pi_{i-1}(Y_p^{2(mp-1)p-2}:p)\to\pi_{i-1}(Q_2^{2mp-3}:p)$ *is an epimorphism, which holds if* $i<2(mp^2-p-1)p-4$. *Then the following relation holds*:

$$H^{(2)}(\Delta\alpha) = x\cdot g'_*(\alpha(2(mp-1)p-2)\circ S\beta)$$

for some integer $x\not\equiv 0 \pmod p$.

Proof. Choose a map $G: K(mp-1, 2) \to Q_4^{2mp-3}$ of Proposition 3.6, which is an extension of g'. We have the following (homotopy) commutative diagram:

$$\begin{array}{ccccc} Y_p^{2(mp-1)p-2} & \xrightarrow{i} & K(mp-1,2) & \xrightarrow{\pi} & Y_p^{2mp^2-4} \\ \downarrow g' & & \downarrow G & & \downarrow g'' \\ Q_2^{2mp-3} & \xrightarrow{i} & Q_4^{2mp-3} & \xrightarrow{j} & \Omega^2 Q_2^{2mp-1}. \end{array}$$

Compare g'' with the map $\Omega^2 g: Y_p^{2mp^2-4} \to \Omega^2 Q_2^{2mp-1}$ induced by g. Both maps satisfy the condition of Lemma 2.5, hence they are equivalent up to a homotopy equivalence of $Y_p^{2mp^2-4}$ representing $y\cdot\iota_Y$ for some $y \not\equiv 0 \pmod p$. Thus we may assume that $\Omega^2 g = g''$ without loss of generality. By Proposition 4.5, the attaching map h in $K(mp-1, 2) = Y_p^{2(mp-1)p-2} \cup_h CY_p^{2mp^2-5}$ represents $-\alpha\delta(2(mp-1)p-2)$. By the assumption $\alpha\delta(2(mp-1)p-2)\circ\beta = 0$, there exists a coextension $\tilde\beta \in \pi_{i-1}(K(mp-1, 2): p)$ of β. Then $j_* G_* \tilde\beta = g''_* \pi_* \tilde\beta = g''_*(S\beta)$.

Now we have the following commutative and exact diagram:

$$\begin{array}{ccccc} \pi_{i+2}(S^{2mp-1}) & \xrightarrow{H^{(2)}} & \pi_{i-1}(Q_2^{2mp-3}) & \xrightarrow{p_*} & \pi_{i-1}(S^{2mp-3}) \\ \downarrow S^2 & & \downarrow i_* & & \| \\ \pi_{i+4}(S^{2mp+1}) & \xrightarrow{H^{(4)}} & \pi_{i-1}(Q_4^{2mp-3}) & \xrightarrow{p'_*} & \pi_{i-1}(S^{2mp-3}) \\ \downarrow H^{(2)} & & \downarrow j_* & & \\ \pi_{i+1}(Q_2^{2mp-1}) & \underset{\approx}{\xrightarrow{\Omega^2}} & \pi_{i-1}(\Omega^2 Q^{mp-1}). & & \end{array}$$

By use of (1.3), we have $\Omega^2(g_* S^3\beta) = (\Omega^2 g)_*(S\beta) = g''_*(S\beta) = j_* G_* \tilde\beta$. By the first assumption,

$$j_* G_* \tilde\beta = \Omega^2(g_* S^3\beta) = \Omega^2 H^{(2)}\alpha = j_* H^{(4)}\alpha .$$

By the exactness and by the last assumption of the theorem, there exists an element γ' of $\pi_{i-1}(Y_p^{2(mp-1)p-2}: p)$ such that $H^{(4)}\alpha = G_*\tilde\beta + i_* g'_* \gamma'$. Put $\gamma = \tilde\beta + i_*\gamma'$, then $H^{(4)}\alpha = G_*\gamma$. Applying Lemma 9.1, Lemma 9.2 and (9.3), we have

$$\begin{aligned} H^{(2)}(\Delta\alpha) &= \bar\Delta H^{(4)}\alpha = \bar\Delta G_*\gamma = g'_* D_* \pi_*(\tilde\beta + i_*\gamma') \\ &= g'_* D_*(S\beta) = x\cdot g'_*(\alpha(2(mp-1)p-2)\circ S\beta) . \end{aligned}$$

Finally consider the homomorphism g'_* for $i < 2(mp^2-p-1)-4$.

Let α be an arbitrary element of $\pi_{i-1}(Q_2^{2mp-3}:p)$ and consider the exact sequence (2.5). By (2.7) and (2.8), $p\cdot I(\alpha)=0$. By (2.8), there exists an element β of $\pi_{i-2}(S^{2(mp-1)p-3}:p)$ such that $S^4\beta=I\alpha$ and $p\cdot\beta=0$. Let $\tilde{\beta}\in\pi_{i-1}(Y_p^{2(mp-1)p-2}:p)$ be a coextension of β. By Lemma 2.5, $I\alpha=S^4\beta=S^3(\pi_*\tilde{\beta})=y'\cdot Ig'_*\tilde{\beta}$ for some $y'\not\equiv 0$ (mod p). By the exactness of (2.5), $\alpha=y'g'_*\beta+I'\gamma'$ for some $\gamma'\in\pi_{i+1}(S^{2(mp-1)p-1}:p)$. By (2.8) we can put $\gamma'=S^2\gamma$. By use of Lemma 2.5, we have $\alpha=y'\cdot g'_*\beta+I'S^2\gamma=g'_*(y'\cdot\beta+x\cdot i_*\gamma)$. This shows that g'_* is an epimorphism if $i<2(mp^2-p-1)p-4$. q.e.d.

The following two corollaries are important in Section 11.

Corollary 9.4. *Assume* $i<2(mp^2-p-1)p-4$. *If*

$$H^{(2)}\alpha = I'(S^5\gamma)$$

for $\alpha\in\pi_{i+4}(S^{2mp+1}:p)$ *and* $\gamma\in\pi_{i-2}(S^{2mp^2-6}:p)$, *then we have*

$$H_p(\Delta\alpha) = IH^{(2)}(\Delta\alpha) = x\cdot\alpha_1(2(mp-1)p+1)\circ S^4\gamma$$

for some integer $x\not\equiv 0$ (mod p).

Proof. By Lemma 2.5, $H^{(2)}\alpha=x'\cdot g_*S^3(i_*\gamma)$ for $x'\not\equiv 0$ (mod p) and for the inclusion $i:S^{2mp^2-6}\to Y_p^{2mp^2-5}$. Put $\beta=x'\cdot i_*\gamma$. Then $H^{(2)}\alpha=g_*(S^3\beta)$. We have also $\alpha\delta(2(mp-1)p-2)\circ\beta=x'\cdot\alpha(2(mp-1)p-2)\circ i_*\pi_*i_*\gamma=0$. By use of Theorem 9.3 and Lemma 2.5 we have

$$\begin{aligned} H_p(\Delta\alpha) &= xx'\cdot I(g'_*\alpha(2(mp-1)p-2)\circ S(i_*\gamma)) && \text{for } x\not\equiv 0 \pmod p\\ &= xx'y\cdot S^3(\pi_*i^*\alpha(2(mp-1)p-2)\circ S\gamma) && \text{for } y\not\equiv 0 \pmod p\\ &= xx'y\cdot\alpha_1(2(mp-1)p+1)\circ S^4\gamma\,. \end{aligned}$$

q.e.d.

Corollary 9.5. *Assume* $1\leq k<mp^2-p-1$. *If*

$$H^{(2)}\alpha = I'(\alpha'_k(2mp^2-1))$$

for $\alpha\in\pi_{i+4}(S^{2mp+1}:p)$, $i=2mp^2-5+2k(p-1)$, *then*

$$H^{(2)}(\Delta\alpha) = x\cdot I'(\alpha'_{k+1}(2(mp-1)p-1))$$

for some integer $x\not\equiv 0$ (*mod* p).

Proof. Remark that $\alpha'_k(t)$ exists if $t\geq 2k+1$ or if $k\not\equiv 0$ (mod p) and $t\geq 6$, and defined by the relation $i_*\alpha'_k(t)=i^*(\alpha^{k-1}\delta\alpha(t+1))$ in

Proposition 4.4. We have $\alpha(t+1)\cdot i_*\alpha'_k(t+2p-3)=\alpha(t+1)\cdot i^*(\alpha^{k-1}\delta\alpha(t+2p-1))=i^*(\alpha^k\delta\alpha(t+1))=i_*\alpha'_{k+1}(t)$. Then the corollary is an easy consequence of Theorem 9.3. q.e.d.

Remark that unstable elements of the first type in Proposition 6.2 may be taken as the element α in one of the above two corollaries.

10. Unstable elements of the third and the fourth types.

We start from the following remarks. By (1.3), the homomorphism $H^{(k)}:\pi_{i+k}(S^{n+k})\to\pi_{i-1}(Q^n_k)$ in the exact sequence (1.7) satisfies

(10.1). $H^{(k)}(\alpha\circ S^{k+1}\beta)=H^{(k)}\alpha\circ\beta$ *for* $\alpha\in\pi_{i+k}(S^{n+k})$, $\beta\in\pi_{j-1}(S^{i-1})$.

It follows

(10.2). *If* $S^{2r}\gamma=p_*\gamma'$ *for* $\gamma\in\pi_i(S^{2m-1}:p)$, $\gamma'\in\pi_{i+2r}(Q_2^{2m+2r-1}:p)$ *and* $p_*:\pi_{i+2r}(Q_2^{2m+2r-1}:p)\to\pi_{i+2r}(S^{2m+2r-1}:p)$, *then we have* $S^{2r}(\gamma\circ S^3\beta)=p_*(\gamma'\circ S^{2r+3}\beta)$ *for* $\beta\in\pi_j(S^{i-3}:p)$. *So, if* $H^{(2)}(\gamma\circ S^3\beta)=H^{(2)}\gamma\circ\beta\neq0$ *and* $S^{2r}(\gamma\circ S^3\beta)=p_*(\gamma')\circ S^{2r+3}\beta\neq0$ *and* $r=p-2$ (*resp.* $r=p$) *then* $\gamma\circ S^3\beta$ *is an unstable element of the third* (*resp. the fourth*) *type.*

Next we prepare two lemmas.

Lemma 10.1. *If* $m\leq p$, *then the complex* $K(m,p)$ *of Proposition* 3.6 *can be chosen such that the cells* $e^{2mp-3+2j(p-1)}$, $j=0,1,\cdots,p-1$, *together with the base point* $*$ *form a subcomplex* $K_0(m,p)=S^{2mp-3}\cup e^{2mp-3+2(p-1)}\cup\cdots\cup e^{2mp-3+2(p-1)^2}$ *of* $K(m,p)$.

Proof. We shall prove by induction on $j\leq p$ that $K_0(m,j)=S^{2mp-3}\cup\cdots e^{2mp-3+2(j-1)(p-1)}$ is a subcomplex of $K(m,j)$ by changing $K(m,j)$ in its homotopy type. The case $j=1$ is trivial. Assume $K(m,j-1)$ has the subcomplex $K_0(m,j-1)$, $l<j\leq p$. $K(m,j)$ is obtained by attaching a cone $CY_p^{2mp-3+2(j-1)(p-1)}$ by a map h: $Y_p^{2mp-3+2(j-1)(p-1)}\to K(m,j-1)$. Let $h_0:S^{2mp-4+2(j-1)(p-1)}\to K(m,j)$ be the restriction of h. h_0 represents an element of $\pi_{2mp-4+2(j-1)(p-1)}(K(m,j-1):p)$. By (6.1), we have $\pi_{2mp-4+2(j-1)(p-1)}(K_0(m,j-1)\cup K(m,i),K_0(m,j-1)\cup K(m,i-1):p)\approx\pi_{2mp-4+2(j-1)(p-1)}(S^{2mp-2+2i(p-1)}:p)\approx(\pi^S_{2(j-i-1)(p-1)-2}:p)=0$ for $1\leq i\leq j-1$. By use of homotopy exact sequences, it follows that $\pi_{2mp-4+2(j-1)(p-1)}(K(m,j-1),K_0(m,j):p)=0$

and the injection homomorphism $\pi_{2mp-4+2(j-1)(p-1)}(K_0(m, j-1) : p) \to \pi_{2mp-4+2(j-1)(p-1)}(K(m, j-1) : p)$ is an epimorphism. Thus h_0 is homotopic to a map $h'_0 : S^{2mp-4+2(j-1)(p-1)} \to K_0(m, j-1)$. Extending the homotopy, we have a map h' which is homotopic to h and is an extension of h'_0. Change the attaching map h to h', then $K(m, j)$ is changed in its homotopy type and $K_0(m, j) = K_0(m, j-1) \cup e^{2mp-3+2(j-1)(p-1)}$ is a subcomplex of $K(m, j)$. By induction on j the lemma is proved. q.ed.

Lemma 10. 2. *Assume that* $\mathcal{P}^{p-1} \neq 0$ *in a complex* $K_0 = S^N \cup e^{N+2(p-1)} \cup \cdots \cup e^{N+2(p-1)^2}$ *and* $N > 2(p-1)^2 - 3$, *for example* $K_0 = K_0(m, p)$ *for* $m \equiv 0 \pmod p$, $m \geq p$ *or* $K_0 = K(m, p)/K_0(m, p)$ *for* $m \equiv 1 \pmod p$, $m > p$. *Let* $h : S^{N+2(p-1)^2-1} \to N_0^{N+2(p-2)(p-1)}$ *be the attaching map of the top cell* $e^{N+2(p-1)^2}$ *and let* $i : S^N \to K_0^{N+2(p-2)(p-1)}$ *be the inclusion. Then we have*

$$h_*(\alpha_1(N+2(p-1)^2-1)) = x \cdot i_*(\beta_1(N))$$

for some integer $x \not\equiv 0 \pmod p$. (See [10: Lemma 4. 10].)

Proof. By Corollary 8. 4, for a generator a_0 of $H^{2mp-3}(Q_{2p}^{2m-1} : Z_p)$ we have $\mathcal{P}^{p-1}a_0 = a_{p-1} \neq 0$ if $m \equiv 0 \pmod p$ and $\mathcal{P}^{p-1}\Delta a_0 = \Delta a_{p-1} \neq 0$ if $m \equiv 1 \pmod p$. By Proposition 3. 6, the same is true for $H^*(K(m, p); Z_p)$. It follows that $\mathcal{P}^{p-1} \neq 0$ in $K_0(m, p)$ if $m \equiv 0 \pmod p$ and in $K(m, p)/K_0(m, p)$ if $m \equiv 1 \pmod p$. As in the proof of the previous lemma, we see $\pi_{N+2p(p-1)-2}(K_0^{N+2(p-2)(p-1)}, S^N : p) = 0$. Thus $h_*(\alpha_1(N+2(p-1)^2-1))$ belongs to $i_*\pi_{N+2p(p-1)-2}(S^N)$ generated by $i_*\beta_1(N)$. Put $h_*(\alpha_1(N+2(p-1)^2-1)) = x \cdot i_*\beta_1(N)$ for some integer x. Assume $x \equiv 0 \pmod p$, then $h_*(\alpha_1(N+2(p-1)^2-1)) = 0$ and there exists an extension $\bar{h} : S^{N+2(p-1)^2-1} \cup_{\alpha_1} e^{N+2p(p-1)-1} \to K_0^{N+2(p-2)(p-1)}$ of h. Consider the mapping-cone of $\bar{h}$, then it is easily seen that $\mathcal{P}^1\mathcal{P}^{p-1} \neq 0$ in the mapping-cone. But this contradicts to Adem's relation $\mathcal{P}^1\mathcal{P}^{p-1} = 0$. Thus $x \not\equiv 0 \pmod p$. q.e.d.

The following four theorems indicate the existence of unstable elements of the third type.

Theorem 10. 3. *Assume that* $m \equiv 0 \pmod p$ *and* $m \geq p$. *Then there exists an element* γ *of* $\pi_{2mp-2+2p(p-1)}(S^{2m+1} : p)$ *such that*

$H^{(2)}\gamma = x\cdot I'(\beta_1(2mp-1))$ *for some integer* $x\not\equiv 0 \pmod p$,
$S^{2p-4}\gamma = p_* I'(\alpha_1(2(m+p-1)p-1))$ *and* $S^{2p-2}\gamma = 0$.

Thus for an arbitrary element β *of* $\pi_i(S^{2mp-5+2p(p-1)})$ *we have* $H^{(2)}(\gamma\circ S^3\beta) = x\cdot I'\beta_1(2mp-1)\circ\beta$ *and* $S^{2p-4}(\gamma\circ S^3\beta) = p_* I'(\alpha_1(2(m+p-1)p-1)\circ S^{2p-1}\beta)$.

Proof. Choose a complex $K(m, p)$ as in Lemma 10.1 and apply Proposition 3.6, then we have the following commutative diagram:

$$\text{(10.3)}\quad \begin{array}{ccccc}
\pi_i(Y_p^{2mp-3+2(p-1)^2}) & \xrightarrow{h_{p-1*}} & \pi_i(K(m, p-1)) & \xleftarrow{i'_*} & \pi_i(Y_p^{2mp-2}) \\
\downarrow g'_* & & \downarrow G_{p-1*} & & \downarrow g_* \\
\pi_i(\Omega^{2p-1}Q_2^{2m+2p-3}) & \xrightarrow{d_*} & \pi_i(Q_{2p-2}^{2m-1}) & \xleftarrow{i_*} & \pi_i(Q_2^{2m-1}) \\
\approx\uparrow \Omega^{2p-1} & & \uparrow H^{(2p-2)} & & \\
\pi_{i+2p-1}(Q_2^{2m+2p-3}) & \xrightarrow{p_*} & \pi_{i+2p-1}(S^{2m+2p-3}) & &
\end{array}$$

$$(i = 2mp-5+2p(p-1)).$$

By Lemma 10.2 with $h = h_{p-1}|S^{2mp-4+2(p-1)^2}$, we have

$$h_{p-1*}(i_{1*}\alpha_1(2mp-4+2(p-1)^2)) = x'\cdot i'_*(i_{2*}\beta_1(2mp-3))$$

for some $x'\not\equiv 0 \pmod p$, where $i_1 : S^{2mp-4+2(p-1)^2} \to Y_p^{2mp-3+2(p-1)^2}$ and $i_2 : S^{2mp-3} \to Y_p^{2mp-2}$ are the inclusions. By Lemma 2.5, $(\Omega^{2p-1})^{-1} g'_* i_{1*}\alpha_1(2mp-4+2(p-1)^2) = y\cdot I'\alpha_1(2(m+p-1)p-1)$ and $g_* i_{2*}\beta_1(2mp-3) = y'\cdot I'\beta_1(2mp-1)$ for $y, y'\not\equiv 0 \pmod p$. From the commutativity of the above diagram it follows

$$H^{(2p-2)}(p_* I'\alpha_1(2(m+p-1)p-1)) = x\cdot i_* I'\beta_1(2mp-1)$$

for some integer $x\not\equiv 0 \pmod p$. Next the following diagram is exact and commutative:

$$\text{(10.4)}\quad \begin{array}{ccccccc}
\pi_{i+3}(Q_{2p-4}^{2m+1}) & \xrightarrow{p_*} & \pi_{i+3}(S^{2m+1}) & \xrightarrow{S^{2p-4}} & \pi_{i+2p-1}(S^{2m+2p-3}) & \xrightarrow{H^{(2p-4)}} & \pi_{i+2}(Q_{2p-4}^{2m+1}) \\
\approx\downarrow \Omega^3 & & \downarrow H^{(2)} & & \downarrow H^{(2p-2)} & & \approx\downarrow \Omega^2 \\
\pi_i(\Omega^3 Q_{2p-4}^{2m-1}) & \xrightarrow{d''_*} & \pi_i(Q_2^{2m-1}) & \xrightarrow{i_*} & \pi_i(Q_{2p-2}^{2m-1}) & \xrightarrow{j_*} & \pi_i(\Omega^2 Q_{2p-4}^{2m+1}).
\end{array}$$

Since $H^{(2p-4)}(p_* I'\alpha_1(2(m+p-1)p-1)) = x\cdot\Omega^{-2} j_* i_* I'\beta_1(2mp-1) = 0$, there exists an element γ' of $\pi_{i+3}(S^{2m+1} : p)$ such that $S^{2p-4}\gamma' = p_* I'\alpha_1(2(m+p-1)p-1)$. By the commutativity of the above diagram, we have $i_*(H^{(2)}\gamma'-x\cdot I'\beta_1(2mp-1))=0$. Thus there exists an element γ'' such that $d''_*\gamma''=H^{(2)}\gamma'-x\cdot I'\beta_1(2mp-1)$. Put $\gamma=\gamma'-p_*\gamma''$, then we obtain

$$H^{(2)}\gamma = x\cdot I'\beta_1(2mp-1) \quad\text{and}\quad S^{2p-4}\gamma = p_*I'\alpha_1(2(m+p-1)p-1)\,.$$

By the exactness of (1.7), $S^{2p-2}\gamma=0$. The remaining part of the theorem is a direct consequence of (10.2). q.e.d.

Theorem 10.4. *Assume* $m\equiv 0 \pmod p$ *and* $m\geq p$ *and let* $\gamma\in\pi_{2mp-2+2p(p-1)}(S^{2m+1}:p)$ *be the element of Theorem* 10.3.

(i). *If* $0\leq r$, $1\leq s$ *and* $r+s<p-1$, *then the composition* $\gamma\circ\beta_1^r\beta_s(2mp-2+2p(p-1))$ *is an unstable element of the third type, i.e., by putting* $\gamma'=\gamma\circ\beta_1^r\beta_s(2mp-2+2p(p-1))$ *we have* $H^{(2)}\gamma'\neq 0$, $S^{2p-4}\gamma'\neq 0$ *and* $S^{2p-2}\gamma'=0$. *The elements* $S^{2j}\gamma'$, $0\leq j\leq p-2$, *generate direct summands isomorphic to* Z_p.

(ii). *The element* γ *is an unstable element of the third type, i.e.,* $H^{(2)}\gamma\neq 0$, $S^{2p-4}\gamma\neq 0$ *and* $S^{2p-2}\gamma=0$. *Let the order of* γ *be* p^t, *then* $1\leq t\leq p-1$, $p^{j-1}\cdot S^{2p-2j-2}\gamma=x_j\cdot p_*I'\alpha_j(2(m+p-j)p-1)$ *for* $1\leq j\leq p-1$ *and for some integer* $x_j\not\equiv 0 \pmod p$ *and the order of* $S^{2p-2j-2}\gamma$ *is* $p^{\mathrm{Min}(t,j)}$. *Thus* $p_*I'\alpha_j(2(m+p-j)p-1)\neq 0$ *for* $1\leq j\leq t$ *and* $=0$ *for* $t<j\leq p-1$.

Proof. (i). The element $\gamma'=\gamma\circ\beta_1^r\beta_s(2mp-2+2p(p-1))$ belongs to $\pi_{2m+1+k}(S^{2m+1}:p)$ for $k=2(m+(r+s+1)p+s-1)(p-1)-2r-5$. Since $\beta_1^r\beta_s(2mp-2+2p(p-1))$ is a stable element of order p, we have $p\cdot\gamma'=0$. By Theorem 10.3, $H^{(2)}\gamma'=x\cdot I'\beta_1^{r+1}\beta_s(2mp-1)$ for some $x\not\equiv 0 \pmod p$. Thus γ' is of order p and not divisible by p since the same is true for $H^{(2)}\gamma$, by Theorem 2.2, where we have $2m+1+k<2p^2m-5$ from the assumption $m\geq p$ and Theorem 2.2 can be applied for our case. Now, it is sufficient to prove that $S^{2p-4}:\pi_{2m+1+k}(S^{2m+1}:p)\to\pi_{2m+2p-3+k}(S^{2m+2p-3}:p)$ is an isomorphism onto. By (6.4), $\pi_{2m+2j+1+k}(Q_2^{2m+2j+1}:p)=0$ for $0\leq j<p-2$. We have also, by (6.4), $\pi_{2m+2j+k}(Q_2^{2m+2j+1}:p)=0$ for $0\leq j<p-2$ if $(r,s)\neq(p-3,1)$ and $\approx Z_p$ generated by $\bar{Q}^{m+j+1}(\alpha_{(p-1)p-j-2})$ if $(r,s)=(p-3,1)$. $\bar{Q}^{m+j+1}(\alpha_{(p-1)p-j-2})$ is not in the $H^{(2)}$-image since $p_*\bar{Q}^{m+j+1}(\alpha_{(p-1)p-j-2})\neq 0$ by Lemma 6.1, (iii). Then, from the exactness of the sequence (1.7), it follows that S^{2p-4} is an isomorphism onto.

(ii). The fact $H^{(2)}\gamma \neq 0$ is proved as above. Since $S^{2p-4}(\gamma\circ\beta_1(2mp-2+2p(p-1)) \neq 0$ by (i), we have $S^{2p-4}\gamma \neq 0$. By Theorem 10. 3, $p_* I'\alpha_1(2(m+p-1)p-1) = S^{2p-4}\gamma \neq 0$. Apply Theorem 5. 3, (ii) to $\varepsilon = I'\alpha_1(2(m+p-1)p-1)$, then there exists an element γ_1 such that $S^2\gamma_1 = p_*\varepsilon = S^{2p-4}\gamma$ and $p\cdot\gamma_1 = x\cdot p_* I'\alpha_2(2(m+p-2)p-1)$ for some $x \not\equiv 0 \pmod p$. By the exactness of (1. 7), we have $\gamma_1 = S^{2p-6}\gamma + y\cdot p_* I'\alpha_2(2(m+p-2)p-1)$ for some integer y. It follows $p\cdot S^{2p-6}\gamma = x\cdot p_* I'\alpha_2(2(m+p-2)p-1)$. Repeating this process (ii) is proved.

q.e.d.

Before proving the next theorem, we need some remaks on the concept of the *coextension*. Let $f: Y \to X$ be a map and construct a mapping cone

$$X^* = X \cup_f CY$$

of f. Let $\bar{f}: (CY, Y) \to (X^*, X)$ be a characteristic map, i.e., $\bar{f} \mid Y = f$ and $\bar{f}$ is a homoemorphism of $CY - Y$ onto $X^* - X$. A coextension

$$\tilde{\beta} \in \pi(SZ; X)$$

of $\beta \in \pi(Z; Y)$, with the relation $f_*\beta = 0$, is defined as follows. Let $g: Z \to Y$ be a representative of β. Represent each point of SZ and CX by pairs (z, t), $z \in Z$, $t \in I$ and (y, s), $y \in Y$, $z \in I$. Then $\tilde{\beta}$ is represented by a map $\tilde{g}: SZ \to X$ given by $\tilde{g}(z, t) = (g(z), 2t)$ for $0 \leq t \leq 1/2$ and $\tilde{g}(z, t) \in X$ for $1/2 \leq t \leq 1$. We see that $\pi\circ\tilde{g}$ is homotopic to Sg for a map $\pi: X^* \to SY$ shrinking X. Consider the relativization

$$j_*: \pi(SZ; X^*) = \pi(CZ, Z; X^*, *) \to \pi(CZ, Z; X^*, X).$$

Then from the above definition we have

(10.5). *An element γ of $\pi(SZ; X^*)$ is a coextension of $\beta \in \pi(Z; Y)$ if and only if the following relation hold:*

$$j_*(\gamma) = \bar{f}_*(\partial^{-1}\beta),$$

where $\bar{f}_*: \pi(CZ, Z; CY, Y) \to \pi(CZ, Z; X^*, X)$ is induced by $\bar{f}$ and $\partial: \pi(CZ, Z; CY, Y) \overset{\approx}{\to} \pi(Z; Y)$ is the boundary may (restriction).

The map $\bar{f}$ defines canonically a map $\Omega\bar{f}: Y \to \Omega(X^*, X)$. Then we have

(10. 5)$'$. *$\gamma\in\pi(SZ;X^*)$ is a coextension of $\beta\in\pi(Z;Y)$ if and only if*

$$\Omega(j_*\gamma)=(\Omega\bar{f})_*\beta,$$

where $\Omega:\pi(CZ,Z;X^*,X)\to\pi(Z;\Omega(X^*,X))$ is one-to-one map of (1. 1).

These (10. 5) and (10. 5)$'$ can be taken as the definition of the coextension.

Lemma 10. 5. *Assume that X is arcwise connected. Let $f: S^{r-1}\to X$ be a map, $X^*=X\cup_f e^r$ a mapping cone of f and $\pi:X^*\to S^r$ be a map shrinking X. Then for arbitrary element α of $\pi_i(X^*)$, its suspension $S\alpha\in\pi_{i+1}(SX^*)$ is a coextension of $(\pm\iota_r)\circ\pi_*\alpha\in\pi_i(S^r)$. Thus, if $\pi_*\alpha=0$ then $S\alpha$ is in the image of the injection homomorphism $(Si)_*:\pi_{i+1}(SX)\to\pi_{i+1}(SX^*)$.*

Proof. The canonical inclusion $\Omega j\circ i_0: X^*\to\Omega SX^*\subset\Omega(SX^*,SX)$ can be extended over a map $i_1:X^*\cup CX\to\Omega(SX^*,SX)$ since $i_0X\subset\Omega(SX,SX)$ and $\Omega(SX,SX)$ is contractible to a point. π defines a homotopy equivalence of $X^*\cup CX$ onto $S^r=SS^{r-1}$ (by shrinking CX). Let $h:S^r\to X^*\cup CX$ be a homotopy inverse. It is easily seen that $h_1=i_1\circ h: S^r\to\Omega(SX^*,SX)$ represents a generator of $\pi_r(\Omega(SX^*,SX))\approx\pi_{r+1}(SX^*,SX)\approx Z$. The map $\Omega(S\bar{f}):S^r\to\Omega(SX^*,SX)$ induced by the characteristic map $S\bar{f}:CS^r\to SX^*$ of the $(r+1)$-cell in SX^* represents also a generator of $\pi_r(\Omega(SX^*,SX))$. Thus $\Omega(S\bar{f})$ is homotopic to h_1 up to sign. For $\alpha\in\pi_i(X^*)$ we have

$$\Omega((Sj)_*S\alpha)=(\Omega j)_*\Omega(S\alpha)=(\Omega j)_*i_{0*}\alpha=h_{1*}\pi_*\alpha=(\pm\Omega S\bar{f})_*(\pi_*\alpha)$$

by use of (1. 2). This shows, by (10. 5)$'$, that $S\alpha$ is a coextension of $(\pm\iota_r)\circ\pi_*\alpha$. If $\pi_*\alpha=0$, then $(Sj)_*S\alpha=\Omega^{-1}((\pm\Omega S\bar{f})_*\pi_*\alpha)=0$. By the exactness of the homotopy sequence of the pair (SX^*,SX), we have that $S\alpha$ is in the image of $(Si)_*$. q.e.d.

Theorem 10. 6. *Let $m\equiv 1 \pmod p$ and $m\geq p+1$. Assume that $\pi_{i-2j}(S^{2mp+2(j-1)(p-2)-1}:p)=0$ for $1\leq j<p-1$. Then for an arbitrary element β of $\pi_{i-2p+4}(S^{2mp-2+2(p-1)^2}:p)$, there exists elements $\gamma\in\pi_{i+3}(S^{2m+1}:p)$ and $\gamma'\in\pi_{i+2p-1}(Q_2^{2m+2p-3}:p)$ such that*

$H_p\gamma = IH^{(2)}\gamma = x\cdot\beta_1(2mp+1)\circ S^{2p-1}\beta$ *for some integer* $x\not\equiv 0 \pmod p$, $S^{2p-4}\gamma = p_*\gamma'$, $I\gamma' = \alpha_1(2(m+p-1)p+1)\circ S^{4p-2}\beta$ *and* $S^{2p-2}\gamma = 0$.

Proof. Choose a complex $K(m, p)$ as in Lemma 10.1 and let $h_{p-1}: Y_p^{2mp-3+2(p-1)^2} \to K(m, p-1)$ be the attaching map as in Proposition 3.6. Consider the subcomplex $K_0(m, p-1) \cup Y_p^{2mp-2}$ of $K(m, p-1)$ and let i_0 be the inclusion of this subcomplex. Since the complex $K_0(m, p-1) \cup Y_p^{2mp-2}$ is in a stable range, we may assume that it is a mapping cone

$$K_0(m, p-1) \cup Y_p^{2mp-2} = S^{2mp-3} \cup C(M \vee S^{2mp-3}),$$

where $M \vee S^{2mp-3}$ is a one point union of a complex $M = S^{2mp-4+2(p-1)} \cup \cdots \cup e^{2mp-4+2(p-2)(p-1)}$ and S^{2mp-3}. Also we may assume that $M = S^{2p-5}M_0$ for a complex $M_0 = S^{2mp-1} \cup \cdots \cup e^{2mp-1+2(p-3)(p-1)}$.

First we prove the following (10.6) for a coextension $i^*\alpha(2mp-3+2(p-1)^2)$ of $\alpha_1(2mp-4+2(p-1)^2)$ given in Section 4.

(10.6). $$h_{p-1*}(i^*\alpha(2mp-3+2(p-1)^2)) = i_{0*}(\varepsilon')$$

for a coextension $\varepsilon' \in \pi_{2mp-4+2p(p-1)}(K_0(m, p-1) \cup Y_p^{2mp-2} : p)$ *of an element* $\nu \oplus x\cdot\beta_1(2mp-3)$, $x \not\equiv 0 \pmod p$, *where* $\nu \in \pi_{2mp-5+2p(p-1)}(M)$ *and* $\oplus$ *indicates the direct sum decomposition*: $\pi_t(M) \oplus \pi_t(S^{2mp-3}) \approx \pi_t(M \vee S^{2mp-3})$, $t = 2mp-5+2p(p-1)$.

Since $m \geq p+1$, the homotopy groups considered here are stable. In particular, $\pi_{t+1}(K(m, p-1), K_0(m, p-1) \cup Y_p^{2mp-2}) \approx \pi_{t+1}(K(m, p-1)/(K_0(m, p-1) \cup Y_p^{2mp-2}))$ and this has a trivial p-primary component by a similar reason as in the proof of Lemma 10.1. We have also $\pi_{t+1}(K_0(m, p-1) \cup Y_p^{2mp-2}, S^{2mp-3}) \approx \pi_t(M \vee S^{2mp-3})$ and this shows that every element of $\pi_{t+1}(K_0(m, p-1), Y_p^{2mp-2})$ is a coextension of an element of $\pi_t(M \vee S^{2mp-3})$. It follows the relation of (10.6) for a coextension ε' of $\nu \oplus \beta'$, $\beta' \in \pi_{t-1}(S^{2mp-3} : p)$. To show $\beta' = x\cdot\beta_1(2mp-3)$, we shrink the subcomplex $K_0(m, p-1)$ of $K(m, p-1)$, then Lemma 10.2 implies $\beta' = x\cdot\beta_1(2mp-3)$.

Next let $i': Y_p^{2mp-2} \to K(m, p-1)$ be the inclusion. Then we have

(10.7). $h_{p-1*}(i^*\alpha(2mp-3+2(p-1)^2)\circ S^{2p-4}\beta) = i'_*(\varepsilon)$ *for a coextension* $\varepsilon \in \pi_t(Y_p^{2mp-2} : p)$ *of* $x\cdot\beta_1(2mp-3)\circ S^{2p-5}\beta$, $x \not\equiv 0 \pmod p$).

To prove this it is sufficient to show $\nu\circ S^{2p-5}\beta=0$. Since $m\geq p+1$, $S^{2p-5}:\pi_{2mp-2+2(p-1)^2}(M_0)\rightarrow\pi_{2mp-5+2p(p-1)}(M)$ is an isomorphism onto. Let $\nu=S^{2p-5}\nu'$ for some ν'. Consider a map $\pi':M_0\rightarrow S^{2mp-1+2(p-3)(p-1)}$ which shrinks lower dimensional cells, then $\pi'_*(\nu'\circ\beta)$ belongs to $\pi_{i-2p+4}(S^{2mp-1+2(p-3)(p-1)}:p)$ which vanishes by the assumption of the theorem. By Lemma 10. 5, $S(\nu'\circ\beta)=i_*\nu''$ for some $\nu''\in\pi_{i-2p+5}(SM_0^{2mp-1+2(p-4)(p-1)}:p)$. If $p=3$, $\nu\circ S^{2p-5}\beta=S(\nu'\circ\beta)=i_*\nu''=0$. If $p>3$, we consider $S\nu''$ and repeat the process, then the relation $\nu\circ S^{2p-5}\beta=0$ is proved as the image of $\pi_{i-1}(*:p)=0$.

Now considering the commutative diagram (10. 3), we have

$$H^{(2p-2)}p_*(\Omega^{-(2p-1)}g'_*i^*\alpha(2mp-3+2(p-1)^2)\circ S^{2p-4}\beta)=i_*g_*\varepsilon\,.$$

Putting $\gamma'=x'\cdot\Omega^{-(2p-1)}g'_*i^*\alpha(2mp-3+2(p-1)^2)\circ S^{2p-4}\beta$ for suitable $x'\not\equiv 0\pmod p$, we have by Lemma 2. 5

$$\begin{aligned}I\gamma'&=S^{2p+2}(\pi_*i^*\alpha(2mp-3+2(p-1)^2)\circ S^{2p-4}\beta)\\&=\alpha_1(2(m+p-1)p+1)\circ S^{4p-2}\beta\,.\end{aligned}$$

Next considering the diagram (10. 4), we have

$$H^{(2p-4)}(p_*\gamma')=\Omega^{-2}j_*H^{(2p-2)}(p_*\gamma')=\Omega^{-2}j_*i_*g_*(\varepsilon)=0\,.$$

Thus there exists an element γ_1 of $\pi_{i+3}(S^{2m+1}:p)$ such that $S^{2p-4}\gamma_1=p_*\gamma'$. As in the proof of Theorem 10. 3, modifying γ_1 by $\gamma=\gamma_1-p_*\gamma_2$ for some $\gamma_2\in\pi_{i+3}(Q_{2p-1}^{2m+1}:p)$, we have

$$H^{(2)}\gamma=g_*\varepsilon\,,\quad S^{2p-4}\gamma=p_*\gamma'\quad\text{and}\quad S^{2p-2}\gamma=0\,.$$

Since ε is a coextension of $x\cdot\beta_1(2mp-3)\circ S^{2p-5}\beta$ by (10. 7), we have using Lemma 2. 5

$$H_p\gamma=IH^{(2)}\gamma=Ig_*\varepsilon=y\cdot S^3\pi_*\varepsilon=xy\cdot\beta_1(2mp+1)\circ S^{2p-1}\beta\,,$$

for some $y\not\equiv 0\pmod p$. Changing xy to x, the theorem is proved. q.e.d.

The proof of the following theorem is similar to one of Theorem 10. 4, using Theorem 10. 6 in place of Theorem 10. 3.

Theorem 10. 7. *Assume* $m\equiv 1\pmod p$ *and* $m\geq p+1$. *If* $0\leq r$, $1\leq s$ *and* $r+s<p-1$, *then there exist elements*

$$\gamma\in\pi_{2mp+2((r+s+1)p+s-1)(p-1)-2r-3}(S^{2m+1}:p)$$

and $$\gamma'\in\pi_{2mp+2((r+s+1)p+s)(p-1)-2r-5}(Q_2^{2m+2p-3}:p)$$

such that

$H_p\gamma = IH^{(2)}\gamma = x\cdot\beta_1^{r+1}\beta_s(2mp+1) \neq 0$ *for some integer* $x \not\equiv 0 \pmod p$, $S^{2p-4}\gamma = p_*\gamma' \neq 0$, $I\gamma' = \alpha_1\beta_1^r\beta_s(2(m+p-1)p+1)$ *and* $S^{2p-2}\gamma = 0$.

Thus γ *is an unstable element of the third type. The elements* $S^{2j}\gamma$, $0 \leq j \leq p-2$, *generate direct summands isomorphic to* Z_p.

The corresponding results for $m=1$ will be seen in the next section.

For unstable elements of the fourth type, we have the following

Theorem 10.8. *Assume* $m \equiv 0 \pmod p$, $m \not\equiv (p-2)p \pmod{p^2}$ *and* $m \geq 2p$. *Then there exist elements* $\gamma \in \pi_{2mp+2(2p+1)(p-1)-2}(S^{2m+1}: p)$ *and* $\gamma' \in \pi_{2mp+2(2p+2)(p-1)}(Q_2^{2m+2p+1}: p)$ *such that*

$$H^{(2)}\gamma = x\cdot I'\beta_2(2mp-1) \quad \textit{for some integer} \quad x \not\equiv 0 \pmod p,$$
$$S^{2p}\gamma = p_*\gamma', \quad I\gamma' = \beta_1(2(m+p+1)p+1) \quad \textit{and} \quad S^{2p+2}\gamma = 0.$$

Proof. By Theorem 2.2, there exists an element γ' such that $I\gamma' = \beta_1(2(m+p+1)p+1)$. Let $t = 2mp+2(2p+1)(p-1)-2$ and consider the exact sequences

$$\pi_{t+2j-2}(S^{2m+2j-1}) \xrightarrow{S^2} \pi_{t+2j}(S^{2m+2j+1}) \xrightarrow{H^{(2)}} \pi_{t+2j-3}(Q_2^{2m+2j-1}) \xrightarrow{p_*} \cdots$$

for $j=0, 1, 2, \cdots, p$. By (6.4), $\pi_{t+2j-3}(Q_2^{2m+2j-1}: p) = 0$ if $1 \leq j < p$, $\approx Z_p$ generated by $Q^m(\beta_2) = I'\beta_2(2mp-1)$ if $j=0$ and $\approx Z_p$ generated by $\bar{Q}^{m+p}(\alpha_1\beta_1)$ if $j=p$. $\bar{Q}^{m+p}(\alpha_1\beta_1)$ is characterized by the relation $I\bar{Q}^{m+p}(\alpha_1\beta_1) = \alpha_1\beta_1(2(m+p)p+1)$. Then, by use of Theorem 10.7, $p_*\bar{Q}^{m+p}(\alpha_1\beta_1) \neq 0$. From the exactness of the above sequences, the above S^2 are epimorphisms of the p-primary components for $1 \leq j \leq p$. Thus there exists an element $\gamma \in \pi_t(S^{2m+1}: p)$ such that $S^{2p}\gamma = p_*\gamma'$. We can put $H^{(2)}\gamma = x\cdot I'\beta_2(2mp-1)$ for some integer x.

We assume $x \equiv 0 \pmod p$ and lead to a contradiction. From this assumption it follows $\gamma = S^2\gamma_0$ for some γ_0. Consider the following exact and commutative diagram:

$$\begin{array}{ccccc}
 & & & & \pi_{t-3}(\Omega Q_{2p+4}^{2m-1}) \\
 & & & & \downarrow \Omega j_* \\
 & & \pi_{t+2p}(Q_2^{2m+2p+1}) & \xrightarrow[\approx]{\Omega^{2p+3}} & \pi_{t-3}(\Omega^{2p+3}Q_2^{2m+2p+1}) \\
 & & \downarrow p_* & & \downarrow d_* \\
\pi_{t-2}(S^{2m-1}) & \xrightarrow{S^{2p+2}} & \pi_{t+2p}(S^{2m+2p+1}) & \xrightarrow{H^{(2p+2)}} & \pi_{t-3}(Q_{2p+2}^{2m-1}).
\end{array}$$

We have $d_*\Omega^{2p+3}\gamma' = H^{(2p+2)}p_*\gamma' = H^{(2p+2)}S^{2p+2}\gamma_0 = 0$. Thus there exists $\varepsilon' \in \pi_{t-3}(\Omega Q_{2p+4}^{2m-1} : p)$ such that $(\Omega j)_*\varepsilon' = \Omega^{2p+3}\gamma'$. Consider $\varepsilon = \Omega^{-1}\varepsilon' \in \pi_{t-2}(Q_{2p+4}^{2m-1} : p)$, then $j_*\varepsilon = \Omega^{2p+2}\gamma'$ for $j_* : \pi_{t-2}(Q_{2p+4}^{2m-1}) \to \pi_{t-2}(\Omega^{2p+2}Q_2^{2m+2p+1})$.

Since $m \geq 2p$, by Proposition 3.6 and Lemma 2.5, we may replace Q_{2p+4}^{2m-1} and $\Omega^{2p+2}Q_2^{2m+2p+1}$ by $K(m, p+2)$ and $Y_p^{2(m+p)p-2p-1}$ respectively and we may consider that $\Omega^{2p+2}\gamma'$ is a coextension of $x'\cdot\beta_1(2(m+p)p-2p-2)$, $x' \not\equiv 0 \pmod p$. As a characterization of the element β_1, we know [13] [10] that in a mapping cone $Y_p^{2(m+p)p-2p-1} \cup e^{t-1}$ of $\Omega^{2p+2}\gamma'$ we have $\mathcal{P}^p H^{2(m+p)p-2p-2}(\ ; Z_p) \neq 0$, hence the same is true in a mapping cone $K(m, p+2) \cup e^{t-1}$ of ε. By identifying $H^*(K(m, p+2); Z_p)$ with $H^*(Q_{2p+4}^{2m-1}; Z_p)$ in lower dimensions this is indicated by $\mathcal{P}^p a_{p+1} \neq 0$ in $K(m, p+2) \cup e^{t-1}$. By Corollary 8.4, we have $\mathcal{P}^{p+1}a_0 = \binom{m(p-1)-1}{p+1} a_{p+1} = ((m/p)+1)a_{p+1}$, $\mathcal{P}^p\mathcal{P}^1 a_0 = -\mathcal{P}^p a_1 = -\binom{(m+1)(p-1)}{p} a_{p+1} = (m/p)a_{p+1}$. On the other hand, by Adem's relation, $\mathcal{P}^p\mathcal{P}^{p+1} = \mathcal{P}^{2p+1} + \mathcal{P}^{2p}\mathcal{P}^1 = \mathcal{P}^1\mathcal{P}^{2p} + ((1/2)\mathcal{P}^p\mathcal{P}^p - \mathcal{P}^{2p-1}\mathcal{P}^1)\mathcal{P}^1$, i.e., $\mathcal{P}^p(2\mathcal{P}^{p+1} - \mathcal{P}^p\mathcal{P}^1) = \mathcal{P}^1(2\mathcal{P}^{2p} + \mathcal{P}^{2p-2}\mathcal{P}^1\mathcal{P}^1)$. We have $(2\mathcal{P}^{2p} + \mathcal{P}^{2p-2}\mathcal{P}^1\mathcal{P}^1)a_0 = 0$ since there is no cell of the corresponding dimension. Thus $0 = \mathcal{P}^p(2\mathcal{P}^{p+1} - \mathcal{P}^p\mathcal{P}^1)a_0 = ((m/p)+2)\mathcal{P}^p a_{p+1}$. This contradicts to $\mathcal{P}^p a_{p+1} \neq 0$ since $m \not\equiv -2p \pmod{p^2}$. We conclude $x \not\equiv 0 \pmod p$. q.e.d.

In the following section we shall see the above theorem holds for $m = p \geq 5$.

It is an open question whether the above theorem holds for β_s and β_{s+1} instead of β_1 and β_2 respectively.

11. Unstable groups—II.

The main theorem of this section is briefly stated as follows.

Theorem 11.1. *For $m \geq 1$ and $k < 2p^2(p-1)-3$, we have the following direct sum decomposition:*

$$\pi_{2m+1+k}(S^{2m+1} : p) = A(m, k) + B(m, k) + \sum_{t=1}^{4} U_t(m, k),$$

where the subgroups $A(m, k)$ and $B(m, k)$ are mapped isomorphically into the stable group π_k^S under S^∞ and the subgroups $U_t(m, k)$ are *generated by unstable elements of the t-th type.* (*The precise definition of these subgroups will be given in the sequel.*)

The fundamental tool of the proof is the following two exact sequences:

$$(11.1)=(1.7) \quad \cdots \xrightarrow{H^{(2)}} \pi_{2m-1+k}(Q_2^{2m-1}:p) \xrightarrow{p_*} \pi_{2m-1+k}(S^{2m-1}:p) \xrightarrow{S^2} \pi_{2m+1+k}(S^{2m+1}:p) \xrightarrow{H^{(2)}} \pi_{2m-2+k}(Q_2^{2m-1}:p) \longrightarrow \cdots,$$

$$(11.2)=(2.5) \quad \cdots \xrightarrow{I} \pi_{2m+3+k}(S^{2mp+1}:p) \xrightarrow{\Delta} \pi_{2m+1+k}(S^{2mp-1}:p) \xrightarrow{I'} \pi_{2m-1+k}(Q_2^{2m-1}:p) \xrightarrow{I} \pi_{2m+2+k}(S^{2mp+1}:p) \longrightarrow \cdots.$$

We shall use the notation $Q^m(\gamma)$ and $\bar{Q}^m(\gamma)$ of (6.3), i.e. $Q^m(\gamma)$ is an element of $\pi_i(Q_2^{2m-1}:p)$ such that $Q^m(\gamma)=I'(\gamma')$ and $S^\infty\gamma'=\gamma$ for some γ'; $\bar{Q}^m(\gamma)$ is an element of $\pi_i(Q_2^{2m-1}:p)$ such that $S^\infty I\bar{Q}^m(\gamma)=\gamma$.

In the following we always assume $m\geq 1$ and $k<2p^2(p-1)-3$.

We start from the definition of $A(m, k)$. We have seen in Section 4 that there exists $\alpha_r(2m+1)=S^{2m-2}\alpha_r(3)$ for each $r\geq 1$ which is of order p and satisfies $S^\infty\alpha_r(2m+1)=\alpha_r$. Also we have seen in Lemma 7.3 that there exists $\alpha'_{sp}(2m+1)=S^{2m-4}\alpha'_{sp}(5)$ for $m\geq 2$ and $1\leq s<p$ which is is order p^2 and $S^\infty\alpha'_{sp}(2m+1)=\alpha'_{sp}$. Remark that we can define $\alpha'_r(2m+1)=x\cdot\alpha_r(2m+1)$ for $r\not\equiv 0 \pmod p$ for some $x\not\equiv 0 \pmod p$. (See (6.2)). By use of these elements $A(m, k)$ is defined as follows.

(11.3). *$A(m, 2sp(p-1)-1)\approx Z_{p^2}$ generated by $\alpha'_{sp}(2m+1)$ for $m\geq 2$ and $1\leq s<p$,*
$A(m, 2r(p-1)-1)\approx Z_p$ generated by $\alpha_r(2m+1)$ (or $\alpha'_r(2m+1)$) for $m=1$ and for $r\not\equiv 0 \pmod p$,
$A(m, k) = 0$ for $k\not\equiv -1 \pmod{2p-2}$.

In order to define $B(m, k)$ we prove

Lemma 11.2. *There exists an element $\beta_1(2p-1)\in\pi_{2p-1+2p(p-1)-2}(S^{2p-1}:p)$ which is of order p^2 and satisfies $S^\infty\beta_1(2p-1)=\beta_1$ and $H^{(2)}\beta_1(2p-1)=x\cdot Q^{p-1}(\alpha_1)$, $x\not\equiv 0 \pmod p$. The order of $\beta_1(2m+1)=S^{2m-2p+2}\beta_1(2p-1)$ is p for $m\geq p$.*

For $2 \leq s < p$, *there exists an element* $\beta_s(2p+3) \in \pi_{2p+3+2(sp+s-1)\times(p-1)-2}(S^{2p+3}: p)$ *of order* p *staisfying* $S^\infty \beta_s(2p+3) = \beta_s$.

The order of $\alpha_1\beta_1(3) = \alpha_1(3) \circ S\beta_1(2p-1)$ *is* p. *For* $2 \leq s < p$, *there exists an element* $\alpha_1\beta_s(5) \in \pi_{2(sp+s)(p-1)+2}(S^5: p)$ *of order* p *such that* $S^2(\alpha_1\beta_s(5)) = \alpha_1(7) \circ S\beta_s(2p+3)$.

Proof. The first two assertion were seen in the proof of Theorem 7.1. The third assertion for $\beta_s(2p+3)$ follows from Corollary 6.4, (ii).

By use of (1.3), (iv), we have $p(\alpha_1\beta_1(3)) = p \cdot \alpha_1(3) \circ S\beta_1(2p-1) = 0$. Since $S^\infty(\alpha_1\beta_1(3)) = \alpha_1\beta_1 \neq 0$, the order of $\alpha_1\beta_1(3)$ is p. According to (1.9), decempose $S\alpha_1(5) \circ \beta_s(2p+2)$ into a direct sum $S(\alpha_1\beta_s(5)) + [\iota_6, \iota_6] \circ \gamma$. Since $p(S\alpha_1(5) \circ \beta_s(2p+3)) = S\alpha_1(5) \circ p \cdot \beta_s(2p+3) = 0$, we have $p \cdot \alpha_1\beta_s(5) = 0$. As in (1.10), (ii), we have $S^2(\alpha_1\beta_s(5)) = S(S\alpha_1(5) \circ \beta_s(2p+3)) = \alpha_1(7) \circ S\beta_s(2p+3)$. q.e.d.

We denote $\beta_s(2m+1) = S^{2m-2p-2}\beta_s(2p+3)$ for $m \geq p+1$. As compositions of $\beta_1(\ )$ and $\beta_s(\ )$, we define $\beta_1^r\beta_s(2m+1)$ for $m \geq p-1$ if $r \geq 1$ or $s=1$ as is seen in the proof of Lemma 6.1. We also define $\alpha_1\beta_1^r\beta_s(2m+1)$ for $m \geq 1$ if $r \geq 1$ or $s=1$ and for $m \geq 2$ if $r=0$, $s \geq 2$ by $\alpha_1\beta_1^r\beta_s(2m+1) = \alpha_1(2m+1) \circ \beta_1^r\beta_s(2m+2p-2)$ and by the element $\alpha_1\beta_s(5)$ of Lemma 11.2. We define $B(m, k)$ as follows.

(11.4). $B(m, 2((r+s)p+s-1)(p-1)-2(r+1)) \approx Z_p$ *generated by*

$\beta_1^r\beta_s(2m+1)$ *for* $m \geq p-1$ *if* $r \geq 1$ *and* $s \geq 1$,
for $m \geq p$ *if* $r=0$ *and* $s=1$,
for $m \geq p+1$ *if* $r=0$ *and* $s \geq 2$.

$B(m, 2((r+s)p+s)(p-1)-2(r+1)-1) \approx Z_p$ *generated by*

$\alpha_1\beta_1^r\beta_s(2m+1)$ *for* $m \geq 1$ *if* $r \geq 1$ *or* $s=1$,
for $m \geq 2$ *if* $r=0$ *and* $s \geq 2$.

For the other cases we put $B(m, k) = 0$.

Lemma 11.3. *The subgroups* $A(m, k) + B(m, k)$ *are direct factors of the groups* $\pi_{2m+1+k}(S^{2m+1}: p)$ *for* $m \geq 2$ *and* $k < 2p^2(p-1)-3$.

This follows easily from (6.1) and the above definitions.

The homomorphism S^2 maps $A(mp-1, k) + B(mp-1, k)$ isomorphically onto $A(mp, k) + B(mp, k)$ except the case $m=1$, $k=2p(p-1)-2$. Then the homomorphism Δ in the sequence (11.2) is determined by the formula (2.7): $\Delta S^2\alpha = p\cdot\alpha$. From the cxactness of (11.2), we have

(11.5). *$Q^m(\iota)$, $Q^m(\alpha_r')$ and $\bar{Q}^m(\alpha_r)$ exist for $r\geq 1$. $\bar{Q}^m(\beta_1^r\beta_s)$ and $Q^m(\beta_1^r\beta_s)$ exist for $m\geq 2$ and for $m=1$ if $r\geq 1$. $Q^1(\beta_1)$ exists. $\bar{Q}^m(\alpha_1\beta_1^r\beta_s)$ and $Q^m(\alpha_1\beta_1^r\beta_s)$ exist for $m\geq 1$.*

Remark that $\bar{Q}^1(\beta_1)$ does not exist since $\Delta\beta_1(2p+1)\not\equiv 0$ as is seen in the proof of Theorem 7.2. We shall see also that $\bar{Q}^1(\beta_s)$ and $Q^1(\beta_s)$ do not exist for $s\geq 2$.

Note that in meta-stable cases the above elements of (11.5) are independent generators of order p as is seen in (6.4), but for smaller values of m the non-triviality of these elements has to be checked in the inductive proof of Theorem 11.1.

The definition of $U_t(m, k)$ starts from the case $t=4$.

(11.6). $U_4(lp+j, 2((s+l)p+s-1)(p-1)-3)\approx Z_p$
generated by an element $S^{2j}u_4(l, \beta_s)$ for $l\geq 1$, $s\geq 2$, $s+l<p$ and $j=0, 1, 2, \cdots, p$.

For the other cases we put $U_4(m, k)=0$. The element $u_4(l, \beta_s)$ is required to satisfy

$$H^{(2)}(u_4(l, \beta_s)) = x\cdot Q^{lp}(\beta_s), \qquad x\not\equiv 0 \pmod p,$$
$$S^{2p}(u_4(l, \beta_s)) = p_*\bar{Q}^{lp+p+1}(\beta_{s-1}) \quad \text{and} \quad S^{2p+2}(u_4(l, \beta_s)) = 0.$$

Note that we know the existence of such $u_4(l, \beta_s)$ only for the case $l\geq 2$ and $s=2$ in Theorem 10.8.

(11.7). $U_3(lp+j, 2((r+s+l)p+s-1)(p-1)-2(r+1)-1)\approx Z_p$
generated by an element $S^{2j}u_3(l, \beta_1^r\beta_s)$ for $r\geq 0$, $s\geq 1$, $l\geq 1$, and $j=0, 1, \cdots, p-2$ except the case $r=0$, $s\geq 2$.
$U_3(lp+1+j, 2((r+s+l)p+s)(p-1)-2(r+1))\approx Z_p$
generated by an element $S^{2j}\bar{u}_3(l, \beta_1^r\beta_s)$ for $r\geq 1$, $s\geq 1$, $l\geq 0$, and $j=0, 1, \cdots, p-2$.

For the other cases we put $U_3(m, k)=0$. The above generators are required to satisfy, for some x, $x'\not\equiv 0 \pmod p$,

$$H^{(2)}u_3(l, \beta_1^r\beta_s) = x\cdot Q^{lp}(\beta_1^r\beta_s),\ H^{(2)}\bar{u}_3(l, \beta_1^r\beta_s) = x'\cdot\bar{Q}^{lp+1}(\beta_1^r\beta_s),$$

$S^{2p-4}u_3(l, \beta_1^r\beta_s) = p_*Q^{lp+p-1}(\alpha_1\beta_1^{r-1}\beta_s)$ $(=p_*Q^{lp+p-1}(\alpha_1)$ if $r=0, s=1)$,
$S^{2p-4}\bar{u}_3(l, \beta_1^r\beta_s) = p_*\bar{Q}^{lp+p}(\alpha_1\beta_1^{r-1}\beta_s)$.

Note that except $\bar{u}_3(0, \beta_1^r\beta_s)$, the existence of the elements $u_3(l, \beta_1^r\beta_s)$ and $\bar{u}_3(l, \beta_1^r\beta_s)$ has been obtained in Theorem 10.4 and Theorem 10.7 respectively.

(11.8). *$U_2(m, 2sp(p-1)-2)\approx Z_{p^2}$ generated by $\gamma_s(2m+1)$ for $2\leq m < sp-1$ and for $m=p-1$, $s=1(\gamma_1(2p-1)=\beta_1(2p-1))$.*
$U_2(1, 2sp(p-1)-2)\approx Z_p$ generated by $\gamma_s(3)$.
$U_2(sp-1, 2sp(p-1)-2)\approx Z_p$ generated by $S^2\gamma_s(2sp-3)$, $s\geq 2$.

For the other cases we put $U_2(m, k)=0$. These elements $\gamma_s(2m+1)$ are required to satisfy

$S^2\gamma_s(2m+1)=p\cdot\gamma_s(2m+3)$ *for* $1\leq m\leq sp-2$ *and for* $m=p-1, s=1$,
$H^{(2)}\gamma_s(2m+1) = x_m\cdot Q^m(\alpha'_{sp-m})$ *for some* $x_m \not\equiv 0 \pmod p$,
$H^{(2)}\gamma_1(2p-1) = x\cdot Q^{p-1}(\alpha_1)$ *for some* $x\not\equiv 0 \pmod p$.

Note that the above fact is known for $s=1,2$ by Proposition 8.8 and the results of Section 7.

(11.9). *$U_1(m, 2(p^2-p+m)(p-1)-2)\approx Z_p+Z_p$ generated by $p_*\bar{Q}^{m+1}(\alpha_{p^2-p-1})$ and $p_*Q^{m+1}(\beta_1^{p-1})$ for $1\leq m<p-1$.*
$U_1(m, 2r(p-1)-2)\approx Z_p$ generated by $p_*\bar{Q}^{m+1}(\alpha_{r-m-1})$ (by $p_*Q^{m+1}(\iota)$ if $m=r-1$) for $1\leq m<r, r\not\equiv 0 \pmod p$) and $r-m\neq p^2-p$.
$U_1(m, 2((r+s)p+s+m)(p-1)-2(r+2))\approx Z_p$ generated by $p_*Q^{m+1}(\beta_1^r\beta_s)$ for $m\not\equiv -1 \pmod p$, $r\geq 0$, $s\geq 1$ except the case $(r, s)=(p-2, 1)$ and the case $m=1$, $r=0$, $s\geq 2$.
$U_1(m, 2((r+s)p+s+m)(p-1)-2(r+1)-1)\approx Z_p$ generated by $p_*\bar{Q}^{m+1}(\beta_1^r\beta_s)$ for $m\not\equiv 0 \pmod p$, $r\geq 0$, $s\geq 1$ except the case $m=1$, $r=0$, $s\geq 2$.
$U_1(m, 2(tp+t)(p-1)-4)\approx Z_p$ for $2\leq m<t$.

For the other cases we put $U_1(m, k)=0$.

Lemma 11.4. *Assume that Theorem* 11.1 *is true for* $\pi_{2m+1+i}(S^{2m+1}:p)$, $i<k$. *Then for* $j<k+2p-4$, *the groups* $\pi_{2m-1+j}(Q_2^{2m-1}:p)$ *are generated by the elements in* (11.5) *and the following elements of the corresponding dimensions*:

$$I'(p_*Q^{mp}(\iota)),\quad 1<m<p,$$
$$Q^m(u_4(\beta_s)) = I'(S^{2p-2}u_4(m-1, \beta_s)),\quad m\geq 2,\ s\geq 2,\ m+s<p$$

and $\bar{Q}^m(u_4(\beta_s))$ *satisfying* $I(\bar{Q}^m u_4(\beta_s))=S^{2p}u_4(m-1, \beta_s))$, $m\geq 2$, $s\geq 2$, $m+s<p$. *These elements are independent in the following sense*: *if* t *is the number of the above elements contained in the group* $\pi_{2m-1+j}(Q_2^{2m-1}:p)$ *then the order of the group is* p^t.

Proof. Since $S^2: U_4(mp-1, i)\rightarrow U_4(mp, i)$ is an isomorphism by the definition (11.6), we have the existence of $Q^m(u_4(\beta_s))$ and $\bar{Q}^m(u_4(\beta_s))$ by use of (2.7). For the case $1<m<p$, $p_*Q^{mp}(\iota)$ is a generator of $\pi_{2mp^2-3}(S^{2mp-1}:p)\approx Z_p$ and $\pi_{2mp^2-1}(S^{2mp+1}:p)=0$. Thus $I'(p_*Q^{mp}(\iota))$ exists. Remark that for the case $m=1$, $p_*Q^p(\iota)=p\cdot\beta_1(2p-1)$ is cancelled with $\beta_1(2p+1)$ and gives none.

Since the exact sequence (11.2) indicates the independence of the elements in the lemma, it is sufficient to prove that the generators of $\sum_{t=1}^3 U_t(mp, i)$ and $\sum_{t=1}^3 U_t(mp-1, i)$ are cancelled by Δ, excepting the generators $\beta_1(2p+1)$, $\beta_1(2p-1)=\gamma_1(2p-1)$ and $p_*Q^{mp}(\iota)$. By checking the generators, we see that the following pairs are the candidates which are cancelled by Δ:

(i) $(p_*Q^{mp+1}(\iota),\ p_*\bar{Q}^{mp}(\alpha_1))$,
(ii) $(p_*\bar{Q}^{mp+1}(\alpha_{r-1}),\ p_*\bar{Q}^{mp}(\alpha_r))$ for $r\not\equiv 0 \pmod p$, $r\geq 2$,
(iii) $(\gamma_s(2mp+1),\ \gamma_s(2mp-1))$ for $1\leq m<s$,
(iv) $(u_3(m, \beta_1^r\beta_s),\ p_*\bar{Q}^{mp}(\beta_1^r\beta_s))$ for $r\geq 1$ or $s=1$,
(v) $(u_4(m, \beta_s),\ p_*\bar{Q}^{mp}(\beta_s))$ for $s\geq 2$,
(vi) $(p_*Q^{mp+1}(\beta_1^r\beta^s),\ S^{2p-4}\bar{u}_3(m-1, \beta_1^{r-1}\beta_s))$ for $r\geq 0$, $s\geq 1$, $m\geq 2$,

By Lemma 6.1., (iii) we have $H^{(2)}p_*\bar{Q}^{mp+1}(\alpha_{r-1})=x'\cdot Q^{mp}(\alpha'_r)$ and $H^{(2)}p_*\bar{Q}^{mp}(\alpha_r)=x''\cdot Q^{mp-1}(\alpha'_{r+1})$ for some $x', x''\not\equiv 0 \pmod p$. Then it follows from Corollary 9.5 that $H^{(2)}\Delta(p_*\bar{Q}^{mp+1}(\alpha_{r-1}))=xx'\cdot Q^{mp-1}(\alpha'_{r+1})=(xx'/x'')\cdot H^{(2)}p_*\bar{Q}^{mp}(\alpha_r)$. By the exactness of the sequence (11.1) we have

$$\Delta(p_*\bar{Q}^{mp+1}(\alpha_{r-1}))\equiv y\cdot p_*\bar{Q}^{mp}(\alpha_r) \qquad \text{mod } Im.\, S^2$$

for some $y\not\equiv 0 \pmod p$). This shows that the pair (i) is cancelled by Δ. The proof for the pairs (ii), (iii), (iv) and (v) is similar to the above, by use of Corollaries 9.5, 9.4, Lemma 6.1 and the relations in (11.6)-(11.8). Consider the pair (vi). By (11.7), $S^{2p-4}\bar{u}_3(m-1, \beta_1^{r+1}\beta_s)=p_*\bar{Q}^{mp}(\alpha_1\beta_1^r\beta_s)$. Compare $\bar{Q}^{mp}(\alpha_1\beta_1^r\beta_s)$ and the composition $\bar{Q}^{mp}(\beta_1^r\beta_s)\circ\alpha_1(t)$ for some suitable t. The I-images of these two elements coincide, hence the difference is in the I'-image which vanishes in our case. Thus $S^{2p-4}\bar{u}_3(m-1, \beta_1^{r+1}\beta_s) = p_*\bar{Q}^{mp}(\beta_1^r\beta_s)\circ\alpha_1(t)$. Similarly, $p_*Q^{mp+1}(\beta_1^r\beta_s)$ coincides with $u_3(m, \beta_1^r\beta_s)\circ\alpha_1(t-3)$ or $u_4(m, \beta_s)\circ\alpha_1(t-3)$ up to non-zero contant. By the commutativity of Δ with the composition, the case (vi) follows from the cases (iv) and (v). q.e.d.

Lemma 11.5. *Up to some non-zero constants, we have the following relations:*

$$H^{(2)}\alpha_1(3)=Q^1(\iota),\ H^{(2)}\alpha_r(3)=\bar{Q}^1(\alpha_{r-1}) \quad \textit{for} \quad 2\leq r<p^2,$$
$$H^{(2)}\alpha'_{sp}(5)=\bar{Q}^2(\alpha_{sp-2}) \quad \text{for} \quad 1\leq s<p,$$
$$H^{(2)}(\beta_1^r\beta_s(2p-1))=Q^{p-1}(\alpha_1\beta_1^{r-1}\beta_s) \quad \textit{for} \quad r\geq 1 \textit{ or } s=1$$
and $$H^{(2)}(\alpha_1\beta_1^r\beta_s(3))=Q^1(\beta_1^r\beta_s) \quad \textit{for} \quad r\geq 1 \textit{ or } s=1.$$

Proof. First remark

(11.10). *In Lemma* 11.4 *of the case* $m=1$, *the groups* $\pi_{1+j}(Q_2^1:p)$ *are isomorphic to* 0, Z_p *or* Z_p+Z_p. $\pi_{1+j}(Q_2^1:p)\approx Z_p+Z_p$ *only for the cases* $j=2(p^2-p)(p-1)$, $2(p^2-p+1)(p-1)-1$, $2(p^2-p+1)(p-1)$, $2p^2(p-1)-2$ *and* $=2p^2(p-1)-1$.

The first half of (11.10) is obtained just by checking the numbers of generators. Consider the last half. For the second and third cases of j, we see that the group are isomorphic to Z_p+Z_p by I' and I respectively. For the first case of j, the group $\pi_{1+j}(Q_2^1:p)\approx\pi_{4+j}(S^3:p)$ contains $Z_p+Z_p\approx A(1, j-1)+B(1, j-1)$. For the remaining two cases the groups are generated by $\bar{Q}^1(\alpha_1\beta_{p-1})$, $Q^1(\beta_1^p)$ and $\bar{Q}^1(\beta_1^p)$, $Q^1(\alpha_{p^2-1})$ respectively. The elements $Q^1(\beta_1^p)$ and $Q^1(\alpha_{p^2-1})$ are of order p. Also the other two elements are of order p since they are represented by some suitable composition. Thus (11.10) is proved. Then the relations for $\alpha_r(3)$ is obvious. (The relation for $\alpha_r(3)$ is true for general r which can be proved by use of Lemma 2.7 easily.) (11.10) also shows that $\alpha'_{sp}(5)$ is not contained in S^2-image since it is of order p^2. Thus $H^{(2)}\alpha'_{sp}(5)\neq 0$. From Lemma 11.4 we can check that the only possibility is $H^{(2)}\alpha_{sp}(5)=x\cdot\bar{Q}^2(\alpha_{sp-2})$, $x\not\equiv 0 \pmod p$. The relation for $\beta_1^r\beta_s$ follows from the relation $H^{(2)}\beta_1(2p-1)=x\cdot Q^{p-1}(\alpha_1)$, $x\not\equiv 0 \pmod p$, of (11.8). The last relation follows from (2.13). q.e.d.

Now we consider the structure of the groups $\pi_{2m-1+j}(Q_2^{2m-1}:p)$ of Lemma 11.4. It is directly checked that the orders of the groups are at most p^2. Consider the cases that the orders are p^2. For metastable cases the groups are isomorphic to Z_p+Z_p as is seen in (6.4). The possibility to be isomorphic to Z_{p^2} occurs for the cases of the first five ones of (6.4) of lower m and the cases that the generators listed in Lemma 11.4 overlapping to some other ones. Let $m\geq 2$. Then the first case of (6.4) is meta-stable. By a similar reason to the proof of (11.10), the group splits for the fourth and the fifth cases of (6.4). The same is true for the third case of (6.4) since the generator $\bar{Q}^m(\alpha_{(p-1)p-1})$ can be obtained as an image of $i^*\alpha^{(p-1)p-1}(2mp-2)$ which is of order p. Together with (11.10), we obtain

(11.11). *The group* $\pi_{2m-1+j}(Q_2^{2m-1}:p)$ *in Lemma* 11.4 *is isomorphic to* 0, Z_p *or* Z_p+Z_p *except the cases that the groups are generated by the followings*:

$$\{Q^m(\alpha_{sp+s-1}),\ \bar{Q}^m(\beta_s)\} \qquad \text{for } 2\leq m\leq s,$$
$$\{\bar{Q}^2(\beta_1\beta_s),\ Q^2(u_4(\beta_s))\} \qquad \text{for } s\geq 2.$$

We prove Theorem 11.1.

Proof of Theorem 11.1. We define a subgroup $\pi'(m, k)$ of $\pi_{2m+1+k}(S^{2m+1}:p)$ by

$$\pi'(m, k) = A(m, k)+B'(m, k)+U_1'(m, k)+U_3'(m, k),$$

where $B'(m, k)$ is obtained from $B(m, k)$ by omitting the generators $\beta_s(2m+1)$, $s\geq 1$, and $\alpha_1\beta_s(2m+1)$, $s\geq 2$; $U_1'(m, k)$ is obtained from $U_1(m, k)$ by putting $U_1'(m, 2(tp+t)(p-1)-4)=0$ and $U_3'(m, k)$ is obtained from $U_3(m, k)$ by omitting the generators $S^{2j}u_3(l, \beta_1)$ and

$S^{2j}\bar{u}_3(0, \beta_1^r\beta_s)$. The generators of $\pi'(m, k)$ satisfy the required conditions and the group $\pi'(m, k)$ is a direct factor of $\pi_{2m+1+k}(S^{2m+1}:p)$. This is shown by use of Lemma 6.1, (11.11), Lemma 11.2, Lemma 11.3, Lemma 11.5, Theorem 10,4 and Theorem 10.7. Put $\bar{\pi}(m, k) = \pi_{2m+1+k}(S^{2m+1}:p)/\pi'(m, k)$, then

$$\pi_{2m+1+k}(S^{2m+1}:p) \approx \pi'(m, k)+\bar{\pi}(m, k).$$

We shall determine the group $\bar{\pi}(m, k)$. Denote by $Q'(m, k)$ a subgroup of $\pi_{2m-1+k}(Q_2^{2m-1}:p)$ generated by $H^{(2)}\pi'(m, k+1)$ and a maximal subgroup $Q_0(m, k)$ which is mapped monomorphically into $\pi'(m-1, k)$ under p_*. The subgroup $Q_0(m, k)$ is generated by corresponding elements of (11.5) which appear in (11.9) and in the last two relations of (11.7). Then we have an exact sequence

$$\cdots \to Q'(m, k) \xrightarrow{p_*} \pi'(m-1, k) \xrightarrow{S^2} \pi'(m, k) \xrightarrow{H^{(2)}} Q'(m, k-1) \to \cdots.$$

we put $P(m, k)=\pi_{2m-1+k}(Q_2^{2m-1}:p)/Q'(m, k)$, then we obtain an exact sequence

(11.12)

$$\cdots \to P(m, k) \xrightarrow{p_*} \bar{\pi}(m-1, k) \xrightarrow{S^2} \bar{\pi}(m, k) \xrightarrow{H^{(2)}} P(m, k-1) \to \cdots$$

from (11.1). The group $P(m, k)$ is generated by the corresponding one of the following elements:

(i) $Q^m(\alpha'_{p-m})$ for $1\leq m\leq p-1$; $\bar{Q}^m(\alpha_{p-m})$ for $3\leq m\leq p-1$ and $Q^p(\iota)$,

(ii) $Q^m(\alpha'_{sp-m})$ for $1\leq m\leq sp-2$, $\bar{Q}^m(\alpha_{sp-m})$ $3\leq m\leq sp-1$ and $Q^{sp}(\iota)$, where $s\geq 2$,

(iii) $Q^{sp-1}(\alpha_1)$ and $Q^{sp-p}(\beta_1)$ for $s\geq 2$,

(iv) $\bar{Q}^1(\beta_1^r\beta_s)$ and $\bar{Q}^p(\alpha_1\beta_1^{r-1}\beta_s)$ for $r\geq 1$, $s\geq 1$,

(v) $Q^{lp-1}(\beta_s)$ and $\bar{Q}^{lp+p+1}(\beta_{s-1})$ for $l\geq 1$, $s\geq 2$,

(vi) the elements listed in Lemma 11.4,

(vii) $\bar{Q}^{p+1}(\beta_s)$ for $s\geq 1$,

First consider the case $k=2sp(p-1)-1$, $s\geq 1$. In this case, we see that $P(m, k)=0$ for all m. Thus $\bar{\pi}(m, k)$ is mapped isomorphically into $\pi(m+1, k)$. In the stable range we see that $\bar{\pi}(m, k)=0$. It follows that $\bar{\pi}(m, 2sp(p-1)-1)=0$ for all m. Next consider the case $k=2sp(p-1)-2$, $s\geq 1$. For the case $s=1$ we quote Theorem 7.1. Let $s \geq 2$. By the result just obtained we have exact sequences

$$0 \to P(m, k) \to \pi(m-1, k) \to \pi(m, k) \to P(m, k-1) \to \pi(m-1, k-1)$$

for $m=1, 2, \cdots$ ($\pi(0, k)=\pi(0, k-1)=0$). We see $\pi(m, k)=0$ for sufficiently large m. The elements of (ii) and the first element of (iii) are in the exact sequence. By Theorem 10.4, (ii) $p_* Q^{sp-1}(\alpha_1) \neq 0$. Thus we can omit $Q^{sp-1}(\alpha_1)$ in computing $\pi(m, k)$. By counting the number of the generators of (ii), we have that the order of $\pi(m, k)$ is p^2 if $2 \leq m \leq sp-2$ and is p if $m=1$ or $m=sp-1$. The cyclicity of the groups $\pi(m, k)$ for $2 \leq m \leq sp-2$ is obtained by use of Theorem 5.4, (i), as in the proof of Theorem 7.1. Then we have that $\pi(m, k)$ is isomorphic to $U_2(m, k)$ and generated by the element $\gamma_s(2m+1)$ of (11.8).

The remaining cases are computed rather simply. We mention that the elements of (iv) and (v) produce the elements $\bar{u}_3(0, \beta_1^r \beta_s)$ of (11.7) and $u_4(l, \beta_s)$ of (11.6) respectively, the elements of (vi) produce the groups $U_1(m, 2(tp+t)(p-1)-4)$ in (11.9) and the elements $\alpha_1 \beta_{t+1}(2m+1)$, and the element $\bar{Q}^{p+1}(\beta_s)$ produces β_{s+1}. The details are left to the readers.

Finally we remark that above discussion has been done by the induction on $k < 2p^2(p-1)-3$, starting from the assumption of Lemma 11.4. q.e.d.

In the above proof we have

(11.12). *Up to some non-zero constants the following relations hold*:

$$H^{(2)}\beta_s(2p+3) = \bar{Q}^{p+1}(\beta_{s-1}) \quad \text{for} \quad 2 \leq s < p,$$
$$H^{(2)}\alpha_1\beta_2(5) = I'(p_* Q^{2p}(\iota))$$
and $$H^{(2)}\alpha_1\beta_s(5) = I(\bar{Q}^2 u_4(\beta_s)) \quad \text{for} \quad 3 \leq s < p.$$

12. Meta-stable groups—II.

In the results of the previous section, we have seen the existence of an element $\gamma_s(2sp-3) \in \pi_{2sp^2-5}(S^{2sp-3} : p)$ for $1 \leq s < p$ such that $H^{(2)}\gamma_s(2sp-3) = x \cdot Q^{sp-2}(\alpha_2')$ for some $x \not\equiv 0 \pmod p$, $S^2\gamma_s(2sp-3) \neq 0$ and $S^4\gamma_s(2sp-3)=0$. The kernel of the homomorphism $S^2 : \pi_{2sp^2-3}(S^{2sp-1} : p) \to \pi_{2sp^2-1}(S^{2sp+1} : p)$ is generated by the element

$p_*Q^{sp}(\iota)=p_*I'(\iota_{2sp^2-1})$. It follows that $p_*I'(\iota_{2sp^2-1})=S^2(x'\cdot\gamma_s(2sp-3))$ of some integer x'. In the proof of Proposition 8.8, we see that the existence of such an element $\gamma^{(1)}=x'\cdot\gamma_s(2sp-3)$ implies the assertion of Proposition 8.8 for $h\equiv sp \pmod{p^2}$. Thus we have the following

Theorem 12.1. *For each positive integer s with $s\not\equiv 0 \pmod p$, there exists a sequence $\{\gamma^{(t)}\in\pi_{2sp^2-2t-3}(S^{2sp-2t-1}:p)\,;\ t=1,2,\cdots,[(sp^2-p-2)/(p+1)]\}$ satisfying the following relations.*

$$S^2\gamma^{(1)} = p_*I'(\iota_{2sp^2-1}) = p_*Q^{sp}(\iota)\,,$$
$$S^2\gamma^{(t)} = p\cdot\gamma^{(t-1)} \qquad \textit{for} \quad t\geq 2\,,$$
$$\textit{and} \quad H^{(2)}\gamma^{(t)} = x_t\cdot I'\alpha'_{t+1}(2(sp-t-1)p-1) = x_t\cdot Q^{sp-t-1}(\alpha'_{t+1}) \not\equiv 0$$

for some $x_t\not\equiv 0 \pmod p$. If $t<\mathrm{Min}([(sp^2-p-2)/(p+1)],p^2)$, we have that the order of $H^{(4)}\gamma^{(t)}$ is p^2.

Next we prove the following

Theorem 12.2. *For each positive integer s with $s\not\equiv 0 \pmod p$, there exists a sequence $\{\gamma_s(2m+1)\in\pi_{2m+2sp(p-1)-1}(S^{2m+1}:p)\,;\ \mathrm{Max}(1,sp-p^2)\leq m\leq sp-2\}$ satisfying the following relations:*

$$S^2\gamma_s(2sp-3) = p_*Q^{sp}(\iota) \not\equiv 0\,,$$
$$S^2\gamma_s(2m+1) = p\cdot\gamma_s(2m+3) = y_m\cdot p_*\bar{Q}^{m+2}(\alpha_{sp-m-2}) \not\equiv 0 \quad \textit{for} \quad m<sp-2$$
and
$$H^{(2)}\gamma_s(2m+1)=x_m\cdot Q^m(\alpha'_{sp-m})\not\equiv 0\ (\mathrm{mod}\ Q^m(\alpha_1\beta_1^{p-1}) \quad \textit{if}\ sp-m=p^2-p),$$

where $x_m, y_m\not\equiv 0 \pmod p$. The order of $\gamma_s(2m+1)$ is p^2 if $\mathrm{Max}(1,sp-p^2)<m\leq sp-2$.

Proof. Apply Theorem 5.3, (i) for $m=k=sp$ and $2\leq m\leq sp-1$, then we have elements $\varepsilon_m\in\pi_{2m+2sp(p-1)-1}(Q_2^{2m+1}:p)$, $\varepsilon'_m\in\pi_{2m+2sp(p-1)-3}(Q_2^{2m-1}:p)$ and $\gamma_m\in\pi_{2m+2sp(p-1)-3}(S^{2m-1}:p)$ satisfying

$$p_*\varepsilon_m = S^2\gamma_m,\ p_*\varepsilon'_m= p\cdot\gamma_m,\ I(\varepsilon'_m)=x'_m\cdot\alpha_{sp-m}(2mp+1),\ x'_m\not\equiv 0 \pmod p,$$
$$\varepsilon_{sp-1} = Q^{sp}(\iota) \quad \text{and} \quad I(\varepsilon_m)=\alpha_{sp-m-1}(2(m+1)p+1), \quad m<sp-1\,.$$

By the exactness of (11.2)=(2.5), $I(x'_m\cdot\varepsilon_{m-1}-\varepsilon'_m)=0$ implies $\varepsilon'_m\equiv x'_m\cdot\varepsilon_{m-1} \bmod I'\pi_{2m+2sp(p-1)-1}(S^{2mp-1}:p)$. Thus

$$p\cdot\gamma_m\equiv x'_m\cdot S^2\gamma_{m-1} \bmod p_*I'(\pi_{2m+2sp(p-1)-1}(S^{2mp-1}:p))\,.$$

If $sp-m<p^2$ and $sp-m\neq p^2-p-1$, the group $I'(\pi_{2m+2sp(p-1)-1}(S^{2mp-1}:p))$ vanishes by (6.1) and Lemma 11.4. If $sp-m=p^2-p-1$, this group is generated by $I'\beta_1^{p-1}(2mp-1)=Q^m(\beta_1^{p-1})$, and then $p\cdot\gamma_m=x'_m\cdot S^2\gamma_{m-1}+z\cdot p_*Q^m(\beta_1^{p-1})$ for some integer z. Apply the homomorphism $H^{(2)}$ to the both sides of the last relation, then $p\cdot H^{(2)}\gamma_m=z\cdot H^{(2)}p_*Q^m(\beta_1^{p-1})=zz'\cdot Q^{m-1}(\alpha_1\beta_1^{p-1})$ for some $z'\not\equiv 0$ (mod p), by the exactness of (11.1) and by Lemma 6.1, (ii). We have $p\cdot H^{(2)}\gamma_m=0$ and $Q^{m-1}(\alpha_1\beta_1^{p-1})\neq 0$ by Theorem 2.2 and Lemma 11.4. It follows that $z\equiv 0$ (mod p) and $p\cdot\gamma_m=x'_m\cdot S^2\gamma_{m-1}$ for $sp-m<p^2$. By putting $\gamma_s(2sp-3)=\gamma_{sp-1}$ and $\gamma_s(2m+1)=(\Pi_{j=m+2}^{sp-2}x'_j)\gamma_{m+1}$ for $1\leq m<sp-2$, we have

(12.1). *There exists a sequence* $\{\gamma_s(2m+1)\in\pi_{2m+2sp(p-1)-1}(S^{2m+1}:p)\,;\ m=1,2,\cdots,sp-2\}$ *satisfying*

$$S^2\gamma_s(2s-3)=p_*Q^{sp}(\iota)\,,$$

$$S^2\gamma_s(2m+1)=y_m\cdot p_*\bar{Q}^{m+2}(\alpha_{sp-m-2})\,,\quad \textit{for}\quad 1\leq m<sp-2$$

and $S^2\gamma_s(2m+1)=p\cdot\gamma_s(2m+3)$ *for* $\mathrm{Max}(1,sp-p^2-1)\leq m<sp-2$,

where $y_m\not\equiv 0$ (mod p).

Now, we compare the element $S^2\gamma_s(2m+1)$ with the element $S^2\gamma^{(sp-m-1)}$ of Theorem 12.1. For $m=sp-2$, we have $S^2\gamma_s(2sp-3)=p^*Q^{sp}(\iota)=S^2\gamma^{(1)}$. By the exactness of the sequence (12.1), $S^2\gamma_s(2m+1)=S^2\gamma^{(sp-m-1)}$ implies $\gamma_s(2m+1)\equiv\gamma^{(sp-m-1)}$ mod $p_*\pi_{2m+2sp\times(p-1)-1}(Q_2^{2m+1}:p)$ and $S^2\gamma_s(2m-1)=p\cdot\gamma_s(2m+1)\equiv p\cdot\gamma^{(sp-m-1)}=S^2\gamma^{(sp-m-2)}$ mod $p_*(p\cdot\pi_{2m+2sp(p-1)-1}(Q_2^{2m+1}:p))$. If $s<p$, $p\cdot\pi_{2m+2sp(p-1)-1}(Q_2^{2m+1}:p)=0$ by (11.11). If $s>p$, then $m\leq sp-p^2$ implies $2m+2sp(p-1)-1<2(m+1)p^2-5$, hence $p\cdot\pi_{2m+2sp(p-1)-1}(Q_2^{2m+1}:p)=0$ by Theorem 2.2. By induction on decreasing m we have

(12.2). $S^2\gamma_s(2m+1)=S^2\gamma^{(sp-m-1)}$ *and* $\gamma_s(2m+1)\equiv\gamma^{(sp-m-1)}$ mod $p_*\pi_{2m+2sp(p-1)-1}(Q_2^{2m+1}:p)$ *for* $\mathrm{Max}(1,sp-p^2-1)\leq m\leq sp-2$.

By Theorem 12.1, we have then

$$H^{(2)}\gamma_s(2m+1)\equiv H^{(2)}\gamma^{(sp-m-1)}=x_m\cdot Q^m(\alpha'_{sp-m})\neq 0$$

mod $H^{(2)}p_*\pi_{2m+2sp(p-1)-1}(Q_2^{2m+1}:p)$. For $m\geq sp-p^2$, by (6.1) and Lemma 11.4, the group $\pi_{2m+2sp(p-1)-1}(Q_2^{2m+1}:p)$ is generated by

$Q^{m+1}(\alpha_{sp-m-1})$, $Q^{m+1}(\beta_1^{p-1})$ and $\bar{Q}^{m+1}(\alpha_1\beta_1^{p-1})$. Then, as is seen in the proof of Proposition 8.8, we have

$$H^{(2)}\gamma_s(2m+1)=x_m\cdot Q^m(\alpha'_{sp-m}) \quad \text{for} \quad m\neq sp-p^2-p \text{ and } \operatorname{Max}(1, sp-p^2)\leq m\leq sp-2$$

and

$$H^{(2)}\gamma_s(2m+1)\equiv x_m\cdot Q^m(\alpha'_{sp-m}) \bmod Q^m(\alpha_1\beta_1^{p-1}) \quad \text{if} \quad m=sp-p^2-p\geq 1.$$

We see also that $p_*\pi_{2m+2sp(p-1)-1}(Q_2^{2m+1}:p)$ does not contain $\gamma_s(2m+1)$. Thus $S^2\gamma_s(2m+1)=p\cdot\gamma_s(2m+3)\neq 0$. Obviously $p^2\cdot\gamma_s(2m+1)=p\cdot S^2\gamma_s(2m-1)=py_{m-1}\bar{Q}^{m+2}(\alpha_{sp-m-1})=0$. Thus the order of $\gamma_s(2m+1)$ is p^2 for $\operatorname{Max}(1, sp-p^2)<m\leq sp-2$. q.e.d.

By use of the exact sequence (3.3) we can see that for $\operatorname{Max}(1, sp-p^2-1)<m\leq sp-2$ the element $H^{(4)}\gamma_s(2m+1)$ generates a direct summand isomorphic to Z_{p^2}. The following corollary follows.

Corollary 12.3. *The elements* $\gamma_s(2m+1)$ *of Theorem* 12.2 *generates a direct summand* $U_2(2m+1, 2sp(p-1)-2)$ *of* $\pi_{2m+2sp(p-1)-1}(S^{2m+1}:p)$ *isomorphic to* Z_{p^2} *if* $\operatorname{Max}(1, sp-p^2-1)<m<sp-2$. *If* $s\neq 1$, $S^2\gamma_s(2sp-3)$ *generates a direct summand isomorphic to* Z_p.

The above last assertion follows from the fact that $S^2:\pi_{2sp^2-5}(S^{2sp-3}:p)\to\pi_{2sp^2-3}(S^{2sp-1}:p)$ is an epimorphism for $s>1$ which is a consequence of the result $p_*I'\alpha_1(2(sp-1)p-1)=S^{2sp-4}\gamma\neq 0$ in Theorem 10.4, (ii).

Lemma 12.4. *The following elements in* (11.7) *generate direct summands isomorphic to* Z_p:

$$S^{2j}u_3(l, \beta_1) \quad \text{for} \quad l\not\equiv -1 \pmod p,\ 0\leq j\leq p-2,$$
$$S^{2j}u_3(l, \beta_1^p) \quad \text{for} \quad 0\leq j\leq p-2.$$

Proof. $u_3(l, \beta_1)$ is the element γ of Theorem 10.4 for $m=lp$ and belongs to $\pi_{2m+1+k}(S^{2m+1}:p)$ for $k=2(m+p)(p-1)-3$. By Theorem 12.2, for $1<j\leq p-1$ the elements $I'\alpha_j(2(m+p-j)p-1)=Q^{m+p-j}(\alpha_j)$ are in the $H^{(2)}$-images. Thus $p_*I'\alpha_j(2(m+p-j)p-1)=0$. By (6.1) and by the exactness of (12.1), this result implies that $S^{2p-4}:\pi_{2m+1+k}(S^{2m+1}:p)\to\pi_{2m+2p+k-3}(S^{2m+2p-3}:p)$ is an isomorphism onto. We have also, by (ii) of Theorem 10.4, that the

orders of $S^{2j}u_3(l, \beta_1)$ are p. These elements $S^{2j}u_3(l, \beta_1)$ generate direct summands isomorphic to Z_p since $H^{(2)}$ maps $u_3(l, \beta_1)$ to a generator of $\pi_{2m+4+k}(Q_2^{2m-1}: p) \approx Z_p$.

We may assert that $u_3(l, \beta_1^p) = u_3(l, \beta_1) \circ \beta_1^{p-1}(2m+1+k) \in \pi_{2m+1+h}(S^{2m+1}: p)$, $h = 2(m+1+p)(p-1)-3$. By a similar reason, but using Lemma 6.1, (iii) in place of Theorem 12.2, we have that $S^{2p-6}: \pi_{2m+3+h}(S^{2m+3}: p) \to \pi_{2m+2p+h-3}(S^{2m+2p-3}: p)$ is an isomorphism onto. Consider the exact sequence

$$\pi_{2m+1+h}(Q_2^{2m+1}: p) \xrightarrow{p_*} \pi_{2m+1+h}(S^{2m+1}: p) \xrightarrow{S^2} \pi_{2m+3+h}(S^{2m+3}: p) \xrightarrow{H^{(2)}} \pi_{2m+h}(Q_2^{2m+1}: p).$$

This $H^{(2)}$ is trivial since $\pi_{2m+h}(Q_2^{2m+1}: p)$ is generated by $Q^{m+1}(\beta_{p-1})$ and $p_* Q^{m+1}(\beta_{p-1}) = x \cdot Q^m(\alpha_1 \beta_{p-1}) \neq 0$ by Lemma 6.1, (ii). The group $\pi_{2m+1+h}(Q_2^{2m+1}: p)$ is generated by $\bar{Q}^m(\beta_{p-1})$ and $H^{(2)} p_* \bar{Q}^m(\beta_{p-1}) = 0$ by Theorem 5.1, (i). Since $H^{(2)} u_3(l, \beta_1^p) = x' \cdot Q^m(\beta_1^p)$, $x \not\equiv 0 \pmod p$, generates a direct summand of $\pi_{2m+1+h}(Q_2^{2m-1}: p)$, we concludes that $u_3(l, \beta_1^p)$ and $S^2 u_3(l, \beta_1^p)$ generate direct summans isomorphic to Z_p, and so does $S^{2j}u_3(l, \beta_1^p)$ for $0 \leq j \leq p-2$. q.e.d.

We define subgroups $U_1(m, k)$, $U_2(m, k)$ and $U_3(m, k)$ of $\pi_{2m+1+k}(S^{2m+1}: p)$ as in the previous section. We define also $U_4'(m, k)$ as a subgroup of $U_4(m, k)$ generated by $S^{2j}u_4(l, \beta_2)$. Then we have the following

Theorem 12.5. *Let $k \geq 2p^2(p-1)$, then the group $\pi_{2m+1+k}(S^{2m+1}: p)$ is isomorphic to a direct sum*

$$(\pi_k^S: p) + \sum_{t=1}^{3} U_t(m, k) + U_4'(m, k)$$

if the pair (m, k) satisfies the following conditions:

(i) $k \not\equiv -1, -2, -3 \pmod{2p^2(p-1)}$,

(ii) $m \geq (s-p+1)p+1$ *if* $2sp(p-1)-4 \leq k < 2(s+1)p(p-1)-4$,

(iii) $m \geq (s-r)p+2$ *if* $k = 2(sp+r)(p-1)-2$ *and* $2 \leq r < p-1$,

(iv) $m \geq (s-r)p+1$ *if* $k = 2(sp+r)(p-1)-3$ *and* $2 \leq r < p-1$,

(v) $m \geq (s-r-1)p-1$ *if* $k = 2(sp+r)(p-1)-4$ *and* $2 \leq r < p-1$.

By Lemma 6.1, Theorem 10.4, Theorem 10.7, Corollary 12.3

and Lemma 12.4, the subgroup $\sum_{t=1}^{3} U_t(m, k)$ of $\pi_{2m+1+k}(S^{2m+1} : p)$ is a direct summand under the above conditions. We have also that the subgroup $U_4'(m, k) \approx Z_p$ is a direct summand by use of Theorem 10.8 and Lemma 6.1, (iii). Then the method to prove the above theorem is similar to that of Corollary 6.4 in Section 6, and the details are left to the readers.

We finish this paper with the following two remarks on the above theorem. If we can prove the existence of $U_4(s-r-1, \beta_{r+1})$, as a generalization of Theorem 10.8, then the conditions (iii), (iv) and (v) can be removed replacing $U_4'(m, k)$ by $U_4(m, k)$. The condition (ii) may be weakened until $m \geq [(k+4)/2(p-1)] - p^2 + 1$ if $u_3(s-p,\ )$ and $\bar{u}_3(s-p,\ )$ have no influence over the group $\pi_{2m+1+k}(S^{2m+1} : p)$.

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