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Characterizations of canonical differentials

Dedicated to Prof. K. NOSHIRO on his 60th birthday

By

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Introduction

Under the systematical study of Abelian differentials on open Riemann surfaces Ahlfors introduced the notion of distinguished differentials and obtained Abel's theorem ([1~3]). On the other hand, by generalizing the normalized potentials (R. Nevanlinna [10]) Kusunoki [6] defined the (semiexact) canonical differentials and developed the theory of Abelian integrals on open Riemann surfaces ([5~8]). Meanwhile, M. Mori [9] pointed out that these two classes of differentials are essentially the same, more precisely, a distinguished (real) harmonic differential is the real part of a semiexact canonical differential and vice versa.

In the present paper we shall give further some characteristic properties of canonical differentials. First, in §2 we shall show the following characterization: let φ be a semiexact meromorphic differential on open Riemann surface R, then φ is a semiexact canonical differential if and only if (i) there is a compact set F on R such that $du=Re\,\varphi$ is exact on $R-F$ and $\|du\|_{R-F}<\infty$ (ii) for any regular compact region $K(\supset F)$ and any semiexact analytic differential $dU+i^*dU$ with a finite norm on $R-K$, we have

$$\langle du, dU\rangle_{R-K} = \int_{\partial K} u^*dU$$

This definition of canonical differentials is superficially quite different from the original one and that of distinguished differentials, in the sense that last two definitions express rather constructively the form of their elements. Moreover this enables us to give a new interpretation for canonical potentials which are thought to have constant values on the ideal boundary. Actually we shall prove in §3 that every canonical potential on R has a constant value quasieverywhere on each connected component of the Kuramochi boundary Δ of R (Theorem 2). Finally we shall discuss about the converse of this theorem.

§1. Preliminaries

1. *Normalized solution of Dirichlet problem.* Let R be an open Riemann surface and G be a non-compact subregion of R, whose relative boundary Γ consists of a finite number of Jordan closed curves. For the simplicity we assume in the sequel that Γ is analytic.

Proposition 1. *Let u be a function which is harmonic on G and continuous on $G \cup \Gamma$. Then the following statements are equivalent:*

1) *u can be expressed as*

$$u(p) = \int_{\Gamma} u(q) d\omega(q, p), \qquad p \in G$$

where $d\omega$ is the harmonic measure on G with respect to the arc element of Γ.

2) *Let $\{R_n\}$ be a regular exhaustion of R and u_n be harmonic functions on $R_n \cap G$ such that $u_n = u$ on Γ, $u_n = 0$ on $\partial R_n \cap G$, then we have*

$$u(p) = \lim_{n \to \infty} u_n(p), \qquad p \in G$$

where the convergence is uniform on every compact set on G.

3) *Let H_u^G $(= \underline{H}_u^G = \overline{H}_u^G)$ be the solution of Dirichlet problem on G by Perron-Brelot's method for the boundary function which is $= u$ on Γ and $= 0$ at the ideal boundary of R, then we have*

$$u(p) = H_u^G(p), \qquad p \in G$$

Such a function u is called a *normalized solution* (of Dirichlet problem) on G.

The proof is omitted.

Proposition 1 shows that the definitions of normalized solution by R. Nevanlinna [10] (p. 320) and Constantinescu-Cornea [4]

(p. 21) are identical. It is known ([10]) that a normalized solution u on G is bounded ($\leq \max_{\Gamma}|u|$) on G and has a finite Dirichlet norm on G:

$$||du||_G^2 = \langle du, du \rangle_G = \int_{\Gamma} u^* du$$

Let $\hat{u}$ be a smooth (e.g. C^∞) function on R such that $\hat{u}=u$ on G and $\hat{u}$ vanishes near the ideal boundary of R belonging to $R-G$, then it is easily seen that $\hat{u}$ is a Dirichlet potential ([4]) on R.

2. *Canonical potentials and canonical differentials.* Let B be a regular canonical region (Ahlfors-Sario [3]) on R and T_B the set of harmonic functions u on R (which may have a finite number of singularities and additive periods in B) such that on each component G_i of $R-B$, the function $u-c_i$ with some constant c_i is a normalized solution on G_i. Then the set $T = \bigcup_B T_B$ is a real vector space and can be written as

$$T = T_0 + T_1, \quad \text{where} \quad T_0 = \{u \in T; u \in HD(R)\}.$$

The $HD(R)$ is a Hilbert space of single-valued harmonic functions with finite Dirichlet norms on R, where two functions with constant difference are identified. Let $\bar{T}_0$ be the completion of T_0 in $HD(R)$, and denote

$$\tilde{T} = \bar{T}_0 + T_1.$$

We call $u \in \tilde{T}$ a *canonical potential* on R. A meromorphic differential φ on R is called a *canonical differential* if $Re\int\varphi$ is a canonical potential on R and the sum of residues of φ vanishes. Note that the condition for residues is automatically satisfied if φ is semiexact, i.e. φ has no periods along every dividing cycle on R.

3. Let R be an open Riemann surface and R^* the Kuramochi compactification of R. We denote by $\Delta = R^* - R$ the Kuramochi boundary of R. For each ideal boundary point e of the Kerékjártó-Stoilow compactification of R we set

$$\Delta_e = \bigcap_{U} \overline{U \cap R}$$

where U represent the neighborhoods of e and the closure is taken on R^*. Then we know (cf. [4]) that Δ_e are mutually disjoint closed connected sets on R^* and

$$\Delta = \bigcup_e \Delta_e$$

§ 2. Characterizatoins of canonical differentials

4. For the canonical differentials on an arbitrary open Riemann surface R we have the following characterizations.

THEOREM 1. *Let $\varphi = du + i{}^*du$ be a meromorphic differential on R which may have a finite number of singularities, then the following three statements are equivalent:*

(I) *φ is a semiexact canonical differential on R.*

(II) *$du = Re\,\varphi$ is a distinguished (real) harmonic differential ([3]) on R.*

(III) 1°) *φ is semiexact on R.*

There exists a compact set F on R such that

2°) *$u \in HD(R-F)$, i.e. du is (real) harmonic, exact on $R-F$ and $\|du\|_{R-F} < \infty$.*

3°) *For any regular region $G(\supset F)$ with analytic boundary Γ and any function $U \in HD(R-G)$ such that $\int_\gamma {}^*dU = 0$ for every dividing curve γ on $R-G$, we have*

$$\langle du, dU\rangle_{R-G} = \int_{R-G} du \wedge {}^*dU = \int_\Gamma u\,{}^*dU\,. \tag{4.1}$$

The property (4.1) is obviously equivalent with that

$$\lim_{n\to\infty} \int_{\partial R_n} u\,{}^*dU = 0 \tag{4.2}$$

for any regular exhaustion $\{R_n\}$ of R.

We shall denote by $\mathfrak{R} = \mathfrak{R}(R)$ the real vector space of semiexact canonical differentials on R.

PROOF. (I)$\Leftrightarrow$(II): M. Mori [9] Theorem 1.

(I)$\Rightarrow$(III): For a canonical potential $u \in \tilde{T}$ on R (4.2) was proved by Kusunoki [6] Lemma 4. The semiexactness of φ is unnecessary.

(III)$\Rightarrow$(I): The sum of residues of φ vanishes on account of 1°). Hence we can construct a semiexact canonical differential $\omega_\varphi \in \mathfrak{R}(R)$ having on F the same singularities and real periods as φ, i.e. ω_φ is given by a finite linear combination of semiexact canonical differentials $\varphi^*_{A_n}$, $\varphi^*_{B_n}$ (of the first kind), $\psi^{(\mu)*}_P$, $\tilde{\psi}^{(\mu)*}_P$ (second kind) and ϕ^*_{PQ}, $\tilde{\phi}^*_{PQ}$ (third kind) (cf. [6] Theorems 1 and 5). Let

$$dv = Re(\varphi - \omega_\varphi) \tag{4.3}$$

then $dv + i^*dv$ is semiexact and $v \in HD(R)$. While (4.1) holds for $du = Re\,\varphi$ and $Re\,\omega_\varphi$, hence for $du = dv$. It follows by (4.2) that

$$\|dv\|^2_{R_n} = \langle dv, dv\rangle_{R_n} = \int_{\partial R_n} v^*dv \to 0 \qquad (n \to \infty)$$

which implies $dv = 0$, hence $\varphi = \omega_\varphi \in \mathfrak{R}(R)$.

The definition of canonical differentials by (III) is superficially quite different from the original one and that of distinguished differentials, in the sense that last two definitions express rather constructively the form of their elements. (cf. [11])

5. For particular classes of Riemann surfaces the definition of semiexact canonical differentials becomes very simple. Namely,

(a) If $R \in O_{KD}$ (Sario's class), a meromorphic differential φ belongs to $\mathfrak{R}$ if and only if the properties

(III) 1°) and 2°)

are fulfiled ([6] p. 251). Moreover then $i\varphi \in \mathfrak{R}$ ($\Gamma_{he} \cap \Gamma_{hse}{}^* \subset \Gamma_{he}{}^*$, if $R \in O_{KD}$. cf. ([9]) i.e. the space $\mathfrak{R}$ is a complex vector space and O_{KD} is the largest class of Riemann surface where $\mathfrak{R}$ becomes so.

(b) If $R \in O_{HD}$, φ with vanishing sum of residues belongs to $\mathfrak{R}$ (cf. [5] Lemma 1) if and only if the condition (III) 2°) is satisfied.

(c) If $R \in O_G$, $\varphi \in \mathfrak{R}$ if and only if a property

(III) 2°)

holds.

6. By $\mathfrak{R}_0 = \mathfrak{R}_0(R)$ we denote the space of *exact* canonical differentials on R. $\mathfrak{R}_0 \subset \mathfrak{R}$. Let f be a meromorphic function on R

such that $df \in \mathfrak{K}_0$ and $q(<\infty)$ be the number of poles (counted with multiplicities) of f, then it is known ([6]) that f is at most q-valent on R if the genus of R is finite, moreover that if f is regular everywhere on any R, then f reduces to a constant.

Here we note further the following property.

PROPOSITION 2. *Let R be an any open Riemann surface and $df \in \mathfrak{K}_0$. Let q be the number of poles (counted with multiplicities) of f on R. If $w=f$ is at most q-valent on R, then the projection E on the w-plane of the boundary of covering surface $S=f(R)$ has area zero, and S is exactly of q-sheeted over the complement of E.*

PROOF. From our assumption the set E is compact. Let $D(\supset E)$ be an open set with (piecewise) analytic boundaries. We may assume that the boundary ∂D does not contain any branch points of S and has a positive finite distance from E. Let D_i be a connected component of S over D and $G_i = f^{-1}(D_i)$. G_i is a domain (non-compact or compact) on R and ∂G_i consists of analytic curves which separate the poles of f from the ideal boundary of R provided that G_i is non-compact. Since $df = du + idv$ belongs to $\mathfrak{K}_0$, we have by (4.1) for each i

$$\int_{G_i} |f'|^2 dxdy = \|du\|^2_{G_i} = \int_{\partial G_i} udv = \int_{c_i} udv$$

where c_i is the boundary of D_i over ∂D. The first term gives the area of D_i and the last line integral is equal to $s_i \times$(area of D), s_i being the maximum number of sheets of D_i. This implies that the area of E is zero. The remaing part of our claim is trivial.

§ 3. Canonical potentials and Kuramochi boundary

In the sequel we are much indebted to Constantinescu-Cornea [4].

7. First we shall prove the following

THEOREM 2. *Let u be a canonical potential on R, then u has a constant value quasi everywhere on each component Δ_e of the Kuramochi boundary Δ of R.*

This is a consequence from the following slightly general result.

THEOREM 2$'$. *Let $u \in HD(R-F)$ (F: compact set) such that for any regular region $G(\supset F)$ with analytic boundary Γ and any function $U \in HD(R-G)$ such that $\int_\gamma {}^*dU = 0$ for every dividing curve γ on $R-G$, we have*

$$\langle du, dU \rangle_{R-G} = \int_\Gamma u^*dU . \tag{7.1}$$

Then u has a quasicontinuous extension onto Δ so that the extended u is a constant quasi everywhere ("quasi überall") on each Δ_e.

PROOF. We may assume that R is hyperbolic, otherwise the conclusion is trivial. Now we extend the function u onto G as a C^∞-function on R and denote the extended function by $\hat{u}$. Since $\hat{u}$ is a Dirichlet function on R it has a quasicontinuous extension onto Δ ([4] p. 191). Suppose that $u=\hat{u}$ is not a constant quasi everywhere on Δ_e, then there exist two closed subsets E_1 and E_2 of Δ_e such that both are of positive capacity and

$$\inf_{E_2} u > \sup_{E_1} u .$$

And there exist two measures $\mu_i (i=1, 2)$ on E_i with finite energy and $\mu_i(E_i)=1$. Since $\mu=\mu_2-\mu_1$ becomes a signed measure on Δ with total measure zero, there exists a function $v \in HD(R)$ which has the (generalized) normal derivative μ on Δ, that is,

$$\langle dv, df \rangle = \int_\Delta f d\mu$$

for any Dirichlet function f on R. Hence if we take $f=\hat{u}$ we have

$$\langle dv, d\hat{u} \rangle = \int_\Delta u d\mu = \int_\Delta u d\mu_2 - \int_\Delta u d\mu_1$$
$$\geq \inf_{E_2} u - \sup_{E_1} u > 0 .$$

While, we can prove contrary that

$$\langle dv, d\hat{u} \rangle = 0 . \tag{7.2}$$

To see this we first show that

$$\int_\gamma {}^*dv = 0 \quad \text{for every dividing curve } \gamma . \tag{7.3}$$

Since $v \in HD(R)$ has the normal derivative μ on Δ with $\mu(\Delta)=0$, v can be written as

$$(7.4) \qquad 2\pi(v-v^{K_0}) = \tilde{p}^{\mu} = \int \tilde{g}_a d\mu$$

with a closed disc K_0 on R ([4] p. 218). ($\tilde{g}_a$ is an N-Green function in Kuramochi's terminology). We may assume $K_0 \cap \gamma = \phi$. Take a compact region $B \supset K_0 \cup \gamma$ such that ∂B consists of analytic curves and one component γ' of ∂B is homologous to the dividing curve γ. Let R_1 be a non-compact region on $R-B$ whose relative boundary is γ', and $R_2 = R-(R_1 \cup \gamma')$. Let h be a C^∞-function on R such that $h=1$ on $R_1 \cup \gamma'$ and $h=0$ on $K_0 \cup (R_2 - B)$. Then by Green formula we have

$$(7.5) \qquad \int_\gamma {}^*dv^{K_0} = \int_{\gamma'} {}^*dv^{K_0} = \int_{\partial(B-K_0)} h^*dv^{K_0} = \langle dh, dv^{K_0}\rangle_{B-K_0}$$
$$= \langle dh, dv^{K_0}\rangle = 0\,.$$

Moreover, since

$$\int_\gamma {}^*d\tilde{g}_a = c_\gamma = \begin{cases} 2\pi\,, & \text{if } \gamma \text{ separates } K_0 \text{ from } \Delta_e \\ 0\,, & \text{otherwise} \end{cases}$$

we have

$$(7.6) \qquad \int_\gamma {}^*d\tilde{p}^{\mu} = \int_\Delta \left(\int_\gamma {}^*d\tilde{g}_a\right) d\mu = c_\gamma \int_\Delta d\mu = 0\,.$$

The conclusion (7.6) can also be obtained immediately from Hilfssatz 17.12 [4]. Hence (7.3) holds by (7.4$\sim$6), i.e. $dv + i^*dv$ is semiexact on R. It follows by (7.1) that

$$\langle du, dv\rangle_{R-G} = \int_\Gamma u^*dv$$

While, by Green's formula

$$\langle d\hat{u}, dv\rangle_G = -\int_\Gamma \hat{u}^*dv = -\int_\Gamma u^*dv\,.$$

It follows that

$$\langle dv, d\hat{u}\rangle = \langle d\hat{u}, dv\rangle_G + \langle du, dv\rangle_{R-G} = 0$$

which completes the proof.

8. Here we note on the boundary values on Δ of special canonical potentials. Let γ be a dividing curve which divides R into R_1 and R_2. Let $\Delta^i = \Delta \cap \bar{R}_i (i=1, 2)$ where the closure is taken on

R^*, then the generalized harmonic measure ω_γ associated with γ is a canonical potential ([6], [10]). Suppose $\omega_\gamma \not\equiv$ const. and $\inf_{R_1} \omega_\gamma = 0$. Let $\hat{\omega}_\gamma$ be a C^∞-function on R such that $\hat{\omega}_\gamma = \omega_\gamma$ on R_1 and $\hat{\omega}_\gamma$ vanishes near the ideal boundary Δ^2. Then $\hat{\omega}_\gamma$ is a Dirichlet potential (sec. 2) on R, hence $\hat{\omega}_\gamma = \omega_\gamma = 0$ on Δ^1, q.e. (=quasi everywhere). By considering $1-\omega_\gamma$ we know analogously that $\omega_\gamma = 1$ q.e. on Δ^2.

9. Finally in this section we shall study on the converse of Theorem 2.

PROPOSITION 3. *Let $\varphi = du + i^*du$ be a semiexact meromorphic differential on R such that (α) $u \in HD(R-F)$ with a compact set F on R (β) u has a constant value quasieverywhere on each component Δ_e of the Kuramochi boundary Δ of R, then φ can be written as*

$$\varphi = \omega + \psi : \ \omega \in \mathfrak{K}, \quad \psi \in \mathfrak{C}$$

where $\mathfrak{C} = \{du\,;\ u \in KD(R)$ and u has a property (β) on $\Delta\}$

$$KD = KD(R) = \{u \in HD(R)\,;\ {}^*du \text{ is semiexact on } R\}$$

This is a direct consequence of Theorem 2 if we take $\omega = \omega_\varphi$ (cf. (4.3)).

Thus together with Theorem 2 we know that the properties (α) and (β) characterize the semiexact canonical differentials provided that $\mathfrak{C}$ is empty. $\mathfrak{C}$ vanishes, of course if $R \in O_{KD}$, but I don't know whether it is true for the general case. In the following we shall give a sufficient condition for vanishing of $\mathfrak{C}$. Let

$$\mathfrak{A} = \{du\,;\ u \in HD(R) \text{ and } u \text{ has a normal derivative on } \Delta\}\,.$$

The class $\mathfrak{A}$ is known to be a dense subset of $d\ HD$ ([4] p. 220).

PROPOSITION 4. *The class $\mathfrak{C}$ vanishes if and only if $\mathfrak{B} = \mathfrak{A} \cap dKD$ is dense in dKD. More precisely, we have an orthogonal decomposition*

$$dKD = [\mathfrak{B}] \oplus \mathfrak{C}$$

, $[\mathfrak{B}]$ being the closure of $\mathfrak{B}$ in dKD.

PROOF. That any element belonging to the orthogonal complement $\mathfrak{B}^\perp$ in $d\,KD$ of $\mathfrak{B}$ possesses a property (β) can be seen from the proof of Theorem 2. Hence $\mathfrak{B}^{\perp}=[\mathfrak{B}]^{\perp}\subset\mathfrak{C}$. So it suffices to show that any $dv\in[\mathfrak{B}]\cap\mathfrak{C}$ is identically zero. Let dv_0 be any element of $\mathfrak{B}$ and μ_0 be the normal derivative of v_0 on Δ.

$$\langle dv, dv_0\rangle = \int_{\Delta} v d\mu_0 . \tag{9.1}$$

We claim that the integral on the right hand side vanishes. Let $v=v^{+}-v^{-}$ on Δ ($v^{+}, v^{-}\geq 0$ q.e.). Then it suffices to prove that

$$\int_{\Delta} v^{M} d\mu_0 = 0 \tag{9.2}$$

for $v^{M}=\min(v^{+}, M)$, $\min(v^{-}, M)$ (M: positive constant). Note that $v^{M}=$const., say $\alpha(e)$, on each Δ_e quasi everywhere. $0\leq\alpha(e)\leq M$. Let

$$E_n = \left\{\Delta_e\subset\Delta\,;\,|\mu_0(\Delta_e)|\geq\frac{1}{n}\right\} \qquad \{n = 1, 2, \cdots)$$

Each E_n is a finite set and

$$\mu_0(\Delta_e) = 0 \quad \text{for any } \Delta_e\in\Delta-\bigcup_n E_n$$

Let $E_n=\Delta_{e_1}\cup\Delta_{e_2}\cup\cdots\cup\Delta_{e_\nu}$, then there exists a canonical region R_n on R such that $\partial R_n=\gamma_1\cup\gamma_2\cup\cdots\cup\gamma_p$ separates $\Delta_{e_1},\cdots,\Delta_{e_\nu}$ mutually on R^*. Suppose ∂R_n divide Δ so that

$$\Delta = \Delta^1\cup\Delta^2\cup\cdots\cup\Delta^p \qquad (p\geq\nu)$$

and each Δ^j contains Δ_{e_j} ($j=1,\cdots,\nu$). By a linear combination of $\omega_{\gamma_1},\cdots,\omega_{\gamma_p}$ we can construct a canonical potential $\omega_n\in T_0$ (cf. see. 8) that

$$\omega_n = \begin{cases} \alpha(e_j) & \text{q.e. on } \Delta^j \ (j=1,\cdots,\nu) \\ 0 & \text{q.e. on } \Delta^k \ (k=\nu+1,\cdots,p) \end{cases}$$

Then $0\leq\omega_n\leq M$ and $\omega_n\to v^M (n\to\infty)$ on Δ except a set of μ_0-measure zero. Since $v_0\in KD(R)$ and $\omega_n\in T_0$,

$$0 = \langle d\omega_n, dv_0\rangle = \int_{\Delta}\omega_n d\mu_0 .$$

For $n\to\infty$ we have (9.2) by Lebesgue's bounded convergence theorem. Thus $\langle dv, dv_0\rangle=0$.

Now since $dv\in[\mathfrak{B}]$, there exists a sequence $\{dv_m\}$ such that

$$||dv_m - dv|| \to 0\,, \qquad dv_m \in \mathfrak{B}\,.$$

By what already proved we have $\langle dv_m, dv\rangle = 0$. Hence

$$\langle dv, dv\rangle = \lim_{m\to\infty} \langle dv_m, dv\rangle = 0$$

i.e. $dv \equiv 0$, which completes the proof.

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