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# Rational sections and Chern classes of vector bundles*

by

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Let $\mathcal{E}$ be a quasi-coherent sheaf, of finite type, on an integral prescheme $X$, and denote by $\mathbf{V}(\mathcal{E})$, $\mathbf{P}(\mathcal{E})$ the vector and projective fibres of $\mathcal{E}$ respectively. Then each non-zero rational section $\omega$ of $\mathbf{V}(\mathcal{E})$ over $X$ defines a rational section $\bar{\omega}$ of $\mathbf{P}(\mathcal{E})$ over $X$ (section 2), and we can construct a closed subscheme $\langle\omega\rangle$ of $X$ whose points are the non-regular points of $\bar{\omega}$ (Prop. 5). Denote by $[\omega]$ the $X$-prescheme obtained by blowing up centered at $\langle\omega\rangle$. On the other hand we can construct a quasi-coherent fractional Ideal $\mathcal{O}_X(\omega)$ of the sheaf of rational functions $\mathcal{R}(X)$ of $X$ which is invertible when $X$ is *UFD* (Cor. of Prop. 4) and which corresponds to the Cartier divisor of the rational section $\omega$.

In this note, we shall prove some relations between these schemes or sheaves (Th. 1. 2). In the case that $X$ is a non-singular quasi-projective algebraic scheme, they give an explicite formula of Chern classes of vector bundles of rank 2 (Cor. of Th. 2′). And, as a special case, if $X$ is a surface and $\mathbf{V}(\mathcal{E})$ is the bundles of simple differentials, then our formula proves that the Severi-series of $X$ coincides with the second Chern class $c_2(X)$ of $X$ (last Remark).

**1. Rational maps and rational functions** (EGA. I. 7) Let $X$ and $Y$ be $S$-preschemes, and $\mathfrak{U}_X$ the set of dense open subsets of $X$; then the family of sets of $S$-mophisms $(\mathrm{Hom}_S(U, Y))_{U\in\mathfrak{U}_X}$

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forms an inductive system (with natural restriction of morphisms), and each element of the set $\mathrm{Rat}_S(X, Y) = \varinjlim_{U \in \mathfrak{U}_X} \mathrm{Hom}_S(U, Y)$ is called a *rational map* from $X$ to $Y$ over $S$ (or a *rational S-map* form $X$ to $Y$). We shall call the rational $S$-maps from $S$ to $X$ the *rational sections* of $S$-prescheme $X$: $\mathrm{Rat}_S(S, X) = \Gamma_{rat}(X/S)$. Let $\mathcal{F}$ be a sheaf (of sets) on a prescheme $X$, for each open subset $U$ of $X$, put $\Gamma_{rat}(U, \mathcal{F}) = \varinjlim_{V \in \mathfrak{U}_U} \Gamma(V, \mathcal{F})$; each element of $\Gamma_{rat}(U, \mathcal{F})$ is called a *rational section* of $\mathcal{F}$ on $U$. It is easy to see that, for tow open subsets $U$ and $V$ of $X$, if $V \subset U$ and $V$ is dense in $U$, then $\Gamma_{rat}(U, \mathcal{F}) = \Gamma_{rat}(V, \mathcal{F})$, and that, if $U$ is irreducible, then $\Gamma_{rat}(U, \mathcal{F})$ is nothing but the stalk at the generic point $x$ of $U$. In case of $\mathcal{F} = \mathcal{O}_X$, the structure sheaf of $X$, the rational sections of $\mathcal{O}_X$ on $U$ are called the *rational functions* of $X$ on $U$, and we denote $R(U) = \Gamma_{rat}(U, \mathcal{O}_X)$. The sheaf associated with the presheaf $U \rightsquigarrow R(U)$ is called the *sheaf of rational functions* on $X$ and we denote it $\mathcal{R}(X)$. The canonical map $\Gamma(U, \mathcal{O}_X) \to R(U)$ defines the canonical homomorphism $\iota: \mathcal{O}_X \to \mathcal{R}(X)$, and, by means of it, $\mathcal{R}(X)$ is considered as an $\mathcal{O}_X$-Algebra.

Let $\mathcal{F}$ be an $\mathcal{O}_X$-Module, $U$ a dense open subset of $X$ and $f$: $\mathcal{F} \mid U \to \mathcal{O}_X \mid U$ an $(\mathcal{O}_X \mid U)$-homomorphism. Then, for each open subset $W$ of $X$, consider the following $\Gamma(W, \mathcal{O}_X)$-homomorphism obtained as the composition map:

$$(1) \quad \bar{f}(W): \Gamma(W, \mathcal{F}) \xrightarrow{\text{rest.}} \Gamma(W \cap U, \mathcal{F}) \xrightarrow{f(W \cap U)} \Gamma(W \cap U, \mathcal{O}_X) \xrightarrow{\iota(W \cap U)} \Gamma(W \cap U, \mathcal{R}(X)) \xrightarrow{(\text{rest.})^{-1}} \Gamma(W, \mathcal{R}(X))$$

(note that $W \cap U$ is dense in $W$, hence the restriction $\Gamma(W, \mathcal{R}(X)) \to \Gamma(W \cap U, \mathcal{R}(X))$ is an isomorphism). Obviously $\bar{f}(W)$ commutes with the restriction maps of the sections of $\mathcal{F}$ and $\mathcal{R}(X)$, hence the collection $(\bar{f}(W))_{W \subset X}$ gives an $\mathcal{O}_X$-homomorphism $\bar{f}: \mathcal{F} \to \mathcal{R}(X)$, and, thus, we get a map

$$\alpha_U: \mathrm{Hom}_{\mathcal{O}_X \mid U}(\mathcal{F} \mid U, \mathcal{O}_X \mid U) = \Gamma(U, \check{\mathcal{F}}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{R}(X)),$$

($\check{\mathcal{F}} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$, the dual of the $\mathcal{O}_X$-Module $\mathcal{F}$). Moreover, it is clear that $\alpha_U$ is a $\Gamma(X, \mathcal{O}_X)$-homomorphism and commutes with the restriction map $\alpha_V^U$: $\Gamma(U, \check{\mathcal{F}}) \to \Gamma(V, \check{\mathcal{F}})$ $(U, V \in \mathfrak{U}_X, U \supset V)$ $: \alpha_U = \alpha_V . \alpha_V^U$. Therefore, passing to the inductive limit, we have the canonical $\Gamma(X, \mathcal{R}(X))$-homomorphism

$$\alpha: \Gamma_{\mathrm{rat}}(X, \check{\mathcal{F}}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{R}(X)).$$

The following proposition is well known (EGA. I. 7. 3).

**Proposition 1.** *Let $X$ be an integral (i.e. reduced and irreducible) prescheme. Then,* (i) *$\mathcal{R}(X)$ is a quasi-coherent $\mathcal{O}_X$-Module,* (ii) *$\mathcal{R}(X)$ is a constant sheaf, hence $\Gamma(U, \mathcal{R}(X)) = R(U) = R(X)$, for each open subset $U$ of $X$,* (iii) *the canonical homomorphism $\iota: \mathcal{O}_X \to \mathcal{R}(X)$ is injective,* (iv), *for each point $x$ of $X$, $\mathcal{R}(X)_x = R(X)$ is the quotient field of $\mathcal{O}_{X,x}$, and at the generic point $\bar{x}$, $\mathcal{R}(X)_x = R(X) = \mathcal{O}_{X,\bar{x}}$, and* (v), *for any quasi-coherent $\mathcal{O}_X$-Module $\mathcal{F}$, $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{R}(X) = R(X)^{(I)}$* (direct sum).

**Corollary.** *If $X$ is integral, then, for each $\mathcal{O}_X$-Module $\mathcal{F}$ of finite type, the canonical homomorphism $\alpha: \Gamma_{\mathrm{rat}}(X, \check{\mathcal{F}}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{R}(X))$ is injective. Moreover, if $\mathcal{F}$ is quasi-coherent, then $\alpha$ is an isomorphism.*

*Proof*) Since $X$ is irreducible, $\Gamma_{\mathrm{rat}}(X, \check{\mathcal{F}}) = \check{\mathcal{F}}_{\bar{x}} = \mathrm{Hom}_{\mathcal{O}_{X,\bar{x}}}(\mathcal{F}_{\bar{x}}, \mathcal{O}_{X,\bar{x}})$, where $\bar{x}$ is the generic point of $X$. Since $\mathcal{R}(X)_{\bar{x}} = \mathcal{O}_{X,\bar{x}}$ (Prop. 1 (iv)), by the definition of $\alpha$, it is easy to check that the composition map

$$\Gamma_{\mathrm{rat}}(X, \check{\mathcal{F}}) = \check{\mathcal{F}}_{\bar{x}} \xrightarrow{\alpha} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{R}(X)) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X,\bar{x}}}(\mathcal{F}_{\bar{x}}, \mathcal{R}(X)_{\bar{x}}) = \check{\mathcal{F}}_{\bar{x}}$$

is the identity map, where the last arrow is the map which corresponds each sheaf homomorphism $f$ to its restriction $f_{\bar{x}}$ at the generic point $\bar{x}$. Hence $\alpha$ is injective. Moreover, assume that $\mathcal{F}$ is quasi-coherent. When that is so, in order to prove that $\alpha$ is surjective, it is sufficient to prove that the last arrow is injective, i.e., for $f \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{R}(X))$, $f_{\bar{x}} = 0$ implies $f = 0$. To show this, we may assume $X$ to be affine. Let $X=\mathrm{Spec}(A)$, $\mathcal{F}=\widetilde{M}$, $M$ is an A-module; then $A$ is integral and $\mathcal{R}(X)$ is the shef associated with the quotient field $K$ of $A$, and $f:\mathcal{F}\to\mathcal{R}(X)$ corresponds to an $A$-homomorphism $\varphi: M\to K$. But, by tensoring $K$, $\varphi$ can be decomposed into $M\xrightarrow{u} M\otimes_A K\xrightarrow{v=\varphi\otimes 1} K$ and $v$ is exactly the same to $f_x:\mathcal{F}_x\to\mathcal{O}_{X,x}=K$, hence, $f_x=v=0$ implies $\varphi=v.u=0$.

## 2. Rational sections of vector- and projective fibres.

Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_X$-Module of finite type and denote by $\mathbf{S}(\mathcal{E})$ the symmetric $\mathcal{O}_X$-Algebra of $\mathcal{E}$ (EGA. II. 1. 7. 4). And put $\mathbf{V}(\mathcal{E})=\mathrm{Spec}\,(\mathbf{S}(\mathcal{E}))$ (resp. $\mathbf{P}(\mathcal{E})=\mathrm{Proj}\,(\mathbf{S}(\mathcal{E}))$); $\mathbf{V}(\mathcal{E})$ (resp. $\mathbf{P}(\mathcal{E})$) is called the *vector* (resp. *projective*) *fibre* over $X$ defined by $\mathcal{E}$ (EGA. II. 1. 7. 8, 4. 1. 1).

**Proposition 2.** *Let $X$ be a prescheme. For each quasi-coherent $\mathcal{O}_X$-Module $\mathcal{E}$ of finite type,* (i) *we have a canonical isomorphism*

$$\Gamma_{\mathrm{rat}}(\mathbf{V}(\mathcal{E})/X)\xrightarrow{\sim}\Gamma_{\mathrm{rat}}(X,\check{\mathcal{E}}),$$

*and moreover* (ii), *if $X$ is integral, the canonical homomorphism $\iota:\check{\mathcal{E}}\to\check{\mathcal{E}}\otimes_{\mathcal{O}_X}\mathcal{R}(X)$ induces a canonical isomorphism*

$$\bar{\iota}:\Gamma_{\mathrm{rat}}(X,\check{\mathcal{E}})\xrightarrow{\sim}\Gamma(X,\check{\mathcal{E}}\otimes_{\mathcal{O}_X}\mathcal{R}(X)).$$

*Proof*) (i) $\Gamma_{\mathrm{rat}}(\mathbf{V}(\mathcal{E})/X)=\varinjlim \mathrm{Hom}_x(U,V(\mathcal{E}))\cong\varinjlim \mathrm{Hom}_{\mathcal{O}_X|U}(\mathcal{E}|U,\mathcal{O}_X|U)=\Gamma_{\mathrm{rat}}(X,\check{\mathcal{E}})$ (EGA. II. 1. 7. 8, 1. 7. 9). (ii) Assume $X$ to be integral. Since $\check{\mathcal{E}}\otimes_{\mathcal{O}_X}\mathcal{R}(X)$ is a constant sheaf (Prop. 1,(v)), for any pair of open subsets $U$, $V$ of $X$ such that $U\supset V$, we get a commutative diagram:

$$(2)\qquad \begin{array}{ccc} \Gamma(U,\check{\mathcal{E}}) & \xrightarrow{\iota(U)} & \Gamma(U,\check{\mathcal{E}}\otimes\mathcal{R}(X)) \\ \downarrow & & \downarrow\wr \\ \Gamma(V,\check{\mathcal{E}}) & \xrightarrow{\iota(V)} & \Gamma(V,\check{\mathcal{E}}\otimes\mathcal{R}(X)) \end{array} \quad \begin{array}{c}\xleftarrow{\sim}\\ \xleftarrow{\sim}\end{array} \Gamma(X,\check{\mathcal{E}}\otimes\mathcal{R}(X))$$

Passing to the direct limit, this defines our $\bar{\iota}$. Consider the following canonical $\mathcal{R}(X)$-homomorphism

$$\beta:\ \check{\mathcal{E}}\otimes_{\mathcal{O}_X}\mathcal{R}(X)=\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{O}_X)\otimes_{\mathcal{O}_X}\mathcal{R}(X)\to\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{R}(X)),$$

obtained by tensoring $\mathcal{R}(X)$ to the natural $\mathcal{O}_X$-homomorphism $\check{\mathcal{E}}=$

$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{R}(X))$. Taking the global sections, we get an $R(X)$-homomorphism

$$\beta(X): \Gamma(X, \check{\mathcal{E}} \otimes_{\mathcal{O}_X} \mathcal{R}(X)) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{R}(X)).$$

It is easy to see that $\alpha = \beta \cdot \bar{\iota}$, where $\alpha$ is the canonical homomorphism defined in the section 1. In our case $\alpha$ is an isomorphism (Cor. of Prop. 1), hence $\bar{\iota}$ is injective. Moreover $\bar{\iota}$ is surjective; in fact, for any $s \in \Gamma(X, \check{\mathcal{E}} \otimes \mathcal{R}(X))$, at the generic point $x$, $s_x \in (\check{\mathcal{E}} \otimes \mathcal{R}(X))_x = \check{\mathcal{E}}_x$, hence there exist an opuset $U$ and an $(\mathcal{O}_X | U)$-homomorphism $t$: $\mathcal{E} | U \to \mathcal{O}_X | U$ such that $t \cdot \iota(U) = (s | U)$ in $\Gamma(U, \check{\mathcal{E}} \otimes \mathcal{R}(X))$. Q. E.D.

*Remark.* In the above proof, we may replace $X$ by any open subset $U$ of $X$, hence $\beta(U)$: $\Gamma(U, \check{\mathcal{E}} \otimes \mathcal{R}(X)) \to \Gamma(U, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{R}(X)))$ is a $\Gamma(U, \mathcal{R}(X))$-isomorphism. Therefore we have the following

**Corollary.** *For any quasi-coherent $\mathcal{O}_X$-Module $\mathcal{E}$, of finite type, on a integral prescheme $X$, we have a canonical $\mathcal{R}(X)$-isomorphism*

$$\beta: \check{\mathcal{E}} \otimes_{\mathcal{O}_X} \mathcal{R}(X) \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{R}(X)).$$

3. Now we shall give some fundamental notions and notations needed for our study. From now on, we shall assume the base prescheme $X$ to be integral. Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_X$-Module of finite type; then, by Cor. of Prop. 1 and Prop. 2, we have canonical isomorphisms

$$\Gamma_{\mathrm{rat}}(\mathbf{V}(\mathcal{E})/X) \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{R}(X)) \simeq \Gamma(X, \check{\mathcal{E}} \otimes_{\mathcal{O}_X} \mathcal{R}(X)).$$

For each rational section $\omega \in \Gamma_{\mathrm{rat}}(\mathbf{V}(\mathcal{E})/X)$, we denote by $\omega_1^*$ and $\omega_2^*$ the images of $\omega$ under these isomorphisms in $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{R}(X))$ and in $\Gamma(X, \check{\mathcal{E}} \otimes \mathcal{R}(X))$, respectively. Now fix a rational section $\omega \in \Gamma_{\mathrm{rat}}(\mathbf{V}(\mathcal{E})/X)$. Then, the $\mathcal{O}_X$-homomorphism of $\mathcal{O}_X$-Modules $\omega_1^*$: $\mathcal{E} \to \mathcal{R}(X)$ can be uniquely extended to a homomorphism of graded $\mathcal{O}_X$-Algebras (of homogeneous degree 0)

$$\omega^*: \mathbf{S}(\mathcal{E}) \to \mathcal{R}(X)[T] = \mathcal{R}(X) \otimes_Z Z[T].$$

Put $I(\omega)=$ *Image of* $\omega_1^*$ and $J(\omega)=$ *Kernel of* $\omega^*$; then $I(\omega)$ is a quasi-coherent fractional Ideal of $\mathcal{R}(X)$, and *Image of* $\omega^*\cong\bigoplus_{n\geq 0}I(\omega)^n$. $J(\omega)=\bigoplus_{n\geq 0}J_n(\omega)$ is a quasi-coherent homogeneous of $\mathbf{S}(\mathcal{E})$, and it is generated by the component of degree 1: $J(\omega)=J_1(\omega)\cdot \mathbf{S}(\mathcal{E})$.

Thus, we have exact sequences

$$(3) \qquad 0\to J(\omega)\to \mathbf{S}(\mathcal{E})\to\bigoplus_{n\geq 0}I(\omega)^n\to 0,$$

and

$$(3') \qquad 0\to J_1(\omega)\to\mathcal{E}\to I(\omega)\to 0.$$

Put $[\omega]=\mathrm{Proj}\bigoplus_{n\geq 0}(I(\omega)^n)$; then $[\omega]$ is a closed subscheme of $\boldsymbol{P}(\mathcal{E})=\mathrm{Proj}(\boldsymbol{S}(\mathcal{E})$; $[\omega]\xrightarrow{i}\boldsymbol{P}(\mathcal{E})$, and it is the $X$-prescheme obtained by blowing up the fractional Ideal $I(\omega)$ of $\mathcal{R}(X)$ (EGA. II. 8. 1. 3), and the canonical projection $[\omega]\xrightarrow{i}\boldsymbol{P}(\mathcal{E})\xrightarrow{\pi}X$ is birational (EGA. II. 8. 1. 4). Note that $[\omega]$ is not empty if and only if $\omega_1^*\neq 0$, i.e., $\omega\neq 0$.

If $\omega\neq 0$, there exists an open subset $U$ of $X$ such that $\omega^*$ induces a homomorphism $\mathbf{S}(\mathcal{E})|U\to(\mathcal{O}_X|U)[T]$ (take a defining homomorphism $\mathcal{E}|U\to\mathcal{O}_X|U$ of $\omega_1^*$ (Cor. of Prop. 1) and extend it to $\mathbf{S}(\mathcal{E}|U)=\mathbf{S}(\mathcal{E})|U\to(\mathcal{O}_X|U)[T]$). This gives a *rational* map $U=\mathrm{Proj}((\mathcal{O}_X|U)[T])\to\mathrm{Proj}(\boldsymbol{S}(\mathcal{E})|U)\to\boldsymbol{P}(\mathcal{E})$ (cf. EGA. II. 2. 8. 1), hence a rational map

$$\bar{\omega}:X\to\boldsymbol{P}(\mathcal{E}).$$

By the definition of rational maps and the fact that $X$ is integral, this does not depend on the choice of $U$. And, since $\omega^*$ is an $\mathcal{O}_X$-homomorphism, the rational map $\bar{\omega}$ is a rational section of the projective fibre $\boldsymbol{P}(\mathcal{E})/X$, and, obviously, it can be decomposed into $X\to[\omega]\xrightarrow{i}\boldsymbol{P}(\mathcal{E})$. $\bar{\omega}$ is called the *induced section* of $\omega$ to the projective fibre $\boldsymbol{P}(\mathcal{E})/X$, and $[\omega]$ is called the *image* of $\bar{\omega}$ or the projective image of $\omega$. Hence we get a correspondence

$$-:[\Gamma_{\mathrm{rat}}(V(\mathcal{E})/X)]-\{0\}\to\Gamma_{\mathrm{rat}}(P(\mathcal{E})/X).$$

**Proposition 3.** *Let* $\bar{\omega}_0$: $U\to\mathbf{P}(\mathcal{E})$ *be an X-morphism which* *represents* $\bar{\omega}$, *then the closure of the image* $\bar{\omega}_{\mathfrak{I}}(U)$ *in* $\mathbf{P}(E)$ *coincides with* $[\omega]$.

*Proof*) Since $\pi\cdot i$: $[\omega]\to X$ is birational, $\bar{\omega}_{\mathfrak{I}}$: $U\to[\omega]$ is also birational, hence *the closure of* $\bar{\omega}_{\mathfrak{I}}(U)$ *in* $\mathbf{p}(\mathcal{E})=$*the closure of* $\bar{\omega}_{\mathfrak{I}}(U)$ *in* $[\omega]=[\omega]$. Q. E. D.

To each open subset $U$ of $X$, associate a subset $\mathcal{G}(U)$ of $\Gamma(U, \mathcal{R}(X))=R(X)$ consisting of the rational functions $f$ such that $f\cdot\Gamma(U, I(\omega))\subset\Gamma(U, \mathcal{O}_X)$. This correspondence $U$ to $\mathcal{G}(U)$ gives, with natural restrictions, a presheaf $\mathcal{G}$ of sub-$\mathcal{O}_X$-modules of $\mathcal{R}(X)$. The sheaf associated with $\mathcal{G}$ is called the *sheaf of* $\omega$ and denoted by $\mathcal{O}_X(\omega)$.

**Proposition 4.** (i) *The presheaf* $\mathcal{G}$ *is a sheaf, i. e.,* $\mathcal{G}=\mathcal{O}_X(\omega)$. (ii) *For each open subset* $U$ *of* $X$, $\mathcal{G}(U)=\Gamma(U, \mathcal{O}_X(\omega))$ *coincides with the set of rational functions* $f\in R(X)$ *such that* $f\cdot(\omega_2^*\,|\,U)$ $\in$*Image of* $(\Gamma(U, \check{\mathcal{E}})\to\Gamma(U, \check{\mathcal{E}}\otimes\mathcal{R}(X))$.

*Proof*) (i) Easy. (ii) By the isomorphism $\beta$: $\check{\mathcal{E}}\otimes\mathcal{R}(X)\stackrel{\sim}{\to}\mathcal{H}om(\mathcal{E}, \mathcal{R}(X))$ (Cor. of Prop. 2), $f\cdot(\omega_2^*\,|\,U)$ corresponds to $f\cdot(\omega_1^*\,|\,U)$, hence, by the commutative diagram

$$\begin{array}{ccc} \check{\mathcal{E}} & = & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \\ \downarrow & & \downarrow \\ \check{\mathcal{E}}\otimes_{\mathcal{O}_X}\mathcal{R}(X) & \xrightarrow[\sim]{\beta} & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{R}(X)), \end{array}$$

it is easy to see that $f(\omega_2^*\,|\,U)\in Im[\Gamma(U, \check{\mathcal{E}})\to\Gamma(U, \check{\mathcal{E}}\otimes\mathcal{R}(X))]$ if and only if $Im[f\cdot(\omega_1^*\,|\,U)]\subset\mathcal{O}_X\,|\,U$. On the other hand $Im[f\cdot(\omega_1^*\,|\,U)]$ $=f\cdot Im(\omega_1^*\,|\,U)=f\cdot(I(\omega)\,|\,U)$. This proves (ii). Q. E. D.

**Corollary.** *If, for each point* $x$ *of* $X$, $\mathcal{O}_{X,x}$ *is an unique factrization domain (in this case, we shall say that* $X$ *is UFD), then* $\mathcal{O}_X(\omega)$ *is an invertible sheaf on* $X$.

*Proof*) Let $x$ be a point of $X$. Then

$$\mathcal{O}_X(\omega)_x=\{f\in R(X) \text{ such that } f\cdot I(\omega)_x\subset\mathcal{O}_{X,x}\}.$$

Let $a_i\in R(X)$ $(i=1, \cdots, r)$ be a set of generators of $I(\omega)_x$ over $\mathcal{O}_{X,x}$.

Since $R(X)$ is the quotient field of $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,x}$ is an unique factorization domain, we may write $a_i=gc_i$ such that $g\in R(X)$, $c_i\in \mathcal{O}_{X,x}$ and $c_i$'s have no common factors in $\mathcal{O}_{X,x}$. Then, for $f\in R(X)$, $f$ is in $\mathcal{O}_X(\omega)_x$ if and only if $f\cdot g\cdot c_i$ is in $\mathcal{O}_{X,x}$ for every $i$. This proves that $\mathcal{O}_X(\omega)_x=(1/g)\mathcal{O}_{X,x}\cong\mathcal{O}_{X,x}$. Hence, $\mathcal{O}_X(\omega)$ is invertible. Q. E. D.

*Remark.* The fractional invertible Ideal $\mathcal{O}_X(\omega)$ of $\mathcal{R}(X)$ defines a Carier diviser $(\omega)$ on $X$, and $g$ (of the above proof) is its local equation at $x$ (cf. [7]). This $(\omega)$ is called the *divisor of the rational section* $\omega$.

From now on we shall assume that our integral prescheme $X$ is *UFD*. By definition, $\mathcal{O}_X(\omega)I(\omega)(\subset\mathcal{O}_X)$ is an quasi-coherent Ideal of $\mathcal{O}_X$, we denote it $\bar{I}(\omega)=\mathcal{O}_X(\omega)I(\omega)$. Then, since $\mathcal{O}_X(\omega)$ is invertible, we have a canoical isomorphism of $X$-preschemes (EGA. II. 3. 1. 8)

$$g:[\omega]_1=\mathrm{P}roj(\oplus_{n\geq 0}\bar{I}(\omega)^n)\ \tilde{\to}\ [\omega]=\mathrm{Proj}(\oplus_{n\geq 0}I(\omega)^n),$$

and (EGA. II. 3. 2. 10)

$$(4)\qquad g_*(\mathcal{O}_{[\omega]_1}(n))\cong\mathcal{O}_{[\omega]}(n)\otimes_{\mathcal{O}_X}\mathcal{O}_X(\omega)^n.$$

By means of $g$, we shall identify $[\omega]_1$ and $[\omega]$. Moreover, we shall denote by $\langle\omega\rangle$ the closed sub-prescheme of $X$ defined by the quasi-coherent Ieal $\bar{I}(\omega)$ of $\mathcal{O}_X$ ($\mathcal{O}_{<\omega>}=\mathcal{O}_X/\bar{I}(\omega)$), then $[\omega]=[\omega]_1$ is the $X$-prescheme obtained by the blowing up centered at $\langle\omega\rangle$.

**4. Some results.** We shall give here some relations among th sheaves and preschemes defined in the above section.

**Proposition 5.** *The underlying space of the closed sub-prescheme* $\langle\omega\rangle$ *of* $X$ *is the set of points of* $X$ *at which the rational section* $\bar{\omega}$ *is not defined*, i.e., $X-\langle\omega\rangle$ *is the domain of definition of* $\bar{\omega}$[1].

*Proof*) Since the question is local, we may assume that $X=$

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(1) We shall say that a rational map $f:X\to Y$ is defined at $x\in X$, if there exist an open nbd. $U$ of $x$ and a morphism $f_0:U\to Y$ which represents $f$, and the set of points of $X$ at which $f$ is defined is called the domain of definition of $f$.

Spec$(A)$ is affine and that $\mathcal{E}=\widetilde{E}$ is generated by its global section $E$ which is of finite type over $A$. Let $e_1, \cdots, e_n$ be a set of generators of $E$ over $A$ and put $\alpha_i=\omega_1^*(e_i)\in R(X)$ ($\omega_1^*: E\to R(X)$). Let $x$ be a point of $X$ and $\mathfrak{p}$ the corresponding prime ideal of $A$; and write $\alpha_i=g\cdot a_i$ where $g\in R(X)$, $a_i\in A$ and $a_i$'s have no common divisors in $A_{\mathfrak{p}}=\mathcal{O}_{X,x}$. Consider the following commutative diagram:

$$\begin{array}{ccl} E & \xrightarrow{\omega_1^*} & R(X) \\ {\scriptstyle \tau_1^*}\downarrow & \nearrow & \text{multiplication by } g, \\ A & & \end{array}$$

where $\tau_1^*$ is the $A$-homomorphism defined by $\tau_1^*(e_i)=a_i$. It can be extended to the following commutative diagram of graded $A$-algebras:

$$\begin{array}{ccl} \mathbf{S}_A(E) & \xrightarrow{\omega^*} & A[g\cdot T]\subset R(X)[T] \\ {\scriptstyle \tau^*}\downarrow & \nearrow {\scriptstyle \mu^*} & \\ A[T] & & \end{array}$$

and, passing to the associated projective fibres, we get the following:

$$\begin{array}{ccl} \boldsymbol{p}(E)=\mathrm{Proj}(\boldsymbol{S}_A(E)) & \xleftarrow{\bar{\omega}} & \mathrm{Proj}(A[g\,T]) \\ {\scriptstyle \tau}\downarrow & \swarrow {\scriptstyle \mu} & \\ \mathrm{Proj}(A[T])\mu, & & \end{array}$$

where $\bar{\omega}$, $\tau$ are rational maps. While Proj$(A[T])$ and Proj$(A[g\cdot T])$ can be canonically identified with $X$, and, by means of this identification, $\mu$ is the identity morphism of $X$(EGA. II. 3. 1. 7 and 3. 1. 8). Hence $\tau=\bar{\omega}$, in this sense. Now, since $I(\omega)_x=\mathcal{O}_X(\omega)_x=\Sigma_i a_i\mathcal{O}_{X,x}$ (Cf. Proof of Cor. of Prop. 4), we see that

$$x\in\langle\omega\rangle \Leftrightarrow \bar{I}(\omega)_x\neq\mathcal{O}_{X,x} \Leftrightarrow \text{ all } a_i\text{'s in } \mathfrak{p} \Leftrightarrow (\tau_1^*)^{-1}\mathfrak{p}=E$$
$$\Leftrightarrow \tau=\bar{\omega} \text{ is not defined at } x. \qquad \text{Q. E. D.}$$

The prescheme structure of $\langle\omega\rangle$ (i. e., the sheaf $\mathcal{O}_{<\omega>}$) may involve more detailed nature of the singular part of the rational section $\omega$(or $\bar{\omega}$). The following two theorems will tell us some of these aspects.

**Theorem 1.** *The Ideal $I(\omega)\cdot\mathcal{O}_{[\omega]}$ of the closed sub-prescheme*

*$i^{-1}\pi^{-1}\langle\omega\rangle=[\omega]\times_X\langle\omega\rangle$ in $\mathcal{O}_{[\omega]}$ is isomorphic to the $\mathcal{O}_{[\omega]}$-Module $i^*(\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}(1)\otimes_{\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}}\pi^*\mathcal{O}_X(\omega))=(\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}(1)\otimes_{\mathcal{O}_X}\mathcal{O}_X(\omega))\mid[\omega]$, i.e., we getan exact sequence of $\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}$-Modules*

(5) $$0\to i^*(\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}(1)\otimes_{\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}}\pi^*\mathcal{O}_X(\omega))\to\mathcal{O}_{[\omega]}\to i^*\pi^*\mathcal{O}_{\langle\omega\rangle}\to 0.$$

*Proof*) We have an exact sequence (EGA. II. 8. 1. 8)

$$0\to\mathcal{O}_{[\omega]_1}(1)\to\mathcal{O}_{[\omega]_1}\to\mathcal{O}_{[\omega]X_X\langle\omega\rangle}\to 0.$$

By this and the isomorphism (4), we get our assertion. Q. E. D.

**Proposition 6.** *If $\mathcal{E}$ is locally free of rank 2, then $J_1(\omega)$ is an invertible $\mathcal{O}_X$-Module and $\widetilde{J(\omega)}$ (the Ideal of $[\omega]$ in $\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}$) is also an invertible $\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}$-Module.*

*Proof*) At any point $x$ of $X$, we have an exact sequence

$$0\to J_1(\omega)_x\to\mathcal{E}_x\xrightarrow{\omega_1^*}I(\omega)_x\to 0.$$

Take a basis $(e_1,e_2)$ of $\mathcal{E}_x$ over $\mathcal{O}_{X,x}$, and put $\alpha_i=\omega_1^*(e_i)$ $(i=1,2)$; then, if we write $\alpha_i=g\cdot a_i$ as in the proof of Cor. of Prop. 4, we get the following commutative diagram

$$\begin{array}{ccccccccc}0\to & a_1a_2\mathcal{O}_x & \xrightarrow{\lambda} & \mathcal{O}_x\oplus\mathcal{O}_x & \xrightarrow{\mu} & a_1\mathcal{O}_x+a_2\mathcal{O}_x & \to 0\\ & & & \downarrow\wr f & & \downarrow\wr h & \\ 0\to & J_1(\omega)_x & \to & \mathcal{E}_x & \to & I(\omega)_x & \to 0,\end{array}$$

where $\lambda(a_1a_2c)=(a_2c,-a_1c)$, $\mu(c,d)=a_1c+a_2c$, $f(c,d)=c\cdot e_1+d\cdot e_2$ and $h(a_1c+a_2d)=g\cdot(a_1c+a_2d)$. And it is easy to see that the upper horizontal sequence is exact, hence $J_1(\omega)_x\cong a_1a_2\mathcal{O}_x\cong\mathcal{O}_x$, i. e., $J_1(\omega)$ is invertible. Moreover, since $J(\omega)=J_1(\omega)\cdot\boldsymbol{S}(\mathcal{E})$, $\widetilde{J(\omega)}=J_1(\omega)\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}$. This proves that $\widetilde{J(\omega)}$ is invertible. Q. E. D.

*Remark.* When $\mathcal{E}$ is locally free of rank 2, by the above proposition, we may regard $[\omega]$ as a Cartier divisor on $\boldsymbol{P}(\mathcal{E})$, and $\widetilde{J(\omega)}=\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}(-[\omega])$, the invertible $\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}$-Module corresponding to the Cartier divisor $-[\omega]$.

**Theorem 2.** *When $\mathcal{E}$ is locally free of rank 2,*

$$J_1(\omega)\otimes_{\mathcal{O}_X}\Lambda^2\check{\mathcal{E}}\cong\mathcal{O}_X(\omega)\qquad(\omega\neq 0).$$

*Proof*) Take an open covering $(U_\alpha)_{\alpha\in I}$ of $X$ such that $\mathcal{E}|U_\alpha \overset{\varphi_\alpha}{\longleftarrow} \mathcal{O}_X^2|U$, for each $\alpha\in I$. Let $t^\alpha=(t_1^\alpha, t_2^\alpha)$ be the basis of $\mathcal{E}|U_\alpha$ over $\mathcal{O}_X|U_\alpha$, determined by $\varphi_\alpha$, and $\tau^\alpha=(\tau_1^\alpha, \tau_2^\alpha)$ the dual basis of $\check{\mathcal{E}}|U_\alpha$ of $t^\alpha$, then $\tau_1^\alpha\wedge\tau_2^\alpha$ is a basis of $\wedge^2\check{\mathcal{E}}|U_\alpha$. When that is so, the homomorphism $\omega_1^*:\mathcal{E}\to\mathcal{R}(X)$ can be expressed, locally on $U_\alpha$, as

$$\omega_1^*=A_1^\alpha\cdot\tau_1^\alpha+A_2^\alpha\cdot\tau_2^\alpha,\ \ A_i^\alpha\in\Gamma(U_\alpha,\mathcal{R}(X))=R(X).$$

Note that, $\omega\neq 0$ (i. e., $\omega_1^*\neq 0$) implies $A_i^\alpha\neq 0$ for $i=1$ or 2, and that, for $e=\Sigma b_i^\alpha\cdot t_i^\alpha\in\Gamma(U_\alpha,\mathcal{E})$, $e\in\Gamma(U_\alpha, J_1(\omega))$ if and only if $\Sigma A_i^\alpha\cdot b_i^\alpha=0$. Consider the map

$$(J_1(\omega)\otimes_{\mathcal{O}_X}\wedge^2\check{\mathcal{E}})|U_\alpha\to\mathcal{R}(X)|U_\alpha$$

given by the correspondence

$$(b_1^\alpha\cdot t_1^\alpha+b_2^\alpha\cdot t_2^\alpha)\otimes c^\alpha\tau_1^\alpha\wedge\tau_2^\alpha \rightsquigarrow b_1^\alpha c^\alpha/A_2^\alpha=-b_2^\alpha c^\alpha/A_1^\alpha=k^\alpha.$$

At any point $x$ of $U_\alpha$, let $A_i^\alpha=g\cdot a_i$, $g\in R(X)$, $a_i\in\mathcal{O}_x$ such that $a_1$ and $a_2$ are relatively prime to each other in $\mathcal{O}_x$. Then $a_1\cdot b_1+a_2\cdot b_2=0$, hence $b_1/a_2=-b_2/a_1$ is in $\mathcal{O}_x$. Therefore $k^\alpha$ is an element of $(1/g)\cdot\mathcal{O}_x=\mathcal{O}_X(\omega)_x$ (cf. the proof of Cor. of Prop. 4). This means that the above map induces an $(\mathcal{O}_X|U_\alpha)$-homomorphism

$$\Phi_\alpha:(J_1(\omega)\otimes_{\mathcal{O}_X}\wedge^2(\check{\mathcal{E}})|U_\alpha\to\mathcal{O}_X(\omega)|U_\alpha,$$

and it is easy to see that this is an isomorphism. If $G^{\alpha\beta}=(G_{ij}^{\alpha\beta})$ are the transition matrices of $\mathcal{E}$ (with respect to $\varphi_\alpha$), then $(G^{\alpha\beta})^{-1}=G^{\beta\alpha}$ and $\det(G^{\alpha\beta})^{-1}=\det(G^{\beta\alpha})$ are the transition matrices and functions of $\check{\mathcal{E}}$ and $\wedge^2\check{\mathcal{E}}$, respectively. Hence,

$$c^\alpha=\det(G^{\beta\alpha})\cdot c^\beta,\ \ b_1^\alpha=G_{11}^{\alpha\beta}\cdot b_1+G_{12}^{\alpha\beta}\cdot b_2^\beta,$$

and

$$A_2^\alpha=\det(G^{\beta\alpha})\cdot(-G_{12}^{\alpha\beta}\cdot A_1^\beta+G_{11}^{\alpha\beta}\cdot A_2^\beta),$$

therefore, by easy calculation, we get the identity $k^\alpha=k^\beta$. This shows that the $\Phi_\alpha$'s can be patched together and give a global isomorphism

$$\Phi:J_1(\omega)\otimes\wedge^2\check{\mathcal{E}}\xrightarrow{\sim}\mathcal{O}_X(\omega).\qquad \text{Q. E. D.}$$

**Corollary.** *Under the same assumptions in Th.* 2,

$$\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}([\omega])\cong\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}(1)\otimes_{\mathcal{O}_X}\wedge^2\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{O}_X(\omega)^{-1}$$

(cf. Remark of Cor. of Prop. 6).

*Proof*) Since $J(\omega)=J_1(\omega)\cdot S(\mathcal{E})\cong J_1(\omega)\otimes_{\mathcal{O}_X}S(\mathcal{E})(-1)$, we get, by Th. 2, an isomorphism

$$J(\omega)\otimes_{\mathcal{O}_X}\wedge^2\check{\mathcal{E}}\cong S(\mathcal{E})(-1)\otimes_{\mathcal{O}_X}\mathcal{O}_X(\omega).$$

Hence, passing to the associated sheaves on $\boldsymbol{P}(\mathcal{E})$, we get an isomorphism

$$\widetilde{J(\omega)}\otimes_{\mathcal{O}_X}\wedge^2\check{\mathcal{E}}\cong\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}(-1)\otimes_{\mathcal{O}_X}\mathcal{O}_X(\omega).$$

On the other hand $\widetilde{J(\omega)}=\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}([\omega])^{-1}$, therefore, combining these two isomorphisms, we get our assertion. Q. E. D.

**5. The case of algebraic schemes.** Let $X$ be an algebraic scheme over an algebraically closed field $k$. We denote, for each non-negative integer $p$, by $X^p$ the set of points $x$ of $X$ such that $\operatorname{codim}_X x=\dim\mathcal{O}_{X,x}=p$, and by $Z^p(X)$ the free abelian group generated by the irreducible closed subsets $\overline{\{x\}}$ of $X$, where $x$ are in $X^p$, and we shall say each element of $Z^p(x)$ a *cycle on $X$ of codimension $p$.*

Let $C^p(X)$ $(p\geqq 0)$ be the abelian category of coherent $\mathcal{O}_X$-Modules whose supports are of codimension$\geqq p$, and

$$\gamma_p: C^p(X)\to K^p(X)$$

the universal solution in the category of abelian groups satisfying the following axiom (i. e., the Grothendieck group of $C^p(X)$):

(Additivity) If $0\to F'\to F\to F''\to 0$ *is exact in* $C^p(X)$, *then* $\gamma_p(\mathcal{F})=\gamma_p(\mathcal{F}')+\gamma_p(\mathcal{F}'')$.

The immersion $C^p(X)\to C^q(X)$ (for $p\geqq q$) determines a canonical homomorphism $K^p(X)\to K^q(X)$. By means of this homomorphism, we shall consider that every element of $K^p(X)$ lies on $K^q(X)$, especially on $K^0(X)=K(X)$. Defining the product by

$$\gamma(\mathcal{F})\cdot\gamma(\mathcal{G})=\Sigma_{p\geq 0}(-1)^p\gamma(\mathcal{T}or^{\mathcal{O}_X}{}_p(\mathcal{F},\mathcal{G})),\ \mathcal{F},\mathcal{G}\in ObC^0(X),$$

$K^0(X)=K(X)$ has a ring structure (cf. Borel-Serre[1]).

For any $\mathcal{F}\in ObC^p(X)$, put

$$z_p(\mathcal{F}) = \Sigma_{x\in X^p} \text{length}_{\mathcal{O}_x}(\mathcal{F}_x) \cdot \overline{\{x\}} \in Z^p(X),$$

and call it the *cycle of codimension $p$ associated to $\mathcal{F}$* (cf. Serre [8]). Since the map $z_p: C^p(X) \to Z^p(X)$ is clearly additive, it defines a group homomorphism $z_p: K^p(X) \to Z^p(X)$ such that $z_p(\gamma_p(\mathcal{F})) = z_p(\mathcal{F})$. We denote, for any closed subscheme $Y$ of $X$, $z_p(\mathcal{O}_Y) = Y_p$ ($p \leqq \operatorname{codim} Y$), it is easy to show that, if $Y$ is reduced and irreducible of codimension $p$, $z_p(\mathcal{O}_Y) = Y_p = Y$, i. e., the underlying space of $Y$ with multiplicity 1. Moreover, if $X$ is regular (i. e., non-singular), the Cartier divisors on $X$ are identified to the elements of $Z^1(X)$ (i. e., the Weil divisors), hence we have a bijective canonical correcpondence between $Z^1(X)$ and the set of invertible sub-$\mathcal{O}_X$-Modules of $\mathcal{R}(X)$ ($D \rightsquigarrow \mathcal{O}_X(D)$), and it is easy to see that, for any positive divisor $D \in Z^1(X)$, $z_1(\mathcal{O}_D) = D_1 = D$, where $\mathcal{O}_D = \mathcal{O}_X/\mathcal{O}_X(-D)$ (Cf. Mumford. [7]). The following theorem has been proved by Serre which is very usefull for our study.

**Serre's Intersection Theory** (Serre [8], Prop. 1 of V, c.). *Assume the algebraic scheme $X$ to be regular. For elements $\xi \in K^p(X)$ and $\eta \in K^q(X)$ such that $\xi, \eta \in K^{p+q}(X)$, the cycles $z_p(\xi)$ and $z_q(\eta)$ intersect properly to each other and*

$$z_{p+q}(\xi \cdot \eta) = X_p(\xi) \cdot z_q(\eta) \text{ (the intersection product in usual sense).}$$

**Lemma 1.** *Assume $X$ to be regular. For closed subscheme $Y$ of $X$ and any divisor $D$ on $X$, if we have an exact sequence of coherent $\mathcal{O}_X$-Modules*

$$\text{(6)} \qquad 0 \to \mathcal{O}_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \mathcal{O}_Y \to \mathcal{G} \to 0$$

*then there exsets a divisor $D' \in Z^1(X)$, linearly equivalent to $D$, such that the intersection product $D' \cdot Y_p$ is defined and, for any $p \leqq \operatorname{codim}_X Y$,*

$$z_{p+1}(\mathcal{G}) = D' \cdot Y_p.$$

*Proof*) Take a $D' \in Z^1(X)$ which is linearly equivalent to $D$

and intersets properly with Supp$(Y)$. Let $D'=E_1-E_2$, $E_i>0$ and they have no common components. Then, since exact sequences

$$0\to\mathcal{O}_X(-E_i)\to\mathcal{O}_X\to\mathcal{O}_{E_i}\to 0\quad (i=1,2)$$

are locally free resolution of $\mathcal{O}_{E_i}$, we get

$$0\to\mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{O}_{E_i},\mathcal{O}_Y)\to\mathcal{O}_X(-E_i)\otimes_{\mathcal{O}_X}\mathcal{O}_Y\to\mathcal{O}_Y\to\mathcal{O}_{E_i}\otimes_{\mathcal{O}_X}\mathcal{O}_Y\to 0\ \text{(exact)}$$
$$\text{and}\quad \mathcal{T}or_p^{\mathcal{O}_X}(\mathcal{O}_{E_i},\mathcal{O}_Y)=0,\ \text{if}\ p\geqq 2.$$

Since $E_i$ intersect properly with Supp(Y) and $\mathrm{Supp}(\mathcal{O}_{E_i}\otimes\mathcal{O}_Y)\subset\mathrm{Supp}(\mathcal{O}_{E_i})\cap\mathrm{Supp}(\mathcal{O}_Y)$, we have codim. $\mathrm{Supp}(\mathcal{O}_{E_i}\otimes_Y)\geqq$ codim. $\mathrm{Supp}(\mathcal{O}_Y)-1$. Hence, by Serre's intersection theory, the intersection product $z_1(\mathcal{O}_{E_i})\cdot z_p(\mathcal{O}_Y)=E_i\cdot Y_p$ is defined and is equal to

$$z_{p+1}(\mathcal{O}_{E_i}\otimes\mathcal{O}_Y)-z_{p+1}(\mathcal{T}or_1^{\mathcal{O}_X}(\mathcal{O}_{E_i}\otimes\mathcal{O}_Y)$$
$$=z_{p+1}(\mathcal{O}_Y)-z_{p+1}(\mathcal{O}_X(-E_i)\otimes 0_Y).$$

Therefore

$$D'\cdot Y_p=E_1\cdot Y_p-E_2\cdot Y_p=z_{p+1}(\mathcal{O}_X(-E_2)\otimes\mathcal{O}_Y)$$
$$-z_{p+1}(\mathcal{O}_X(-E)\otimes\mathcal{O}_Y),$$

while $\mathcal{O}_X(-D)\cong\mathcal{O}_X(-D')\cong\mathcal{O}_X(-E_1)\otimes\mathcal{O}_X(E_2)$, hence, by tensoring $\mathcal{O}_X(-E_2)$ to the exact sequence (6), we get an exact sequence

$$0\to\mathcal{O}_X(-E_1)\otimes\mathcal{O}_Y\to\mathcal{O}_X(-E_2)\otimes\mathcal{O}_Y\to\mathcal{O}_X(-E_2)\otimes\mathcal{G}\to 0,$$

and, taking $z_{p+1}$,

$$z_{p+1}(\mathcal{O}_X(-E_2)\otimes\mathcal{O}_Y)-z_{p+1}(\mathcal{O}_X(-E_1)\otimes\mathcal{O}_Y$$
$$=z_{p+1}(\mathcal{O}_X(-E_2)\otimes\mathcal{G})=z_{p+1}(\mathcal{G}).$$

Thus we get the proof. Q. E. D.

Now we shall apply this result to Th. 1.

**Theorem 1$'$.** *Let $X$ be a regular algebraic scheme, $\mathcal{E}$ a locally free $\mathcal{O}_X$-Module of rank $p+1$ and $H$ a divisor on $\boldsymbol{P}(\mathcal{E})$ such that $\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}(H)\cong\mathcal{O}_{P(\mathcal{E})}(1)$. Then, for any non-zero rational section $\omega\in\Gamma_{\mathrm{rat}}(V(\mathcal{E})/X)$, there exists a divisor $D$ on $\boldsymbol{P}(\mathcal{E})$ such that it is linearly equivalent to $H+\pi^{-1}(\omega)$ and that the intersection product $D\cdot z_p(\mathcal{O}_{[\omega]})=D\cdot[\omega]_p$ is defined and is equal to*

$-z_{p+1}(\mathcal{O}_i - i^{-1}\pi^{-1}\langle\omega\rangle) = -(i^{-1}\pi^{-1}\langle\omega\rangle)_{p+1}$, *i. e., in the Chow ring* $A(\boldsymbol{P}(\mathcal{E}))$ *of* $\boldsymbol{P}(\mathcal{E})$ *(if X is quasi-projective,* cf. [2]),

$$(i^{-1}\pi^{-1}\langle\omega\rangle)_{p+1} = (-H - \pi^*(\omega)) \cdot [\omega]_p.$$

*Moreover, if* $\mathcal{E}$ *is of rank* 2,

$$(\pi^{-1}\langle\omega\rangle)_2 = -D \cdot [\omega],\ i.e.,\ (\pi^{-1}\langle\omega\rangle)_2 = (-H - \pi^{-1}(\omega)) \cdot [\omega] \text{ in } A(\boldsymbol{P}(\mathcal{E})).$$

*Proof*) Note that, $\boldsymbol{P}(\mathcal{E})$ is also a regular algebraic scheme and that the projection $\pi: \boldsymbol{P}(\mathcal{E}) \to X$ is flat; then it is easy to see that $z_p(\pi^*\mathcal{O}_X(\omega)) = \pi^{-1}(\omega)$. Then the first part is straightly obtained applying Lemma 1 to the exact sequence (5). The second part is an immediate consequence of the following lemma.

**Lemma 2.** *Under the same assumption in Th.* 1′, *if* $\mathcal{E}$ *is of rank* 2,

(i) *codim. Supp* $(\mathcal{O}_{<\omega>}) \geqq 2$, *and* (ii) $i^*\pi^*\mathcal{O}_{<\omega>} \cong \pi^*\mathcal{O}_{<\omega>}$.

*Proof*) (i) For any point $x$ of $X$, $\bar{I}(\omega)_x$ is generated by relatively prime two elements of $\mathcal{O}_x$, hence $\dim \mathcal{O}_{<\omega>,x} = \dim(\mathcal{O}_x/\bar{I}(\omega)_x) \leqslant \dim \mathcal{O}_x - 2$. This proves (i).

(ii) Since $i^*\pi^*\mathcal{O}_{<\omega>} = \mathcal{O}_{\boldsymbol{P}(\mathcal{E})}/\bar{I}(\omega).\mathcal{O}_{\boldsymbol{P}(\mathcal{E})}) \otimes \mathcal{O}_{[\omega]}$, in order to get our assertion, it is sufficient to prove that

$$J(\omega)(\boldsymbol{S}(\mathcal{E})/\bar{I}(\omega) \cdot \boldsymbol{S}(\mathcal{E})) = (J(\omega) + \bar{I}(\omega) \cdot \boldsymbol{S}(\mathcal{E}))/\bar{I}(\omega) \cdot \boldsymbol{S}(\mathcal{E}) = 0,$$
$$i.\,e.,\ J(\omega) \subset \bar{I}(\omega) \cdot \boldsymbol{S}(\mathcal{E}).$$

At any point $x$ of $X$, any element $e \in J_1(\omega)_x$ is expressed as

$$e = b_1 t_1 + b_2 t_2,\ b_i \in \mathcal{O}_x, \text{ such that } b_1 a_1 + b_2 a_2 = 0$$

(with the notations used in the proof of Th. 2). Since $\bar{I}(\omega)_x = a_1\mathcal{O}_x + a_2\mathcal{O}_x$ and $a_1$ and $a_2$ are relatively prime to each other,

$$e\boldsymbol{S}_{m-1}(\mathcal{E}) \subset b_1\boldsymbol{S}_m(\mathcal{E}) + b_2\boldsymbol{S}_m(\mathcal{E}) \subset a_1\boldsymbol{S}_m(\mathcal{E}) + a_2\boldsymbol{S}_m(\mathcal{E}) = \bar{I}(\omega)_x\boldsymbol{S}_m(\mathcal{E}).$$

This proves $J_1(\omega) \cdot \boldsymbol{S}_{m-1}(\mathcal{E}) \subset \bar{I}(\omega)\boldsymbol{S}_m(\mathcal{E})$, i. e., $I(\omega) = J_1(\omega)\boldsymbol{S}(\mathcal{E}) \subset \bar{I}(\omega)\boldsymbol{S}(\mathcal{E})$. Q. E. D.

In the algebraic scheme case, Cor. of Th. 2 also can be transelated as follows

**Theorem** 2′. *Under the same assumptions in Th.* 1′, *if* $\mathcal{E}$ *is of rank* 2, *the divisor* $[\omega]$ *is linearly equivalent to the divisor* $H+\pi^{-1}K-\pi^{-1}(\omega)$, *where* $K$ *is a divisor on* $X$ *such that* $\mathcal{O}_X(K)\cong\bigwedge^2\check{\mathcal{E}}$.

**Corollary.** *Under the same assumptions in Th.* 2′, *if* $X$ *is quasi-projective, for any locally free* $\mathcal{O}_X$*-Module* $\mathcal{E}$ *of rank* 2, *the first Chern class* $c_1(\mathcal{E})$ *of* $\mathcal{E}$ *is equal to* $cl_X(\bigwedge^2\mathcal{E})$, *and the second Chern class* $c_2(\mathcal{E})$ *of* $\mathcal{E}$ *is equal to* $\langle\omega\rangle-c_1(\mathcal{E})\cdot(\omega)-(\omega)^2$, *where* $\omega$ *is non-zero rational section of the vector fibre* $\boldsymbol{V}(\mathcal{E})/X$ (*The Chern classes are in the sense of Grothendieck*, cf. [4], [5]).

*Proof*) Combine the results of Th. 1' and 2', we get an equality in the Chow-ring of $\boldsymbol{P}(\mathcal{E})$

$$H^2+\pi^*K\cdot H+\pi^*(\langle\omega\rangle+(\omega)\cdot K-(\omega)^2)=0.$$

This identity shows, by the definition (cf. [4], [5]), that

$$c_1(\mathcal{E})=-c_1(\check{\mathcal{E}})=-K=-cl_X(\bigwedge^2\check{\mathcal{E}})=cl_X(\bigwedge^2\check{\mathcal{E}}), \text{ and}$$
$$c_2(\mathcal{E})=c_2(\mathcal{E})=\langle\omega\rangle+(\omega)\cdot K-(\omega)^2$$
$$=\langle\omega\rangle-(\omega)\cdot c_1(\mathcal{E})-(\omega)^2. \qquad \text{Q. E. D.}$$

*Remark.* We shall now apply the result to the case of surfaces. Let $X=F$ be a non-singular projective surface and $\mathcal{E}=\mathcal{T}_F=\mathcal{H}om_{\mathcal{O}_F}(\Omega_F^1,\mathcal{O}_F)$ the tangential sheaf on $F$. Then, for any linear differential form $\omega$ on $F$ (i.e., an element of $\Gamma(F,\Omega_F^1\otimes\mathcal{R}(F))$), we can express it, at any point $x$ of $F$, as $\omega=h(f\cdot dt_1+g\cdot dt_2)$ ($t$'s are local parameters at $x$) where $h$, $f$ and $g$ are rational functions on $F$ such that $f$ and $g$ are regular at $x$ and are relatively prime in $\mathcal{O}_{F,x}$. Denote by $m_x$ the intersection multiplicity of the divisors $(f)$ and $(g)$ at $x$, and put $\langle\omega\rangle=\Sigma_x m_x\cdot x$; then the 0-cycle $\langle\omega\rangle$ is just the same thing of ours. And the second Chern class

$$c_2(\mathcal{T}_F)=c_2(F)=\langle\omega\rangle+(\omega)\cdot K-(\omega)^2$$

($K=cl(\wedge^2\check{\mathfrak{T}}_F)=cl(\Omega_F^2)=$the canonical divisor class on $F$) is called the Severi-series which has been defined by F. Severi in [9], and used by J. Igusa, in [6], in order to prove the in-equality $B_2 \geqq \rho$ where $B_2$ is the second Betti number of the surface $F$ and $\rho$ is the Picard number of $F$[2)].

## Appendix

Let $V \xrightarrow{\pi} \mathrm{Spec}(k)$ be a non-singular projective algebraic variety of dimension $n$ and $\Omega_V^p=\wedge^p\Omega_V^1$ the sheaf of germs of holomorphic $p$-forms on $V$. Then we get

$$c_n(V)=\sum_{p,q}(-1)^{p+q}h^{p,q},\quad h^{p,q}=\dim_k H^q(V,\Omega_V^p).$$

In fact, let

$$c_t(V)=\sum_{i=1}^n c_i t^i=\sum_{i=0}^n(-1)^i c_i(\Omega_V^1)t^i=\prod_{i=1}^n(1+\alpha_i t)$$

be the Chern polynomial of $V$. Then we have

$$c_t(\Omega_V^p)=\sum_{i=0}^n c_i(\Omega)_V^p)t^i=\sum_{1\leqq i_1<i_2<\cdots<i_p\leqq n}(1-(\alpha_{i_1}+\alpha_{i_2}+\cdots+\alpha_{i_p})t)$$

(cf. [5]). Hence

$$ch(\Omega_V^p)=\sum_{1\leqq i_1<i_2<\cdots i_p\leqq n}\exp(-\alpha_{i_1}-\alpha_{i_2}-\cdots-\alpha_{i_p}).$$

Applying this result to the theorem of Riemann-roch ([1]) we get

$$\begin{aligned}\chi(V,\Omega_V^p)&=\pi_*(ch(\Omega_V^p)\cdot T(V))\\ &=\pi_*(\Sigma\, exp(-\alpha_{i_1}-\alpha_{i_2}-\cdots-\alpha_{i_p})\cdot\Pi(\alpha_i/1-\exp(-\alpha_i)))\\ \text{(put)}\quad &=T_n^p(c_1,c_2,\cdots,c_n).\end{aligned}$$

Therefore, the polynomial

$$\sum_{p=0}^n\chi(V,\Omega_V^p)y^p=\sum_{p=0}^n T_n^p(c_1,\cdots,c_n)y^p(=T_n(c_1,\cdots,c_n))$$

is the n-th term of the "*m-Folge*" belonging to the power series

$$Q(y,x)=x(y+1)/(1-\exp(-x(y+1))$$

2) Igusa difined $B_2$ by the classical fact $\Sigma(-1)^i B_i=c_2(F)$. On the other hand we can show $c_2(F)=\Sigma(-1)^{p+q}h^{p,q}(F)$ by means of the Riemonn-Roch theorem of Grothendieck ([1]) (see Appendix).

(cf. [10] p. 16, note that $\pi_*(\quad)=\kappa_n[\quad]$).

This proves that

$$c_n=\sum_{p=0}^{n}(-1)^p T_n^p(c_1,\cdots,c_n)=\Sigma(-1)^p\chi(V,\Omega_V^p)$$
$$=\sum_{p,q=0}^{n}(-1)^{p+q}h^{p,q}.$$

(cf. ibid. the formula (16) of Chap. 1, sect. 8, p. 17).

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