J. Math. Kyoto Univ.
6-2 (1967) 131-176

The Homotopy groups of Lie groups of low rank

By

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(Received November 29, 1966)

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§1. Introduction

The compact simply connected simple Lie groups are classified as follows:

$A_n = SU(n+1)$, $B_n = Spin(2n+1)$, $C_n = Sp(n)$, $D_n = Spin(2n)$
G_2, F_4, E_6, E_7, E_8,

where $A_1 = B_1 = C_1$, that is, $SU(2) = Spin(3) = Sp(1)$,
$B_2 = C_2$, that is, $Spin(5) = Sp(2)$,
and $A_3 = D_3$, that is, $SU(4) = Spin(6)$.
The first four types are called the classical Lie groups, and the last five are called the exceptional Lie groups.

The purpose of this paper is to determine the first 23 homotopy groups of G_2, F_4, and of B_n and D_n of low rank.

This paper is divided into two parts. The first part consists of §2 and §3. In §2 we calculate the cohomology groups of the 3-connective fibre space over G_2 and F_4. In §3, we compute the odd primary components of the homotopy groups of G_2 and F_4 by the killing-homotopy method [6].

We study in §4 some properties in the homotopy theory of the fibre spaces, especially, of the bundles. These are used in §6 for the determination of $\pi_i(G_2)$.

Section 5 is an intermediate one. It is the preparation for the second part, which consists of §6, §7, §8 and §9. In §6 we determine the 2-primary components of $\pi_i(G_2)$ by making use of the exact sequence associated with the well-known fibering $G_2/SU(3)=S^6$. F_4 operates transitively on the octonionic projective plane Π, and the isotropy group is isomorphic to $Spin(9)$. Hence $F_4/Spin(9)=\Pi$. The homotopy groups of Π will be determined in §7. The 2-primary components of $\pi_i(F_4)$ will be computed in §8 by making use of the exact sequence associated with the homogeneous space F_4/G_2.

The last section, §9, is devoted to the determination of the homotopy groups of spinor groups of low rank.

The results are summarized in the following table:

$\pi_i(G)$

G \ i	1	2	3	4	5	6	7	8	9	10	11
$Spin(7)$	0	0	∞	0	0	0	∞	$(2)^2$	$(2)^2$	8	$\infty+2$
$Spin(9)$	0	0	∞	0	0	0	∞	$(2)^2$	$(2)^2$	8	$\infty+2$
G_2	0	0	∞	0	0	3	0	2	6	0	$\infty+2$
F_4	0	0	∞	0	0	0	0	2	2	0	$\infty+2$
Π	0	0	0	0	0	0	0	∞	2	2	24

G \ i	12	13	14	15	16	17	18
$Spin(7)$	0	2	$2520+8+2$	$(2)^4$	$(2)^7$	$(8)^2+(2)^2$	$945+16+8+2$
$Spin(9)$	0	2	$8+2$	$\infty+(2)^3$	$(2)^6$	$8+(2)^2$	$2835+16+8+2$
G_2	0	0	$168+2$	2	$6+(2)^2$	$8+2$	240
F_4	0	0	2	∞	$(2)^2$	2	$720+3$
Π	0	0	2	120	$(2)^3$	$(2)^4$	$24+2$

G \ i	19	20	21	22	23
$Spin(7)$	2	$(2)^2$	$24+4$	$10395+(8)^2+(2)^4$	$G+(2)^5$
$Spin(9)$	2	2	12	$11!/32+8+(2)^2$	$G+(2)^2$
G_2	6	2	0	$1386+8$	$G+2$
F_4	2	0	$(3)^2$	27 *or* 9	$G+\infty$
Π	$504+2$	0	6	4	$\infty+120+(2)^2$

where $G=4$ *or* $(2)^2$.

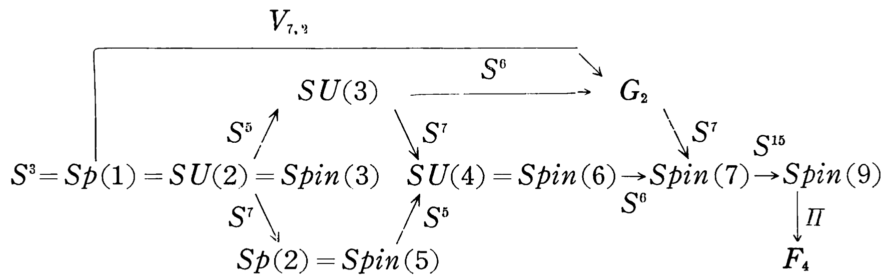
In the above table an integer n indicates a cyclic group Z_n of order n, the symbol "∞" an infinite cyclic group Z, the symbol "+" the direct sum of the groups, and $(2)^k$ indicates the direct sum of k-copies of Z_2.

For the other spinor groups of low rank we have the isomorphisms

$$\pi_i(Spin(3)) \cong \pi_i(S^3),$$
$$\pi_i(Spin(4)) \cong \pi_i(Spin(3) \times S^3),$$
$$\pi_i(Spin(5)) \cong \pi_i(Sp(2)),$$
$$\pi_i(Spin(6)) \cong \pi_i(SU(4)),$$
$$\pi_i(Spin(8)) \cong \pi_i(Spin(7) \times S^7),$$

so that the homotopy may be obtained from the known results; [13], [14], [15], [16], [18].

For the convenience of the reader we indicate the various fiberings used in this paper in the following diagram.



Here $F \overset{B}{\to} E$ denotes the fibering $E \to B$ with fibre F.

All spaces considered in the present work are those which have the homotopy groups of finite type. Let X be such a space. Then $\pi_i(X)$ is isomorphic to the direct sums of free parts F and p-primary components of $\pi_i(X)$ for every prime p. We denote by $\pi_i(X : p)$ the direct sums of a certain subgroup F' and the p-primary components of $\pi_i(X)$, where the index $[F : F']$ is prime to p. Given an exact sequence for such spaces A, B and C:

$$\cdots \longrightarrow \pi_i(A) \longrightarrow \pi_i(B) \longrightarrow \pi_i(C) \longrightarrow \cdots,$$

we can form the following exact one in our cases by suitable choice of $\pi_i(\ : p)$:

$$\cdots \longrightarrow \pi_i(A:p) \longrightarrow \pi_i(B:p) \longrightarrow \pi_i(C:p) \longrightarrow \cdots.$$

The notations and the terminologies of [14], [15] and [18] are carried over to the present work.

The author wishes to thank Professor A. Komatu for his encouragement and Professor H. Toda for his advices and criticism throughout the preparation of the manuscript.

§2. The cohomology of the 3-connective fibre space of G_2 and F_4.

Borel [3] calculated the cohomology groups of G_2 and F_4 and their results are stated as follows.

Theorem 2.1.

(i) $H^*(G_2;\ Z_2) \cong Z_2[x_3]/(x_3^4) \otimes \Lambda(Sq^2x_3)$.

$H^*(G_2;\ Z_p) \cong \Lambda(x_3, x_{11})$ *for each prime* $p \geq 3$,

where $x_3^2 = Sq^1Sq^2x_3$, Sq^ix_3 *is trivial for the other cases*, $\mathcal{P}_5^1 x_3 = x_{11}$ *and* $\mathcal{P}_p^i$ *is trivial for the other cases.*

(ii) $H^*(F_4;\ Z_2) \cong Z_2[x_3]/(x_3^4) \otimes \Lambda(Sq^2x_3, x_{15}, Sq^8x_{15})$.

$H^*(F_4;\ Z_3) \cong Z_3[\delta\mathcal{P}^1x_3]/((\delta\mathcal{P}^1x_3)^3) \otimes \Lambda(x_3, \mathcal{P}^1x_3, x_{11}, \mathcal{P}^1x_{11})$.

$H^*(F_4;\ Z_p) \cong \Lambda(x_3, x_{11}, x_{15}, x_{23})$ *for each prime* $p \geq 5$,

where $\mathcal{P}_5^1 x_3 = x_{11}$ *and* $\mathcal{P}_7^1 x_3 = x_{15}$.

Note that the following relations hold:

(2.1) $\quad Sq^4Sq^2x_3 = 0 \quad in \quad H^*(F_4;\ Z_2)$.

(2.2) $\quad \mathcal{P}^3\mathcal{P}^1x_3 = 0 \quad in \quad H^*(F_4;\ Z_3)$.

(2.1) follows from Théorème 19.2 of [3] and (2.2) follows from the fact that there are no primitive elements in $H^{19}(F_4;\ Z_3)$.

Recently Kumpel [12] has proved the following

Proposition 2.2.

(i) $\mathcal{P}_5^1 x_{15} = x_{23} \quad in \quad H^*(F_4;\ Z_5)$.

(ii) $\mathcal{P}_7^1 x_{11} = x_{23} \quad in \quad H^*(F_4;\ Z_7)$.

(iii) $\mathcal{P}_{11}^1 x_3 = x_{23} \quad in \quad H^*(F_4;\ Z_{11})$.

Denote by $\widetilde{G}_2$ the 3-connective fibre space over G_2, so that,

$$\pi_i(\widetilde{G}_2)\cong\begin{cases}\pi_i(G_2) & for \quad i\geq 4\\ 0 & for \quad i<4.\end{cases}$$

Then we have two fiberings

$$K(Z, 2)\longrightarrow\widetilde{G}_2\longrightarrow G_2, \tag{2.3}$$

$$\widetilde{G}_2\longrightarrow G_2'\longrightarrow K(Z, 3), \tag{2.4}$$

where G_2' has the same homotopy type as G_2 and $K(Z, m)$ is the Eilenberg-MacLane space of type (Z, m).

Let $\{E_r^*\}$ be the cohomology spectral sequence with Z_2-coefficient associated with (2. 3). Then we have

$$E_2^*=H^*(G_2;\ Z_2)\otimes H^*(Z, 2;\ Z_2)$$
$$\cong(Z_2[x_3]/(x_3^4)\otimes\Lambda(Sq^2x_3))\otimes Z_2[u].$$

Clearly $d_2=0$ and we have $E_2^*\cong E_3^*$. We have $d_3(1\otimes u)=x_3$, since $\widetilde{G}_2$ is a 3-connective fibering over G_2. This implies

$$E_4^*=H(E_3^*)\cong Z_2[1\otimes u^2]\otimes\Lambda(Sq^2x_3\otimes 1,\ x_3^3\otimes u).$$

d_4 is trivial by the dimensional reason, and hence $E_5^*\cong E_4^*$. Next we get $d_5(1\otimes u^2)=Sq^2x_3\otimes 1$, since the transgression commutes with Sq^2 and since $Sq^2u=u^2$. It follows that

$$E_6^*=H(E_5^*)\cong Z_2[1\otimes u^4]\otimes\Lambda(Sq^2x_3\otimes u^2,\ x_3^3\otimes u).$$

By the dimensional reason $d_r=0$ for $r\geq 6$ and hence $E_\infty^*\cong E_6^*$. As E_∞^* is associated to $H^*(\widetilde{G}_2;\ Z_2)$, we have obtained

$$H^*(\widetilde{G}_2;\ Z_2)\cong Z_2[y_8]\otimes\Lambda(y_9,\ y_{11}).$$

To investigate the relations among these elements we consider the spectral sequence $\{E_r^*\}$ associated with (2. 4). Then

$$E_2^*=H^*(Z, 3;\ Z_2)\otimes H^*(G_2;\ Z_2).$$

It is known that

$$H^*(Z, 3;\ Z_2)\cong Z_2[u,\ Sq^2u,\ Sq^4Sq^2u, \cdots],$$

where u is a fundamental class of $H^3(Z, 3;\ Z_2)$. It is easy to see that $d_r(1\otimes y_8)=0$ for $r\leq 8$, whence $E_9^{0,8}\neq 0$. Let p be the projection $G_2'\to K(Z,3)$. Then we have $p^*Sq^4Sq^2u=Sq^4Sq^2x_3=0$ by Theorem 2.1, whence $Sq^4Sq^2u\otimes 1$ must be a d_r-image, that is, $d_9(1\otimes y_8)=Sq^4Sq^2u\otimes 1$. By the Adem's relation, $Sq^1Sq^4Sq^2u=Sq^5Sq^2u=(Sq^2u)^2$. As Sq^1y_8 is also transgressive, so we have

$$d_{10}(1\otimes Sq^1y_8)=Sq^1(Sq^4Sq^2u)\otimes 1=(Sq^2u)^2\otimes 1.$$

Here $(Sq^2u)^2\otimes 1\neq 0$ in E_{10}^*, since it is not a d_r-image for $r\leq 9$. Thus $Sq^1y_8=y_9$. Moreover, by Adem's relation we have $Sq^2Sq^1Sq^4Sq^2u=Sq^2Sq^5Sq^2u=Sq^6Sq^1Sq^2u=Sq^6Sq^3u=u^4$. As $Sq^2Sq^1y_8$ is also transgressive, we have

$$d_{12}(1\otimes Sq^2Sq^1y_8)=Sq^2Sq^1Sq^4Sq^2u\otimes 1=u^4\otimes 1.$$

The fact that $u^4\otimes 1\neq 0$ in E_{12}^* implies the relation $y_{11}=Sq^2Sq^1y_8$. Thus we have shown

(2.5) $$H^*(\widetilde{G}_2;\ Z_2)\cong Z_2[y_8]\otimes\Lambda(Sq^1y_8,\ Sq^2Sq^1y_8).$$

Next we will calculate $H^*(\widetilde{G}_2;\ Z_p)$ for $p\neq 2,5$. For this, we consider the spectral sequence over Z_p associated with (2.3). We have

$$E_2^*=H^*(G_2;\ Z_p)\otimes H^*(Z,2;\ Z_p)\cong\Lambda(x_3,\ x_{11})\otimes Z_p[u].$$

Clearly $d_2=0$, whence $E_3^*\cong E_2^*$. We may choose $u\in H^2(Z,2;\ Z_p)$ so that $d_3(1\otimes u)=x_3\otimes 1$. Then

$$E_4^*\cong Z_p[1\otimes u^p]\otimes\Lambda(x_3\otimes u^{p-1},\ x_{11}\otimes 1).$$

Obviously, $d_r=0$ for $r\geq 4$. Hence $E_\infty^*\cong E_4^*$.

Thus

$$H^*(\widetilde{G}_2;\ Z_p)\cong Z_p[y_{2p}]\otimes\Lambda(y_{11},\ y_{2p+1}).$$

One can easily see that $\delta y_{2p}=y_{2p+1}$ by the same argument as above. Thus we have shown

(2.6) $$H^*(\widetilde{G}_2;\ Z_p)\cong Z_p[y_{2p}]\otimes\Lambda(y_{11},\ \delta y_{2p})\quad \textit{for}\quad p\neq 2,5.$$

Finally consider the case $p=5$. The calculation of the spectral sequence is quite similar to that of the case $p \neq 2,5$ until E_4^*. Namely, for the spectral sequence of (2.3), we have

$$E_4^* \cong Z_5[1 \otimes u^5] \otimes \Lambda(x_3 \otimes u^4, x_{11} \otimes 1).$$

Obviously $d_r = 0$ for $4 \leq r \leq 10$ and hence $E_4^* \cong E_{11}^*$. The relation $\mathcal{P}^1 x_3 = x_{11}$ implies $d_{11}(1 \otimes u^5) = x_{11} \otimes 1$, since $u^5 = \mathcal{P}^1 u$ is also transgressive. It follows

$$E_{12}^* \cong Z_5[1 \otimes u^{25}] \otimes \Lambda(x_3 \otimes u^4, x_{11} \otimes u^{20}).$$

By the dimensional reason $d_r = 0$ for $r \geq 12$, and hence $E_\infty^* \cong E_{12}^*$. Thus we obtain

(2.7) $$H^*(\widetilde{G}_2;\ Z_5) \cong Z_5[y_{50}] \otimes \Lambda(y_{11}, y_{51}).$$

The relation $y_{51} = \delta y_{50}$ is easily seen.

Thus we have shown the following

Theorem 2.3. *Let* G_2 *be the* 3-*connective fibering over* G_2. *Then we have*

(i) $H^*(\widetilde{G}_2;\ Z_2) \cong Z_2[y_8] \otimes \Lambda(Sq^1 y_8, Sq^2 Sq^1 y_8)$.

(ii) $H^*(\widetilde{G}_2;\ Z_5) \cong Z_5[y_{50}] \otimes \Lambda(y_{11}, \delta y_{50})$.

(iii) $H^*(\widetilde{G}_2;\ Z_p) \cong Z_p[y_{2p}] \otimes \Lambda(y_{11}, \delta y_{2p})$ *for a prime* $p \neq 2,5$, *where* $\deg. y_i = i$.

Next we study the cohomology of the 3-connective fibering $\widetilde{F}_4$ over F_4. We have two fibering:

(2.8) $$K(Z, 2) \longrightarrow \widetilde{F}_4 \longrightarrow F_4,$$

(2.9) $$\widetilde{F}_4 \longrightarrow F_4' \longrightarrow K(Z, 3),$$

where F_4' has the same homotopy type as F_4.

First we consider the spectral sequence $\{E_r^*\}$ over Z_2 associated with (2.8).

Then

$$E_2^* = H^*(F_4;\ Z_2) \otimes H^*(Z, 2;\ Z_2)$$
$$\cong (Z_2[x_3]/(x_3^4) \otimes \Lambda(Sq^2 x_3, x_{15}, Sq^8 x_{15})) \otimes Z_2[u].$$

As the degree of x_{15} is 15, so the computation of this spectral sequence is done by the same way as that of G_2. That is,

$$E_{15}^* \cong Z_2[1\otimes u^4]\otimes \Lambda(Sq^2x_3\otimes u^2,\ x_3^3\otimes u,\ x_{15}\otimes 1,\ Sq^8x_{15}\otimes 1).$$

But by the dimensional reason it is easily seen that $d_r=0$ for $r\geq 15$. Thus $E_\infty^* \cong E_{15}^*$, and hence

$$H^*(\widetilde{F}_4;\ Z_2)\cong Z_2[y_8]\otimes \Lambda(y_9,\ y_{11},\ y_{15},\ Sq^8y_{15}).$$

By the same argument as that of G_2, one can obtain the relations $y_9=Sq^1y_8$ and $Sq^2Sq^1y_8=y_{11}$. Thus

$$(2.10)\qquad H^*(\widetilde{F}_4;\ Z_2)\cong Z_2[y_8]\otimes \Lambda(Sq^1y_8,\ Sq^2Sq^1y_8,\ y_{15},\ Sq^8y_{15}).$$

Now we introduce the transgression theorem due to Kudo [11]. Let $\{E_r^*\}$ be the cohomology spectral sequence over Z_p associated with a fibre space (E, p, B, F) in the sense of Serre. For $\alpha\in E_{r+1}^{a,b}$, let $\theta=\theta(\alpha)$ be defined as follows:

$$d_{\rho+1}(\alpha)=0 \quad \text{for} \quad \rho=r, r+1, \cdots, \theta-1,$$
$$\neq 0 \quad \text{for} \quad \rho=\theta.$$

α is called *trangressive* if $\theta(\alpha)\geq b=DF(\alpha)$ (the fibre degree). If α is transgressive, there exists a base element $\beta\in E_2^{a+b+1,0}$ such that $d_{b+1}(\alpha)=\beta$.

Theorem 2.4. (Kudo) *Let $\alpha\in E_2^{0,2k}$ be transgressive, then we have*

(i) $\mathcal{P}^k\alpha=\alpha^p$ *and* $\tau\alpha\otimes\alpha^{p-1}$ *are also transgressive*

(ii) *the following relations hold*:

$$(2.11)\qquad d_{2pk+1}(1\otimes\alpha^p)=\mathcal{P}^k\tau\alpha\otimes 1,$$

$$(2.12)\qquad d_{2(p-1)k+1}(\tau\alpha\otimes\alpha^{p-1})=-\delta\mathcal{P}^k\tau\alpha\otimes 1,$$

where δ denotes the Bockstein operator associated with an exact sequence $0\longrightarrow Z_p\longrightarrow Z_{p^2}\longrightarrow Z_p\longrightarrow 0.$

For the proof see [11].

Let us consider the spectral sequence $\{E_r^*\}$ with Z_3-coefficient associated with (2.8). Then

$$E_2^* = H^*(F_4;\ Z_3) \otimes H^*(Z, 2;\ Z_3)$$
$$\cong (Z_3[\delta\mathcal{P}^1 x_3]/((\delta\mathcal{P}^1 x_3)^3)$$
$$\otimes \Lambda(x_3, \mathcal{P}^1 x_3, x_{11}, \mathcal{P}^1 x_{11})) \otimes Z_3[u].$$

Clearly $d_2 = 0$ and hence $E_3^* \cong E_2^*$. We may choose $u \in H^2(Z, 2;\ Z_3)$ so that $d_3(1 \otimes u) = x_3 \otimes 1$. It follows that

$$E_4^* \cong Z_3[1 \otimes u^3, \delta\mathcal{P}^1 x_3 \otimes 1]/((\delta\mathcal{P}^1 x_3 \otimes 1)^3)$$
$$\otimes \Lambda(x_3 \otimes u^2, \mathcal{P}^1 x_3 \otimes 1, x_{11} \otimes 1, \mathcal{P}^1 x_{11} \otimes 1).$$

Clearly $d_2 = 0$ and hence $E_3^* \cong E_2^*$. It follows from Theorem 2.4 that $d_5(x_3 \otimes u^2) = \delta\mathcal{P}^1 x_3 \otimes 1$. Hence

$$E_6^* \cong Z_3[1 \otimes u^3] \otimes \Lambda((\delta\mathcal{P}^1 x_3)^2 x_3 \otimes u^2, \mathcal{P}^1 x_3 \otimes 1, x_{11} \otimes 1, \mathcal{P}^1 x_{11} \otimes 1).$$

$E_7^* \cong E_6^*$, since $d_6 = 0$. As the transgression commutes with $\mathcal{P}^1$, we get $d_7(1 \otimes u^3) = \mathcal{P}^1 x_3 \otimes 1$ and hence

$$E_8^* \cong Z_3[1 \otimes u^9] \otimes \Lambda(\mathcal{P}^1 x_3 \otimes u^6, (\delta\mathcal{P}^1 x_3)^2 x_3 \otimes u^2, x_{11} \otimes 1, \mathcal{P}^1 x_{11} \otimes 1).$$

By the dimensional reason it is seen that $d_r = 0$ for $r \geq 8$, and hence $E_\infty^* \cong E_8^*$. Thus we obtain

$$H^*(\widetilde{F}_4;\ Z_3) \cong Z_3[y_{18}] \otimes \Lambda(y_{11}, \mathcal{P}^1 y_{11}, y_{19}, y_{23}).$$

In order to see the relations among y_{18}, y_{19} and y_{23}, we consider the spectral sequence $\{E_r^*\}$ associated with (2.9).

Then

$$E_2^* = H^*(Z, 3;\ Z_3) \otimes H^*(\widetilde{F}_4;\ Z_3).$$

According to Cartan [5],

$$H^*(Z, 3;\ Z_3) \cong \Lambda(u, \mathcal{P}^1 u, \mathcal{P}^3\mathcal{P}^1 u, \cdots) \otimes Z_3[\delta\mathcal{P}^1 u, \delta\mathcal{P}^3\mathcal{P}^1 u, \cdots].$$

It is easy to see that $d_r(1 \otimes y_{18}) = 0$ for $r \leq 18$. Then $E_{19}^{0,18} \neq 0$. Let p be the projection $F_4' \to K(Z, 3)$. Then the element $x_3(\delta\mathcal{P}^1 x_3)^2$ of $H^{19}(F_4;\ Z_3)$ is the p^*-image of $u(\delta\mathcal{P}^1 u)^2$. On the other hand the element $\mathcal{P}^3\mathcal{P}^1 u \otimes 1$ is not a d_r-image for $r < 19$. Thus it must be a d_{19}-image, since $\mathcal{P}^3\mathcal{P}^1 x_3 = 0$ by (2.2). By changing the coefficient of y_{18}, if necessary, we have $d_{19}(1 \otimes y_{18}) = \mathcal{P}^3\mathcal{P}^1 u \otimes 1$. As δy_{18} is also transgressive, we have $d_{20}(1\otimes\delta y_{18})=\delta\mathcal{P}^3\mathcal{P}^1u\otimes 1$. Here $\delta\mathcal{P}^3\mathcal{P}^1u\otimes 1$ is not a d_r-image for $r<20$, whence $\delta\mathcal{P}^3\mathcal{P}^1u\otimes 1\neq 0$. This shows $\delta y_{18}\neq 0$, and so $\delta y_{18}=y_{19}$. Similarly $\mathcal{P}^1\delta y_{18}$ is transgressive and so $d_{23}(1\otimes\mathcal{P}^1\delta y_{18})=\mathcal{P}^1\delta\mathcal{P}^3\mathcal{P}^1u\otimes 1$, where $\mathcal{P}^1\delta\mathcal{P}^3\mathcal{P}^1u=\mathcal{P}^4\delta\mathcal{P}^1u=(\delta\mathcal{P}^1u)^3$ by the Adem's relation. It is easily seen that $(\delta\mathcal{P}^1u)^3\otimes 1$ is not a d_r-image for $r<23$ and hence $(\delta\mathcal{P}^1u)^3\otimes 1\neq 0$ in E_{23}^*, which indicates $\mathcal{P}^1\delta y_{18}\neq 0$. Thus $\mathcal{P}^1\delta y_{18}=y_{23}$.

Next we will show $\mathcal{P}^2y_{11}=\mathcal{P}^3y_{11}=0$. Note that $p^*x_{11}=y_{11}$ for the projection $p:\widetilde{F}_4\to F_4$ of (2.8). The elements of the degree 19 in $H^*(F_4;\ Z_3)$ are $(\delta\mathcal{P}^1x_3)x_{11}$ and $(\delta\mathcal{P}^1x_3)^2x_3$. These two elements are mapped to zero by p^*. Hence $\mathcal{P}^2y_{11}=p^*(\mathcal{P}^2x_{11})=0$. Similarly $\mathcal{P}^3y_{11}=0$ follows.

Thus we have shown

(2.13) $\qquad H^*(\widetilde{F}_4;\ Z_3)\cong Z_3[y_{18}]\otimes\Lambda(y_{11},\mathcal{P}^1y_{11},\delta y_{18},\mathcal{P}^1\delta y_{18})$,

where $\qquad \mathcal{P}^2y_{11}=\mathcal{P}^3y_{11}=0$.

Consider the spectral sequence $\{E_r^*\}$ over Z_5 associated with (2.8). Then we have

$$E_2^*=H^*(F_4;\ Z_5)\otimes H^*(Z,2;\ Z_5)$$
$$\cong\Lambda(x_3,\mathcal{P}^1x_3,x_{15},\mathcal{P}^1x_{15})\otimes Z_5[u].$$

Clearly $d_2=0$ and hence $E_2^*\cong E_3^*$. We may choose the fundamental class $u\in H^2(Z,2;\ Z_5)$ so that $d_3(1\otimes u)=x_3\otimes 1$. It follows that

$$E_4^*\cong Z_5[1\otimes u^5]\otimes\Lambda(x_3\otimes u^4,\mathcal{P}^1x_3\otimes 1,x_{15}\otimes 1,\mathcal{P}^1x_{15}\otimes 1).$$

By the dimensional reason $d_r=0$ for $4\leq r\leq 10$ and hence $E_{11}^*\cong E_4^*$. There we obtain $d_{11}(1\otimes u^5)=\mathcal{P}^1x_3\otimes 1$, because the transgression commutes with $\mathcal{P}^1$. Therefore

$$E_{12}^*\cong Z_5[1\otimes u^{25}]\otimes\Lambda(x_3\otimes u^4,\mathcal{P}^1x_3\otimes u^{20},x_{15}\otimes 1,\mathcal{P}^1x_{15}\otimes 1).$$

It follows from the dimensional reason that d_r is trivial for $r\geq 12$, and hence $E_\infty^*\cong E_{12}^*$. Thus we get

$$H^*(\widetilde{F}_4;\ Z_5)=Z_5[y_{50}]\otimes\Lambda(y_{11},y_{15},\mathcal{P}^1y_{15},y_{51}).$$

It is easily checked by the spectral theory associated with (2. 9) that $\delta y_{50}=y_{51}$.

Thus we have shown

$$(2.14) \qquad H^*(\widetilde{F}_4;\ Z_5)\cong Z_5[y_{50}]\otimes\Lambda(y_{11},y_{15},\mathcal{P}^1y_{15},\delta y_{50}).$$

The same calculation as that for the case $p=5$ shows

$$(2.15) \qquad H^*(\widetilde{F}_4;\ Z_7)\cong Z_7[y_{98}]\otimes\Lambda(y_{11},y_{15},\mathcal{P}^1y_{11},\delta y_{98}).$$

$$(2.16) \qquad H^*(\widetilde{F}_4;\ Z_{11})\cong Z_{11}[y_{242}]\otimes\Lambda(y_{11},y_{15},y_{23},\delta y_{242}).$$

The calculation for the case $p\geq 13$ is easier than the other cases, since there are no relations among generators of $H^*(F_4;\ Z_p)$. The results are stated as follows.

$$(2.17) \qquad H^*(\widetilde{F}_4;\ Z_p)\cong Z_p[y_{2p}]\otimes\Lambda(y_{11},y_{15},y_{23},\delta y_{2p}).$$

Summing up these results,

Theorem 2. 5. *Let $\widetilde{F}_4$ be the 3-connective fibering over F_4. Then we have*

(i) $H^*(\widetilde{F}_4;\ Z_2)\cong Z_2[y_8]\otimes\Lambda(Sq^1y_8,Sq^2Sq^1y_8,y_{15},Sq^8y_{15})$.

(ii) $H^*(\widetilde{F}_4;\ Z_3)\cong Z_3[y_{18}]\otimes\Lambda(y_{11},\mathcal{P}^1y_{11},\delta y_{18},\mathcal{P}^1\delta y_{18})$, *where* $\mathcal{P}^2y_{11}=\mathcal{P}^3y_{11}=0$.

(iii) $H^*(\widetilde{F}_4;\ Z_p)\cong Z_p[y_{2p^2}]\otimes\Lambda(y_{11},y_{15},y_{23},\delta y_{2p^2})$ *for* $p=5,7,11$, *where* $\mathcal{P}^1_5y_{15}=y_{23}$, $\mathcal{P}^1_7y_{11}=y_{23}$.

(iv) $H^*(\widetilde{F}_4;\ Z_p)\cong Z_p[y_{2p}]\otimes\Lambda(y_{11},y_{15},y_{23},\delta y_{2p})$ *for* $p\geq 13$.

In the above deg. $y_i=i$.

Theorem 2. 3 and 2. 5 give much informations for the homotopy groups of G_2 and F_4. In the below we will investigate them.

§3. The odd primary components of $\pi_i(G_2)$ and $\pi_i(F_4)$.

Let G be a compact connected, simply connected, simple Lie group. According to Hopf, we have

$$H^*(G;\ R)=\Lambda_R(x_{n_1},x_{n_2}\cdots,x_{n_l}),$$

where deg. $x_{n_i}=n_i$: odd$(1\leq i\leq l)$, $l=\text{rank}\, G$ and $n=\dim G=\sum_{i=1}^{l} n_i$. We set $X(G)=S^{n_1}\times\cdots\cdots\times S^{n_l}$.

Serre defines a prime p to be *regular* for G if there exists a map f: $X(G)\to G$ such that $f_*: H_i(X(G); Z_p)\to H_i(G; Z_p)$ is an isomorphism for $i\geq 0$.

Put $N(G)=(\dim. G/\text{rank}\, G)-1$. Then the following theorem is due to Serre [17] and Kumpel [12].

Theorem 3.1. *A prime p is regular for G if and only if $p\geq N(G)$.*

For the cases G_2 and F_4, we have

$$H^*(G_2; R)=\Lambda_R(x_3, x_{11}),$$
$$H^*(F_4; R)=\Lambda_R(x_3, x_{11}, x_{15}, x_{23}).$$

Hence $N(G_2)=6$ and $N(F_4)=12$. It follows from these facts

Corollary 3.2.

$\pi_i(G_2: p)\cong\pi_i(S^3\times S^{11}: p)$ *for each prime* $p\geq 7$.

$\pi_i(F_4: p)\cong\pi_i(S^3\times S^{11}\times S^{15}\times S^{23}: p)$ *for each prime* $p\geq 13$.

In the below we will compute $\pi_i(G_2: p)$ for $p=3,5$ and $\pi_i(F_4: p)$ for $p=3, 5, 7, 11$ by making use of the Serre's $\mathcal{C}$-theory [17].

(I) $\pi_i(G_2: p)$ $p=3$ *and* 5.

It follows immediately from (i) of Theorem 2.1

(3.1) $$\pi_i(G_2: p)\cong\pi_i(S^3: p)$$
for $i\leq 9$ *and for each prime* $p\geq 3$.

The 5-components of $\pi_i(G_2)$ are deduced immediately from (ii) of Theorem 2.3 and the results are the following

Proposition 3.3.

$$\pi_i(G_2: 5)\cong\pi_i(S^{11}: 5) \quad \textit{for} \quad 3<i<49.$$

Further results are seen in [19].

In order to calculate the 3-components of $\pi_i(G_2)$, we consider the fibration $G_2/S^3=V_{7,2}$. Associated with it we have the exact sequence:

$$\cdots\to\pi_{11}(S^3:3)\to\pi_{11}(G_2:3)\to\pi_{11}(V_{7,2}:3)\xrightarrow{\Delta'}\pi_{10}(S^3:3)\to\pi_{10}(G_2:3)\to\cdots$$

where $\pi_{11}(S^3:3)=0$ and $\pi_{10}(S^3:3)\cong Z_3$ by [18]. And $\pi_{10}(G_2:3)=0$, since we have $\pi_{10}(Spin(7):3)\cong\pi_{10}(Sp(3):3)=0$ by [8] in the following exact sequence which is associated with the fibering $Spin(7)/G_2=S^7$:

$$0=\pi_{11}(S^7)\to\pi_{10}(G_2)\to\pi_{10}(Spin(7))\to\cdots.$$

Next we need

Lemma 3. 4.

$$\pi_{11}(V_{7,2}:3)\cong Z.$$

This follows from the exact sequence associated with the fibering $V_{7,2}/S^5=S^6$:

$$\cdots\to\pi_{11}(S^5)\to\pi_{11}(V_{7,2})\to\pi_{11}(S^6)\to\pi_{10}(S^5)\to\cdots,$$

where $\pi_{11}(S^5:3)=\pi_{10}(S^5:3)=0$ and $\pi_{11}(S^6:3)\cong Z$ by [18].

We choose a map $f: S^{11}\to V_{7,2}$ representing a generator of $\pi_{11}(V_{7,2}:3)\cong Z$, then $f^*: H^*(V_{7,2}; Z_3)\cong H^*(S^{11}; Z_3)$. We consider the induced bundle f^*G_2 of f from the bundle $G_2/S^3=V_{7,2}$.

$$\begin{array}{ccccc}\pi_{11}(V_{7,2}:3) & \xrightarrow{\Delta'} & \pi_{10}(S^3:3) & \longrightarrow & 0\\ \uparrow f_* & & \| & & \\ \pi_{11}(S^{11}:3) & \xrightarrow{\Delta} & \pi_{10}(S^3:3)\cong Z_3. & & \end{array}$$

The characteristic class of the bundle (f^*G_2, p, S^{11}, S^3), $\Delta\iota_{11}$, equals to $\Delta' f_*\iota_{11}$ by the commutativity of the above diagram, where Δ' is the boundary homomorphism of $G_2/S^3=V_{7,2}$. So $\Delta\iota_{11}$ is a generator of $\pi_{10}(S^3:3)\cong Z_3$, since the map f induces an isomorphism $f_*: \pi_{11}(S^{11}:3)\cong\pi_{11}(V_{7,2}:3)$. Consider the homomorphism between the exact sequences associated with $G_2/S^3=V_{7,2}$ and $f^*G_2/S^3=S^{11}$. Then the homomorphism is identical on $\pi_i(S^3)$ and $\mathcal{C}_3$-isomorphic between $\pi_i(V_{7,2})$ and $\pi_i(S^{11})$. Hence it is also $\mathcal{C}_3$-isomorphic between $\pi_i(G_2)$ and $\pi_i(f^*G_2)$. Thus we have

(3.2) $$\pi_i(G_2 : 3) \cong \pi_i(f^* G_2 : 3).$$

In order to calculate this group we need some results in [18]. ($\pi_i(G_2 : 3)$ for $i \leq 9$ are obtained from the known results of [18]).

i	10	11	12	13	14	15	16	17	18	19	20	21	22	23
$\pi_{i+1}(S^{11} : 3)$	Z	0	0	Z_3	0	0	0	Z_3	0	0	Z_3	Z_9	0	Z_3
gen.	ι_{11}			α_1				α_2			β_1	α_3'		$\alpha_1\beta_1$
$\pi_i(S^3 : 3)$	Z_3	0	0	Z_3	Z_3	0	Z_3	Z_3	Z_3	Z_3	Z_3	Z_3	Z_3	0
gen.	α_2			$\alpha_2\alpha_1$	α_3		$\alpha_1\beta_1$	α_2^2	α_4	$\alpha_1^2\beta_1$	$\alpha_2\beta_1$	$\alpha_2\alpha_3'$	α_5	

In the above table, the generators of $\pi_i(S^3 : 3)$ for $i = 10, 14, 16, 18$ and 22 are given in Chapter XIII of [18]. The other generators are checked as follows.

Consider the exact sequence in Proposition 13.3 of [18];

$$\cdots \to \pi_{i+2}(S^7 : 3) \xrightarrow{\Delta} \pi_i(S^5 : 3) \xrightarrow{G} \pi_{i+1}(S^3 : 3) \xrightarrow{H} \pi_{i+1}(S^7 : 3) \to \cdots,$$

where $G(\beta) = \alpha_1 \circ S\beta$ for $\beta \in \pi_i(S^5 : 3)$.

Note that $H(\alpha_2) = \alpha_1$. Then we have $H(\alpha_2\beta_1) = \alpha_1\beta_1 \neq 0$. Thus $\alpha_2\beta_1 \neq 0$. Moreover we have

$$\alpha_2\alpha_3' \in \{\alpha_1, 3\iota, \alpha_1\}\alpha_3' = -\alpha_1\{3\iota, \alpha_1, \alpha_3'\} \equiv -\alpha_1\{\alpha_3', \alpha_1, 3\iota\} \ni -\alpha_1\alpha_4.$$

Hence $\alpha_2\alpha_3' \equiv -\alpha_1\alpha_4 \bmod \{3\alpha_1\pi_{21}(S^6 : 3) \oplus \alpha_1\pi_{10}(S^6 : 3)\alpha_3'\} = 0$. Here we have $\alpha_1\alpha_4 \neq 0$, since α_4 is not a Δ-image. Thus $\alpha_2\alpha_3' \neq 0$. We have the relation $\alpha_2^2 = -\alpha_1\alpha_3'$, since $\alpha_2^2 \in \{\alpha_1, 3\iota, \alpha_1\}\alpha_2 = -\alpha_1\{3\iota, \alpha_1, \alpha_2\} \ni \alpha_1\alpha_3'$ mod 0. So $\pi_{13}(S^3 : 3)$ is generated by $\alpha_2\alpha_1$. Similarly it follows from the relation $\alpha_1\alpha_2 = -\alpha_2\alpha_1$ that $\pi_{13}(S^3 : 3)$ is generated by $\alpha_2\alpha_1$. We have $\alpha_1^2\beta_1 = G(\alpha_1\beta_1)$ and $\alpha_1\beta_1$ is not a Δ-image. Hence $\alpha_1^2\beta_1 \neq 0$. So $\pi_{19}(S^3 : 3)$ is generated by $\alpha_1^2\beta_1$.

Now the characteristic class of the bundle f^*G_2 is $\Delta\iota_{11} = \alpha_2$. By making use of the above table one can obtain

(3.3) $$\pi_i(G_2 : 3) \cong \pi_i(f^* G_2 : 3) \cong \begin{cases} Z_3 & \text{for} \quad i = (6, 9,)14, 16, 18, 19 \\ Z_9 & \text{for} \quad i = 22 \\ Z & \text{for} \quad i = (3,)11 \\ 0 & \text{otherwise for} \quad i < 24. \end{cases}$$

The only difficulty to determine $\pi_i(f^*G_2\colon 3)$ will be found in the case $i=22$. In this case one has the extension

$$0\longrightarrow Z_3\xrightarrow{i_*}\pi_{22}(f^*G_2:3)\xrightarrow{p_*}Z_3\longrightarrow 0.$$

It follows from Theorem 4.3 in §4 that for an arbitrary element δ of $\{\alpha_2, \alpha_3, 3\iota\}\subset\pi_{22}(S^3\colon 3)$, there exists an element $\varepsilon\in\pi_{22}(f^*G_2\colon 3)$ such that $p_*\varepsilon=\alpha_3$ and $i_*\delta=3\varepsilon$. Consider the stable secondary composition $\langle\alpha_2, \alpha_3, 3\iota\rangle=S^\infty\{\alpha_2, \alpha_3, 3\iota\}$. Then we have

$$\begin{aligned}\langle\alpha_2, \alpha_3, 3\iota\rangle &=\langle\langle\alpha_1, \alpha_1, 3\iota\rangle, \alpha_3, 3\iota\rangle\\ &=\pm\langle\alpha_1, \langle\alpha_1, 3\iota, \alpha_3\rangle, 3\iota\rangle\\ &=\langle\alpha_1, \alpha_4, 3\iota\rangle\\ &=\alpha_5.\end{aligned}$$

Hence the order of ε in the above is 9. Thus we have shown

$$\pi_{22}(f^*G_2\colon 3)\cong Z_9.$$

Remark 3.5. Analogously one can calculate the 5-components of $\pi_i(G_2)$.

(II) $\pi_i(F_4\colon p)$ $p=3, 5, 7$ *and* 11.

Hereafter we denote by $F_4^{(n)}$ the $(n-1)$-connective fibre space over F_4, so that

$$\pi_i(F_4^{(n)})\cong\begin{cases}\pi_i(F_4) & \text{for } i\geq n\\ 0 & \text{for } i<n.\end{cases}$$

For example $F_4^{(4)}=\widetilde{F}_4$.

It follows directly from (iii) of Theorem 2.5 that

(3.4) $\quad\pi_i(F_4\colon 11)\cong\pi_i(S^{11}\times S^{15}\times S^{23}\colon 11)$ *for* $3<i<241$.

Consider the cohomology spectral sequence over Z_7 associated with the following fibering: $K(Z, 10)\rightarrow F_4^{(12)}\rightarrow\widetilde{F}_4$. Then

$$\begin{aligned}E_2^* &= H^*(F_4;\ Z_7)\otimes H^*(Z, 10;\ Z_7)\\ &\cong Z_7[y_{98}]\otimes\Lambda(y_{11}, y_{15}, \mathcal{P}^1y_{11}, \delta y_{98})\\ &\quad\otimes Z_7[u, \mathcal{P}^1u, \mathcal{P}^2u, \cdots]\otimes\Lambda(\delta\mathcal{P}^1u, \delta\mathcal{P}^2u, \cdots).\end{aligned}$$

Obviously $E_2^* \cong E_{10}^*$. We choose $u \in H^{10}(Z, 10; Z_7)$ so that $d_{10}(1 \otimes u) = y_{11} \otimes 1$. Hence

$$E_{11}^* \cong Z_7[1 \otimes \mathcal{P}^1 u, 1 \otimes \mathcal{P}^2 u, \cdots] \otimes \Lambda(y_{15} \otimes 1, \mathcal{P}^1 y_{11} \otimes 1, 1 \otimes \delta \mathcal{P}^1 u, 1 \otimes \delta \mathcal{P}^2 u, \cdots) \quad \text{for} \quad \text{dim.} < 70.$$

By the dimensional reason $d_r = 0$ for $11 \leq r < 23$, whence $E_{11}^* \cong E_{23}^*$. Here we have $d_{23}(1 \otimes \mathcal{P}^1 u) = \mathcal{P}^1 y_{11} \otimes 1$, since the transgression commutes with $\mathcal{P}^1$. Thus

$$E_{24}^* \cong Z_7[1 \otimes \mathcal{P}^2 u, \cdots] \otimes \Lambda(y_{15} \otimes 1, 1 \otimes \delta \mathcal{P}^1 u, 1 \otimes \delta \mathcal{P}^2 u, \cdots) \quad \text{for} \quad \text{dim.} < 70.$$

It is easily seen that $d_r = 0$ for $r \geq 24$, and hence $E_{24}^* \cong E_\infty^*$ (dim.< <70). The degree of the elements $\delta \mathcal{P}^1 u$ and $\mathcal{P}^2 u$ are 23 and 34 respectively. So we obtain that

$$H^*(F_4^{(12)}; Z_7) = \{z_{15}, z_{23}\} \quad \text{for} \quad \text{dim.} < 34,$$

where { } represents the additive basis.

It follows that

$$(3.5) \qquad \pi_i(F_4; 7) \cong \pi_i(S^{15} \times S^{23}: 7) \quad \textit{for} \quad 11 < i \leq 32.$$

Recall that $H^*(\widetilde{F}_4: Z_5) = Z_5[y_{50}] \otimes \Lambda(y_{11}, y_{15}, \mathcal{P}^1 y_{15}, \delta y_{50})$. Let f be a map: $S^{11} \to \widetilde{F}_4$ representing a generator of $\pi_{11}(F_4: 5) \cong Z$. We may regard this map as a fibering. Let F be its fibre. Then it is easily obtained that

$$H^*(F; Z_5) = \{z_{14}, \mathcal{P}^1 z_{14}\} \qquad \text{for dim.} < 25.$$

Associated with it we have the exact sequence

$$\cdots \to \pi_i(S^{11}: 5) \to \pi_i(F_4: 5) \to \pi_{i-1}(F: 5) \to \pi_{i-1}(S^{11}: 5) \to \cdots.$$

Here we have $\pi_i(F: 5) \cong \begin{cases} Z & \text{for} \quad i = 14 \quad \text{and} \quad 22 \\ 0 & \text{otherwise for} \quad i < 24. \end{cases}$

It follows directly that

$$(3.6) \qquad \pi_i(F_4: 5) \cong \begin{cases} Z & \textit{for} \quad i = 11, 15, 23 \\ Z_5 & \textit{for} \quad i = 18 \\ 0 & \textit{otherwise for} \quad 3 < i < 23. \end{cases}$$

As to the 3-components of $\pi_i(F_4)$ we need more computations.

Consider the spectral sequence with Z_3-coefficient associated with a fibering $K(Z,10)\overset{i}{\rightarrow}F_4^{(12)}\overset{p}{\rightarrow}\widetilde{F}_4$. Then we have

$$E_2^*\cong H^*(\widetilde{F}_4;\ Z_3)\otimes H^*(Z,10;\ Z_3)$$
$$\cong Z_3[y_{18}]\otimes\Lambda(y_{11},\mathcal{P}^1y_{11},\delta y_{18},\mathcal{P}^1\delta y_{18})$$
$$\otimes Z_3[u,\mathcal{P}^1u,\mathcal{P}^2u,\mathcal{P}^3u,\cdots]\otimes\Lambda(\delta\mathcal{P}^1u,\delta\mathcal{P}^2u,\delta\mathcal{P}^3u,\cdots).$$

We choose an element $u\in H^{10}(Z,10;\ Z_3)$ so that it may satisfy the relation $d_{11}(1\otimes u)=y_{11}\otimes 1$. (Obviously $d_r=0$ for $r<11$, and hence $E_{11}^*\cong E_2^*$). The element $\mathcal{P}^1u$ is also transgressive and $d_{15}(1\otimes\mathcal{P}^1u)=\mathcal{P}^1y_{11}\otimes 1$ holds. The other elements of $E_r^{0,*}$ are d_r-cocycle for $r\geq 11$. Hence we obtain

$$E_\infty^*\cong Z_3[y_{18}\otimes 1,1\otimes\mathcal{P}^2u,1\otimes\mathcal{P}^3u,\cdots]$$
$$\otimes\Lambda(\delta y_{18}\otimes 1,\mathcal{P}^1\delta y_{18}\otimes 1,1\otimes\delta\mathcal{P}^1u,1\otimes\delta\mathcal{P}^2u,1\otimes\delta\mathcal{P}^3u,\cdots)$$
$$\text{for dim.}<30,$$

where $1\otimes\mathcal{P}^iu$ and $1\otimes\delta\mathcal{P}^iu$ are of degree $4i+10(i\geq 2)$ and $4i+11$ $(i>1)$ respectively. Thus

$$H^*(F_4^{(12)};\ Z_3)=\{z_{18},\delta z_{18},\mathcal{P}^1\delta z_{18},a_{18},a_{22},b_{15},b_{19},b_{23}\}\quad\text{for dim.}<26,$$

where a_{18}, a_{22} correspond to $1\otimes\mathcal{P}^2u$, $1\otimes\mathcal{P}^3u$ and b_{15}, b_{19}, b_{23} to $1\otimes\delta\mathcal{P}^1u$, $1\otimes\delta\mathcal{P}^2u$, $1\otimes\delta\mathcal{P}^3u$ respectively. Here we have the relations as follows:

$$i^*(\mathcal{P}^1b_{15}-b_{19})=0,\quad\text{and hence}\quad\mathcal{P}^1b_{15}\equiv b_{19}\quad\text{mod }\delta y_{18}.$$
$$i^*(\mathcal{P}^2b_{15}-b_{23})=0,\quad\text{and hence}\quad\mathcal{P}^2b_{15}\equiv b_{23}\quad\text{mod }\mathcal{P}^1\delta y_{18}.$$
$$i^*(\delta a_{18}-b_{19})=0,\quad\text{and hence}\quad\delta a_{18}\equiv b_{19}\quad\text{mod }\delta y_{18}.$$
$$i^*(\delta a_{22}-b_{23})=0,\quad\text{and hence}\quad\delta a_{22}\equiv b_{23}\quad\text{mod }\mathcal{P}^1\delta y_{18}.$$

But it is easily seen that one may choose appropriately a_{18}, b_{19} and b_{22} so that the next relations hold:

$$\mathcal{P}^1b_{15}=b_{19}=\delta a_{18}\tag{3.7}$$
$$\mathcal{P}^2b_{15}=b_{23}+A\mathcal{P}^1\delta y_{18},\quad b_{23}=\delta a_{22}.\ (A=0,1,2.)$$

(We cannot determine whether or not A is zero.) Thus we have shown

(3.8) $$H^*(F_4^{(12)}:Z_3)=\{y_{18},\delta y_{18},\mathcal{P}^1\delta y_{18},b_{15},\mathcal{P}^1b_{15},b_{23},a_{18},a_{22}\}$$

for dim.<26, *where the relations* (3.7) *hold.*

It follows from (3.8) that

$$\pi_i(F_4:3)=0 \quad \text{for} \quad 12\leq i\leq 14$$
$$\cong Z \quad \text{for} \quad i=15.$$

Case 1. $A=0$.

By calculating the spectral sequence associated with fiberings $F_4^{(16)}\to F_4^{(12)}$ and $F_4^{(19)}\to F_4^{(16)}$ one may easily obtain that

(3.9) $$H^*(F_4^{(16)}:Z_3)=\{y_{18},\delta y_{18},\mathcal{P}^1\delta y_{18},a_{18},\delta_2a_{18},a_{22},\delta_2a_{22}=\mathcal{P}^1\delta_2a_{18}\}$$

(3.10) $$H^*(F_4^{(19)}:Z_3)=\{d_{21},\delta d_{21},d_{23},e_{21},\delta e_{21},a_{22},\delta_3a_{22}\}$$
for dim.<26,

where δ_n is the Bockstein operation associated with an exact sequence

$$0\to Z_3\to Z_{3^{n+1}}\to Z_{3^n}\to 0. \quad (\delta_1=\delta)$$

It follows (3.9) and (3.10) that

$$\pi_i(F_4:3)\cong\begin{cases}0 & i=16,17,19,20\\ Z_3\oplus Z_9 & i=18\\ Z_3\oplus Z_3 & i=21\\ Z_{27} & i=22\\ Z & i=23.\end{cases}$$

Case 2. $A\neq 0$.

Similarly one may easily obtain that

(3.9)′ $$H^*(F_4^{(16)}:Z_3)=\{y_{18},\delta y_{18},\mathcal{P}^1\delta y_{18}=\delta a_{22},a_{18},\delta_2a_{18},\mathcal{P}^1\delta a_{18},a_{22}\}$$

(3.10)′ $$H^*(F_4^{(19)}:Z_3)=\{d_{21},\delta d_{21},a_{22},\delta_2a_{22},e_{21},\delta e_{21},e_{23}\}$$
for dim.<26.

It follows from (3.9)′ and (3.10)′ that

$$\pi_i(F_4:3)=\begin{cases}0 & i=16,17,19,20\\ Z_3\oplus Z_9 & i=18\\ Z_3\oplus Z_3 & i=12\end{cases}$$

$$\begin{cases} Z_9 & i=22 \\ Z & i=23. \end{cases}$$

In any way we have shown

$$(3.11) \qquad \pi_i(F_4: 3) = \begin{cases} Z_3 \oplus Z_9 & i=18 \\ Z_3 \oplus Z_3 & i=21 \\ Z_9 \text{ or } Z_{27} & i=22 \\ Z & i=3, 11, 15, 23 \\ 0 & \text{otherwise for } \; i<24. \end{cases}$$

§4. Some properties in the fibre theory.

We denote by $\pi(A, B; C, D)$ the set of the homotopy classes of maps $f: (A, B, a_0) \to (C, D, c_0)$ for topological pairs (A, B, a_0) and (C, D, c_0).

Let X be a CW-complex with a base point x_0. Let $S^n X = X \wedge S^n$ the smashed product of X and the unit n-sphere S^n and let $CS^n X$ be the cone over $S^n X$.

Then for an arbitrary topological pair (A, B, a_0) we have the following exact sequence:

$$(4.1) \qquad \cdots \to \pi(S^{n+1}X, A) \xrightarrow{j_*} \pi(CS^n X, S^n X; A, B) \xrightarrow{\partial} \pi(S^n X, B) \xrightarrow{i_*} \cdots.$$

Let (E, p, B) be a fibre space with a fibre F in the sense of Serre, that is, it has a covering homotopy property. Then we have a one-to-one correspondence

$$(4.2) \qquad p_*: \pi(CX, X; E, F) \cong \pi(SX, B).$$

Define a boundary homomorphism $\Delta: \pi(S^{n+1}X, B) \to \pi(S^n X, F)$ by the commutativity of the following diagram.

$$\begin{array}{ccccccc} \cdots \to \pi(S^{n+1}X, E) & \xrightarrow{j_*} & \pi(CS^n X, S^n X; E, F) & \xrightarrow{\partial} & \pi(S^n X, F) \to \cdots \\ & \searrow^{p_*} & \Vert \wr \, p_* & \nearrow_{\Delta} & \\ & & \pi(S^{n+1}X, B) & & \end{array}$$

For this boundary homomorphism Δ, we have

Proposition 4.1. *Let Y be another CW-complex with a base* *point* y_0. *Then*

$\Delta(\alpha\circ S\beta)=(\Delta\alpha)\circ\beta$ *for* $\alpha\in\pi(S^{n+1}X, B)$ *and* $\beta\in\pi(S^n Y, S^n X)$.

Here S is a suspension homomorphism given by the commutativity of the diagram:

$$\begin{array}{ccc} \pi(S^n Y, S^n X) & \xrightarrow{S} & \pi(S^{n+1}Y, S^{n+1}X) \\ \cong\nwarrow\partial & & \nearrow p_* \\ & \pi(CS^n Y, S^n Y;\ CS^n X, S^n X) & \end{array}$$

where p pinches $S^n X$.

As to the secondary composition (the definition is referred to [18]) we have the following

Proposition 4.2. *Assume that* $\alpha\circ S\beta=\beta\circ\gamma=0$ *for* $\alpha\in\pi(S^{n+1}X, B)$ $\beta\in\pi(S^n Y, S^n X)$ *and* $\gamma\in\pi(S^n Z, S^n Y)$, *where* X, Y, Z *are CW-complexes with base points. Then we have*

$$\Delta\{\alpha, S\beta, S\gamma\}_1\subset\{\Delta\alpha, \beta, \gamma\}.$$

The proof may be found in §5 of [15].

Theorem 4.3. *Assume that* $\alpha\in\pi(S^{i+1}X, B)$, $\beta\in\pi(S^j Y, S^i X)$ *and* $\gamma\in\pi(S^k Z, S^j Y)$ *satisfy the conditions* $(\Delta\alpha)\circ\beta=0$ *and* $\beta\circ\gamma=0$. *Then for an arbitrary element* δ *of* $\{\Delta\alpha, \beta, \gamma\}\subset\pi(S^{k+1}Z, F)$, *there exists an element* $\varepsilon\in\pi(S^{j+1}Y, E)$ *such that* $p_*\varepsilon=\alpha\circ S\beta$ *and* $i_*\delta=\varepsilon\circ S\gamma$.

This is a generalization of Theorem 2.1 of [14] but proved by the quite similar manner.

Let G be a compact Lie group. For a principal G-bundle $(E, p, S^{i+1}=E/G)$ the element $\Delta\iota_{i+1}=\chi(E)\in\pi_i(G)$ is called *the characteristic class* of the bundle and it determines the bundle up to equivalence.

Theorem 4.4. *Let* $j\geq 2$ *and let* C_p *be the class of finite abelian groups without* p*-torsion* (p *a prime*).
Suppose that $q\chi(E)=q'\chi(E')$ *for two* G*-bundles* E, E' *with the same base and for* q, q' *prime to* p.

Then $\pi_j(E)$ *and* $\pi_j(E')$ *are* C_p*-isomorphic to each other for all* j.

This is Lemma 2.3 of [14]. The following is a direct consequence of this theorem.

Corollary 4.5. *If the order of* $\chi(E)$ *is finite and prime to* p, *then we obtain*

$$\pi_j(E) \underset{C_p}{\cong} \pi_j(S^{i+1}) \oplus \pi_j(G).$$

Proposition 4.6. *In a fibre space* (E, p, B, F) *we suppose that* ΩB *has the homotopy type of a* CW*-complex. Then there exists a map* $h: \Omega B \to F$ *such that the following diagram is commutative*:

$$\begin{array}{ccc} \pi_{i+1}(B) & \xrightarrow{\Delta} & \pi_i(F) \\ \Omega \, \wr\!\Vert & \nearrow h_* & \\ \pi_i(\Omega B) & & \end{array}$$

where Δ *is the boundary homomorphism.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccc} \pi_i(\Omega(E, F)) & & \overset{\Omega}{\cong} & & \pi_{i+1}(E, F) \\ & l_* \searrow & & \swarrow \partial & \\ \cong \Big\downarrow (\Omega p)_* & & \pi_i(F) & & \cong \Big\downarrow p_* \\ & & & \nwarrow \Delta & \\ \pi_i(\Omega B) & & \overset{\Omega}{\cong} & & \pi_{i+1}(B) \end{array}$$

where l is the projection of the canonical fibering $\Omega(E, F) \to F$. There exists a map $b: \Omega B \to \Omega(E, F)$ such that b_* is the inverse of $(\Omega p)_*$, since $\Omega p: \Omega(E,F) \to \Omega B$ is the singular homotopy equivalence and ΩB has the homotopy type of a CW-complex. Put $h = l \circ b$.

q. e. d.

As is well known [3], the exceptional Lie group G_2 contains the subgroup $SU(3)$ such that

(4.3) $$G_2/SU(3) = S^6.$$

According to [14], $\pi_5(SU(3))$ is isomorphic to Z and generated by such an element $[2\iota_5]$ that $p_*[2\iota_5]=2\iota_5$ for the projection $p\colon SU(3)\to S^5=SU(3)/SU(2)$. The characteristic class of the bundle (4.3) is then $\Delta\iota_6=[2\iota_5]$, since $\pi_5(G_2)=0$, which follows from Theorem 2.3.

It follows from Theorem 4.3

Corollary 4.7. *Assume that* $[2\iota_5]\circ\beta=n\beta=0$ *for* $\beta\in\pi_j(S^5)$ *and an integer* $n\geq 2$. *Then for an arbitrary element* δ *in* $\{[2\iota_5], \beta, n\iota_j\}\subset\pi_{j+1}(SU(3))$, *there exists an element* ε *in* $\pi_{j+1}(G_2)$ *such that* $p_*\varepsilon=S\beta$ *and* $i_*\delta=n\varepsilon$.

It is well known that the classifying space B_{S^3} of S^3 may be considered as the infinite quaternion projective space $QP^\infty=S^4\cup e^8\cup\cdots$ and that $B_{SU(3)}$ has the cell structure $S^4\cup e^6\cup\cdots$, where e^6 is attached to S^4 by a generator η_4 of $\pi_5(S^4)\cong Z_2$.

In the homotopy class of a generator of $\pi_6(B_{SU(3)})\cong Z$ we choose a map $f\colon S^6\to B_{SU(3)}$ so that the diagram may commute.

$$
\begin{array}{ccc}
\pi_{i+1}(S^6) & \xrightarrow{\ \Delta\ } & \pi_i(SU(3)) \\
\downarrow{\scriptstyle f_*} & \nearrow{\scriptstyle \Delta_{SU(3)}} & \\
\pi_{i+1}(B_{SU(3)}) & &
\end{array}
$$

where $\Delta_{SU(3)}$ is the boundary homomorphism in the exact sequence of the universal bundle of $SU(3)$.

It is easily seen that f represents a coextension of $2\iota_5$.

Consider the following commutative diagram.

$$
\begin{array}{ccccc}
\pi_{i+1}(S^6) & \xleftarrow{\ \Delta\ } & \pi_i(SU(3)) & \xleftarrow{\ i_*\ } & \pi_i(S^3) \\
{\scriptstyle f_*}\searrow & & \cong\uparrow{\scriptstyle \Delta_{SU(3)}} & & \cong\uparrow{\scriptstyle \Delta_{S^3}} \\
 & & \pi_{i+1}(B_{SU(3)}) & \xleftarrow{\ i_{1*}\ } & \pi_{i+1}(B_{S^3}) \\
 & & {\scriptstyle i_{0*}}\nwarrow & & \nearrow{\scriptstyle i_{2*}} \\
 & & & \pi_{i+1}(S^4) &
\end{array}
$$

where i_0, i_1, i_2 are inclusions and Δ_{S^3} is the boundary homomorphism of the universal bundle of S^3.

We note here that the next formula holds:

(4.4) $\Delta_{S^3}(i_{2*}(S\alpha))=\alpha$ *for any* $\alpha\in\pi_i(S^3)$.

Suppose that $\alpha\in\pi_i(S^5)$ satisfies $2\iota_5\circ\alpha=0$. Then the secondary composition $\{\eta_4, 2\iota_5, \alpha\}$ is well defined. According to Proposition 1.8 of [18] $-i_{2*}\{\eta_4, 2\iota_5, \alpha\}$ coincides with the set of all compositions Coext.$(2\iota_5)\circ S\alpha=f_*(S\alpha)$. Therefore $\Delta(S\alpha)=\Delta_{SU(3)}(f_*(S\alpha))$ belongs to $-\Delta_{SU(3)}i_{0*}\{\eta_4, 2\iota_5, \alpha\}$ which is equal to $i_*\Delta_{S^3}i_{2*}\{\eta_4, 2\iota_5, \alpha\}$ by the commutativity of the above diagram. Thus we have shown

Proposition 4.8. *For any element* $\alpha\in\pi_i(S^5)$ *satisfying* $2\iota_5\circ\alpha=0$,

$$\Delta(S\alpha)\in i_*\circ\Delta_{S^3}i_{2*}\{\eta_4, 2\iota_5, \alpha\}$$
$$\mod i_*\pi_5(S^3)\circ\alpha+i_*\circ\Delta_{S^3}\circ i_{2*}(\eta_4\circ\pi_{i+1}(S^5)).$$

Corollary 4.9. *Suppose that* $\alpha\in\pi_{i-2}(S_3)$ *satisfies* $2\alpha=0$. *Then*

$$H(i_*^{-1}\circ\Delta\circ S^3\alpha)\ni S^2\alpha \mod H(\Delta_{S^3}i_{2*}\eta_4\circ\pi_{i+1}(S^5)),$$

where H is the Hopf homomorphism: $\pi_i(S^3)\to\pi_i(S^5)$.

Proof. The above proposition says that $i_*^{-1}(\Delta S^3\alpha)$ is a subset of $\Delta_{S^3}i_{2*}\{\eta_4, 2\iota, S^2\alpha\}$. On the other hand, the secondary composition $\{\eta_3, 2\iota_4, S\alpha\}_1$ is equal to $\Delta_{S^3}i_{2*}\{\eta_4, 2\iota_5, S^2\alpha\}_2$ by (3.4), which is a subset of $\Delta_{S^3}i_{2*}\{\eta_3, 2\iota_4, S^2\alpha\}$. Thus we obtain

$$i_*^{-1}(\Delta S^3\alpha)\equiv\{\eta_3, 2\iota_4, S\alpha\}_1 \mod \Delta_{S^3}i_{2*}\eta_4\circ\pi_{i+1}(S^5)+\pi_5(S^3)\circ S^2\alpha.$$

Hence we have that

$$H(i_*^{-1}\circ\Delta\circ S^3\alpha)\equiv H\{\eta_3, 2\iota_4, S\alpha\}_1 \mod H(\Delta_{S^3}i_{2*}\eta_4\circ\pi_{i+1}(S^5))$$
$$=-\Delta^{-1}(2\eta_2)\circ S^2\alpha \quad \text{by Proposition 2.6 of [18]}$$
$$=S^2\alpha.$$

q. e. d.

Remark 4.10. It is easily checked that $\eta_4\circ\pi_{i+1}(S^5)\subset S\pi_i(S^3)$ for $i\leq 26$. and hence $H(\Delta_{S^3}i_{2*}\eta_4\circ\pi_{i+1}(S^5))$ is easily obtained by making use of (4.4) and the relations in [18] etc.

§5. Some lemmas.

This section is a preparation for the following one. Let X_{15}

be a cell complex $S^{15}\cup e^{23}$ where e^{23} is attached to S^{15} by a generator σ_{15} of $\pi_{22}^{15}\cong Z_{15}$.

Lemma. 5. 1. *First few groups $\pi_i(X_{15}\colon 2)$ are listed as follows.*

i	$i\leq 14$	15	16	17	18	19	20	21	22	23
$\pi_i(X_{15}\colon 2)$	0	Z	Z_2	Z_2	Z_8	0	0	Z_2	0	$Z\oplus Z_2$
gen		ι_{15}	η_{15}	η_{15}^2	ν_{15}			ν_{15}^2		$\langle 16\iota_{23}\rangle$, ε_{15}

where $p_*\langle 16\iota_{23}\rangle = 16\iota_{23}\in\pi_{23}(S^{23})$ for a shrinking map $p\colon S^{15}\cup e^{23}\to S^{23}$.

Proof. Clearly $\pi_i(X_{15})\cong\pi_i(S^{15})$ for $i\leq 21$. We have the next exact sequence, since $\pi_i(X_{15}, S^{15})\cong\pi_i(S^{23})$ for $i\leq 36$.

$$\cdots\longrightarrow\pi_{24}^{23}\xrightarrow{\Delta}\pi_{23}^{15}\xrightarrow{i_*}\pi_{23}(X_{15})\xrightarrow{p_*}\pi_{23}^{23}\xrightarrow{\Delta}\pi_{22}^{15}\longrightarrow\pi_{22}(X_{15})\longrightarrow\pi_{22}^{23}=0,$$

where $\pi_{24}^{23}\cong Z_2=\{\eta_{23}\}$, $\pi_{23}^{15}\cong Z_2\oplus Z_2=\{\varepsilon_{15}, \bar{\nu}_{15}\}$, $\pi_{23}^{23}\cong Z=\{\iota_{23}\}$ and $\pi_{22}^{15}\cong Z_{16}=\{\sigma_{15}\}$. By the difinition of X_{15}, $\Delta\colon\pi_{23}^{23}\to\pi_{22}^{15}$ is epimorphic and hence $\pi_{22}(X_{15})=0$. It follows from Proposition 4.1 that $\Delta\eta_{23}=\sigma_{15}\eta_{22}=\varepsilon_{15}+\bar{\nu}_{15}$ and its cokernel is Z_2. Thus $\pi_{23}(X_{15})=Z\oplus Z_2=\{\langle 16\iota_{23}\rangle, \varepsilon_{15}\}$.

Consider the Stiefel manifold $V_{7,2}$ of orthogonal 2-frames in euclidean 7-space. There associates a fibering

$$S^5\to V_{7,2}\to S^6,$$

whose characteristic class is $2\iota_5$. Let S_∞^5 be the reduced product space of S^5 in the sense of James [18]. This space S_∞^5 has a cell structure $S^5\cup e^{10}\cup\cdots$, where e^{10} is attached to S^5 by the Whitehead product $[\iota_5, \iota_5]=\nu_5\eta_8$, which is of order 2. Then we have the following

Lemma 5. 2. *There exists a map $f\colon S_\infty^5\to S^5$ such that $f|S^5$ has a mapping degree 2 and the following diagram is commutative:*

$$\begin{array}{ccc}\pi_{i+1}(S^6) & \xrightarrow{\Delta} & \pi_i(SU(3)) \\ \Omega_1\Big\| & \overset{\Delta'}{\searrow} & \Big\downarrow p_* \\ \pi_i(S_\infty^5) & \xrightarrow{f_*} & \pi_i(S^5),\end{array}$$

where Δ and Δ' are the boundary homomorphisms associated with the fibering $G_2/SU(3)=S^6$ and $V_{7,2}/S^5=S^6$ respectively and p is the projection: $SU(3)\to S^5=SU(3)/SU(2)$.

Proof. By Proposition 4.6 there exists a map $h: \Omega S^6\to S^5$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi_{i+1}(S^6) & \xrightarrow{\Delta'} & \pi_i(S^5) \\ \Omega\big\| & \nearrow h_* & \\ \pi_i(\Omega S^6) & & \end{array}$$

Let $i: S^5_\infty\to\Omega S^6$ be a canonical injection. We set $f=h\circ i: S^5_\infty\to\Omega S^6\to S^5$. Then the commutativity of the lemma is clear, since $\Omega_1=\Omega\circ i_*: \pi_i(S^5_\infty)\to\pi_i(\Omega S^6)\to\pi_{i+1}(S^6)$.

The map $f|S^5$ represents an element $f_*\iota_5$, where $\iota_5\in\pi_5(S^5)$ is identified with its image in $\pi_5(S^5_\infty)$. By the commutativity, we have $f_*\iota_5=\Delta'\Omega_1\iota_5$. Here $\Omega_1\iota_5$ is obviously equal to ι_6. Hence $f_*\iota_5=2\iota_6$, since $\Delta'\iota_6=2\iota_5$ (the characteristic class of the bundle $V_{7,2}/S^5=S^6$). q. e. d.

Remark that the restriction $f|S^5\cup e^{10}$ is an extension of $2\iota_5$ in $S^5\cup e^{10}$ whose attaching element is $[\iota_5,\iota_5]=\nu_5\eta_8$.

Let us recall that $\pi_{10}(SU(3):2)\cong Z_2$ and generated by $[\nu_5\eta_8^2]$, where $[\nu_5\eta_8^2]$ is such an element that $p_*[\nu_5\eta_8^2]=\nu_5\eta_8^2$ for the projection $p: SU(3)\to S^5$ ([14]). Then we have the following

Corollary 5.3. *For the boundary homomorphism $\Delta: \pi_{11}(S^6)\to\pi_{10}(SU(3))$ we have $\Delta(\Delta\iota_{13})=[\nu_5\eta_8^2]$.*

Proof. First we show that $\Omega_1(\Delta\iota_{13})$ is a coextension of $2\iota_9$ in $S^5\cup e^{10}$. For this it is sufficient to show $q_*(\Omega_1\Delta\iota_{13})=2\iota_{10}$ for the pinching map $q: S^5\cup e^{10}\to S^{10}$. The restriction of h_5 (for the definition see [18]) on $S^5\cup e^{10}$ is the map q. By Proposition 2.7 of [18] we have $H(\Delta\iota_{13})=2\iota_{11}$. By the definition of H this is equivalent to

$$\Omega_1^{-1}h_{5*}\Omega_1(\Delta\iota_{13})=2\iota_{11}.$$

Hence $$h_{5*}\Omega_1(\Delta\iota_{13})=\Omega_1(2\iota_{11})=2\iota_{10}.$$

Thus $q_{*}\Omega_1(\Delta\iota_{13})=2\iota_{10}$.

It is already seen that the map $f|S^5\cup e^{10}$ is an extension of $2\iota_5$. So $f_*\Omega_1(\Delta\iota_{13})$, which equals $p_*\Delta(\Delta\iota_{13})$, belongs to $\{2\iota_5, \nu_5\eta_8, 2\iota_9\}$ by Proposition 1.7 of [18]. This secondary composition is $\nu_5\eta_8^2$ by Corollary 3.7 of [18]. Thus we have shown $p_*\Delta(\Delta\iota_{13})=\nu_5\eta_8^2$, which implies the corollary. q. e. d.

Next we consider some elements in π_n^i. We have relations $2\eta_4=0$, $2\rho^{IV}=0, 8\varepsilon'=0, 16\sigma_{13}=0, 8\bar{\nu}_6=0$ [18]. So the secondary composition $\{\eta_4, 2\iota_5, \rho^{IV}\}, \{\varepsilon', 8\iota_{13}, 2\sigma_{13}\}$ and $\{\bar{\nu}_6, 8\iota_{14}, 2\sigma_{14}\}$ are well defined. We will prove

Lemma 5.4.

(i) $H(\bar{\varepsilon}')=\bar{\varepsilon}_5$.

(ii) $\{\eta_4, 2\iota_5, \rho^{IV}\}\equiv\bar{\mu}_4$ mod $\{\eta_4\mu_5\sigma_{14}, 2S\bar{\varepsilon}'\}$.

(iii) $\{\varepsilon', 8\iota_{13}, 2\sigma_{13}\}\equiv\mu'\sigma_{14}$ mod $\{\nu'\bar{\varepsilon}_6, \eta_3\bar{\mu}_4\}$.

(iv) $\{\bar{\nu}_6, 8\iota_{14}, 2\sigma_{14}\}\equiv\zeta'$ mod $\{\eta_6\bar{\varepsilon}_7\}$.

Proof.

(i) We apply Lemma 5.2 of [18] for the element $\bar{\varepsilon}_3\in\pi_{18}^3$. Then $H(\beta)=\bar{\varepsilon}_5$ for an arbitrary element β of $\{\eta_3, 2\iota_4, \bar{\varepsilon}_4\}_1$. Such a β belongs to π_{20}^3 and $2\beta=\eta_3^2\bar{\varepsilon}_5=2\bar{\varepsilon}'$. Hence we have $\beta\equiv\bar{\varepsilon}'$ mod $\{\bar{\mu}_3, \eta_3\mu_4\sigma_{13}, 2\bar{\varepsilon}'\}$. Note that $\bar{\mu}_3$ survives in the stable range. On the other hand we have

$$S^\infty\bar{\varepsilon}'=2\nu\kappa=0$$

and
$$S^\infty\beta\in\langle\eta, 2\iota, \bar{\varepsilon}\rangle=\langle\eta, 2\iota, \eta\kappa\rangle$$
$$\supset\langle\eta, 2\iota, \eta\rangle\kappa$$
$$\ni 2\nu\kappa=0 \quad \text{mod} \quad \{\eta\eta^*, \eta\mu\sigma\}.$$

Thus $\bar{\varepsilon}'\equiv\beta$ mod $\{\eta_3\mu_4\sigma_{13}, 2\bar{\varepsilon}'\}$, whence

$$H(\bar{\varepsilon}')\equiv H(\beta)=\bar{\varepsilon}_5 \quad \text{mod} \quad \{H(\eta_3\mu_4\sigma_{13}), 2H(\bar{\varepsilon}')\}=0.$$

(ii) We have

$$\{\eta_4, 2\iota_5, \rho^{IV}\}=\{\eta_4, 2\iota_5, \{\sigma^{III}, 2\iota_{12}, 8\sigma_{12}\}\}$$
$$\equiv-\{\eta_4, \{2\iota_5, \sigma^{III}, 2\iota_{12}\}, 8\sigma_{13}\}-\{\{\eta_4, 2\iota_5, \sigma^{III}\}, 2\iota_{13}, 8\sigma_{13}\}$$

by Proposition 1. 5 of [18]

$\equiv \{\mu_4, 2\iota_{13}, 8\sigma_{13}\}$, since $\{2\iota_5, \sigma^{III}, 2\iota_{12}\}=0$

$\equiv \bar{\mu}_4 \mod G$,

where $G=\eta_4\circ\pi_{21}^5+\{\eta_4\circ S\rho^{IV}\}+\pi_{14}^4\cdot 8\sigma_{14}+\mu_4\circ\pi_{21}^{13}=\{\eta_4\mu_5\sigma_{14}, 2S\bar{\varepsilon}'\}$.

(iii) We have

$H\{\varepsilon', 8\iota_{13}, 2\sigma_{13}\}\subset\{H(\varepsilon'), 8\iota_{13}, 2\sigma_{13}\}=\{\varepsilon_5, 8\iota_{13}, 2\sigma_{13}\}$

by Proposition 2. 3 of [18].

Moreover we have the following relations in the stable secondary composition (note that the equality holds, since the largest composition $\langle\eta\sigma, 8\iota, 2\sigma\rangle$ is a coset of $\{\eta\sigma\varepsilon, 2\mu\sigma\}=0$).

$$\begin{aligned}\langle\varepsilon, 8\iota, 2\sigma\rangle &= \langle\bar{\nu}+\eta\sigma, 8\iota, 2\sigma\rangle\\ &= \langle\bar{\nu}, 8\iota, 2\sigma\rangle+\langle\eta\sigma, 8\iota, 2\sigma\rangle\\ &= \langle\eta\sigma, 8\iota, 2\sigma\rangle, \quad \text{since } \langle\bar{\nu}, 8\iota, 2\sigma\rangle=0\\ &= \sigma\langle\eta, 8\iota, 2\sigma\rangle\\ &= \sigma\mu \quad \text{by the definition of } \mu.\end{aligned}$$

Hence $\{\varepsilon_5, 8\iota_{13}, 2\sigma_{13}\}\equiv\mu_5\sigma_{14}=H(\mu'\sigma_{14}) \mod \{\varepsilon_5^2=\eta_5\bar{\varepsilon}_6\}$, since the kernel of $S^\infty: \pi_{21}^5\to(G_{16}: 2)$ is generated by $\eta_5\bar{\varepsilon}_6$. Thus

$$\{\varepsilon', 8\iota_{13}, 2\sigma_{13}\}\equiv\mu'\sigma_{14} \mod \{\nu'\bar{\varepsilon}_6, \eta_3\bar{\mu}_4\}.$$

(iv) We have $H\{\bar{\nu}_6, 8\iota_{14}, 2\sigma_{14}\}\subset\{H(\bar{\nu}_6), 8\iota_{14}, 2\sigma_{14}\}\equiv\{\nu_{11}, 8\iota_{14}, 2\sigma_{14}\}$ by Proposition 2. 3 of [18]. According to (9. 2) of [18], $\zeta_{11}=H(\zeta')$ is equal to $\{\nu_{11}, 8\iota_{14}, 2\sigma_{14}\}$. The kernel of $H: \pi_{22}^6\to\pi_{22}^{11}$ is generated by $\eta_6\bar{\varepsilon}_7$ and $\mu_6\sigma_{15}$. Thus we have

$$\zeta'\equiv\{\bar{\nu}_6, 8\iota_{14}, 2\sigma_{14}\} \mod \{\eta_6\bar{\varepsilon}_7, \mu_6\sigma_{15}\}.$$

Though $\mu_6\sigma_{15}$ survives in the stable range, but ζ', $\eta_6\bar{\varepsilon}_7$ and $\{\bar{\nu}_6, 8\iota_{14}, 2\sigma_{14}\}$ do not. For, $S^\infty\zeta'=2\sigma\eta\varepsilon=0$, $S^\infty\eta_6\bar{\varepsilon}_7=\varepsilon^2=0$ and $S^\infty\{\bar{\nu}_6, 8\iota_{14}, 2\sigma_{14}\}=\langle\bar{\nu}, 8\iota, 2\sigma\rangle=0$. Hence $\zeta'\equiv\{\bar{\nu}_6, 8\iota_{14}, 2\sigma_{14}\} \mod \{\eta_6\varepsilon_7\}$. q. e. d.

Next we will prove the following lemma which is due to Toda.

Lemma 5. 5.

$$\pi_{14}(F_4)\cong Z_2.$$

Proof.

Consider the following commutative diagram where the horizontal sequences are exact ((11. 4) of [18]).

$$\begin{array}{ccccccccc}
\longrightarrow \pi_{15}(SO(17)) & \longrightarrow & \pi_{15}(V_{17,8}) & \xrightarrow{\Delta} & \pi_{14}(SO(9)) & \xrightarrow{i_*} & \pi_{14}(SO(17)) & \longrightarrow \\
\downarrow J & & \downarrow \cong & & \downarrow J & & \downarrow J & \\
\xrightarrow{S^8} \pi_{32}(S^{17}) & \longrightarrow & \pi_{24}(\Omega^8 S^{17}, S^9) & \xrightarrow{\partial} & \pi_{23}(S^9) & \xrightarrow{S^8} & \pi_{31}(S^{17}) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccccc}
\longrightarrow \pi_{14}(V_{17,8}) & \longrightarrow & \pi_{13}(SO(9)) & \longrightarrow & 0 & & \\
\downarrow \cong & & \downarrow J & & & & \\
\longrightarrow \pi_{23}(\Omega^8 S^{17}, S^9) & \longrightarrow & \pi_{22}(S^9) & \xrightarrow{S^8} & \pi_{30}(S^{17}) & \longrightarrow & 0,
\end{array}$$

where

$$\pi_{15}(SO(17)) \cong Z, \ \pi_{14}(SO(17)) \cong 0, \ \pi_{32}(S^{17}) \cong Z_{480} \oplus Z_2,$$

$$\pi_{23}(S^9) = Z_{16} \oplus Z_4, \ \pi_{31}(S^{17}) \cong Z_2 \oplus Z_2, \ \pi_{22}(S^9) \cong Z_6, \ \pi_{30}(S^{17}) \cong Z_3$$

and S^8: $\pi_i(S^9) \to \pi_{i+8}(S^{17})$ are epimorphic for $i=22$, 23 and Cokernel of S^8: $\pi_{24}(S^9) \to \pi_{32}(S^{17})$ is isomorphic to Z_2 ([18]). It follows easily from the lower exact sequence that the sequence

$$0 \longrightarrow Z_2 \longrightarrow \pi_{15}(V_{17,8}) \xrightarrow{J} Z_8 \oplus Z_2 \longrightarrow 0 \quad \text{is exact and that}$$

$$\pi_{13}(SO(9)) \cong Z_2 \quad \text{and} \quad J\pi_{13}(SO(9)) \cong Z_2 = \{\sigma_9 \nu_{16}^2\}.$$

As the image of Z_2 in the above sequence into $\pi_{15}(V_{17,8})$ coincides with that of $J\pi_{15}(SO(17)) \cong Z_{480}$, we have $\pi_{14}(SO(9)) = Z_8 \oplus Z_2$ and $J\pi_{14}(SO(9))$ is generated by $\{2\sigma_9^2, 2\kappa_9 = \bar{\nu}_9 \nu_{17}^2\}$.

Thus we have shown that

(5. 1) $\pi_{14}(SO(9)) \cong Z_8 \oplus Z_2$, $\pi_{13}(SO(9)) \cong Z_2$, *and that J-homomorphisms on these groups are monomorphic.*

Let α be a generator of $\pi_7(SO(9)) \cong Z$. Then $J(\alpha) = \sigma_9$, if it is restricted on 2-components. It follows that

$$J(\alpha \cdot \sigma') = \sigma_9 \circ S^9 \sigma' = 2\sigma_9^2 \quad \text{which is of order 8,}$$

$$J(\alpha \cdot \nu_7^2) = \sigma_9 \circ \nu_{16}^2 \quad \text{which is of order 2.}$$

Consider the exact sequence associated with a fibering $F_4/Spin(9) = \Pi$. It follows from Proposition 4. 6 that there exists a map h: $\Omega\Pi \to Spin(9)$ such that the following diagram commutes

$$\begin{array}{ccc} \pi_{i+1}(\Pi) & \xrightarrow{\Delta} & \pi_i(Spin(9)) \\ \Omega \wr\Vert & \nearrow h_* & \\ \pi_i(\Omega\Pi) & & \end{array}$$

Let $f=h\circ i$ be a composition of h and a natural inclusion i: $S^7\to\Omega\Pi$. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc} \pi_{15}(\Pi) & \longrightarrow & \pi_{14}(Spin(9)) & \longrightarrow & \pi_{14}(F_4) & \longrightarrow & \pi_{14}(\Pi) & \longrightarrow & \pi_{13}(Spin(9)) \\ \Vert\wr & \nearrow f_* & & & & & \Vert\wr & \nearrow f_* & \\ \pi_{14}(\Omega\Pi) & & & & & & \pi_{13}(\Omega\Pi) & & \\ \uparrow & & & & & & \uparrow & & \\ \pi_{14}(S^7) & & & & & & \pi_{13}(S^7) & & \end{array}$$

Here $f_*\iota_7$ is a generator of $\pi_7(Spin(9))$, since we have $\pi_7(F_4)=0$ by Theorem 4. 4.

Let P be a covering map $Spin(9)\to SO(9)$. Then we have

$JP_*f_*(\sigma')=J(\alpha\circ\sigma')=2\sigma_9^2$ and hence $f_*\pi_{14}(S^7)\cong Z_8$

and $JP_*f_*(\nu^2)=J(\alpha\circ\nu^2)=\sigma_9\nu_{16}^2$ and hence $f_*: \pi_{13}(S^7)\to\pi_{13}(Spin(9))$ is monomorphic.

Thus we have obtained

$$\pi_{14}(F_4)\cong Z_2. \qquad \text{q. e. d.}$$

§6. The 2-primary components of $\pi_i(G_2)$.

In this section we compute $\pi_i(G_2: 2)$ by making use of the exact sequence associated with the fibering $G_2/SU(3)=S^6$:

$$(6.1) \qquad \cdots\longrightarrow\pi_i(SU(3))\xrightarrow{i_*}\pi_i(G_2)\xrightarrow{p_*}\pi_i(S^6)\xrightarrow{\Delta}\pi_{i-1}(SU(3))\longrightarrow\cdots.$$

Theorem 6. 1. $\pi_i(G_2: 2)$ *are listed as follows*

i	1	2	3	4	5	6	7	8	9	10	11	12	13
$\pi_i(G_2: 2)$	0	0	Z	0	0	0	0	Z_2	Z_2	0	$Z\oplus Z_2$	0	0
gen.			$i_*\iota_3$					$\langle\eta_6^2\rangle$	$\langle\eta_6^2\rangle\eta_8$		$\langle 2\iota_{13}\rangle$, $i_*[\nu_5^2]$		

i	14	15	16
$\pi_i(G_2: 2)$	$Z_8\oplus Z_2$	Z_2	$Z_2\oplus Z_2\oplus Z_2$
gen.	$\langle\bar\nu_6+\varepsilon_6\rangle$, $i_*[\nu_5^2]\nu_{11}$	$\langle\bar\nu_6+\varepsilon_6\rangle\eta_{14}$	$\langle\eta_6^2\rangle\eta_8\sigma_9$, $\langle\eta_6\mu_7\rangle$, $i_*[\nu_5\bar\nu_8]$

i	17	18	19	20	21
$\pi_i(G_2:2)$	$Z_8 \oplus Z_2$	Z_{16}	Z_2	Z_2	0
gen.	$\langle\bar{\nu}_5\nu_{14}\rangle$, $\langle\eta_6^2\rangle\mu_8$	$\langle 2\Delta\iota_{13}\rangle\sigma_{11}$	$i_*[\nu_5\bar{\nu}_8]\nu_{16}$	$\langle\bar{\nu}_6\nu_{14}\rangle\nu_{17}$	

i	22	23
$\pi_i(G_2:2)$	$Z_8 \oplus Z_2$	$G \oplus Z_2$
gen.	$\langle\zeta'+\mu_6\sigma_{15}\rangle$, $\langle\eta_6\bar{\varepsilon}_7\rangle$	$\langle\eta_6\mu_7\rangle\sigma_{16}$

where $G \cong Z_4$ *or* $Z_2 \oplus Z_2$ *and generated by* $\{\langle \Delta S\theta + \nu_6\kappa_9\rangle\}$ *or* $\{\langle \Delta S\theta + \nu_6\kappa_9\rangle, i_*\nu_5\bar{\varepsilon}_3\}$ *respectively.*

We have the following relations

$$4\langle\bar{\nu}_6\nu_{14}\rangle = i_*[\nu_5^2]\nu_{11}^2$$

$$8\langle 2\Delta\iota_{13}\rangle\sigma_{11} = i_*[\nu_5\eta_8\mu_9] \quad \text{mod } \pi_{18}(G_2:3)$$

$$2\langle \Delta S\theta + \nu_6\kappa_9\rangle = i_*[\nu_5\bar{\varepsilon}_3] \quad \textit{in the case} \quad G \cong Z_4.$$

Here the notation $[\alpha]$ means such an element of $\pi_i(SU(3):2)$ that $q_*[\alpha] = \alpha \in \pi_i(S^5:2)$ for the projection $q: SU(3) \to S^5 = SU(3)/SU(2)$, and the notation $\langle\beta\rangle$ means such an element of $\pi_i(G_2:2)$ that $p_*\langle\beta\rangle = \beta \in \pi_i(S^6:2)$ for the projection $p: G_2 \to S^6$.

In order to prove this theorem we need the following results on $\pi_i(S^6:2)$ and $\pi_i(SU(3):2)$ ([13], [14] and [18]).

For simplicity we denote $\pi_i(SU(3):2) = U_i^3$.

(6.2)

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
π_{i+1}^6	0	0	0	0	Z	Z_2	Z_2	Z_8	0	Z	Z_2	Z_4	$Z_8 \oplus Z_2$	$Z_2 \oplus Z_2 \oplus Z_2$
gen.					ι_6	η_6	η_6^2	ν_6		$\Delta\iota_{13}$	ν_6^2	σ''	$\bar{\nu}_6$, ε_6,	ν_6^3, μ_6, $\eta_6\varepsilon_7$
U_i^3	0	0	Z	0	Z	Z_2	0	Z_4	0	Z_2	Z_4	Z_4	Z_2	$Z_4 \oplus Z_2$
gen.			$i_*\iota_3$		$[2\iota_5]$	$i_*\nu'$		$[2\iota_5]\nu_5$		$[\nu_5\eta_8^2]$	$[\nu_5^2]$	$[\sigma''']$	$i_*\varepsilon'$	$[\nu_5^2]\nu_{11}$, $i_*\mu'$

i	15	16	17	18	19
π_{i+1}^6	$Z_8 \oplus Z_2$	$Z_8 \oplus Z_4$	Z_{16}	Z_2	$Z_4 \oplus Z_2$
gen.	$\nu_6\sigma_9$, $\eta_6\mu_7$	ζ_6, $\bar{\nu}_6\nu_{14}$	$\Delta\sigma_{13}$	$\nu_6\sigma_9\nu_{16}$	$\sigma''\sigma_{13}$, $\bar{\nu}_6\nu_{14}^2$
U_i^3	Z_4	$Z_4 \oplus Z_2$	$Z_2 \oplus Z_2$	$Z_2 \oplus Z_2$	$Z_4 \oplus Z_2$
gen.	$[2\iota_5]\nu_5\sigma_8$	$[2\iota_5]\zeta_5$, $[\nu_5\bar{\nu}_8]$	$[\nu_5^2]\nu_{11}^2$, $[\nu_5\eta_8\varepsilon_9]$	$i_*\bar{\varepsilon}_3$, $[\nu_5\eta_8\mu_9]$	$[\sigma''']\sigma_{12}$, $[\nu_5\bar{\nu}_8]\nu_{16}$

i	20	21	22	23
π^6_{i+1}	$Z_4\oplus Z_2$	$Z_8\oplus Z_2\oplus Z_2$	$Z_2\oplus Z_2\oplus Z_2\oplus Z_2$	$Z_2\oplus Z_2\oplus Z_2$
gen.	$\rho_{\mathrm{III}},\bar{\varepsilon}_6$	ζ', $\mu_6\sigma_{15}$, $\eta_6\bar{\varepsilon}_7$	$\Delta S\theta$, $\nu_6\kappa_9$, $\bar{\mu}_6$, $\eta_6\mu_7\sigma_{16}$	$\Delta S\theta\circ\eta_{23}$, $\zeta_6\sigma_{17}$, $\eta_6\bar{\mu}_7$
U^3_i	$Z_4\oplus Z_2$	Z_2	$Z_2\oplus Z_2$	$Z_4\oplus Z_2$
gen.	$[\rho^{\mathrm{IV}}], i^*\bar{\varepsilon}'$	$i_*\mu'\sigma_{14}$	$i_*\bar{\mu}'$, $[2\iota_5]\nu_5\kappa_8$	$[2\iota_5]\zeta_5\sigma_{16}$, $[\nu_5\bar{\varepsilon}_8]$

The exact sequence (6. 1) induces an exact one:

$$(6.3)\qquad 0\longrightarrow \mathrm{Coker.}(\varDelta:\ \pi^6_{i+1}\to U^3_i)\xrightarrow{i_*}\pi_i(G_2:\ 2)$$
$$\xrightarrow{p_*}\mathrm{Ker.}(\varDelta:\ \pi^6_i\to U^3_{i-1})\longrightarrow 0.$$

It follows from Proposition 4. 1

Proposition 6. 2.

$$\varDelta S\alpha=[2\iota_5]\circ\alpha \quad \textit{for} \quad \alpha\in\pi_{i-1}(S^5).$$

Furthermore we will prove

Proposition 6. 3. *For the homomorphism* $\varDelta$: $\pi^6_{i+1}\to U^3_i$ *we have the following table.*

$\alpha=$	η_6	η_6^2	ν_6	$\Delta\iota_{13}$	ν_6^2	σ''	$\bar{\nu}_6$	ε_6	ν_6^3	$\eta_6\varepsilon_7$	μ_6	$\nu_6\sigma_9$
$\varDelta\alpha=$	$i^*\nu'$	0	$[2\iota_5]\nu_5$	$[\nu_5\eta_8^2]$	$2[\nu_5^2]$	$[\sigma^{\mathrm{III}}]$	$i_*\varepsilon'$	$i_*\varepsilon'$	$2[\nu_5^2]\nu_{11}$	$2[\nu_5^2]\nu_{11}$	$i_*\mu'$	$[2\iota_5]\nu_5\sigma_8$

$\alpha=$	$\eta_6\mu_7$	ζ_6	$\bar{\nu}_6\nu_{14}$	$\Delta\sigma_{13}$	$\nu_6\sigma_9\nu_{16}$	$\sigma''\sigma_{13}$	$\bar{\nu}_6\nu_{14}^2$	ρ^{III}	$\bar{\varepsilon}_6$
$\varDelta\alpha=$	0	$[2\iota_5]\zeta_5$	0	$[\nu_5^2]\nu_{11}^2+[\nu_5\eta_8\varepsilon_9]$	$i_*\bar{\varepsilon}_3$	$[\sigma^{\mathrm{III}}]\sigma_{12}$	0	$[\rho^{\mathrm{IV}}]$	$i_*\bar{\varepsilon}'$

$\alpha=$	ζ'	$\mu_6\sigma_{15}$	$\eta_6\bar{\varepsilon}_7$	$\Delta S\theta$	$\nu_6\kappa_9$	$\bar{\mu}_6$	$\eta_6\mu_7\sigma_{16}$	$\Delta S\theta\eta_{23}$	$\zeta_6\sigma_{17}$
$\varDelta\alpha=$	$i_*\mu'\sigma_{14}$	$i_*\mu'\sigma_{14}$	0	$[2\iota_5]\nu_5\kappa_8$	$[2\iota_5]\nu_5\kappa_8$	$i_*\bar{\mu}'$	0	0	$[2\iota_5]\zeta_5\sigma_{16}$

Proof. The cases $\alpha=\nu_6, \nu_6\sigma_9, \zeta_6, \nu_6\kappa_9, \zeta_6\sigma_{17}$ are easily obtained by Proposition 4. 1.

For the cases $\alpha=\eta_6, \varepsilon_6, \mu_6, \bar{\varepsilon}_6, \bar{\mu}_6$ we apply Corollary 4. 8.

$H(i_*^{-1}\varDelta\eta_6)\equiv\eta_5$ mod $H(\eta_3^3)=0$, on the other hand $H(\nu')=\eta_5$ by (5. 3) of [14]. Hence $\varDelta\eta_6=i_*\nu'$. Similarly we have

$H(i_*^{-1}\varDelta\varepsilon_6)\equiv\varepsilon_5=H(\varepsilon')$ mod $H(\{\eta_3\nu_4^3, \eta_3\mu_4, \eta_3^2\varepsilon_5\})=0$ by Lemma 6. 6 of [18], whence $\varDelta\varepsilon_6=i_*\varepsilon'$.

$H(i_*^{-1}\varDelta\mu_6)\equiv\mu_5=H(\mu')$ mod $H(\varepsilon_3\nu_{11}+\nu'\varepsilon_6)$ by (7. 7) of [18], whence $\Delta\mu_6 \equiv i^*\mu'$ mod $i_*(\varepsilon_3\nu_{11}+\nu'\varepsilon_6)=0$.

$H(i_*^{-'}\Delta\bar{\varepsilon}_6)\equiv\bar{\varepsilon}_5=H(\bar{\varepsilon}')$ mod $H(\{\eta_3\nu_4\zeta_7,\ \eta_3\nu_4\bar{\nu}_7\nu_{15}\})=0$ by Lemma 5.3, whence $\Delta\bar{\varepsilon}_6=i_*\bar{\varepsilon}'$.

$H(i_*^{-'}\Delta\bar{\mu}_6)\equiv\bar{\mu}_5=H(\bar{\mu}')$ mod $H(\nu'\mu_6\sigma_{15})$ by Lemma 12.4 of [18], whence $\Delta\bar{\mu}_6\equiv i_*\bar{\mu}'$ mod $i_*\nu'\mu_6\sigma_{15}=0$.

For the cases $\alpha=\nu_6^2,\ \sigma'',\ \rho^{\mathrm{III}}$ we use Proposition 4.8. By Lemma 5.14 of [18] we have $2\sigma''=S\sigma^{\mathrm{III}}$. Hence we can apply Proposition 4.8 for $2\sigma''=S\sigma^{\mathrm{III}}$. We have $2\Delta\sigma''=\Delta S\sigma^{\mathrm{III}}=i_*\Delta_{S^3}i_{2*}\{\eta_4,\ 2\iota_5,\ \sigma^{\mathrm{III}}\}$, which contains $i_*\Delta_{S^3}i_{2*}\mu_4$ by the definition of μ_4. By (4.4) $i_*\Delta_{S^3}i_{2*}\mu_4=i_*\mu_3$, which is equal to $2[\sigma^{\mathrm{III}}]$ by (4.1) of [14]. Thus we have obtained

$$\Delta\sigma''\equiv[\sigma^{\mathrm{III}}] \mod 2[\sigma^{\mathrm{III}}].$$

Similarly,

$$\Delta 2\rho^{\mathrm{III}}=\Delta S\rho^{\mathrm{IV}}=i_*\Delta_{S^3}i_{2*}\{\eta_4,\ 2\iota_5,\ \rho^{\mathrm{IV}}\}\ni i_*\Delta_{S^3}i_{2*}\bar{\mu}_4=i_*\bar{\mu}_3$$

mod $\{2i_*\bar{\varepsilon}',\ i_*\eta_3\mu_4\sigma_{13}\}=0$ by (ii) of Lemma 5.3, whence we have $\Delta\rho^{\mathrm{III}}\equiv[\rho^{\mathrm{IV}}]$ mod $\{2[\rho^{\mathrm{IV}}], i_*\bar{\varepsilon}'\}$, since $i_*\bar{\mu}_3\equiv 2[\rho^{\mathrm{IV}}]$ mod $i^*\bar{\varepsilon}'$.

It follows from Proposition 4.8 that

$$\begin{aligned}\Delta\nu_6^2 &\equiv i_*\Delta_{S^3}i_{2*}\{\eta_4,\ 2\iota_5,\ \nu_5^2\} && \text{mod } \{i_*\eta_3^2\nu_5^2+i_*\Delta_{S^3}i_{2*}\eta_4\sigma^{\mathrm{III}}\}=0\\ &\ni i_*\Delta_{S^3}i_{2*}\varepsilon_4 && \text{by (6.1) of [18]}\end{aligned}$$

which is equal to $i_*\varepsilon_3=2[\nu_5^2]$ by (4.4) and (4.1) of [14]. Thus $\Delta\nu_6^2=2[\nu_5^2]$.

The cases $\alpha=\eta_6^2,\ \bar{\nu}_6,\ \eta_6\varepsilon_7,\ \eta_6\mu_7,\ \bar{\nu}_6\nu_{14},\ \sigma''\sigma_{13},\ \bar{\nu}_6\nu_{14}^2,\ \eta_6\bar{\varepsilon}_7,\ \eta_6\mu_7\sigma_{16},\ \mu_6\sigma_{15}$ and $\eta_6\bar{\mu}_7$ are proved by making use of the relations of elements in U_i^3 and π_i^3 as follows (see §4 of [14]).

$\Delta\eta_6^2=0$, since $U_7^3=0$.

$\Delta\eta_6\varepsilon_7=i_*\nu'\varepsilon_6=i_*\varepsilon_3\nu_{11}=2[\nu_5^2]\nu_{11}$ in U_{14}^3.

$\Delta\nu_6^3=\Delta(\eta_6\bar{\nu}_7)=i_*\nu'\bar{\nu}_6=2[\nu_5^2]\nu_{11}$ in U_{14}^3.

$\Delta\bar{\nu}_6=i_*\bar{\varepsilon}'$, since $\Delta\nu_6^3=\Delta(\eta_6\bar{\nu}_7)\neq 0$ implies $\Delta\bar{\nu}_6\neq 0$.

$\Delta\eta_6\mu_7=i_*\nu'\mu_6=0$ in U_{14}^3.

$\Delta\bar{\nu}_6\nu_{14}=i_*\varepsilon'\nu_{13}=0$, since $\varepsilon'\nu_{14}=0$ in U_{14}^3.

$\Delta\sigma''\sigma_{13}=(\Delta\sigma'')\sigma_{12}=[\sigma^{\mathrm{III}}]\sigma_{12}$, by the above.

$\Delta\bar{\nu}_6\nu_{14}^2=\Delta(\bar{\nu}_6\nu_{14})\nu_{16}=0.$

$\Delta\eta_6\bar{\varepsilon}_7=i_*\nu'\bar{\varepsilon}_6=0,$ since $\nu'\bar{\varepsilon}_6=0$ in U_{21}^3.

$\Delta\eta_6\mu_7\sigma_{16}=\Delta(\eta_6\mu_7)\sigma_{15}=0.$

$\Delta\mu_6\sigma_{15}=i_*\mu'\sigma_{14}.$

$\Delta\eta_6\bar{\mu}_7=i_*\nu'\bar{\mu}_6=0,$ since $\nu'\bar{\mu}_6=0$ in U_{23}^3.

Corollary 5.3 says that $\Delta(\Delta\iota_{13})=[\nu_5\eta_8^2]$. This relation indicates that $\Delta(\Delta\sigma_{13})=[\nu_5^2]\nu_{11}^2+[\nu_5\eta_8\varepsilon_9]$, since $p_*(\Delta(\Delta\sigma_{13}))=\nu_5\eta_8^2\sigma_{10}=\nu_5^4+\nu_5\eta_8\varepsilon_9$.

Next we will prove $\Delta(\nu_6\sigma_9\nu_{16})=i_*\bar{\varepsilon}_3$.

Consider the exact sequence associated with the fibering

$$SU(4)/SU(3)=S^7:\cdots\longrightarrow U_8^3\xrightarrow{i'_*}U_8^4\longrightarrow\pi_8^7\longrightarrow,$$

where $U_8^3\cong Z_4=\{[2\iota_5]\nu_5\}$, $U_8^4\cong Z_8=\{[\nu_5\oplus\eta_7]\}$, $\pi_8^7\cong Z_2=\{\eta_7\}$ (see §4 of [14]). There we obtained already $i'_*[2\iota_5]\nu_5=2[\nu_5\oplus\eta_7]$ and $2[\nu_5\oplus\eta_7]\sigma_8\nu_{15}=i'_*i_*\bar{\varepsilon}_3$. It follows that $i'_*(\Delta\nu_6\sigma_9\nu_{16})=(i'_*\Delta\nu_6)\sigma_8\nu_{15}=2[\nu_5\oplus\eta_7]\sigma_8\nu_{15}=i'_*i_*\bar{\varepsilon}_3$ and hence $\Delta(\nu_6\sigma_9\nu_{16})=i_*\bar{\varepsilon}_3$, since i'_* is monomorphic.

For the cases $\alpha=\zeta'$, $\Delta S\theta$ we apply Proposition 4.2. By (iv) of Lemma 5.3 we have

$$\begin{aligned}\Delta\zeta'&\equiv\Delta\{\bar{\nu}_6,8\iota_{14},2\sigma_{14}\} && \text{mod } \{\Delta\eta_6\bar{\varepsilon}_7=0\}\\ &\subset\{\Delta\bar{\nu}_6,8\iota_{13},2\sigma_{13}\} && \text{by Proposition 3.2}\\ &=\{i_*\varepsilon',8\iota_{13},2\sigma_{13}\}\\ &\supset i_*\{\varepsilon',8\iota_{13},2\sigma_{13}\}\end{aligned}$$

where $\{\varepsilon',8\iota_{13},2\sigma_{13}\}\equiv\mu'\sigma_{14}$ mod $\{\nu'\bar{\varepsilon}_6,\eta_3\bar{\mu}_4\}$. Hence $\Delta\zeta'\equiv i_*\mu'\sigma_{14}$ mod $\{i_*\varepsilon'\varepsilon_{13},i_*\varepsilon'\bar{\nu}_{13},2i_*\mu'\sigma_{14}\}=0$.

It follows from Lemma 12.11 of [18] that

$$p_*\Delta(\Delta S\theta)\in p_*\Delta\{\Delta\sigma_{13},\nu_{18},\eta_{21}\},$$

which is a subset of $\{p_*\Delta(\Delta\sigma_{13}),\nu_{17},\eta_{20}\}=\{\nu_5^4+\nu_5\eta_8\varepsilon_9,\nu_{17},\eta_{20}\}$. Here we have

$$\begin{aligned}\{\nu_5^4+\nu_5\eta_8\varepsilon_9,\nu_{17},\eta_{20}\}&\equiv(\nu_5\bar{\nu}_8+\nu_5\varepsilon_8)\{\eta_{16},\nu_{17},\eta_{20}\} && \text{mod } \{\eta_5\mu_6\sigma_{15}\}\\ &=\nu_5(\bar{\nu}_8+\varepsilon_8)\nu_{16}^2 && \text{by Lemma 5.12 of [18]}\\ &=\nu_5\bar{\nu}_8\nu_{16}^2 && \text{by (7.13) of [18]}\end{aligned}$$

$$=2\nu_5\kappa_8 \qquad \text{by Lemma 10.1 of [18].}$$

Thus $p_*\varDelta(\Delta S\theta)\equiv 2\nu_5\kappa_8 \mod \{\eta_5\mu_6\sigma_{15}\}$. Hence we obtain

$$\varDelta(\Delta S\theta)\equiv[2\iota_5]\nu_5\kappa_8 \mod \{i_*\bar{\mu}'\}.$$

It follows from this relation that $p_*\varDelta(\Delta S\theta\eta_{23})=0$ and hence $\varDelta(\Delta S\theta\cdot\eta_{23})=0$, since $p_*: U_{23}^3\to\pi_{23}^5$ is monomorphic. Thus the proof is completed. q. e. d.

The following lemma follows directly from the table (6.2) and Proposition 6.3.

Lemma 6.4.

i) *The homomorphisms* $\varDelta: \pi_{i+1}^6\to U_i^3$ *are epimorphisms for* $5\leq i\leq 10$ *and* $i=12, 13, 15, 20, 21, 22$. *For the other values of* i, $4<i<24$, *we have the following table of the cokernel of* $\varDelta$.

i	11	14	16	17	18	19	23
Coker-$\varDelta$	Z_2	Z_2	Z_2	Z_2	Z_2	Z_2	Z_2
repr. of gen.	$\langle\nu_5^2\rangle$	$\langle\nu_5^2\rangle\nu_{11}$,	$\langle\nu_5\bar{\nu}_8\rangle$	$\langle\nu_5^2\rangle\nu_{11}^2=\langle\nu_5\eta_8\varepsilon_9\rangle$	$\langle\nu_5\eta_8\mu_9\rangle$	$\langle\nu_5\bar{\nu}_8\rangle\nu_{16}$	$\langle\nu_5\bar{\varepsilon}_8\rangle$

ii) *The homomorphisms* $\varDelta: \pi_i^6\to U_{i-1}^3$ *are monomorphisms for* $i=6, 7, 10, 12, 13, 19, 21$. *For the other values of* i, $4<i<24$, *we have the following table of the kernel of* $\varDelta$.

i	8	9	11	14	15	16	17
Ker-$\varDelta$	Z_2	Z_2	Z	Z_8	Z_2	$Z_2\oplus Z_2$	$Z_2\oplus Z_4$
gen.	η_6	η_6^2	$2\Delta\iota_{13}$	$\bar{\nu}_6+\varepsilon_6$	$(\bar{\nu}_6+\varepsilon_6)\eta_{14}$	$\eta_6^3\sigma_9$, $\eta_6\mu_7$	$\eta_6^2\mu_8$, $\bar{\nu}_6\nu_{14}$

i	18	20	22	23
Ker-$\varDelta$	Z_8	Z_2	$Z_8\oplus Z_2$	$Z_2\oplus Z_2$
gen.	$2\Delta\sigma_{13}$	$\bar{\nu}_6\nu_{14}^2$	$\zeta'+\mu_6\sigma_{15}$, $\eta_6\bar{\varepsilon}_7$	$(\Delta S\theta+\nu_6\kappa_9)$, $\eta_6\mu_7\sigma_{16}$

We prove Theorem 6.1 by dividing into three cases.

Case 1. $5 \leq i \leq 10$, $i=12, 13, 15, 20, 21$ *and* 22.

For these values of i, it follows from the exactness of (6.3) and i) of Lemma 6.4 that $\pi_i(G_2: 2)$ is isomorphic to the kernel of $\Delta: \pi_i^6 \to U_{i-1}^3$ under the projection homomorphism p_*. Thus Theorem 6.1 is established for these values of i by making use of ii) of Lemma 5.2.

Case 2. $i=19$.

For this case, $\pi_i(G_2: 2)$ is isomorphic to the cokernel of Δ: $\pi_{i+1}^6 \to U_i^3$ under the injection homomorphism i_*.

Case 3. $i=11, 14, 16, 17, 18$ *and* 23.

For these values of i, we must determine the extension (6.3).

For the case $i=11$, the kernel of $\Delta: \pi_{11}^6 \to U_{10}^3$ is isomorphic to Z, so the sequence obviously splits:

$$\pi_{11}(G_2: 2) \cong Z \oplus Z_2 = \{\langle 2\Delta\sigma_{13}\rangle, i_*[\nu_5^2]\}.$$

Consider the case $i=14$. Suppose $i_*[\nu_5^2]\nu_{11} = 8\langle\bar{\nu}_6+\varepsilon_6\rangle$, then $i_*[\nu_5^2]\nu_{11}^2 = 8\langle\bar{\nu}_6+\varepsilon_6\rangle\nu_{14}=0$. This contradicts the fact that $i_*[\nu_5^2]\nu_{11}^2 \neq 0$. So there are no relations between $i_*[\nu_5^2]\nu_{11}$ and $\langle\bar{\nu}_6+\varepsilon_6\rangle$, which implies

$$\pi_{14}(G_2: 2) \cong Z_8 \oplus Z_2 = \{\langle\bar{\nu}_6+\varepsilon_6\rangle, i_*[\nu_5^2]\nu_{11}\}.$$

Consider the case $i=16$. Obviously the order of $\langle\eta_6^2\rangle\eta_8\sigma_9$ is 2. We apply Corollary 4.7 for the element $\eta_6\mu_7$. Then for an arbitrary element δ of $\{[2\iota_5], \eta_5\mu_6, 2\iota_{15}\} \subset U_{16}^3$, there exists an element $\langle\eta_6\mu_7\rangle$ in $\pi_{16}(G_2: 2)$ such that $p_*\langle\eta_6\mu_7\rangle = \eta_6\mu_7$ and $i_*\delta = 2\langle\eta_6\mu_7\rangle$. On the other hand we have that $p_*\{[2\iota_5], \eta_5\mu_6, 2\iota_{15}\}$ is a subset of $\{p_*[2\iota_5], \eta_5\mu_6, 2\iota_{15}\} = \{2\iota_5, \eta_5\mu_6, 2\iota_{15}\}$ which contains $4\zeta_5$. This means that the secondary composition $\{[2\iota_5], \eta_5\mu_6, 2\iota_{15}\}$ contains $2[2\iota_5]\zeta_5$. But $2[2\iota_5]\zeta_5$ is already known to be zero in $\pi_{16}(G_2: 2)$. So $\langle\eta_6\mu_7\rangle$ is of order 2, whence

$$\pi_{16}(G_2: 2) \cong Z_2 \oplus Z_2 \oplus Z_2 = \{\langle\eta_6^2\rangle\eta_8\sigma_9, \langle\eta_6\mu_7\rangle, i_*[\nu_5\bar{\nu}_8]\}.$$

As we have the relation $2\bar{\nu}_6\nu_{14}=\nu_6\bar{\nu}_9$ which is a suspension element, we may apply Corollary 4.7 for $2\bar{\nu}_6\nu_{14}$. Corollary 4.7 says that for an arbitrary element δ in $\{[2\iota_5], \nu_5\bar{\nu}_8, 2\iota_{16}\}$, there exists an element $\langle\nu_6\bar{\nu}_9\rangle=2\langle\bar{\nu}_6\nu_{14}\rangle$ such that $p_*\langle\nu_6\bar{\nu}_9\rangle=2\bar{\nu}_6\nu_{14}$ and $i_*\delta=2\langle\nu_6\bar{\nu}_9\rangle=4\langle\bar{\nu}_6\nu_{14}\rangle$. As the secondary composition $\{2\iota_5, \nu_5\bar{\nu}_8, 2\iota_{16}\}$ is equal to $\nu_5\bar{\nu}_8\eta_{16}=\nu_5^4$, so we have $\{[2\iota_5], \nu_5\bar{\nu}_8, 2\iota_{16}\}=[\nu_5^2]\nu_{11}^2$ and hence $i_*[\nu_5^2]\nu_{11}^2=4\langle\bar{\nu}_6\nu_{14}\rangle$. Thus

$$\pi_{17}(G_2: 2)\cong Z_8\oplus Z_2=\{\langle\bar{\nu}_6\nu_{14}\rangle, \langle\eta_6^2\rangle\mu_8\}.$$

Consider the case $i=18$. Since the relation $8\Delta\sigma_{13}=\nu_6\mu_9=S(\nu_5\mu_8)$ holds, we can apply Corollary 4.7 for this element. For an arbitrary element δ of $\{[2\iota_5], \nu_5\mu_8, 2\iota_{17}\}$ there exists an element $\langle\nu_6\mu_9\rangle\in\pi_{18}(G_2: 2)$ such that $p_*\langle\nu_6\mu_9\rangle=\nu_6\mu_9=8\Delta\sigma_{13}$ and $i_*\delta=2\langle\nu_6\mu_9\rangle=8\langle 2\Delta\iota_{13}\rangle\sigma_{11}$. Since $\{2\iota_5, \nu_5\mu_8, 2\iota_{17}\}\equiv\nu_5\eta_8\mu_9 \mod 2\pi_{18}(S^5)$ by Corollary 3.7 of [18], we obtain $\{[2\iota_5], \nu_5\mu_8, 2\iota_{17}\}\equiv[\nu_5\eta_8\mu_9]$. This implies that the order of $\langle 2\Delta\iota_{13}\rangle\sigma_{11}$ is 16, and hence

$$\pi_{18}(G_2: 2)\cong Z_{16}=\{\langle 2\Delta\iota_{13}\rangle\sigma_{11}\}$$

and $\quad i_*[\nu_5\eta_8\mu_9]\equiv 8\langle 2\Delta\iota_{13}\rangle\sigma_{11} \mod \pi_{18}(G_2: 3)$.

Obviously $\langle\eta_6\mu_7\rangle\sigma_{16}$ is of order 2. But we cannot determine the order of $\langle\Delta S\theta+\nu_6\kappa_9\rangle$. In any way

$$\pi_{23}(G_2: 2)\cong Z_2\oplus Z_2\oplus Z_2 \quad \text{or} \quad Z_4\oplus Z_2. \qquad \text{q. e. d.}$$

§7. Homotopy groups of the octonionic projective plane Π.

As is well known the homogeneous space $F_4/Spin(9)$ is the octonionic projective plane Π. It has a cell structure $S^8\cup e^{16}$ in which e^{16} is attached to S^8 by the Hopf-map h_8: $S^{15}\to S^8$.

Let a be a base point of Π. We set $E_{\Pi,a}=\{f: I\to\Pi; f(0)=a, f(1)\in\Pi\}$ with a compact-open topology. Then we have a fibering:

(7.1) $$\Omega\Pi\to E_{\Pi,a}\to\Pi.$$

Obviously $E_{\Pi,a}$ is contractible. We will calculate $H^*(\Omega\Pi)$ by making use of the spectral sequence $\{E_r^*\}$ associated with (7.1).

We have $$E_2^* = H^*(\Pi) \otimes H^*(\Omega\Pi)$$
$$\cong Z[x_8]/(x_8^3) \otimes H^*(\Omega\Pi)$$

First there must exist an element $y_7 \in H^7(\Omega\Pi)$ such that $d_8(1\otimes y_7) = x_8 \otimes 1$, since E_∞^* is trivial. The element $x_8^2 \otimes y_7$ is cocycle, since $d_8(x_3^2 \otimes y_7) = 0$. So $x_8^2 \otimes y_7$ must be killed by a certain element, say, $y_{22} \in H^{22}(\Omega\Pi)$; namely $d_{16}(1\otimes y_{22}) = x_8^2 \otimes y_7$. The third element which will appear in $H^*(\Omega\Pi)$ to kill $x_8^2 \otimes y_7 y_{22}$ is of dimension 44.

Thus we obtain

(7.2) $$H^*(\Omega\Pi) \cong \Lambda(y_7, y_{22}) \quad \textit{for} \quad \text{dim.} < 44.$$

It follows from (7.2) that

(7.3) $$\pi_{i+1}(\Pi) \cong \pi_i(\Omega\Pi) \cong \pi_i(S^7) \quad \textit{for} \quad i \leq 20.$$

Consider the exact sequence of the pair (Π, S^8):

$$\cdots \longrightarrow \pi_i(S^8) \xrightarrow{i_*} \pi_i(\Pi) \xrightarrow{j_*} \pi_i(\Pi, S^8) \xrightarrow{\partial} \pi_{i-1}(S^8) \longrightarrow \cdots.$$

By Blakers-Massey theorem (or Theorem 1.4 of [10]) we have the commutative diagram for $i \leq 22$:

$$\begin{array}{ccc} \pi_i(\Pi, S^8) & \xrightarrow{\partial} & \pi_{i-1}(S^8) \\ \downarrow\cong & S^{-1} & \uparrow h_{8*} \\ \pi_i(S^{16}) & = & \pi_{i-1}(S^{15}). \end{array}$$

First we show that $j_*: \pi_{22}(\Pi) \to \pi_{22}(\Pi, S^8)$ is trivial. For, $h_{8*}S^{-1}(\nu_{16}^2) = \sigma_8 \nu_{15}^2$ is non-trivial for a generator ν_{16}^2 of $\pi_{22}(S^{16}) \cong Z_2 \cong \pi_{22}(\Pi, S^8)$. Thus we have the exact sequence:

$$\cdots \longrightarrow \pi_{24}(\Pi, S^8) \xrightarrow{\partial} \pi_{23}(S^8) \xrightarrow{i_*} \pi_{23}(\Pi) \xrightarrow{j_*} \pi_{23}(\Pi, S^8)$$
$$\xrightarrow{\partial} \pi_{22}(S^8) \xrightarrow{i_*} \pi_{22}(\Pi) \longrightarrow 0.$$

Let $\Sigma \in \pi_{16}(\Pi, S^8)$ be a characteristic map, whence $\partial\Sigma$ is represented by h_8 and it belongs to $\pi_{15}(S^8) \cong Z \oplus Z_{120}$. Then it follows from Theorem 1.4 of [10] that

$$\pi_{23}(\Pi, S^8) \cong \Sigma_* \pi_{23}(CS^{15}, S^{15}) \oplus \{[\iota_8, \Sigma]\}.$$

We have $\partial\Sigma_*\pi_{23}(CS^{15}, S^{15}) = h_{8*}\pi_{22}(S^{15}) \cong Z_{240}$. According to the formula due to Barcus-Barratt (Corollary 7.4 of [1]) we have

$$
\begin{aligned}
(7.4)\qquad \partial[\iota_8, \Sigma] &= [\iota_8, \{h_8\}] \\
&= (2\sigma_8 - S\sigma')\sigma_{15} + [[\iota_8, \iota_8], \iota_8] SH(\{h_8\}) \\
&= 2\sigma_8^2 - S\sigma'\sigma_{15} + [[\iota_8, \iota_8], \iota_8],
\end{aligned}
$$

where $[[\iota_8, \iota_8], \iota_8]$ is non-trivial and belongs to $S\pi_{21}(S^7: 3) \cong Z_3$ by Corollary 2.4 of [9].

Thus $\quad \partial\pi_{23}(\Pi, S^8) \cong Z_{240} \oplus Z_{24}$, and hence

$$\pi_{22}(\Pi) \cong Z_4 = \{\kappa_7\}.$$

Let Ω_7 be a cell complex $S^7 \cup e^{22}$ with an attaching map $\alpha \in \pi_{21}(S^7)$ such that there exists a map $g: \Omega_7 \to \Omega\Pi$ and

$$(7.5)\qquad g_*: \pi_i(\Omega_7) \cong \pi_i(\Omega\Pi) \quad \textit{for } i \leq 27.$$

We should investigate the attaching map $\alpha \in \pi_{21}(S^7)$.

It is easily seen that there is an exact sequence associated with Ω_7 for $i \leq 27$: $(\Delta\iota_{22} = \alpha)$

$$(7.6)\qquad \cdots \longrightarrow \pi_i(S^7) \longrightarrow \pi_i(\Omega_7) \longrightarrow \pi_i(S^{22}) \overset{\Delta}{\longrightarrow} \pi_{i-1}(S^7) \longrightarrow \cdots \quad .$$

Consider the following commutative diagram:

$$
\begin{array}{ccccccccc}
\longrightarrow \pi_{22}(\Omega_7) & \longrightarrow & \pi_{22}(S^{22}) & \overset{\Delta}{\longrightarrow} & \pi_{21}(S^7) & \longrightarrow & \pi_{21}(\Omega_7) & \longrightarrow 0 \\
 & & \wr\Vert & & & & & \\
\wr\Vert & \searrow & \pi_{22}(\Omega_7, S^7) & \nearrow & \Vert & & \wr\Vert & \\
 & & \wr\Vert & & & & & \\
\longrightarrow \pi_{22}(\Omega\Pi) & \longrightarrow & \pi_{22}(\Omega\Pi, S^7) & \longrightarrow & \pi_{21}(S^7) & \longrightarrow & \pi_{21}(\Omega\Pi) & \longrightarrow 0 \\
\Vert & & \downarrow i_* & & \downarrow i_* & & \Vert & \\
\longrightarrow \pi_{22}(\Omega\Pi) & \longrightarrow & \pi_{22}(\Omega\Pi, \Omega S^8) & \longrightarrow & \pi_{21}(\Omega S^8) & \longrightarrow & \pi_{21}(\Omega\Pi) & \longrightarrow 0 \\
\wr\Vert & & \wr\Vert & & \wr\Vert & & \wr\Vert & \\
\longrightarrow \pi_{23}(\Pi) & \longrightarrow & \pi_{23}(\Pi, S^8) & \longrightarrow & \pi_{22}(S^8) & \longrightarrow & \pi_{22}(\Pi) & \longrightarrow 0
\end{array}
$$

where $i: (\Omega\Pi, S^7) \to (\Omega\Pi, \Omega S^8)$ is a natural injection and the third vertical homomorphism $\pi_{21}(S^7) \to \pi_{22}(S^8)$ is a suspension S.

The fact that $\pi_{21}(\Omega_7) \cong \pi_{22}(\Pi) \cong Z_4$ indicates $\{\Delta\iota_{22}\} \cong Z_{24}$, since $\pi_{21}(S^7) \cong Z_{24} \oplus Z_4$. It follows from (7.4) and the commutativity of the diagram that

$$(7.7)\qquad \Delta\iota_{22} = \alpha = -\sigma'\sigma_{14} + [[\iota_8, \iota_8], \iota_8].$$

Thus we have shown

Proposition 7.1. *For* $i\leq 27$, *we have the isomorphisms*

(i) $\pi_{i+1}(\Pi: 2)\cong\pi_i(\Omega\Pi: 2)\cong\pi_i(S^7\underset{\sigma'\sigma_{14}}{\cup}e^{22}: 2)$

(ii) $\pi_{i+1}(\Pi: 3)\cong\pi_i(\Omega\Pi: 3)\cong\pi_i(S^7\underset{\alpha'}{\cup}e^{22}: 3)$

(iii) $\pi_{i+1}(\Pi: p)\cong\pi_i(\Omega\Pi: p)\cong\pi_i(S^7\times S^{22}: p)$ *for any primes* $p\neq 2, 3$, *where* $\alpha'=S^{-1}([[\iota_8, \iota_8], \iota_8])\in\pi_{21}(S^7: 3)$.

Finally we determine $\pi_{23}(\Pi)$. We have the exact sequence:

$$\cdots\longrightarrow\pi_{23}(S^{22})\longrightarrow\pi_{22}(S^7)\longrightarrow\pi_{22}(\Omega_7)\longrightarrow Z\longrightarrow 0,$$

where $\pi_{23}(S^{22})\cong Z_2=\{\eta_{22}\}$, $\pi_{22}(S^7)\cong Z_{120}\oplus Z_2\oplus Z_2\oplus Z_2$ and the generators of $\pi_{22}(S^7: 2)=\{\rho'', \sigma'\bar{\nu}_{14}, \sigma'\varepsilon_{14}, \bar{\varepsilon}_7\}$. By (7.7) we have

$$\Delta\eta_{22}=\sigma'\sigma_{14}\eta_{21}$$
$$=\sigma'\bar{\nu}_{14}+\sigma'\varepsilon_{14} \quad \text{by Lemma 6.4 of [18].}$$

Hence

$$\pi_{23}(\Pi)=\pi_{22}(\Omega_7)=Z\oplus Z_{120}\oplus Z_2\oplus Z_2.$$

Thus we have shown

Theorem 7.2. *The homotopy groups of the octonionic projective plane for* $i\leq 23$ *are stated as follows:*

i	$i\leq 7$	8	9	10	11	12	13	14	15	16
$\pi_i(\Pi)$	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{120}	$Z_2\oplus Z_2\oplus Z_2$

i	17	18	19	20	21	22	23
$\pi_i(\Pi)$	$Z_2\oplus Z_2\oplus Z_2\oplus Z_2$	$Z_{24}\oplus Z_2$	$Z_{504}\oplus Z_2$	0	Z_6	Z_4	$Z\oplus Z_{120}\oplus Z_2\oplus Z_2$

§8. The 2-primary components of $\pi_i(F_4)$.

In this section we compute $\pi_i(F_4: 2)$ by making use of the exact sequence associated with a homogeneous space F_4/G_2:

(8.2) $$\cdots\longrightarrow\pi_i(G_2)\longrightarrow\pi_i(F_4)\longrightarrow\pi_i(F_4/G_2)\longrightarrow\pi_{i-1}(G_2)\longrightarrow\cdots.$$

It follows from Theorem 2.1 that

$$H^*(F_4/G_2;\ Z_2)\cong\Lambda(x_{15}, Sq^8x_{15}).$$

Hence, by the Serre's $\mathcal{C}$-theory [13] the 2-primary components of $\pi_i(F_4/G_2)$ are isomorphic to $\pi_i(X_{15}: 2)$, which are already computed in §5 to some extent.

Thus (8. 1) is reduced to the following

(8. 1)′ $$\cdots\longrightarrow\pi_i(G_2: 2)\xrightarrow{i_*}\pi_i(F_4: 2)\xrightarrow{p_*}\pi_i(X_{15}: 2)\xrightarrow{\Delta}\pi_{i-1}(G_2: 2)\longrightarrow\cdots.$$

As $\pi_i(X_{15})=0$ for $i\leq 14$, it follows directly

(8. 2) $$\pi_i(G_2: 2)\cong\pi_i(F_4: 2) \quad for \quad i\leq 13.$$

Moreover, as to the so-called boundary homomorphism Δ, we have the relation

(8. 3) $$\Delta\iota_{15}=\langle\bar{\nu}_6+\varepsilon_6\rangle+ai_*[\nu_5^2]\nu_{11} \quad where \quad a=0 \quad or \quad 1,$$

since $\pi_{14}(F_4: 2)\cong Z_2$ by Lemma 5. 5.

By making use of (8. 3) one may easily show that Δ: $\pi_{i+1}(X_{15}: 2)\rightarrow\pi_i(G_2: 2)$ is a monomorphism for $i\neq 14$, $i\leq 21$ and that the kernel of Δ is isomorphic to Z for $i=14$. Hence we obtain

(8. 4) $$\pi_i(F_4: 2)\cong Cokernel\ of\ \Delta: \pi_{i+1}(X_{15})\rightarrow\pi_i(G_2: 2) \quad for\ i\neq 15,\ i\leq 22.$$

The easy calculations show that the cokernel of Δ: $\pi_{i+1}(X_{15}: 2)\rightarrow\pi_i(G_2: 2)$ are as follows.

(8. 5)

i	14	15	16	17	18	19	20	21	22	23
	Z_2	0	$Z_2\oplus Z_2$	Z_2	Z_{16}	Z_2	0	0		G

where $G\cong Z_4$ or $Z_2\oplus Z_2$. It follows that $\pi_{15}(F_4: 2)=Z$.

Next consider the case $i=22$:

$$\Delta:\pi_{23}(X_{15}: 2)\rightarrow\pi_{22}(G_2: 2),$$

where $\pi_{23}(X_{15}: 2)\cong Z\oplus Z_2=\{\langle 16\iota_{23}\rangle, \varepsilon_{15}\}$ and $\pi_{22}(G_2: 2)\cong Z_8\oplus Z_2=\{\langle\zeta' + \mu_6\sigma_{15}\rangle, \langle\eta_6\bar{\varepsilon}_7\rangle\}$. Obviously $\Delta\varepsilon_{15}=\langle\eta\,\bar{\varepsilon}_7\rangle$, since $\varepsilon_6^2=\eta_6\bar{\varepsilon}_7$ in $\pi_{22}(S_6: 2)$.

Let X_{14} be a cell complex $S^{14}\cup e^{22}$ with an attaching map $\sigma_{14}\in \pi_{21}(S^{14}: 2)$, a generator. Then $SX_{14}=X_{15}$. Let g be a map representing an element $\bar{\nu}_6+\varepsilon_6$ in $\pi_{14}(S^6: 2)$. Then g may be extended to X_{14}, since $(\bar{\nu}_6+\varepsilon_6)\circ\sigma_{14}=0$ by Lemma 10.7 of [18]. We denote by $\bar{g}$ this extension of g, $\bar{g}: X_{14}\rightarrow S^6$.

Let p be the projection map in the fibering $G_2/SU(3)=S^6$. Then we have a commutative diagram.

$$\begin{array}{ccc} \pi_{22}(X_{14}: 2) & \xrightarrow{S} & \pi_{23}(X_{15}: 2) \\ \downarrow \bar{g}_* & & \downarrow \Delta \\ \pi_{22}(S^6: 2) & \xleftarrow{p_*} & \pi_{22}(G_2: 2) \end{array}$$

The element $\langle 16\iota_{23}\rangle$ may be considered as a coextension: $S^{23}\rightarrow S^{15}\cup e^{22}$ of $16\iota_{22}$. Hence $S^{-1}\langle 16\iota_{23}\rangle$ is also a coextension: $S^{22}\rightarrow S^{14}\cup e^{22}$ of $16\iota_{21}$. Thus the element $p_*\Delta(\langle 16\iota_{23}\rangle)=\bar{g}_*S^{-1}(\langle 16\iota_{23}\rangle)$ forms a secondary composition $\{\bar{\nu}_6+\varepsilon_6, \sigma_{14}, 16\iota_{21}\}$ by Proposition 1.7 of [18]. By applying the Hopf homomorphism H for this secondary composition we have

$$H\{\bar{\nu}_6+\varepsilon_6, \sigma_{14}, 16\iota_{21}\}\subset\{H(\bar{\nu}_6+\varepsilon_6), \sigma_{14}, 16\iota_{21}\}$$
$$=\{\nu_{11}, \sigma_{14}, 16\iota_{21}\} \quad \text{by Lemma 6.1 of [18]},$$

which contains $x\zeta_{11}$ for an odd integer x mod $8G_{11}$. Thus the order of $\{\bar{\nu}_6+\varepsilon_6, \sigma_{14}, 16\iota_{21}\}$, and hence that of $\Delta(\langle 16\iota_{23}\rangle)$, is 8. This implies that $\Delta: \pi_{23}(X_{15})\rightarrow\pi_{22}(G_2: 2)$ is epimorphic. Therefore we obtain

$$\pi_{22}(F_4: 2)=0.$$

We have shown

Theorem 8.1. *The 2-primary components of* $\pi_i(F_4)$ *for* $i\leq 23$.

i	1	2	3	4	5	6	7	8	9	10	11	12
$\pi_i(F_4: 2)$	0	0	Z	0	0	0	0	Z_2	Z_2	0	$Z\oplus Z_2$	0

i	13	14	15	16	17	18	19	20	21	22	23
$\pi_i(F_4: 2)$	0	Z_2	Z	$Z_2\oplus Z_2$	Z_2	Z_{16}	Z_2	0	0	0	G

where $G \cong Z_4$ *or* $Z_2 \oplus Z_2$.

§9. Homotopy groups of spinor groups.

As to the spinor groups of low rank, there exist homeomorphisms as follows:

$$Spin(3) = Sp(1) = SU(2) = S^3,$$
$$Spin(4) = Spin(3) \times S^3 = S^3 \times S^3,$$
$$Spin(5) = Sp(2),$$
$$Spin(6) = SU(4),$$
$$Spin(8) = Spin(7) \times S^7.$$

Thus $\pi_j(Spin(k))$, $k \leq 6$, are obtained from the known results in [13], [14,] [18] for $j \leq 23$.

In this section we calculate $\pi_j(Spin(7))$, which also gives $\pi_j(Spin(8))$, and $\pi_j(Spin(9))$ for $j \leq 23$.

Let p be odd prime for the moment. Then, according to Harris [5], we have the isomorphisms:

(9.1) $\qquad \pi_j(Spin(2n+1):p) \cong \pi_j(Sp(n):p)$ *for all* j.

Hence $\pi_j(Spin(7):p)$ and $\pi_j(Spin(9):p)$ are given by the known results of $\pi_j(Sp(3):p)$ and $\pi_j(Sp(4):p)$ for $j \leq 23$ [15].

So we compute 2-components of these groups.

(I) $\pi_j(Spin(7):2)$.

Consider first the fibration $Spin(7)/G_2 = S^7$. The characteristic class of this fibration belongs to $\pi_6(G_2)$ which is isomorphic to Z_3. Therefore by Corollary 4.5 we have

Proposition 9.1. *For each prime* $p \neq 3$,

$$\pi_j(Spin(7):p) \cong \pi_j(G_2:p) \oplus \pi_j(S^7:p).$$

Thus $\pi_j(Spin(7))$ will be obtained from the known results; Theorem 6.1, [15], [18].

For later use we list their 2-primary components and their generators. (For simplicity we omit the homomorphisms i'_*, the inclusion one, and χ_*, the cross-section one of 2-components.)

(9.2) $\pi_i(Spin(7):2)$

i	1	2	3	4	5	6	7	8	9	10	11	12
	0	0	Z	0	0	0	Z	$Z_2 \oplus Z_2$	$Z_2 \oplus Z_2$	Z_8	$Z \oplus Z_2$	0
gen.			$i_* j_* \iota_3$				ι_7	$\eta_7, \langle \eta_6^2 \rangle$	$\eta_7^2, \langle \eta_6^2 \rangle \eta_8$	ν_7	$\langle 2\Delta\iota_{13} \rangle, i_*[\nu_5^2]$	

i	13	14	15
	Z_2	$Z_8 \oplus Z_2 \oplus Z_2$	$Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$
gen.	ν_7^2	$\sigma_7, \langle \bar{\nu}_6 + \varepsilon_6 \rangle, i_*[\nu_5^2]\nu_{11}$	$\sigma' \eta_{14}, \bar{\nu}_7, \varepsilon_7, \langle \bar{\nu}_6 + \varepsilon_6 \rangle \eta_{14}$

i	16	17
	$Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$	$Z_8 \oplus Z_2 \oplus Z_8 \oplus Z_2$
gen.	$\sigma' \eta_{14}^2, \nu_7^3, \mu_7, \eta_7 \varepsilon_8, \langle \eta_6^2 \rangle \eta_8 \sigma_9, \langle \eta_6 \mu_7 \rangle, i_*[\nu_5 \bar{\nu}_8]$	$\nu_7 \sigma_{10}, \eta_7 \mu_8, \langle \bar{\nu}_6 \nu_{14}, \rangle, \langle \eta_6^2 \rangle \mu_8$

i	18	19	20	21
	$Z_8 \oplus Z_2 \oplus Z_{16}$	Z_2	$Z_2 \oplus Z_2$	$Z_8 \oplus Z_4$
gen.	$\zeta_7, \bar{\nu}_7 \nu_{15}, \langle 2\Delta\iota_{13} \rangle \sigma_{11}$	$i_*[\nu_5 \bar{\nu}_8]\nu_{16}$	$\nu_7 \sigma_{10} \nu_{17}, \langle \bar{\nu}_6 \nu_{14}^2 \rangle$	$\sigma' \sigma_{14}, \kappa_7$

i	22	23
	$Z_8 \oplus Z_2 \oplus Z_2 \oplus Z_8 \oplus Z_2 \oplus Z_2$	$Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus G$
gen.	$\rho'', \sigma' \bar{\nu}_{14}, \sigma' \varepsilon_{14}, \bar{\varepsilon}_7, \langle \zeta' + \mu_6 \sigma_{15} \rangle, \langle \eta_6 \bar{\varepsilon}_7 \rangle$	$\sigma' \mu_{14}, S\zeta', \mu_7 \sigma_{16}\ \eta_7 \bar{\varepsilon}_8, \langle \eta_6 \mu_7 \rangle \sigma_{16}$

where $G \cong Z_4 = \{\langle \Delta S\theta + \nu_6 \kappa_9 \rangle\}$ *or* $\cong Z_2 \oplus Z_2 = \{\langle \Delta S\theta + \nu_6 \kappa_9 \rangle, i_* \nu_5 \bar{\varepsilon}_8\}$.

(II) $\pi_j(Spin(9):2)$

Consider the well known fibration $Spin(9)/Spin(7) = S^{15}$. The characteristic class $\Delta\iota_{15}$ of this fibration belongs to $\pi_{14}(Spin(7))$.

Thus, if one restricts it to the 2-primary components, it is written as follows (cf. (9.2)):

(9.3) $\Delta\iota_{15} = x\langle \bar{\nu}_6 + \varepsilon_6 \rangle + y\sigma' + zi_*[\nu_5^2]\nu_{11}$,

where x, y, z *are integers.*

In order to study the integers x and y, we consider the exact sequence associated with $Spin(9)/Spin(7) = S^{15}$:

$$\longrightarrow \pi_{22}^{15} \xrightarrow{\Delta} \pi_{21}(Spin(7):2) \xrightarrow{i_*} \pi_{21}(Spin(9):2) \xrightarrow{p^*} \pi_{21}^{15}$$

$$\xrightarrow{\Delta} \pi_{20}(Spin(7):2) \longrightarrow \cdots,$$

where $\pi_{21}(Spin(7):2)\cong Z_8\oplus Z_4=\{\sigma'\sigma_{14},\kappa_7\}$, $\pi_{20}(Spin(7):2)\cong Z_2\oplus Z_2=\{\nu_7\sigma_{10}\nu_{17},\langle\bar{\nu}_6\nu_{14}^2\rangle\}$, $\pi_{22}^{15}\cong Z_8=\{\sigma_{15}\}$ and $\pi_{21}^{15}\cong Z_2=\{\nu_{15}^2\}$. It follows from (9.3) that $\Delta\sigma_{15}=y\sigma'\sigma_{14}$ and $\Delta\nu_{15}^2=x\langle\bar{\nu}_6\nu_{14}^2\rangle+y\nu_7\sigma_{10}\nu_{17}$ and hence

$$0\longrightarrow Z_{(8,y)}\oplus Z_4\longrightarrow\pi_{21}(Spin(9):2)\longrightarrow Z_{(x,y,2)}\longrightarrow 0.$$

Here (a,b,c), (d,e) are G. C. M of a, b and c, or d and e respectively. Note that Z_4 is generated by κ_7.

Next consider the exact sequence associated with a fibration $F_4/Spin(9)=\Pi$:

$$\longrightarrow\pi_{22}(\Pi:2)\longrightarrow\pi_{21}(Spin(9):2)\longrightarrow\pi_{21}(F_4:2)\longrightarrow\pi_{21}(\Pi:2)\longrightarrow\cdots.$$

If we take a map f in the proof of Lemma 5.5, the above Δ is equivalent to the homomorphism f_*.

$$\begin{array}{ccc}
\pi_{22}(\Pi:2) & \xrightarrow{\Delta} & \pi_{21}(Spin(9):2) \\
\| \wr & & \nearrow \\
\pi_{21}(\Omega\Pi:2) & & f_* \\
\uparrow i_* & \nearrow & \\
\pi_{21}(S^7:2) & &
\end{array}$$

And a generator κ_7 of $\pi_{22}(\Pi:2)\cong Z_4$ is mapped by it to κ_7 of $\pi_{21}(Spin(9):2)$.

Thus $\pi_{21}(F_4:2)$ has $(8,y)(x,y,2)$ elements at least. On the other hand, according to Theorem 8.1 $\pi_{21}(F_4:2)=0$, which implies $(8,y)(x,y,2)=1$. Hence y must be odd.

If one supposes x even, the cokernel of Δ: $\pi_{21}^{15}\rightarrow\pi_{20}(Spin(7):2)$ is $Z_2=\{\langle\bar{\nu}_6\nu_{14}^2\rangle\}$, and hence we obtain $\pi_{20}(Spin(9):2)\cong Z_2=\{\langle\bar{\nu}_6\nu_{14}^2\rangle\}$. Then the kernel of $\pi_{21}(\Pi:2)\rightarrow\pi_{20}(Spin(9))$ is Z_2 and hence $\pi_{21}(F_4:2)\cong Z_2$. This is also a contradiction. Thus we have shown

Proposition 9.2. *The characteristic class of $Spin(9)/Spin(7)=S^{15}$ is $\Delta\iota_{15}=x\langle\bar{\nu}_6+\varepsilon_6\rangle+y\sigma'+zi_*[\nu_5^2]\nu_{11}$, where x and y are odd integers.*

Now we compute $\pi_j(Spin(9):2)$ by making use of the following exact sequence:

$$\cdots\longrightarrow\pi_j(Spin(7):2)\longrightarrow\pi_j(Spin(9):2)\longrightarrow\pi_j(S^{15}:2)\longrightarrow\cdots.$$

Since $\pi_j(S^{15})=0$ for $j<15$, we obtain

(9.4) $\qquad \pi_j(Spin(7))\cong\pi_j(Spin(9))$ *for* $j\leq 13$.

Furthermore it follows from Proposition 9. 2 and (9. 2) that $\Delta:\pi_{i+1}^{15}\rightarrow\pi_i(Spin(7):2)$ is monomorphic for $15\leq i\leq 23$ and the kernel of Δ for $i=14$ is isomorphic to Z.

Hence we have

$$\pi_j(Spin(9):2)\cong\begin{cases} Z\oplus Coker.\ \Delta(:\pi_{j+1}^{15}\rightarrow\pi_j(Spin(7):2)) & for\ j=15 \\ Coker.\ \Delta(:\pi_{j+1}^{15}\rightarrow\pi_j(Spin(7):2)) & otherwise\ for\ j\leq 23. \end{cases}$$

The cokernel of Δ are easily obtained and their results are as follows.

i	14	15	16	17	18	19	20	21
	$Z_8\oplus Z_2$	$(Z_2)^3$	$(Z_2)^6$	$Z_8\oplus(Z_2)^2$	$Z_{16}\oplus Z_8\oplus Z_2$	Z_2	Z_2	Z_4

i	22	23
	$(Z_8)^2\oplus(Z_2)^2$	$G\oplus(Z_2)^2$

where $(Z_a)^k$ denotes the direct sum of k-copies of Z_a and G is same as in Theorem 7. 1.

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