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# On a Paper of M. Watanabe[1)]

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In the papers [3, 5] M. Watanabe (née Mori) introduced the condition $\Gamma_{he} \cap \Gamma^*_{hse} \subset \Gamma^*_{he}$ which may hold for an open Riemann surface $R$. Several equivalent conditions were found in [3], some of them due to K. Oikawa. In [5] many interesting consequences were obtained under the additional hypothesis that $R$ is of finite genus. These results suggested that such surfaces $R$ may actually be of class $O_{AD}$, and we shall prove here that this conjecture is correct. The idea of the proof is to show that $R$ uniquely determines the Riemann period matrix of a compact prolongation of $R$. By Torelli's theorem $R$ therefore has a unique such prolongation, and an application of Oikawa's theorem [4] then shows that $R \in O_{AD}$.

THEOREM. *If $R$ is a Riemann surface of finite positive genus with the property $\Gamma_{he} \cap \Gamma^*_{hse} \subset \Gamma^*_{he}$ then $R \in O_{AD}$.*

REMARKS. The restriction that $R$ be nonplanar is clearly needed. For surfaces of finite genus it is known that $O_{AD} = O_{KD}$. The converse of the theorem obviously holds for arbitrary surfaces $R \in O_{KD}$.

PROOF. Suppose $R$ satisfies the hypothesis of the theorem and suppose $R \subset S_1$, $R \subset S_2$ where $S_1$, $S_2$ are closed Riemann surfaces of the same genus $g$ as $R$. Let $A_1, B_1, \cdots, A_g, B_g$ be cycles in $R$ which form a cononical homology basis for $S_1$ and $S_2$. For $\mu = 1, 2$

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let $\sigma^{\mu}(A_i)$, $\sigma^{\mu}(B_j)$ denote the reproducing differentials on $S_{\mu}$ $(1 \leq i, j \leq g)$. Thus

$$\int_{A_i} \omega = (\omega, *\sigma^{\mu}(A_i))$$

$$\int_{B_j} \omega = (\omega, *\sigma^{\mu}(B_j))$$

for all harmonic differentials $\omega$ on $S_{\mu}$. Furthermore, the period of $\sigma^{\mu}(\gamma)$ along a cycle $\delta$ is the intersection number $\delta \times \gamma$.

The differentials $\sigma^2(A_i) - \sigma^1(A_i)$ are elements of $\Gamma_{he} \cap \Gamma^*_{hse}$. Hence they are coexact and we see that $*\sigma^2(A_i)$ and $*\sigma^1(A_i)$ have the same period vector. Similarly $*\sigma^2(B_j)$ and $*\sigma^1(B_j)$ have the same period vector. Define a $2g \times 2g$ matrix

$$M_{\mu} = \begin{bmatrix} (\sigma^{\mu}(A_1), \sigma^{\mu}(A_1)) & \cdots & (\sigma^{\mu}(A_1), \sigma^{\mu}(B_g)) \\ \vdots & & \vdots \\ (\sigma^{\mu}(B_g), \sigma^{\mu}(A_1)) & \cdots & (\sigma^{\mu}(B_g), \sigma^{\mu}(B_g)) \end{bmatrix}.$$

Our results so far imply that $M_1 = M_2$. It follows that $S_1$, $S_2$ have the same Riemann period matrix. Indeed, if $Z_{\mu}$ is the Riemann period matrix for $S_{\mu}$ with respect to the given homology basis then an easy calculation shows that

$$Z_{\mu} = A_{\mu}^{-1}(C_{\mu} + \sqrt{-1}\, I)$$

where $I$ is the $g \times g$ identity matrix and the matrices $A_{\mu}$, $C_{\mu}$ are defined by

$$A_{\mu} = \|(\sigma^{\mu}(A_i),\ \sigma^{\mu}(A_j))\|_{1 \leq i, j \leq g}$$

$$C_{\mu} = \|(\sigma^{\mu}(A_i),\ \sigma^{\mu}(B_j))\|_{1 \leq i, j \leq g}\,.$$

By Torelli's theorem $S_1 = S_2$. In [4], Oikawa showed that a Riemann surface with a unique compact prolongation is of class $O_{AD}$ (see also A. Mori [2]). Thus $R \in O_{AD}$ as desired.

REMARK. The full force of Torelli's theorem is not actually needed in the above proof. If we assume that $R \notin O_{AD}$ then the proof of Oikawa's theorem [4] shows that there is a family $\Delta$ of closed Riemann surfaces of genus $g$, each of which is a prolongation of $R$, and such that the surfaces in $\Delta$ when suitably marked form an open subset of the Teichmüller space $T_g$. If we use the fact that the Riemann period matrix gives a nonconstant holomorphic mapping of $T_g$ into $C^{g^2}$ (see Ahlfors [1]) then a contradiction is obtained, which proves the theorem.

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