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Jacobson-Bourbaki Correspondence

By

Edward T. WONG

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P is a field. Considering P as an abelian group with respect to its addition, let R be the ring of all endomorphisms of P into P. R is a right P-vector space in the obvious way. A subring $\mathfrak{A}$ of R containing the identity mapping is called a P-subring of R if $\mathfrak{A}$ is also a P-subspace of R. If $\mathfrak{A}$ is a P-subring of R, let $\Delta_{\mathfrak{A}}=\{\alpha\in R \mid \alpha A=A\alpha\}$, for all $A\in\mathfrak{A}$, the centralizer of $\mathfrak{A}$. $\Delta_{\mathfrak{A}}$ can be considered as a subfield of P. When P is considered as a left $\Delta_{\mathfrak{A}}$ vector space, $\mathfrak{A}$ is a dense ring of linear transformations of P. This follows from the fact that, $P\subset\mathfrak{A}$, $x\in P$, $x\neq 0$, $x\mathfrak{A}=P$; and the general density theorem [2]. The Jacobson-Bourbaki Theorem [1, p. 22] states that: "If the dimension of a P-subring $\mathfrak{A}$ over P is $n<\infty$, then the dimension of P over $\Delta_{\mathfrak{A}}$ is also n and $\mathfrak{A}=\mathfrak{L}_{\Delta_{\mathfrak{A}}}(P, P)$, the complete ring of linear transformations of P over $\Delta_{\mathfrak{A}}$." From this result, one can set up an one-to-one correspondence (Jacobson-Bourbaki Correspondence) between the set of all P-subrings of R which are finite dimensional over P and the set of all subfields of P which are finite co-dimensional in P [1, p. 24]. Furthermore the classical Galois theorem about finite group of automorphisms of a field can be obtained from this approach [1, p. 29].

In this paper, we are going to extend this correspondence further.

A ring S is called a *left* (*right*) *self-injective* ring if S is a left (right) injective module over itself. A left (right) self-injective ring is also called a *left* (*right*) *quasi-Frobenius* ring.

A ring Q is called a *left quotient ring* of a subring T of Q, if $q\in Q$, $q\neq 0$, then $Tq\cap T\neq 0$. *Right quotient* ring is defined in the similar fashion. A left self-injective ring S has no proper left quotient ring. Since if Q is a left quotient ring of S then S is a direct summand of Q as left S-module. Thus if Q contains S properly then Q is not a left quotient ring of S.

It has been proved that the maximal left quotient ring of a ring with zero left singular ideal is left self-injective. [4, 6]. Also the maximal left quotient ring of a left primitive ring with nonzero socle has been determined [3, 4]. For the purpose of self containess of this paper, we shall obtain the latter result for our situation here again.

In our setting, P is a field, R is the ring of all endomorphisms of P into P as an abelian group, and $\mathfrak{A}$ is a P-subring of R. Let $S=\mathfrak{L}_{\Delta_{\mathfrak{A}}}(P, P)$, then $\mathfrak{A}$ is a dense subring of S and hence a primitive (left) ring.

Therem 1. 1. *S is the maximal left quotient ring of $\mathfrak{A}$ if and only if $\mathfrak{A}$ has nonzero socle.*

Proof: If S is a left quotient ring of $\mathfrak{A}$, then $\mathfrak{A}$ must contain linear transformations of finite rank. Hence $\mathfrak{A}$ has nonzero socle.

If $\mathfrak{A}$ has nonzero socle then $\mathfrak{A}$ contains a minimal right ideal $J=e\mathfrak{A}$, where e is an idempotent and a linear transformation of rank 1. Thus there exists a basis B of P over $\Delta_{\mathfrak{A}}$ in which there exists a vector x, $xe=x$, and $ye=0$ for all $y\neq x$ in B. If h is a nonzero element in S, then there exists $p\in P$, $ph=\bar{p}\neq 0$. Since $\mathfrak{A}$ is dense there exist A and C in $\mathfrak{A}$ where $xA=p$ and $xC=\bar{p}$.

$$x((eA)h) = (xA)h) = ph = \bar{p} = x(eC)$$
$$y((eA)h) = y(eC) = 0, y\in B, y\neq x .$$

Thus $(eA)h)=eC\in J$ and $eC\neq 0$. This shows S is a left quotient ring of $\mathfrak{A}$. Since J is a right ideal in $\mathfrak{A}$ and $JA\neq 0$ if $A\neq 0$ in $\mathfrak{A}$. Every left quotient ring of $\mathfrak{A}$ is also a left quotient ring of J.

Let q be an element in a left quotient ring of J. For any $h\in J$, if $hq=0$ then $hq\in J$. If $hq\neq 0$ there exist b and c in J such that $b(hq)=c\neq 0$. Since $bh\neq 0$, $Jbh\neq 0$ and $Jbh=(Jbe)h\subset(e\mathfrak{A}e)h$. $(e\mathfrak{A}e)h$ is an irreducible $e\mathfrak{A}e$-module ($e\mathfrak{A}e$ is a division ring) and Jbh is a nonzero submodule of $(e\mathfrak{A}e)h$. Thus $Jbh=(e\mathfrak{A}e)h$. There exists $d\in J$ such that $dbh=h$. $hq=(dbh)q=d(b(hq))=dc\in J$. This shows $Jq\subset J$ for any element q in a left quotient ring of J.

Since $PJ=Pe\mathfrak{A}=P$, there exists $x_0\in P$, $x_0J=P$ and $x_0h=0$ if and only if $h=0$, $h\in J$. This follows from the fact that J is a minimal right ideal. Thus for each $p\in P$ there exists a unique $h\in J$ where $p=x_0h$. If q is an element in a left quotient ring of J, we define

$$pq = x_0(hq)\,.$$

If p_1 and p_2 are elements of P, $p_1=x_0h_1$, $p_2=x_0h_2$; $p_1+p_2=x_0(h_1+h_2)$ then $(p_1+p_2)q=x_0((h_1+h_2)q)=x_0(h_1q)+x_0(h_2q)=p_1q+p_2q$. If $\alpha\in\Delta_{\mathfrak{A}}$, $\alpha p=\alpha(x_0h)=(x_0h)\alpha=x_0(h\alpha)$. $(\alpha p)q=x_0((h\alpha)q)=x_0((\alpha h)q)=x_0(\alpha(hq))=(x\alpha)(hq)=\alpha(x_0(hq))=\alpha(pq)$.

This shows q can be considered as an element in S and completes the proof of the theorem.

Corollary 1.1.1. *If a P-subring $\mathfrak{A}$ is left self-injective with nonzero socle then $\mathfrak{A}=\mathfrak{L}_{\Delta_{\mathfrak{A}}}(P, P)$.*

Corollary 1.1.2. *The complete ring of linear transformations of vector space over a field is left and right self-injective.*

Proof: V is a vector space over a field P. Let F be the collection of all linear transformations of V into V of finite rank. F is a primitive ring with nonzero socle. Considering V as a right vector space over P, S, the complete ring of linear transformations, is the maximal right quotient ring of F. Since S is a regular ring the left and right singular ideals of S are zero. S is both left and right self-injective.

From this corollary if a P-subring $\mathfrak{A}$ is left-injective with nonzero socle then $\mathfrak{A}$ is both sides self-injective.

Corollary 1.1.3. *$\mathfrak{A}$ is a P-subring. If $[\mathfrak{A}:P]=n<\infty$ then $[P:\Delta_{\mathfrak{A}}]=n$ and $\mathfrak{A}=\mathfrak{F}_{\Delta_{\mathfrak{A}}}(P, P)$.* [1, Theorem 2, p. 22]

Proof: The right ideals of $\mathfrak{A}$ are also P-subspaces of $\mathfrak{A}$.

Therefore if $[\mathfrak{A}:P]<\infty$ then $\mathfrak{A}$ is a primitive ring with minimal condition. By the structure theorem [2], $\mathfrak{A}=\mathfrak{L}_{\Delta_{\mathfrak{A}}}(P, P)$. If $\{p_1,\cdots,p_n\}$ is a linearly independent set of P over $\Delta_{\mathfrak{A}}$, then the set $\{A_1,\cdots,A_n\}$ in $\mathfrak{A}$ where $p_iA_j=\delta_{ij}$ is linearly independent over P. Therefore $[P:\Delta_{\mathfrak{A}}]$ is finite. $[P:\Delta_{\mathfrak{A}}]=n$ follows directly from the relation $[\mathfrak{A}:P][P:\Delta_{\mathfrak{A}}]=[\mathfrak{A}:\Delta_{\mathfrak{A}}]$.

P is a field, let $\mathcal{S}$ = {all subfields of P} and

$\mathcal{A}$ = {all self-injective P-subrings of R with nonzero socle}.

If $\Delta\in\mathcal{S}$, let $A(\Delta)=\mathfrak{L}_{\Delta}(P, P)$. If $\mathfrak{A}$ is a P-subring, let $I(\mathfrak{A})=\Delta_{\mathfrak{A}}$. $A(\Delta)=\mathfrak{A}\in\mathcal{A}$ by Corollary 1.1.2. $I(A(\Delta))=\Delta_{\mathfrak{A}}\supset\Delta$. But $\alpha\in\Delta_{\mathfrak{A}}$, $\alpha\in\mathfrak{L}_{\Delta}(P, P)$ and $\alpha A=A\alpha$ for all $A\in\mathfrak{A}$. α is a scalar transformation. Hence $\alpha\in\Delta$ and $I(A(\Delta))=\Delta$. If $\mathfrak{A}\in\mathcal{A}$, $\mathfrak{A}\subset\mathfrak{L}_{\Delta_{\mathfrak{A}}}(P, P)$. Since $\mathfrak{A}$ is self-injective with nonzero socle, $\mathfrak{A}=\mathfrak{L}_{\Delta_{\mathfrak{A}}}(P, P)$ and hence $A(I(\mathfrak{A}))=\mathfrak{A}$.

Theorem 1.2. *There exists an one-to-one correspondence between the sets $\mathcal{S}$ and $\mathcal{A}$. $\mathfrak{A}\in\mathcal{A}$, $I(\mathfrak{A})\in\mathcal{S}$, and $A(I(\mathfrak{A}))=\mathfrak{A}$, $\Delta\in\mathcal{S}$, $A(\Delta)\in\mathcal{A}$, and $I(A(\Delta))=\Delta$.*

It is clear that $\mathcal{A}$ is also the set of all P-subrings $\mathfrak{A}$ where $\mathfrak{A}=\mathfrak{L}_{\Delta_{\mathfrak{A}}}(P, P)$.

If g is an automorphism of the field P, we denote the image of x in P under g by x^g. When we consider x and g are endomorphisms of P then $xg=gx^g$ (since $p(xg)=(px)g=p^gx^g=p(gx^g)$ for all $p\in P$). If G is a group of automorphisms of P, let

$\mathfrak{A}=GP=\{\sum^n g_iP_i\,|\,g_i\in G, p_i\in P\}$ (here G and P are considered as subsets of R).

By the same proof as in [1, p. 28], $\mathfrak{A}$ is a P-subring, $\{G\}$ is a basis of $\mathfrak{A}$ over P, and $\Delta_{\mathfrak{A}}=I(G)=\{p\in P\,|\,p=p^g$ for all $g\in G\}$, the fixed field of G.

Lemma 2.1. *G_1 and G_2 are groups of automorphisms of a field P. If $G_1\subset G_2$ properly then $\mathfrak{A}_1\subset\mathfrak{A}_2$ properly where $\mathfrak{A}_i=G_iP$, $i=1, 2$.*

Proof: Let $C=G_2-G_1$. C is nonempty. $\mathfrak{A}_2=\mathfrak{A}_1\oplus CP$. $\mathfrak{A}_1\subset\mathfrak{A}_2$ properly.

For a subfield Φ of P, the Galois group $A_P(\Phi)$ of Φ in P is the set of all automorphisms of P which leave each element of Φ fixed. Φ is said to be *Galois* in P or P over Φ *Galois* if $A_P(\Phi)P=\mathfrak{L}_\Phi(P, P)$.*

Let $\mathfrak{Y}=\{G|G$ is a group of automorphisms of P ane $\mathfrak{A}=GP$ is self-injective with nonzero socle$\}$ and
$\mathfrak{F}=\{\Phi|\Phi$ is a subfield of P and Galois in $P\}$.

$G\in\mathfrak{Y}$, $\mathfrak{A}=GP=\mathfrak{L}_\Phi(P, P)$ where $\Phi=I(G)$, the fixed field of G. If $G'=A_P(\Phi)$, the Galois group of Φ in P, then $G\subset G'$. Since $G'\subset\mathfrak{L}_\Phi(P, P)=GP$, $G'P=GP$ and $G=G'$ by Lemma 2.1. This shows $I(G)\in\mathfrak{F}$ and $A_P(I(G))=G$ if $G\in\mathfrak{Y}$. If $\Phi\in\mathfrak{F}$, then $A_P(\Phi)\in\mathfrak{Y}$. Since $A_P(\Phi)P=\mathfrak{L}_\Phi(P, P)$ is self-injective with nonzero socle. Let $G=A_P(\Phi)$ and $\Delta=I(G)$ then $\Delta\supset\Phi$ and $\mathfrak{L}_\Delta(P, P)=\mathfrak{L}_\Phi(P, P)$. $\Delta=\Phi$ follows from the fact that the elements of Δ are scalar transformations in $\mathfrak{L}_\Phi(P, P)$. Therefore if $\Phi\in\mathfrak{F}$, then $A_P(\Phi)\in\mathfrak{Y}$ and $I(A_P(\Phi))=\Phi$.

Theorem 2.2. *There exists a one-to-one correspondence between* $\mathfrak{Y}$ *and* $\mathfrak{F}$. $G\in\mathfrak{Y}$, $I(G)\in\mathfrak{F}$, *and* $A_P(I(G))=G$; $\Phi\in\mathfrak{F}$, $A_P(\Phi)\in\mathfrak{Y}$ *and* $I(A_P(\Phi))=\Phi$.

Corollary 2.2.1. *If G is a finite group of automorphisms of P then $G\in\mathfrak{Y}$. If Φ is a finite co-dimensional subfield of P and $\Phi=I(G)$ for some group of automorphisms G of P then $\Phi\in\mathfrak{F}$* [1, Theorem 5, p. 20].

Proof: If G is finite then $[GP:P]<\infty$. $GP=\mathfrak{L}_{I(G)}(P, P)$ by Corollary 1.1.3. $G\in\mathfrak{Y}$.

If $[P:\Phi]$ is finite and $\Phi=I(G)$ then $A_P(\Phi)\supset G$ and $A_P(\Phi)P\supset GP$. $\mathfrak{L}_\Phi(P, P)=GP$ since GP is dense and $[P:\Phi]$ is finite. But $A_P(\Phi)P\subset\mathfrak{L}_\Phi(P, P)=GP$. $A_P(\Phi)=G$ and $\Phi\in\mathfrak{F}$.

Lemma 2.3. *G is a group of automorphisms of a field P. If*

* Ordinarily we say that Φ is Galois in P if Φ is the fixed field of some group of automorphisms of P.

B is a P-subring of GP and is self-injective (left or right) then $B=G'P$ *where* $G'=G\cap B$ *is a subgroup of G.*

Proof: $h\in B$, $h=g_1p_1+\cdots+g_np_n$, $g_i\in G$, $p_i\in P$. By the same proof as in [1, p. 26], $g_i\in B$.

$GP=B\oplus T$ as B-module by the injectivity of B. If $g\in G\cap B$ and $g^{-1}=b+t$, $b\in B$, $t\in T$. Then $1-bg=tg\in B$, $tg=0$, $t=0$, and $g^{-1}\in B$. This shows $G'=G\cap B$ is a subgroup of G. $B=G'P$ is obvious.

Theorem 2. 4. *G is a group of automorphisms of a filed P. If* $G\in\mathfrak{Y}$ *(GP is self-injective with nonzere socle) then any intermediate field E of P and* Φ, *the fixed field of G, is also in* $\mathfrak{F}$ *(E is Galois in P).*

Proof: Let $S=\mathfrak{L}_E(P, P)$, $S\subset\mathfrak{L}_\Phi(P, P)=GP$. S is self-injective by Corollary 1. 1. 2. Therefore $S=HP$ where $H=G\cap S$, a subgroup of G by the above lemma. $H\subset A_P(E)\subset S$. $HP=A_P(E)P$ and hence $H=A_P(E)$ and $\mathfrak{L}_E(P, P)=A_P(E)P$. E is Galois in P.

Our next natural question is that *under what condition* Φ *will also be Galois in E?* The Galois group $H=A_P(E)$ of E in P has been proved is a subgroup of G. H is a normal subgroup of G if and only if $E^g=E$ for all $g\in G$ (same proof as in [1, p. 30]). We will show that this is also equivalent to Φ is Galois in E.

Let $N=\{g\in G\mid E^g\subset E\}$ and N' be the restrictions of $g\in N$ on E. $N'E\subset\mathfrak{L}_\Phi(E, E)$. Since E is a subspace of P over Φ, for every $h\in\mathfrak{L}_\Phi(E, E)$ there exists an $f\in\mathfrak{L}_\Phi(P, P)$ where $Ef\subset E$ and the restriction f' of f on E equals to h.

From here on we assume G is a group of automorphisms of the field P and GP is self-injective with nonzero socle or equivalently $GP=\mathfrak{L}_\Phi(P, P)$ where Φ is the fixed field of G. Let E be an intermediate field between Φ and P and $H=A_P(E)$, the Galois group of E in P. H is a subgroup of G and $HP=\mathfrak{L}_E(P, P)$.

Lemma 2. 5. $f\in\mathfrak{L}_\Phi(P, P)=GP$, $Ef\subset E$. *If* $f'=g_1'p_1+\cdots+g_n'p_n$ *with* $g_i'\in N'$ *then* $p_i\in E$.

Proof: Of course here we assume g_i' are distinct and $p_i\neq 0$. If the lemma is false, there exists $f_1'=g_1'p_1+\cdots+g_m'p_m$ with

$g_i' \in N'$ and not all $p_i \in E$ (in fact we can say that each $p_i \notin E$). Let m be the minial length among such expressions. If $m \geq 2$, choose $x \in E$ where $x^{g_1'} \neq x^{g_2'}$,

$$xf' = g_1' x^{g_1'} p_1 + g_2' x^{g_2'} p_2 + \cdots + g_m' x^{g_m'} p_m ,$$
$$f' x^{g_1'} = g_1' x^{g_1'} p_1 + g_2' x^{g_1'} p_2 + \cdots + g_m' x^{g_1'} p_m ,$$
$$xf' - f' x^{g_1'} = g_2'(x^{g_2'} - x^{g_1'}) p_2 + \cdots + g_m'(x^{g_m'} - x^{g_1'}) p_m ,$$

$E(xf' - f' x^{g_1'}) \subset E$ and $(x^{g_2'} - x^{g_1'}) p_2 \notin E$. This contradicts to m is minimal. If $m=1$ then $f' = g_1' p_1$. But $1f' = p_1 \in E$. This proves the lemma.

Lemma 2. 6. $g' \in N'$. *If* $g' \in A_E(\Phi)E$ *then* $g' \in A_E(\Phi)$.

Proof: Let $g' = h_1 e_1 + \cdots + h_n e_n$, where $h_i \in A_E(\Phi)$ and $e_i \in E$. If $g' = h_1$ then there is nothing to prove. Otherwise there exists $x \in E$, $x^{g'} \neq x^{h_1}$,

$$g' x^{g'} = h_1 x^{h_1} e_1 + \cdots + h_n x^{h_n} e_n = h_1 x^{g'} e_1 + \cdots + h_n x^{g'} e_n .$$
$$h_1(x^{h'} - x^{g'}) e_1 + \cdots + h_n(x^{h_n} - x^{g'}) e_n = 0. \quad x^{g'} = x^{h_1} .$$

Contradiction. Therefore $g' = h_1 \in A_E(\Phi)$.

Lemma 2. 7. $f \in \mathfrak{L}_\Phi(P, P) = GP$, $f = g_1 p_1 + \cdots + g_n p_n$. *Assume that there exists* $x \in E$, $x^{g_1} \neq x^{g_2}$ *then* $Ef = 0$ *implies all* $p_i = 0$.

Proof: If $n \geq 2$ then

$$xf = g_1 x^{g_1} p_1 + \cdots + g_n x^{g_n} p_n \quad \text{and}$$
$$f x^{g_1} = g_1 x^{g_1} p_1 + \cdots + g_n x^{g_1} p_n .$$
$$0 = E(xf - f x^{g_1}) = E(g_2(x^{g_2} - x^{g_1}) p_2 + \cdots + g_n(x^{g_n} - x^{g_1}) p_n) .$$

By the same technique employed in Lemma 2. 5., result can be obtained immediately.

Corollary 2. 7. 1. *If* $f \in \mathfrak{L}_\Phi(P, P) = GP$, $Ef \subset E$, *and* $(xy)f' = (xf')(yf')$ *for all* x, y *in* E, *then* $f' \in N'E$.

Proof: Suppose $f = g_1 p_1 + \cdots + g_n p_n$, $g_i \in G$ and $p_i \in P$. If $x^{g_1} = x^{g_2} = \cdots = x^{g_n}$ for all x in E, then $xf' = x(g_1 p)$, $x \in E$, where $p = p_1 + \cdots + p_n$. $1f' = p \in E$. If $p = 0$ then $f' = 0$. No problem here. If $p \neq 0$ then $x^{g_1} \in E$ for all $x \in E$, $g_1 \in N$ and $f' = g_1' p \in N'E$. Now suppose $g_1 \neq g_2$ in E,

$$xf' = g_1 x^{g_1} p_1 + \cdots + g_n x^{g_n} p_n\,,$$
$$f' x^{f'} = g_1 x^{f'} p_1 + \cdots + g_n x^{f'} p_n\,,$$
$$0 = E(xf' - f'x^{f'}) = E(g_1(x^{g_1} - x^{f'})p_1 + \cdots + g_n(x^{g_n} - x^{f'})p_n)$$

for all $x \in E$. By the lemma $x^{g_i} = x^{f'}$ for all $x \in E$, $i = 1, \cdots, n$. Contradiction.

If $h \in A_E(\Phi)$, $h \in \mathfrak{L}_\Phi(E, E)$, there exists $f \in \mathfrak{L}_\Phi(P, P)$ such that $Ef \subset E$ and $f' = h$. By the above corollary $h \in N'E$.

Now suppose $H = A_P(E)$ is a normal subgroup of G. Then $N = G$ and N' is a group. It is clear that N' is isomorphic to the factor group G/H. By Lemma 2.5 $\mathfrak{L}_\Phi(E, E) = N'E$. Since N' is a group, $N' \subset A_E(\Phi)$ and $N'E \subset A_E(\Phi)E$. But $A_E(\Phi)E \subset \mathfrak{L}_\Phi(E, E)$. Thus $N'E = A_E(\Phi)E$ and $N' = A_E(\Phi)$. This shows Φ is Galois in E and $A_E(\Phi) \approx G/H$.

If Φ is Galois in E, i.e. $\mathfrak{L}_\Phi(E, E) = A_E(\Phi)E$. Since $N'E \subset \mathfrak{L}_\Phi(E, E)$, $N' \subset A_E(\Phi)E$. By Lemma 2.6, $N' \subset A_E(\Phi)$ and by Corollary 2.7.1 $A_E(\Phi)E = N'E$ and $N' = A_E(\Phi)$. This implies N is a subgroup of G. Since the fixed field of N = the fixed field of N' = the fixed field of $A_E(\Phi) = \Phi$, $N = G$ and H is a normal subgroup of G.

Combining all these we have the following classic Galois Theorem

Theorem 2.6. *G is a group of automorphisms of a field P. Let Φ be the fixed field of G. If GP is a self-injective ring with nonzero socle then*

1. *Any intermediate field of P and Φ is Galois in P.*
2. *If E is an intermediate field of P and Φ then the Galois group $H = A_P(E)$ of E in P is a subgroup of G. Φ is Galois in E if and only if H is a normal subgroup in G. In this case the Galois group $A_E(\Phi)$ of Φ in E is isomorphic to the factor group G/H.*

Oberlin College
Oberlin, Ohio
U. S. A.

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