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Behavior of the Solutions of Certain Heat Equations

By

Frank B. KNIGHT

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Introduction. We shall be concerned with equations which, in the simplest case, have the form

$$0.1)\qquad \left(\frac{1}{2}\frac{\partial^2}{\partial x^2}+V(x)\right)g(t,x)=\frac{\partial}{\partial t}g(t,x)\,,\qquad -\infty<x<\infty\,,$$

and denote the operator $\frac{1}{2}\frac{d^2}{dx^2}+V$ by A_V. Under weak conditions on V there corresponds to 0.1) a strongly continuous semigroup T_t on a suitable space $\tilde{C}$ of continous functions such that the function $T_tf(x)=\int\tilde{p}(t,x,y)f(y)dy$ for $f\in\tilde{C}$ is a solution of 0.1) and $\tilde{p}$ is uniquely determined by T_t and continuity in y. Our primary concern is with the shape and the evolution with t of the kernels $\tilde{p}$.

The kernel $\tilde{p}(t,x,y)$ can be considered as the temperature at time t and point y resulting from a unit source of heat at x when $t=0$. A more detailed interpretation is obtained, however, in terms of a family of measure spaces $(\Omega,\mathcal{F},\mu_x)$ for which the elements of Ω are of the form (η,ρ,w), $0\leq\eta<\rho\leq\infty$, $w=w(t)$, $0\leq t<\rho$, where $w(t)$ is continuous and $T_tf(x)=E_x(f(w(t))\,;\,\eta<t<\rho)$ E_x denoting an integral with respect to μ_x. The functions $w(t)$ represent the paths of particles created at time η and destroyed at time ρ. These space provide, however, only one possible measure-theoretic approach to T_t. They are defined, for bounded, for bounded V,[1] in [3] but are not used in the present paper, except by way of motivation.

The problem to be studied below is that of describing qualitatively the form of the kernels $\tilde{p}$ in terms which apply to all V. It is intuitively clear that the role of V is to modify, in some sense, the "geometry" of the solutions. We shall find that this statement can be made precise in the following way: the role of V is to substitute for the positive constants in the classical case $V=0$ the positive differentiable solutions of $A_V h=0$. A consequence of this study will be to prove that $\lim_{t\to\infty}\tilde{p}(t,x,y)\leq\infty$ exists and is

a) ∞ if three is a non-trivial solution h with 2 or more zeros,

b) 0 if there are two linearly independent positive solutions, and

c) $h(x)h(y)\left(\int h^2dx\right)^{-1}$ if there is only one positive solution with $h(0)=1$.

These three cases are exhaustive. The result itself can be deduced, rather indirectly, from the known eigen-differential expansion of $\tilde{p}$ in the case of V bounded above.[2] But even in this case such a course seems artificial in that the expansion takes no account of the essential positivity features of the problem.

Section 1. The case of bounded V.

We assume that $|V|<M$, and that V is sectionally continuous. In this case the existence of T_t on the space C_0 continuous functions with limit 0 at $\pm\infty$ is known. In fact, the semigroup $e^{-Mt}T_t$, with strong infinitesimal generator of the form A_V-M and domain D consisting of the differentiable functions f such that $(A_V-M)f\in C_0$ (if defined by continuity at the jumps of V) is a special case of [6] and has a kernel $e^{-Mt}\tilde{p}(t,x,y)$ for which the expansion

$$\tilde{p}(t,x,y)=e^{Mt}\int_{-\infty}^{0+}e^{-\gamma t}e(\gamma,x)f(d\gamma)e(\gamma,y)$$

1) The measure for unbounded V is easily defined by monotone passages to the limit in $(V\wedge N)\vee M$.

2) For these expansions we refer to [5] and [6, p. 149].

is known. The following facts are the only consequences of [6] which we shall require:

1. 1) i. $\tilde{p}$ is continuous in (t, x, y), $t>0$, and is symmetric in x and y.

ii. For $t>0$ and fixed x, $\tilde{p}\in D$ and $A_V\tilde{p}=\frac{\partial}{\partial t}\tilde{p}$.

iii. For each finite open interval I containing x, $\lim_{t\to 0}\int_I \tilde{p}(t, x, y)dy=1$.

We shall also use without special mention the Sturm comparison theorem [1, p. 208] and its immediate consequences.

The case in which there is a non-trivial solution of $A_V h=0$ with 2 or more zeros does not lend itself conveniently to our methods, except through the device of replacing V by $V-M$ (which changes the whole aspect of the problem) and we shall be content to show only

Theorem 1.1. *In this case,* $\lim_{t\to\infty}\tilde{p}=\infty$.

Proof. Let a and b be two consecutive zeros of h. Then h is in the domain of the generator A_V of the *absorbing barrier* semigroup on (a, b), i.e. the semigroup determined by the classical boundary conditions 0 at a and b. We can assume that $h>0$ in (a, b); then h is a positive invariant function for this semigroup. It is clear that in any larger interval, the generator $A_V-\lambda$, for some $\lambda>0$, has a positive invariant function, whence, by the symmetry of the kernel, in the larger interval the absorbing barrier kernel for A_V tends to ∞ like $e^{\lambda t}$. Therefore the still larger kernel $\tilde{p}$ satisfies the theorem.

Accordingly, we henceforth assume

Hypothesis 1.1. *There exists a strictly positive differentiable solution of* $A_V h<0$.

Remark: Such a solution always exists except in the situation of theorem 1.1. For example, the solution with values 1 at 0 and 0 at b converges to such a solution as $b\to\infty$ (monotonically in $(-N, 0)$ and in $(0, N)$ for $b>N>0$). See below.

Proposition 1.1. *Every two-point boundary value problem for A_V with positive boundary values has a (unique, positive) solution.*

Proof. Let the given values be c_1 at x_1 and c_2 at x_2, $x_1<x_2$. Let h be a positive solution of $A_V h=0$ with $h(x_1)=c_1$, and let $k(x)$ be the solution with $k(x_1)=0$ and $k'(x_1)=1$. Since $k(x)>0$ for $x>x_1$, a solution to our problem is $h(x)+k(x)(k(x_2))^{-1}(c_2-h(x_2))$. This solution is positive in $[x_1, x_2]$ since it has at most one zero, and it is unique since the difference of two solutions would have two zeres.

Definition 1.1. A function $f>0$ in $[a, b]$, $-\infty\leq a<b\leq+\infty$, is called V-concave if for $a\leq x_1<x_2\leq b$, x_1 and x_2 finite, the solution of $A_V h=0$, $h(x_1)=f(x_1)$, $h(x_2)=f(x_2)$, satisfies $f-h\geq 0$ in $[x_1, x_2]$. It is V-convex if the last inequality is reversed.

It will be shown that this concept replaces the usual one (to which it reduces if $V=0$) when $V\neq 0$.

Definition 1.2. The number of changes of sign of a function f in $[a, b]$ is given by $\sup n: \exists x_1<x_2<\cdots<x_{n+1}$ in $[a, b]$ such that either for all i, $f(x_i)<0<f(x_{i+1})$ or else the inequalities are all reversed, $1\leq i\leq n$.

Proposition 1.2. *For $f\in C_0$, the number $N(t)$ of changes of sign of $T_t f$ is non-increasing in t.*

Proof. It is clearly enough to show that $N(t)\leq N(0)$, and we assume without loss of generality that $N(0)<\infty$. We need the

Lemma 1.1. *Let $f\in C_0$ be non-negative in a finite interval $[a, b]$ with $f(a)>0$ and $f(b)>0$, or in $(-\infty, a]$ with $f(a)>0$, or in $[b, \infty)$ with $f(b)>0$. Then for all t sufficiently small, $T_t f(x)>0$ in $[a, b]$, or in $(-\infty, a]$, or in $[b, \infty)$ respectively.*

Proof. For brevity we treat only the case of $(-\infty, a]$, the others being very similar. By continuity in t, we have ε and $\delta>0$ such that $T_t f(a)\geq\varepsilon$ for $0\leq t\leq\delta$. It follows that for $t\leq\delta$, $T_t f(x)$ for $x\leq a$ is at least as large as the solution of 0.1) with initial value f in $(-\infty, a]$ and boundary value ε at a. This solution exceeds that of the case $V=-M$, which is given by

$$\varepsilon\left(1-\frac{e^{-Mt}}{\sqrt{2\pi t}}\int_{-\infty}^{0}\left(1-\frac{f(y)}{\varepsilon}\right)\left(\exp-\frac{(x-y)^2}{2t}-\exp-\frac{(x-2a+y)^2}{2t}\right)dy\right)$$

and is strictly positive in $(-\infty, a]$ for all $t>0$.

Remark: We shall require this lemma also for $f=g+h$, $g\in C_0$, $A_V h=0$. Now $\int \tilde{p}(t, x, y)h(y)dy$ is well-defined, as one sees if h is positive by considering the absorbing barrier approximation $a\to-\infty$, $b\to\infty$. But if $h(x_0)=0$, then replacing V by $-M$, keeping the value of $h'(x_0)$ fixed, we obtain an h with larger absolute value but exponentially bounded, and hence integrable for $\tilde{p}$ by the comparison $e^{-Mt}p(t, x, y)<\tilde{p}\leq e^{Mt}p(t, x, y)$ where p denotes the (normal) density for the case $V=0$. Thus in each case, by first introducing absorbing barriers at $\pm b$ and letting $b\to\infty$, we can write $T_t h=h$, and then $T_t f=T_t g+h$. The remainder of the argument proceeds as before.

Returning now to proposition 1.2, let $t_0=\inf t>0: N(t)>N(0)$, and note that by continuity $N(t_0)\leq N(0)$ if $t_0<\infty$. In this case there are $x_1<y_0<x_2<y_2<\cdots<x_{N(t_0)}<y_{N(t_0)}<x_{N(t_0)+1}$ such that the x_i satisfy definition 1.2 for $T_{t_0}f$ and this function has no changes of sign in $(-\infty, y_1]$, $[y_1, y_2]$, $\cdots$, $[y_{N(t_0)}, \infty)$. In particular, $T_{t_0}f(y_i)=0$ and the y_i are uniquely determined unless $T_{t_0}f$ vanishes in an entire interval. A contradiction will now be obtained from the fact that there must be an additional change of sign in every period $[t_0, t_0+\varepsilon)$. From the lemma we see that in every subinterval of $[y_i, y_{i+1}]$ such that $T_{t_0}f\neq 0$ at the endpoints there is no change of sign when ε is sufficiently small. This implies that for some i, $1\leq i\leq N(t_0)$, if $[\alpha_i, \beta_i]$ denotes the maximum interval containing y_i in which $T_t f_0=0$, then for some sequence $t_n\downarrow t_0$ $\delta>0$ at least 3 changes of sign of $T_{t_n}f$ in $(\alpha_i-\delta, \beta_i+\delta)$ for all n sufficiently large. We can assume, by choosing a smaller δ if necessary, that $T_{t_0}f(\alpha_i-\delta)\neq 0$ and $T_{t_0}f(\beta_i+\delta)\neq 0$, whence the same holds for $T_t f$, $0<t-t_0<\varepsilon$ small. For $t>t_0$ let $S(t)=\{x\in(\alpha_i-\delta, \beta_i+\delta): T_t f(x)=0\}$, and introduce $\alpha(t)=\min x\in S(t)$ and $\beta(t)=$ man $x\in S(t)$. Then clearly $\lim_{t\downarrow t_0}\alpha(t)=\alpha_i$ and $\lim_{t\downarrow t_0}\beta(t)=\beta_i$ in view of the lemma, and we define $\alpha(t_0)=\alpha_i$, $\beta(t_0)=\beta_i$. On the other hand, for large n there must be at least one change of sign in

3) For $t_0\leq t_1<t_2<t_0+\varepsilon$ let $\tau=\sup\{t\in[t_1, t_2]: c^+(t')\leq c^+(t_1)$ for $t_1\leq t'\leq t\}$. Then $\tau<t_2$ leads to a contradiction.

$(\alpha(t_n), \beta(t_n))$ for $T_{t_n}f$. Now let h be a positive solution of $A_V h=0$ and let $c^+(t)=\inf c: ch\geq T_t f$ in $(\alpha(t), \beta(t))$. If we show that $c^+(t)$ is non-increasing in $[t_0, t_0+\varepsilon)$ then from $c^+(t_0)=0$ we have our contradiction. First we observe that $c^+(t)$ continuous from the right at t_0. It is then sufficient to prove that $c^+(t)$ is decreasing to the right for $t>t_0$, and that $\liminf_{\tau\uparrow t} c^+(\tau)\geq c^+(t)$ for $t_0<t<t_0+\varepsilon$, for by an obvious reasoning[3] this implies that $c^+(t)$ is decreasing. But from $c^+(t)h-T_t f\geq 0$ in $[\alpha(t), \beta(t)]$ and >0 at the endpoints, the lemma implies that it is >0 in $[\alpha(t), \beta(t)]$ at times $t'>t$ when $t'-t<\Delta$ small. Moreover, it is easy to see that $\lim_{t'\downarrow t} \alpha(t')\wedge\alpha(t)=\alpha(t)$, and by continuity this and the preceding remark show that $c^+(t)$ is decreasing to the right for $t_0<t<t_0+\varepsilon$, as asserted. Coming to the second property, one observes that it only needs to be shown that if $\alpha^+(t)=\limsup_{\tau\uparrow t} \alpha(\tau)>\alpha(t)$, then $T_t f=0$ in $[\alpha(t), \alpha^+(t)]$. It is no restriction to assume that $T_{t_0}f(\alpha_i-\delta)>0$, and then clearly $T_t f(x)\geq 0$ for $\alpha_i-\delta\leq x\leq\alpha^+(t)$. Now if the assertion were false, there would be a neighborhood N of $\alpha(t)$ with closure $\bar{N}$ such that $T_t f\geq 0$ in $\bar{N}$ and $T_t f>\varepsilon'>0$ at the endpoints. But then for all $\tau\leq t$ with $t-\tau$ sufficiently small, one would have $T_\tau f>\varepsilon'/2$ at the endpoints while, by definition of $\alpha^+(t)$, there would be arbitrarily small $t-\tau_0$ for which $T_{\tau_0}f>0$ everywhere in $\bar{N}$. (We choose, if necessary, the right endpoint of $\bar{N}$ strictly less than $\alpha^+(t)$). But the argument of the lemma now clearly shows that one would have $T_\tau f>0$ in $\bar{N}$ for $\tau_0\leq\tau\leq t$, contrary to $T_t f(\alpha(t))=0$. This completes the proof.

One can now establish the

Theorem 1.2.[4] *There exist functions* $-\infty\leq x_1(t)<0<x_2(t)\leq\infty$, $t>0$, *such that* $\tilde{p}(t, 0, y)$ *is V-concave in* $[x_1(t), x_2(t)]$ *and V-convex in* $(-\infty, x_1(t)]$ *and in* $[x_2(t), \infty)$. *The same holds for* $\tilde{p}(t, x, y)$, *for each* x, with $x_1(t)<x<x_2(t)$.

Proof. It is enough to find $x_i(t)$ such that $A_V\tilde{p}(t, 0, y)$ is

4) It would be advantageous to know that the $x_i(t)$ are unique and finite, but we could not prove it.

$\begin{cases}\leq 0 \text{ in } [x_1(t), x_2(t)] \\ \geq 0 \text{ in the complement,}\end{cases}$ where A_V operates on y. Indeed by subtracting the invariant solution equal to $\tilde{p}$ at x_1 and at x_2, $x_1(t) \leq x_1 < x_2 \leq x_2(t)$, we are reduced to showing that if $A_V g \leq 0$ in $[x_1, x_2]$, $g(x_1) = g(x_2) = 0$, then $g \geq 0$ in $[x_1, x_2]$. In the opposite case, by first reducing the interval if necessary, we can assume that $g < 0$ in (x_1, x_2). But we then have $\left[\frac{1}{2}\frac{d^2}{dx^2} + \left(V - \frac{A_V g}{g}\right)\right] g = 0$ $\frac{A_V g}{g} \geq 0$ in (x_1, x_2), whence by the Sturm comparison theorem g does not exceed the solution of $A_V h = 0$ in $[x_1, x_2]$ with $h(x_1) = g(x_1) = 0$ and $h'(x_1) = g'(x_1) < 0$. But then this solution would have 2 zeros in $[x_1, x_2]$ contrary to hypothesis 1.1.

We remark next that, as is immediately clear from the eigenfunction expansion of the absorbing barrier kernel in a finite interval (a, b) and the approximation $a \to -\infty$, $b \to +\infty$, we have $A_V \tilde{p}(t, 0, 0) \leq 0$ and hence can assume $x_1(t) < 0 < x_2(t)$[5].

Our problem thus is reduced to showing that $A_V \tilde{p}(t, 0, y)$ can have at most 2 changes of sign. Now let f_n, $n = 1, 2, \cdots$ be functions with the following properties:

i) $f_n(x) \geq 0$, $f_n(x) = 0$ for $|x| \geq \frac{1}{n}$, $\int f_n(x) dx = 1$.

ii) $A_V f_n$ has exactly 2 changes of sign, being negative for $x = 0$ and positive for both a positive and a negative value of x.

The existence of such f_n is easily recognized. For example, one can begin with $\cos \sqrt{c_n} x$ for $|x| < \frac{\pi}{4} c_n^{-1/2}$, $c_n > 2M$, and then piece on two "steep" solutions of $A_V h = 0$, c_n being large and the first derivative being made continuous, which define the function until it is less than a small $\varepsilon > 0$, and then piece on two more sections with large 2nd derivative to reach 0 for $|x| \geq 1/n$. One then multiplies by a positive constant to obtain $\int f_n dx = 1$. A rigorous derivation is also easy, and will be left to the reader.

By proposition 1.2 we know that $A_V T_t f_n$ has at most 2 changes

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5) The eigen-differential expansion shows that, indeed, $A_V p(t, 0, 0) < 0$. We were unable to account for these facts by our methods, and consider them surprising.

af sign. But we also have

$$A_V T_t f_n(y) = \int_{-1/n}^{1/n} A_V \tilde{p}(t, x, y) f_n(x) dx$$
$$= A_V \tilde{p}(t, x_n(y), y)$$

for some $|x_n(y)| < 1/n$, frtm which we obtain $\lim_{n\to\infty} A_V T_t f_n(y) = A_V \tilde{p}(t, 0, y)$ pointwise. This clearly implies that $A_V \tilde{p}(t, 0, y)$ can have at most 2 changes of sign. As before, this number is non-increasing in t.

We turn now to the proof 'of the result mentioned in the introduction, under hypothesis 1.1. The proof, it may be suggested, is of primary interest rather than the result itself.

Theorem 1.3. *For each* (x, y)

$$\lim_{t\to\infty} \tilde{p}(t, x, y) = h(x) h(y) \left(\int h^2 dx \right)^{-1}$$

where h is (any) positive differentiable solution of $A_V h = 0$.

Proof. It is easy to show that unless h is unique up to linear dependence one has $\int h^2 dx = \infty$, and that the above limit is 0. For let k be a positive solution distinct from h with $h(0) = k(0) = 1$. The Wronskian $W = hk' - kh'$ is constant, and we can assume that it is negative. Then from $\frac{d}{dx}(kh^{-1}) = Wh^{-2}$ we get $k(x) = h(x)\left(1 + W \int_0^x h^{-2} dy\right)$. Thus $\int_0^\infty h^{-2} dy \leq -W^{-1} < \infty$ and in view of $A_V h = 0$ we have $\lim_{x\to\infty} h(x) = \infty$. By analogy it follows also that $\lim_{x\to-\infty} k(x) = \infty$. Now the Green function $G(x, y) = \int_0^\infty \tilde{p}(t, x, y) dt$ is finite since, as can be seen from the absorbing barrier approximation, it can be defined from h and k in the usual way. Thus, since $\tilde{p}(t, x, x)$ is decreasing one has $\lim_{t\to\infty} \tilde{p}(t, x, x) = 0$. From the same approximation, and the eighfunction expansion, one sees that $\tilde{p}(t, x, y) \leq \tilde{p}(t, x, x) + \tilde{p}(t, y, y)$. The theorem is thus proved in this case, and we henceforth assume that h is uniquely determined by $h(0) = 1$.

The burden of the proof now rests in applying theorem 1.2 to study in some detail the evolution of $\tilde{p}$ with t. We continue to assume that the second argument of $\tilde{p}$ is 0, but write $\tilde{p}(t, 0, x)$ in place of $\tilde{p}(t, 0, y)$, and then $y=\tilde{p}(t, 0, x)$. The approach to equilibrium of $\tilde{p}$ will be measured by 4 parameters, $\theta_i(t)$ and $y_i(t)$, $i=1$ or 2. Note first that by property 1.1) iii. one has $|x_i(t)|<\infty$ at least for all t sufficiently small. We adopt the convention that $|x_i(t)|=\infty$ whenever possible, i.e. whenever $A_V\tilde{p}\leq 0$ on the whole corresponding half line.

Definition 1.3. Let $h_i(t, x)$ denote the solutions of $A_V h=0$ which are tangent to $\tilde{p}$ at $x_i(t)$, $i=1$ or 2, when $|x_i(t)|<\infty$ (this defines $h_i(t, x)$ uniquely even if the $x_i(t)$ are not unique, as we shall see that no vertical tangents can occur). Let c_i be determined by $c_i h_i(t, 0)=1$ and denote by $\theta_i(t)$ the positive angle from the negative y-axis to the tangent line at $x=0$ of the curve $c_i h_i(t, x)$. Finally, let $y_i(t)$ denote the y-intercept of $h_i(t, x)$, $i=1$ or 2 (Fig. 1).

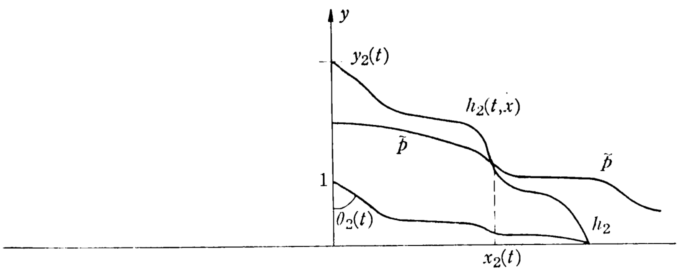

Figure 1

For example, if $V=0$ the $h_2(t, x)$ are straight lines with slopes $-\frac{(2\pi e)^{-1/2}}{t}$ tangent to $\tilde{p}$ at $x_2(t)=t^{1/2}$. In this case $y_2(t)=\left(\frac{2}{\pi e t}\right)^{1/2}$ and $\theta_2(t)=\arctan 2\sqrt{t}$.

Lemma 1.2. *Let* $T_i=\sup t: x_i(t')<\infty,\ 0<t'<t$. *Then* $\theta_i(t)$ *is non-decreasing and* $y_i(t)$ *is non-increasing in* $(0, T_i)$. *If* T_i *is finite, then* $|x_i(t)|=\infty$ *for* $T_i\leq t$[6].

Proof. The lemma, because of symmetry, need only be proved for $i=2$. Let $\theta_2(t, x)$ correspond to the tangent solution h at $(x, \tilde{p}(t, 0, x))$ as $\theta_2(t)$ corresponds at $x=x_2(t)$. It will be shown that

6) It seems doubtful that T_i can be finite (see footnote (4)).

$\theta_2(t, x)$ decreases in $[0, x_2(t)]$ and increases in $[x_2(t), \infty)$. Indeed, for $0 \leq x_0 < x_1 \leq x_2(t)$ the tangent h_0 at x_0 exceeds $\tilde{p}$ for $0 \leq x \leq x_1$ by V-concavity, and therefore for some $c \leq 1$, $ch_0(x_1) = \tilde{p}(t, 0, x_1)$. Then $ch_0 \leq \tilde{p}$ for $x_0 \leq x \leq x_1$ and thus is also less than the tangent h_1 at x_1. Since $ch_0(x_1) = h_1(x_1)$ we have $\theta_2(t, x_0) \geq \theta_2(t, x_1)$ as asserted, and the argument for $x_2(t) \leq x_0 < x_1$ is analogous.

Next, assuming $t < T_2$, we show that $\theta_2(t)$ is non-decreasing to the right, and afterwards that if $t_n \uparrow t$ with $\theta_2(t_n)$ non-decreasing then $\lim_{n\to\infty} \theta_2(t_n) \leq \theta_2(t)$. These two facts show as before that $\theta_2(t)$ is non-decreasing in $(0, T_2)$. Accordingly, consider the one parameter family of curves $ch_2(t, x)$, $0 < c$. Since $\theta_2(t, x)$ has its minimum at $x_2(t)$ one sees that each of these curves intersects $\tilde{p}(t, 0, x)$ at most once or in a single interval where $A_V\tilde{p} = 0$, and conversely, for every x in some neighborhood of $x_2(t)$ containing abscissas of both strict V-concavity and V-convexity points of $\tilde{p}$, $(x, \tilde{p}(t, 0, x))$ is such an intersection point. By proposition 1.2 and the remark following it, $\tilde{p}(t', 0, x)$ for $t' > t$ continues to have at most one intersection point (or interval) with each $ch_2(t, x)$. If, on the other hand, $\theta_2(t') < \theta_2(t)$ for some $0 < t' - t$ arbitrarily small, then it is clear that one could obtain a curve ch_2 such that $ch_2(t, x_2(t')) = \tilde{p}(t', 0, x_2(t'))$ and ch_2 would then have 3 disjoint closed intervals (or points) of intersection with $\tilde{p}(t', 0, x)$. Hence $\theta_2(t') \geq \theta_2(t)$ and $\theta_2(t)$ is non-decreasing to the right. To establish the second fact, one has only to remark that since each $\theta_2(t_n)$ is a minimum of $\theta_2(t_n, x)$, if $\lim_{n\to\infty} \theta_2(t_n) < \theta_2(t)$ we could obtain a contradiction by choosing a small secant at $x_2(t)$ and using the continuity of $\tilde{p}$.

The proof that $y_2(t)$ is non-increasing is somewhat more delicate because $\theta_2(t)$ might remain constant. It is to be noted that if $x_2(t)$ also is or can be chosen constant, then $y_2(t)$ must be constant, in the same time interval. This follows since $\frac{d}{dt}\tilde{p}(t, 0, x_2(t)) = A_V\tilde{p} + \tilde{p}\frac{dx_2(t)}{dt} = 0$, implying that $h_2(t, x)$ does not depend on t.

If, on the other hand, $x_2(t)$ varies, there is no difficulty provided that the motion is toward 0. Indeed, if $0 \leq x' < x_2(t)$ and

$t' = \inf \tau > t : x_2(\tau) \leq x'$, then clearly we can select $x_2(t')$ such that $x_2(t') \leq x'$. But for $t \leq \tau < t'$, $\tilde{p}(\tau, 0, x_2(t'))$ is decreasing in τ, and $\theta_2(\tau)$ is non-decreasing we see that $y_2(t') \leq y_2(t)$.

The situation is more interesting when the distance of $x_2(t)$ from 0 increases. We state the result as the auxiliary

Lemma 1.3. *If $x_2(t) < x''$ and $T_2 > t' = \inf \tau > t : x_2(\tau) \geq x''$, then letting $\underline{h}$ denote the 2-point solution of $A_V \underline{h} = 0$ which satisfies $\tilde{p}(t, 0, x'') = \underline{h}(x'')$, $\tilde{p}(t, 0, x) \leq \underline{h}(x)$ for $x \leq x''$, and $\tilde{p}(t, 0, x') = \underline{h}(x')$ for some $x' < x''$, we have $\theta_2(t') \geq \underline{\theta}$ and $y_2(t') \leq \underline{y}$ where $\underline{\theta}$ and $\underline{y}$ are the corresponding quantities for $\underline{h}$.*

Remark. Clearly, unless $\underline{h}$ and $\tilde{p}$ coincide in $(x_2(t), x'')$, we have $x' < x_2(t)$, $\underline{\theta} > \theta_2(t)$, and $\underline{y} < y_2(t)$, with equality otherwise.

Proof. The proof is complicated by the possibility that h and $\tilde{p}$ may coincide in $(x_2(t), x'')$; let us assume at first that $x_2(t)$ is unique for all t, and show afterwards how this hypothesis may be removed. The graphs of $\underline{h}$ and $\tilde{p}(t, 0, x)$ then intersect in such a way that $\underline{h} \leq \tilde{p}$ for $x'' \leq x$, $\underline{h}$ is negative for large x, and by lemma 1.2 $\underline{h}$ and $\tilde{p}(\tau, 0, x)$ for all $\tau > t$ have exactly one intersection point or interval (i.e. the tangent at x' disappears for $\tau > t$, and $\underline{h} > \tilde{p}$ persists for large $-x$, as can be seen from the uniqueness of h). Now the crucial point is that, since $\tilde{p}(\tau, 0, x'')$ is increasing for $\tau < t'$, this intersection point or interval remains to the left of x''. Accordingly, it must coincide with $x_2(\tau)$ at some $\tau \leq t'$. Let this value be denoted τ_0, then since $\tilde{p}(\tau_0, 0, x) \geq \underline{h}(x)$ for $x \geq x_2(\tau_0)$, we see that $\theta(\tau_0) \geq \underline{\theta}$ and $y(\tau_0) \leq \underline{y}$. Since $\theta(t)$ is non-decreasing, this proves our statement for it. As for $y(t)$, we proceed by induction. Let $\tau_0 = \tau_{0,1} < \tau_{0,2} < \cdots$ be chosen by successive repetition of the above argument; thus $\tau_{0,n+1}$ is based on the solution $\underline{h}_n$ such that $\underline{h}_n(x'') = \tilde{p}(\tau_{0,n}, 0, x'')$ and h_n is tangent to $\tilde{p}(\tau_{0n}, 0, x)$ from above at some smaller x'_n. If $\tau_{0,n} = t'$, at any stage, we are finished. Otherwise, by hypothesis, $x'_n < x_2(\tau_{0,n}) < x''$ and the induction continues. Let $\underline{y} = \underline{y}_1 > \underline{y}_2 > \cdots$ denote the corresponding $\underline{h}$ intercepts; we have $y_2(\tau_{0,n}) \leq \underline{y}_n$ for all n. To justify a passage to $\tau = t'$ we need only show that $y_2(\lim_n \tau_{0,n}) \leq \lim_n y_2(\tau_{0,n})$. But this is obvious from the continuity of $\tilde{p}$, the fact that $\theta_2(t)$

is non-decreasing, and the condition $t' < T_2$.

We consider now the case that $\tilde{p}$ and $\underline{h}$ coincide in $(x_2(t), x'')$; of course one could then choose $x_2(t) = x''$ and $t = t'$, but it may be of some interest to avoid such a restriction. To carry out the argument we must then let $\underline{h}(x'') = \tilde{p}(t, 0, x'') + \varepsilon$ for a small ε, with x' determined as before. Then $\underline{y} < y(t)$ still holds, as does the existence of a unique intersection of $\underline{h}$ and $\tilde{p}(\tau, 0, x)$ for $\tau > t$. To complete the argument one has only to choose ε so small that $\tilde{p}(\tau, 0, x'') \geq \tilde{p}(t, 0, x'') + \varepsilon$ for some $\tau \leq t'$. Indeed, if such an $\varepsilon > 0$ does not exist, then $\tilde{p}(t', 0, x'') \leq \tilde{p}(t, 0, x'')$ and our result is automatic.

Returning to the completion of lemma 1.2, let us dispose of the last sentence concerning $T_2 < \infty$. Indeed, if $x_2(t) = \infty$ then $A_V \tilde{p} \leq 0$ for all $x \geq 0$. This situation must then persist for all $t' \geq t$ since $T_{t_2} A_V \tilde{p}(t_1, 0, x) = A_V \tilde{p}(t_1 + t_2, 0, x)$ and we know that $A_V \tilde{p}(t, 0, 0) < 0$ for all t. Thus $x_2(t) = \infty$ for all $t \geq T_2$.

The remainder of lemma 1.2 is now also easily dispatched. Choosing $t_1 < t_2 < T_2$, we have seen that if $x_2(t)$ is constant then $\tilde{p}(t, 0, x_2(t))$ is constant and $y_2(t)$ is non-decreasing. In the opposite case, we choose $t_1 < \tau_1 < \tau_2 < \cdots < t_2$ such that each τ_{n+1} is of the form $\tau_{n+1} = \inf \tau > \tau_n : x_2(\tau) \leq x' < x_2(\tau_n)$ or else $\tau_{n+1} = \inf \tau > \tau_n : x_2(\tau) \geq x'' > x_2(\tau_n)$. Applying the preceding results, we have $y_2(\tau_1) \geq y_2(\tau_2) \geq \cdots$, and as noted before, $y_2(\lim_n \tau_n) \leq \lim_n y_2(\tau_n)$. This makes possible a passage to $t = t_2$ which then completes the proof. It can now be noted that, as was tacitly assumed, no $\theta_i(t)$ can be 0. Indeed, $x_2(t)$ is clearly small for small t (by 1.1) iii) and therefore must increase, whence lemma 1.3 shows that $\theta_2(t)$ increases strictly.

Before finishing the proof of theorem 1.3, we present definition which provides useful orientation.

Definition 1.4. The effective maximum of $\tilde{p}(t, 0, x)$, t fixed, is given by $C(t) = \inf c : \tilde{p}(t, 0, x) \leq ch(x)$ for all x. Let $[m_1(t), m_2(t)]$ be the point or interval of values of x at which this maximum is assumed. It is easily checked (using $\lim_{|x| \to \infty} \tilde{p}(t, 0, x) = 0$) that $C(t) < \infty$ and $m_2(t) < x_2(t)$ for $t < T_2$, with a similar remark for

$m_1(t)$.[7]

As for the theorem, since $\theta_i(t, x)$ is bounded away from 0 for all x and $t>\varepsilon>0$, $\tilde{p}$ is uniformly equicontinuous in finite intervals of x for such t. If $\lim_{t\to\infty} x_i(t)$ exist, then because $\tilde{p}$ is non-decreasing in $[x_1(t), x_2(t)]$ and non-increasing in the complement, while the effective maximum $C(t)$ is non-increasing (as in lemma 1.1), we can conclude that $\lim_{t\to\infty} \tilde{p}$ also exists. Let us suppose, on the other hand, that $\underline{x}_2=\liminf_{t\to\infty} x_2(t)<x_1<x_2<\limsup_{t\to\infty} x_2(t)=\bar{x}_2$. Now lemma 1.2 has the immediate consequence that the tangent solutions $h_2(t, x)$ converge uniformly in finite intervals as $t\to\infty$ to a continuous limit function $L_2(x)$. Thus for $\varepsilon>0$, one has $|\tilde{p}(t, 0, x_2(t))-L_2(x_2(t))|<\varepsilon$ for all t sufficiently large and such that $x_1\leq x_2(t)\leq x_2$. Moreover, for $x_1<x<x_2$, $\tilde{p}(t, 0, x)$ varies monotonically during the periods when $x_2(t)-x$ has a fixed sign, and therefore has its extrema at times when $x_2(t)-x$ changes sign. But at such times, either $\tilde{p}$ coincides with $h_2(t, x)$ in a neighborhood of x or else $x_2(t)-x$ becomes arbitrarily small within every period $(t-\varepsilon, t+\varepsilon)$. In either case, we have $|\tilde{p}(t, 0, x)-L_2(x)|<\varepsilon$ for such t, and hence for all t sufficiently large. Thus $\lim_{t\to\infty} \tilde{p}(t, 0, x)=L_2(x)$ in $(x_2, \bar{x}_2)$, and since it converges also in the open complement, as before, and is equicontinuous $\lim_{t\to\infty} \tilde{p}(t, 0, x)$ does exist and is continuous.

It remains only to identify this limit with $h(0)h(x)\left(\int h^2 dx\right)^{-1}$, but this is quite straightforward. Indeed, since $\tilde{p}(t, y, x)\leq C_x(\varepsilon)h(y)$ for all $t\geq\varepsilon$, we can apply the dominated convergence theorem to obtain

$$\begin{aligned}\lim_{t\to\infty} \tilde{p}(t, 0, x) &= \lim_{t\to\infty} \tilde{p}(t_1+t, 0, x) \\ &= \lim_{t\to\infty} T_{t_1}\tilde{p}(t, 0, x) = T_{t_1}\lim_{t\to\infty} \tilde{p}(t, 0, x)\,,\end{aligned}$$

and since the limit is non-negative it must be given by ch for some $c\geq 0$. But we also have $\left(\text{by Fatou's lemma if } \int h^2 dx=\infty\right)$

7) It is not asserted that $m_1(t)\leq 0\leq m_2(t)$.

$$h(0) = \lim_{t\to\infty} \int \tilde{p}(t, 0, x)h(x)dx$$
$$= c \int h^2 dx .$$

Since the choice of 0 as the starting point is inessential, the proof is complete.

As a final remark, it is quite easy to show using the methods of theorem 1.2 that for each c, $\tilde{p}(t, 0, x) - ch(x)$ can have at most 2 changes of sign for fixed $t>0$. One need only choose the densities f_n, which approximate the "ε function", such that $f_n - ch$ has this property, and proceed as before by applying T_t.

Section II. The General Case

The upper limit of generality to which our methods apply can be expected to be the equations of the form

$$(2.1) \qquad Ag = \left(\frac{d}{dm}\frac{d^+}{dx^+} + \frac{k(dx)}{m(dx)}\right) g(t, x) = \frac{\partial}{\partial t} g(t, x)$$

where m is a strictly increasing right-continuous function on $(-\infty, \infty)$ and $k(dx)$ is a signed measure of finite variation on finite intervals. If $k(dx)$ is non-positive such equations determine the most general non-singular diffusion in the "natural scale" and which "killing measure" $k(dx)$ [6, p. 107], but the general case cannot be "reduced" to this as in section I by multiplication by e^{-Mt}. The author is grateful to Professor J. L. Doob for calling his attention to the "h-path transformation" which reduces the problem of defining the solutions to the case $k(dx)\equiv 0$, circumventing the obvious (but less general) method based on forming monotone limits of V-sequences.

The operator A on the left of 2.1) is to be interpreted throughout by integration. Thus 2.1) signifies that g is continuous in x and

$$\frac{\partial^+}{\partial x^+} g(t, x) - \frac{\partial^+}{\partial x^+} g(t, 0) + \int_{0+}^{x} g(t, y)k(dy) = \int_{0+}^{x} \frac{\partial}{\partial t} g(t, y)m(dy) .$$

As in section I, we introduce the

Hypothesis 2.1. *There exists a strictly positive, continuous, right-differentiable solution h of*

2.2) $$\frac{d^+}{dx^+}h(x)-\frac{d^+}{dx^+}h(0) = -\int_{0+}^{x} h(y)k(dy).$$

That the role of this hypothesis is the same as in section I will be clear from the sequel, where the Sturm comparison theorem is extended to the operators A and the absorbing barrier processes are considered.

The h-path transformation, as it is involved here, is simply the analytical device of replacing A by operator $\Omega f=h^{-1}A(hf)$ for hf in the domain of A. In the situation of section I for example a direct differentiation shows that $\Omega f=\frac{1}{2}f''+\frac{h'}{h}f'$, which is the generator of one or more diffusion processes with a "drift" determined by $\frac{h'}{h}$. A transtion function p determining a semigroup with generator Ω is obtained from a given $\tilde{p}$ with generator A by the formula

$$p(t, x, y) = h^{-1}(x)\tilde{p}(t, x, y)h(y), \qquad \text{and}$$

conversely this determines a $\tilde{p}$ from a given p. However, because the semigroup T_t corresponding to $\tilde{p}$ so obtained is not necessarily strongly continuous on C_0, nor does it necessarily hold that $T_t h=h$, the methods of section I do not always apply in the general case, even when the generator Ω can be expressed by differentiation. It will be seen that the generator Ω can always be written in a form for which the solutions are known, but that at most one of the semigroups T_t determined in this manner is an instance of those envisaged in section I, and that a further restriction on $k(dx)$ (or equivaletly on h) is necessary in order that there exist even one.

Taking up the general case, we first prove a lemma, following [Feller, 2, p. 109].

Lemma 2.1. *The Sturm comparison theorem is valid for operators of the form A. For fixed k(dx) let h_1 and h_2 be any continuous* *solutions of* 2.2). *Then the Wronskian* $W=h_1\frac{d^+}{dx^+}h_2-h_2\frac{d^+}{dx^+}h_1$ *is constant.*

Proof. Let $k_1(dx)\le k_2(dx)$, and let h_i correspond to k_i, $i=1$ or 2. Then because the differentials satisfy $0=d(h_i^+(x)-h_i^+(0)+\int_{0+}^x h_i(y)k_i(dy))$, where f^+ denotes $\frac{d^+}{dx^+}f$, we obtain

$$0=\int_0^x [h_2(y)d(h_1^+(y)+h_1(y)k_1(dy)) \\ -h_1(y)d(h_2^+(y)+h_2(y)k_2(dy))]\,, \qquad \text{and}$$

therefore

$$\begin{aligned}0 &\le \int_0^x h_2(y)dh_1^+(y)-h_1(y)dh_2^+(y)\\ &= h_2h_1^+-h_1h_2^+\Big]_0^x-\int_0^x h_1^+h_2^+-h_2^+h_1^+\,dy\\ &= h_2h_1^+-h_1h_2^+\Big]_0^x,\end{aligned}$$

with equality only if $k_1=k_2$. In particular this establishes that W is constant. Now if x_1 and x_2 are successive roots of h_1 and if $h_2\neq 0$ in $[x_1,x_2]$ then assuming as we may that both are positive we obtain for $x_1<x\le x_2$ the inequalities $0\le h_2h_1^+-h_1h_2^+]_{x_1}^x\le h_2h_1^+(x)-h_1h_2^+(x)$, and the right side is non-decreasing in x. But since $\liminf_{x\to s_2^-}h_1^+(x)\le 0$ it follows that the right side is zero in (x_1,x_2). This is impossible unless $k_1=k_2$ and $h_1=ch_2$ for a constant c. Thus h_2 must vanish in $[x_1,x_2]$ and the proof is complete.

If follows from this lemma that if h_0 is any solution of 2.2) linearly independent of h then $\frac{d}{dx}h^{-1}h_0=h^{-2}W_0$ has no zeros, and hence $h^{-1}h_0$ is monotone. Let h_0 be fixed with $h_0(0)=0$, $h_0(x)>0$ for $x>0$, and define $u(x)=h^{-1}h_0(x)$ and $d\mu(u)=W_0^{-1}h^2(x)dm(x(u))$. The function $u(x)$ has the role of a scale change, and we set $(u_1,u_2)=(\lim_{x\to-\infty}u(x),\ \lim_{x\to\infty}u(x))$. Then we have the

Theorem 2.1. *For f such that $A\,(hf)$ is continuous, the operator $\Omega f=h^{-1}A(hf)$ satisfies*

$$\Omega f(x)=\frac{d}{d\mu}\frac{d^+}{du^+}f(x(u))\,.$$

Proof. It is sufficient to show that

$$\int_0^x A(hf)dm(x) = \int_0^{u(x)} h(x(u)) \frac{d}{d\mu} \frac{d^+}{du^+} (f(x(u)))dm(x(u)).$$

Setting $f = h^{-1}g$, whence $\frac{d^+}{dx^+}g$ exists, the right side becomes

$$\begin{aligned}
&\int_0^{u(x)} h^{-1}(x(u)) \frac{d}{d\mu}\left(h^2 \frac{d^+}{dx^+}(h^{-1}g)\right) d\mu(u) \\
&= \int_0^{u(x)} h^{-1}(x(u)) \frac{d}{d\mu}\left(h \frac{d^+}{dx^+} g - g \frac{d^+}{dx^+} h\right) d\mu(u) \\
&= \int_0^x h^{-1} d\left(h \frac{d^+}{dx^+} g - g \frac{d^+}{dx^+} h\right) \\
&= h^{-1}(hg^+ - gh^+)]_0^x + \int_0^x \frac{d^+}{dx^+} h \frac{d^+}{dx^+}(h^{-1}g)dx \\
&= g^+]_0^x - \int_0^x h^{-1} g d\left(\frac{d^+}{dx^+} h\right) \\
&= g^+]_0^x - \int_0^x g(x)k(dx),
\end{aligned}$$

where we have used the notation $f^+ = \frac{d^+}{dx^+} f$ when convenient. This completes the proof. The correspondence of semigroups induced by the generators A and Ω follows easily from the definitions.

Theorem 2.2. *For each semigroup T_t^h with generator of the form Ω which is strongly continuous on a space C of bounded continuous functions on (u_1, u_2) with the domain $\mathcal{D}$ of Ω dense in C, the relation $T_t f(x) = h(x) T_t^h(h^{-1}f(u))$, $u = u(x)$, defines a strongly continuous semigroup with generator A on the space C^h of functions f such that $h^{-1}f(u) \in C$. The supremum norm on C corresponds to the norm $\|f\|^h = \inf\{c : |f(x)| \leq ch(x)$ for all $x\}$ and the domain $\mathcal{D}^h$ is $\{f(x) : h^{-1}f(u) \in \mathcal{D}\}$. Conversely, to each semigroup T_t with generator A on a spcace of continuous functions bounded in the h-norm there corresponds by the same definition a semigroup of type T_t^h.*

Proof. Let T_t^h be a semigroup of the specified type. Then clearly we have $\|T_t f\|^h = \|h^{-1}T_t f\| = \|T_t^h(h^{-1}f)\|$ and thus T_t is strongly continuous on C^h. Moreover, the identity

$$\left\|\frac{T_\Delta f-f}{\Delta}-g\right\|^h=\left\|\frac{h^{-1}T_\Delta f-h^{-1}f}{\Delta}-h^{-1}g\right\|$$
$$=\left\|\frac{T^h_\Delta(h^{-1}f)-h^{-1}f}{\Delta}-h^{-1}g\right\|$$

shows that the domains of the generators corresponds as asserted. Finally, for $h^{-1}f\in\mathcal{D}$ we have $\Omega(h^{-1}f)=h^{-1}Af=h^{-1}g$ if the above norms have limit 0 as $\Delta\to 0$. Thus A is the generator of T. Since the steps are clearly reversible, the converse follows immediately.

Corollary. *T_t is positive and $T_t h=h$ if and only if T^h_t is positive and Markovian.*

In accordance with these theorems, we consider the Markov semigroups on (u_1, u_2) with generators of the form $\frac{d}{d\mu}\frac{d^+}{du^+}$. These have been widely studied, and their classification depends upon the boundary type of u_1 and u_2. The corresponding criteria can be easily expressed in terms of the original data.

Theorem 2.3. *Setting $J\pm=\int_0^{\pm\infty}h^{-2}\int_{0\pm}^x h^{-2}dm(y)dx$ and $K^\pm=\int_{0\pm}^{\pm\infty}h^2\int_0^x h^{-2}dydm(x)$, where the signs are chosen equal, the upper boundary is exit if $J+<\infty$ and entrance if $K+<\infty$, with the analogous conclusion for the lower boundary in terms of $J-$ and $K-$.*

Remark: If both $J+=\infty$ and $K+=\infty$ the boundary is sometimes called *natural*, while if both are finite it is called *regular*. This terminology seems to obscure the fact that each "boundary" is naturally dual, depending on on whether the pre- or post-arrival behavior is considered.

Proof. The classification in terms of the variable u is determined by ([5, p. 522]) $J_i=\int_{0\pm}^{u_i}\int_0^u d\mu(v)du$, $K_i=\int_{0\pm}^{u_i}ud\mu(u)$, $i=1$ or 2. Thus, for example, we have

$$J_2=\int_0^\infty W_0h^{-2}\int_{0+}^{u(x)}d\mu(v)dx$$
$$=\int_0^\infty h^{-2}\int_{0+}^x h^2(y)dm(y)dx \qquad \text{and}$$

$$K_2 = \int_{0+}^{\infty} W_0^{-1} h h_0 dm(x)$$
$$= \int_{0+}^{\infty} h^2 \int_0^x h^{-2} dy dm(x) \qquad \text{as required.}$$

This completes the proof.

It is clear by determining first $m(dx)$ and then $k(dx)$ to obtain a given function $h(x)$ that all types of boundaries can occur. However, if (for example) $m(dx) = \frac{1}{2}dx$ as in section 1, then there can be no regular boundaries, as either $\int_0^{\infty} h^2 dy$ or $\int_0^{\infty} \frac{1}{h^2} dy$ must be infinite. The other there types are still possible.

On the other hand, the method of section 1 requires that $T_t h = h$ for all solutions of $Ah = h$. This means that only Markov semigroups T_t^h can be considered. Moreover, our construction of the sequence $\{f_n\}$ requires that there be positive functions vanishing at $x = \pm\infty$ in the domain $\mathcal{D}^h$, and this holds only if all "instantaneous return" processes are excluded. Combining these facts, we see that the method applies only if the minimal process is Markovian, or possibly if there are regular boundaries and "reflecting barrier" boundary conditions (any transition between the boundaries is evidently excluded by $T_t h_0 = h_0$). But finally we have made use of the "absorbing barrier" approximation of T_t wherein T_t is approximated as $n \to \infty$ by the semigroups with absorbing barriers at $\pm n$, and this possibility coincides exactly with the condition that T_t is the minimal process. Thus, howsoever these restrictions might be lifted by using other methods (and it seems possible that they could be for the reflecting barriers) the most natural case is perhaps the present one, and will be assumed.

Hypothesis 2.2. *For any solution $h_0 \not\equiv 0$ of 2.2) with $h_0(0) = 0$, the integrals $\int_{0+}^{\infty} \frac{1}{h_0^2} \left(\int_{0+}^{x} h_0^2 dm(y) \right) dx$ and $\int_{0-}^{-\infty} \frac{1}{h_0^2} \left(\int_0^x h_0^2 dm(y) \right) dx$ are infinite.*

Concerning this hypothesis, we note first that since h_0 vanishes only at 0 the integrals have a meaning. Secondly, since the class of possible solutions can be written $\{ch_0 : c \neq 0\}$ the condition holds for all solutions if it holds for any one of them. Thirdly, since for large c one has $ch_0(1)>h(1)$ and $-ch_0(-1)>h(-1)$, and therefore, by considering the differences $h-ch_0$, the same inequalities hold for $|x|>1$, we can show that for any solution $h>0$ the h-path semigroup has no exit (or regular) boundaries, i.e. the minimal h-path process is Markovian. Indeed, by considering the intersection point of ch_0 and h one sees that the Wronskian $h_0'h-h_0h'=W_0$ is positive and from the equation $h_0(x)=h(x)\times\int_0^x \frac{W_0}{h^2(y)}dy$ it follows that

$$\int_0^\infty \frac{1}{h^2}\left(\int_{0+}^x h^2dm(y)\right)dx \geq \int_0^\infty \frac{1}{h_0^2}\int_{0+}^x h_0^2dm(y)dx = \infty\,,$$

with an analogous result in the form

$$\int_0^{-\infty} \frac{1}{h^2}\left(\int_{0-}^x h^2dm(y)\right)dx \geq \int_0^{-\infty} \frac{1}{h_0^2}\int_{0-}^x h_0^2dm(y)dx = \infty\,.$$

It will next be shown that if there are two positive solutions h and h_1, then the corresponding semigroups T_t obtained from theorem 2.2 are essentially the same. However, since their spaces are different, the identity is phrased in terms of the kernels of the transformations. It is known [5] that for a given h the semigroup T_t^h in the scale u is given by $T_t^hf(u)=\int_{u_1}^{u_2} p(t,u,v)f(v)\times d\mu(v)$ for a unique kernel $p(t,u,v)$ continuous in each variable for $t>0$. In terms of x this becomes $T_tf(x)=\frac{h(x)}{W_0}\int_{-\infty}^\infty p(t,u(x),u(y))h\cdot f(y)dm(y)$. Letting the subscript 1 denote the analogous quantitites defined using h_1 instead of h we will show that

$$\text{2.3)}\qquad \frac{h(x)}{W_0}p(t,u(x),u(y))h(y) = \frac{h_1(x)}{W_1}p_1(t,u_1(x),u_1(y))h_1(y)\,.$$

To prove this it suffices to show that the h_1-path transform of the left side defines a *Markovian* transition function with generator $\frac{d}{d\mu_1}\frac{d^+}{du_1^+}$. Now in terms of the variable u, this h_1-path transformed kernel is $h_1^{-1}h(u)p(t,u,v)h^{-1}h_1(v)$ with respect to the measure

$d\mu$, and the corresponding generator is seen to be $h_1^{-1}h\Omega(h^{-1}h_1 f)$ $=h_1^{-1}A(h_1 f)$ as required. The Markovian property follows if we show that $T_t^h h^{-1}h_1 = h^{-1}h_1$. Now since

$$2.4)\ \Omega(h^{-1}h_1) = \frac{d}{d\mu}\frac{d^+}{du^+}h^{-1}h_1(x(u)) = \frac{d}{d\mu}\left(\frac{W_1}{h^2(x(u))}\frac{dx(u)}{du}\right) = \frac{d}{d\mu}\frac{W_1}{W_0} = 0$$

the "harmonicity" of $h^{-1}h_1$ is demonstrated. Moreover, the minimality of the semigroup T_t^h implies the validity of the "absorbing barrier approximation", i.e. if we define the absorbing barrier kernels in (u_1^0, u_2^0) with generator Ω then $p(t, u, v)$ is the monotone limit of these kernels as $u_1^0 \downarrow u_1$ and $u_2^0 \uparrow u_2$. It thus follows by Fatou's lemma that $T_t^h h^{-1}h_1 \leq h^{-1}h_1$. Therefore, the h_1-path transform is sub-Markovian. However, since the minimal process with generator $h_1^{-1}A(h_1 f)$ is Markovian, this process must be in fact Markovian, and the proof is therefore complete.

We turn. next to the task of showing that $T_t \underline{h} = \underline{h}$ for all (not necessarily positive) solutions of 2.2), where $T_t \underline{h}$ is interpreted by integration of the kernel of T_t. Since this is known for h, it suffices to prove it for h_0. We first consider the processes on $(0, \infty)$ and on $(-\infty, 0)$ separately with 0 as "absorbing barrier". Indeed, let T_t^+ correspond in the manner of theorem 2.2 to the minimal h-path semigroup $(0, u_2')$ with generator $\frac{d}{d\mu}\frac{d^+}{du^+}$ and zero at 0, and let T_t^- correspond to the analogous semigroup on $(u_1, 0)$. We shall show that h_0 is invariant under both T_t^+ and T_t^- in the respective intervals $(0, \infty)$ and $(-\infty, 0)$. To this end, consider the h_0-path transformations of $T_t^{\pm}$. Expressed in the scale u, they becomes h_0/h-path transformations of the h-path transformations of $T_t^{\pm}$. By theorem 2.3, replacing the lower limit 0 of the integration by ± 1, and the upper limits $\pm\infty$ by 0, it follows from the boundedness of h_0' near 0 that 0 is not an exit (nor regular) for these h_0-path processes, and since neither is $\pm\infty$ they must be Markovian. This implies, as before, the invariance of h_0/h for the h-path kernels, and equivalently that of h_0 for $T_t^{\pm}$. To show, finally, the invariance of h_0 for T_t it suffices to demonstrate the invariance of h_0/h for the h-path transform of T_t, which is a minimal Markovian semigroup on the space $C_0(u_1, u_2)$ or continuous functions with limit 0 at u_1 and at u_2. Now if $X(t)$ denotes the corresponding diffusion process [see 6] what we have just shown is that if $T_0=\inf\{t\geq 0 : X(t)=0\}\leq\infty$, then

$$2.5)\qquad E_x(h^{-1}h_0(X(t))\,;\; t\leq T_0) = h^{-1}h_0(x)\,.$$

By the strong Markov property, it thus suffices to show that $E_0(h^{-1}h_0(X(t)))=0$ for all $t\geq 0$. We introduce by induction the sequence of stopping times

$$T_{2n+1} = \inf\{t>T_{2n} : X(t) = \pm 1\}\leq\infty \quad \text{and}$$
$$T_{2(n+1)} = \inf\{t>T_{2n+1} : X(t) = 0\}\leq\infty\,,$$

and set $S_n=\{T_n\leq t<T_{n+1}\}$ and $K=\sup_{|x|\leq 1}|h^{-1}h_0(x)|$. Then we have $E_0(|h^{-1}h_0(X(t))|; S_{2n})<KP(S_{2n})$, and since by 2.5) and the strong Markov property $E_0(|h^{-1}h_0(X(t))|\,;\, S_{2n+1})<KP(S_{n+1})$, hence also $E_0(|h^{-1}h_0(X(t))|)<K$. Next, because $\Omega(h^{-1}h_0)=0$, it follows that if $X(0)=0$ then $h^{-1}h_0(X(t\wedge T_1))$, $0\leq t$, is a bounded martingale. Thus by using 2.5) again and the strong Markov property we get

$$\begin{aligned} 0 &= E_0(h^{-1}h_0(X(t\wedge T_1))) \\ &= E_0(h^{-1}h_0(X(t))\,;\, t<T_1)+P_0\{T_1\leq t,\, X(T_1)=1\}h^{-1}h_0(1) \\ &\quad + P_0\{T_1\leq t,\, X(T_1) = -1\}h^{-1}h_0(-1) \\ &= E_0\{h^{-1}h_0(X(t))\,;\, t<T_2\}\,. \end{aligned}$$

Since $X(T_2)=0$, repetition of this reasoning shows that T_2 may be replaced by T_{2n}, and our result follows by letting n tend to infinity.

We next consider the regularity properties 1.1) i-iii for the kernels 2.3). Properties i and iii are immediate from the form 2.3) of the kernels. It remains to check ii. Since applying A to y we have

$$\begin{aligned} Ap(t, u(x), u(y))h(y) &= h(y)\Omega p(t, u(x), u(y)) \\ &= \frac{\partial}{\partial t}p(t, u(x), u(y))h(y)\,, \end{aligned}$$

the result follows likewise from 2.3.

We are now in a position to extend all of the results of section I to the kernels $\tilde{p}=\frac{h(x)}{W_0}p(t, u(x), u(y))h(y)$. There are only three points requiring further mention, namely, the solution considered in lemma 1.1, the choice of the sequence $\{f_n\}$ in the proof of theorem 1.2, and the non-uniqueness of of the "tangent solutions" at a point where the kernel $\tilde{p}$ is non-differentiable. It is not hard to see that the first two points can be considered for the operators Ω and semigroups T_t^h in place of those given, since the transformation of theorem 2.2 preserves positivity pointwise, and the generators are identical at corresponding functions. Now the existence of $\{f_n\}$ is evident for Ω, and the required property of the solutions in lemma 1.1 is also practically obvious. A rigorous proof can be carried out, at least for functions f in the domain $\mathcal{D}$ or Ω, by first showing the uniqueness of the solution. Thus if there is a solution $f\in\mathcal{D}$ with boundary values 0 at $t=0$ and at $u=a$ and $u=b$, $a<b$, $0<t<t_1$, then $0\leq\int_a^b f^2(t, u)d\mu(u)$ and

$$\frac{d}{dt}\int_a^b f^2 d\mu=\int_a^b 2f(\Omega f)d\mu=2ff^+]_a^b-2\int_a^b (f^+)^2du=-2\int_a^b (f^+)^2du\leq 0.$$

Since at $t=0$ the first integral vanishes, it vanishes for $t<t_1$, along with f itself. Now *a* solution, and hence *the* can be defined by a method of Doob [7] using the "downward directed process" $Z_{u,t}(\tau)=(t-\tau, u+X(\tau))$. One introduces for $a<u<b$ and $0<t<t_1$ the stopping time $T=\inf\tau>0$; either $u+X(\tau)\in\{a, b\}$ of $t-\tau=0$. Then $f(t, u)$ is given by $E_0(B(Z_{u,t}(T)))$ where B is the boundary function. Clearly $f(t, u)>0$ holds for B of the assumed positive character. To extend this reasoning to the interval $(-\infty, a]$ it is only necessary to construct in this manner the solutions f in $(-N, a)$. Since these are equal on their common domain as $-N\to-\infty$, the limit defines the actual solution. Moreover, since as $N-\to-\infty$ the contribution of the boundary at $u=-N$ approaches 0, $-\infty$ being an inaccessible boundary for $X(t)$, the limit solution is given by replacing the boundary at $u=-N$ by the whole $t=0$ axis, and thus is strictly positive as long as this is true at $u=a$. It is easily checked that in the applications made one had $f \in \mathcal{D}$ hence this suffices for the sequel.

Finally, the non-uniqueness of the "tangent solutions" is not serious because there do exist "tangent solutions from the right", and also, by "k-convexity", "tangent solutions from the left". At the point $x_2(t)$ we choose whichever of these has a smaller angle $\theta_2(t)$. At all the other points $x>0$ we adopt the convention that the "tangent solutions" are from the right. For $x<0$ it is not obvious that the angle $\theta_1(t, x)$ can be defined, but in any case we are free to replace x by $-x$ in the original definition of the semi-group and then re-transpose the resulting quantities. The rest of the discussion now applies without change, by appealing when necessary to the form 2.3) of $\tilde{p}$ and the known properties of p. In particular, the discussion of the case when h is non-unique can be easily transferred since in this case, by 2.3), the Green function of p finite. We conclude with the

Theorem 2.4.[8] *The result and proof of theorem* 1.3 *remain valid for the kernels* $\tilde{p}$ *under the hypothesis* 2.1. *and* 2.2.

University of Illinois

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8) The argument commencing the proof perhaps requires a further comment. From hypothesis 2.2 we have $\int_0^\infty \frac{1}{h^2} \int_0^x h^2 dm(y)\, dx = \infty$. Together with $\int_0^\infty \frac{1}{h^2} dx < \infty$ this implies that $\int_0^\infty h^2 dm(y) = \infty$ as required.