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Some remarks on a covering of an abelian variety

By

Masayoshi MIYANISHI

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In [4], the following result was proved:

Let G be a commutative finite group scheme over an algebraically closed field k of arbitrary characteristic and let X be a connected, reduced k-prescheme over which G operates faithfully and gives a geometric quotient A,

$$G \times_k X \underset{pr_2}{\overset{\sigma}{\rightrightarrows}} X \xrightarrow{p_X} A .$$

Then X is an abelian variety and p_X is an isogeny.

The purpose of this paper is to extend this result.

We employ the same notation as in articles of A. Grothendieck, unless explicitly stated otherwise.

Part I.

§ 1. Let k be an algebraically closed field of positive characteristic p. Let G, X and A be a finite group-scheme, an algebraic scheme and an abelian variety respectively, defined over k such that G operates faithfully on X and that A is the geometric quotient of X by G,

$$G \times_k X \underset{pr_2}{\overset{\sigma}{\rightrightarrows}} X \xrightarrow{p_X} A .$$

Then we call X *a Galois covering of A with group G and a field of definition k*, (cf. [4]).

First, we shall give a result on the structure of a finite group-scheme.

Lemma 1. *Let G be a finite group-scheme defined over k. Then there exists a uniquely determined maximal infinitesimal subgroup G_{inf} of G which is invariant in G and defined as the connected component G^0 of G which contains the neutral point e of G. Moreover the quotient of G by G^0 is an étale group over k, isomorphic to G_{red}. Therefore G is the semi-direct product of G_{inf} by G_{red}.*

As for the proof, we refer to SGAD, Exp. VI_A, 2.3 and 5.5.

Let G, X, A be as above, and suppose X is connected and reduced. By Lemma 1, there exists a homomorphism $\alpha : G \to G_{\mathrm{red}}$ with which we can construct an étale covering Y of A with group G_{red},

$$\begin{array}{ccccc} G \times_k X & \rightrightarrows & X & \xrightarrow{p_X} & A \\ {\scriptstyle \alpha \times f}\downarrow & & \downarrow{\scriptstyle f} & & \| \\ G_{\mathrm{red}} \times_k Y & \rightrightarrows & Y & \xrightarrow[p_Y]{} & A \end{array} .$$

Theorem of Lang-Serre [9] shows that Y is an abelian variety, G_{red} is commutative and that p_Y is a separable isogeny. Therefore, if one wants to know whether X is an abelian variety, one can suppose G is infinitesimal, i.e. $G = G_{\mathrm{inf}}$.

Definition 1. *A finite group-scheme G over k is said solvable if G has a central composition series, i.e. a composition series whose successive quotients are commutative.*

The next result follows immediately from the results of [4].

Theorem 1. *Let X be a Galois covering of an abelian variety with group G which is a finite group-scheme whose maximal infinitesimal subgroup is solvable, and with k as the field of definition. Suppose X is connected and reduced. Then X is an abelian variety.*

§ 2. 1. First, we shall recall some definitions and results concerning the derivations and the differential forms which were given in SGAD, Exp II, VII and SGA, etc.. For simplification of notation, we restrict the ground prescheme to the spectrum of the field k.

Definition 2. *Let X be a prescheme over k. We observe the tangent fibre space* $\boldsymbol{T}$ *of X over k as a functor* $T(X/k) = \underline{\mathrm{Hom}}_k(I_k(k), X)$ *from the category* $(\mathbf{Sch}/k)^0$ *to the category* $(\mathbf{Sets})$, *i.e.* $S \in (\mathbf{Sch}/k)^0 \rightsquigarrow \mathrm{Hom}_S(I_S(\mathcal{O}_S), X_S)$, *where $I_S(\mathcal{O}_S)$ is the spectrum of $\mathcal{O}_S$-Algebra $\mathcal{O}_S \oplus \mathcal{O}_S$ in which the multiplication is defined to make $\mathcal{O}_S$ of the right hand side an ideal whose square is zero.*

Then $T(X/k)$ is represented by $\mathbf{Spec}(S(\Omega^1(X/k)))$, in other words $T(X/k) = V(\Omega^1(X/k))$ where $S(\Omega^1(X/k))$ is the symmetric Algebra over X of $\mathcal{O}_X$-Module $\Omega^1(X/k)$ ($\Omega^1(X/k)$= the sheaf of 1-differential forms) and $V(\Omega^1(X/k))(S) = \mathrm{Hom}_{\mathcal{O}_S}(\Omega^1(X/k) \otimes_k \mathcal{O}_S, \mathcal{O}_S)$, for $S \to k$. $T(X/k)$ is considered as a prescheme, affine over X and is endowed with a structure of a commutative X-group (X-group = a group object in $(\mathbf{Sch}/X)$). Also, we have $T(X/k)(X) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1(X/k), \mathcal{O}_X) = \Gamma(X, \mathbf{Hom}_{\mathcal{O}_X}(\Omega^1(X/k), \mathcal{O}_X)) = \Gamma(X, \mathbf{Der}(X/k)) = \mathrm{Der}(X/k)$ where $\mathbf{Der}(X/k)$ is the sheaf of local derivations on X and where $\mathrm{Der}(X/k)$ is the $\Gamma(X, \mathcal{O}_X)$-module of the global derivations on X.

Definition 3. *Let G be a group-scheme over k with the unit* $e: Spec(k) \to G$. *We denote* $e^*(T(G/k))$ *by* $Lie(G/k)$.

Then $\mathrm{Lie}(G/k)$ is representable and $\mathrm{Lie}(G/k)(k)$ is endowed with a structure of k-vector space.

For a group-scheme G over k, we have an exact sequence of k-group-schemes:

$$0 \longrightarrow \mathrm{Lie}(G/k) \xrightarrow{\ i\ } T(G/k) \underset{s}{\overset{q}{\rightleftarrows}} G \longrightarrow 0$$

where s is the unit of G-group $T(G/k)$ and q is the canonical projection of $T(G/k)$ to G. Therefore, $T(G/k)$ is a semi-direct product of $\mathrm{Lie}(G/k)$ by G. And we have $\Gamma(G, T(G/k)) \xrightarrow{\sim} \mathrm{Hom}_k(G, \mathrm{Lie}(G/k))$. The correspondence is given as follows: To a k-morphism $f: G \to \mathrm{Lie}(G/k)$, one associates $s_f: G \to T(G/k)$ such that $s_f(g) = s(g)i(f(g))$ for $g \in G(S)$, $S \to k$. For an element x of $G(k)$, let l_x be the left translation of G by x, i.e. for $g \in G(S)$, $S \to k$, $l_x(g) = x \cdot g$. For l_x, the left translation l_x of x on $\Gamma(G, T(G/k))$ is defined by the translation of structure,

$$\begin{array}{ccc} G & \xrightarrow{t} & T(G/k) \\ {\scriptstyle l_x}\downarrow & & \downarrow{\scriptstyle T(l_x)} \\ G & \xrightarrow{l_x(t)} & T(G/k) \end{array}\quad , \qquad t\in\Gamma(G,\ T(G/k)).$$

Then we see easily that $l_x(s_f)=s_{l_x(f)}$ where $f\in\mathrm{Hom}_k(G,\ \mathrm{Lie}(G/k))$ and $l_x(f)(g)=f(x^{-1}g)$, $g\in G(S)$, $S\to k$. Thus, we can show that the map $\mathrm{Lie}(G/k)(k)\to\Gamma(G,\ T(G/k))$ which associates the section $x \rightsquigarrow x\cdot X$ to $X\in\mathrm{Lie}(G/k)(k)$ is an isomorphism from $\mathrm{Lie}(G/k)(k)$ to the subset of $\Gamma(G,\ T(G/k))$ formed by left-invariant sections.

We can endow $\mathrm{Lie}(G/k)(k)$ (resp. $\Gamma(G,\ T(G/k))=\mathrm{Der}(G/k)$) with the structure of Lie-algebra whose bracket operation is defined by the adjoint representation of G on $\mathrm{Lie}(G/k)$ (resp. the structure of Lie-algebra whose bracket operation is the one of derivations). Since the characteristic p is positive, p-power operations can be naturally defined on $\mathrm{Lie}(G/k)(k)$ and $\mathrm{Der}(G/k)$, (cf. SGAD, Exp. VII). Then the above isomorphism is a morphism of p-Lie algebras.

2. Let X, Y be k-preschemes and let $f: X\to Y$ be a k-morphism of preschemes. Then, by SGA, Exp. II, § 4, the sequence of $\mathcal{O}_X$-Modules

$$(*)\qquad f^*(\Omega^1(Y/k)) \longrightarrow \Omega^1(X/k) \longrightarrow \Omega^1(X/Y) \longrightarrow 0$$

is exact. If X is smooth over k, $\Omega^1(X/k)$ is locally free and its rank at a point x of X is equal to the dimension of X at x. Dualizing the sequence $(*)$ over $\mathcal{O}_X$, we have an exact sequence of $\mathcal{O}_X$-Modules,

$$(**)\ 0\to \mathbf{Hom}_{\mathcal{O}_X}(\Omega^1(X/Y),\mathcal{O}_X)\to \mathbf{Der}(X/k)\to \mathbf{Hom}_{\mathcal{O}_X}(f^*\Omega^1(Y/k),\mathcal{O}_X).$$

We here add a remark which we shall use later:
For $f: X\to Y$ as above, we have an isomorphism $\Omega^1((X\times_Y X, p_2)/X)\xrightarrow{\sim} p_1^*(\Omega^1(X/Y))$ where p_1, p_2 are the projections of $X\times_Y X$ to X.

3. We shall now return to our case. Let X be a Galois covering of an abelian variety A with a group G, infinitesimal and of height $\leqq 1$ and with k as its field of definition. Suppose X is connected and reduced. Then the global derivations on X form a p-Lie algebra $\mathrm{Der}(X/k)$ over k ($\Gamma(X,\mathcal{O}_X)\cong k$), because X is a complete variety. Also, since A is an abelian variety, $\Gamma(A, T(A/k)) \cong \mathrm{Hom}_k(A, \mathrm{Lie}(A/k)) \cong \mathrm{Lie}(A/k)(k)$, i.e. $\mathrm{Der}(A/k)$ is formed by left-invariant derivations of A over k.

Next we shall calculate $\Omega^1(X/A)$. As in n° 2, $p_1^*(\Omega^1(X/A)) \cong \Omega^1(X/A) \times_A X \cong \Omega^1(X \times_A X/X) \underset{\varphi}{\cong} \Omega^1(G \times_k X/X) \cong \Omega^1(G/k) \times_k X$, where an isomorphism φ is defined to be compatible with $\Phi^{-1}: X \times_A X \to G \times_k X$, $\Phi = (\sigma, pr_2)$. Taking the fibres of both terms by morphisms, the diagonal $\Delta_{X/A}: X \to X \times_A X$ and $(e \times_k X: X \cong \mathrm{Spec}(k) \times_k X \to G \times_k X$, we have an isomorphism of $\mathcal{O}_X$-Modules $\Omega^1(X/A) \cong \omega^1(G/k) \times_k X$, where $\omega^1(G/k) = e^*(\Omega^1(G/k))$.

The exact sequence $(*)$ of n° 2 gives an exact sequence,

$$p_X^*(\Omega^1(A/k)) \longrightarrow \Omega^1(X/k) \longrightarrow \omega^1(G/k) \times_k X \longrightarrow 0.$$

The exact sequence $(**)$ of n° 2 gives us an exact sequence,

$$(***) \quad 0 \longrightarrow \mathrm{Lie}(G/k)(k) \times_k X \longrightarrow \mathbf{Der}(X/k) \longrightarrow p_X^*(\mathbf{Der}(A/k)),$$

where $\mathrm{Lie}(G/k)(k) = \mathrm{Hom}_k(\omega^1(G/k), k)$ and where the last term comes from the fact that X is faithfully flat over A. Taking the global sections of the terms of the sequence $(***)$, we have:

Lemma 2. *Let G, X, A be as above. Then we have an exact sequence of k-vector spaces,*

$$0 \longrightarrow \mathrm{Lie}(G/k)(k) \xrightarrow{\tilde{i}} \mathrm{Der}(X/k) \xrightarrow{\tilde{p}_X} \mathrm{Der}(A/k) \cong \mathrm{Lie}(A/k)(k).$$

If we associate to each term the natural structure of p-Lie algebra which we have clarified in n° 1, § 2, the homomorphisms of the sequence are the morphisms of p-Lie algebras.

Proof. The proof of the exactness is obvious by virtue of the fact that $\Gamma(X, p_X^* \mathbf{Der}(A/k)) \cong \Gamma(A, \mathbf{Der}(A/k))$. The assertion concerning p-Lie algebras results from SGAD, Th. 7.2 and the next lemma.

Lemma 3. *Let $j: X \to B$ be a k-morphism from a complete variety X into an abelian variety B. Let* $\mathrm{Der}(X/k)$ *(resp.* $\mathrm{Der}(B/k)$*) be a p-Lie algebra of global derivations on X (resp. B). Then there* *exists a morphism of p-Lie algebras* $\tilde{j}:\mathrm{Der}(X/k)\to\mathrm{Der}(B/k)$ *such that for any open set* U' *of* B, *any element* $f\in\mathcal{O}_B(U')$, *and any element* D *of* $\mathrm{Der}(X/k)$, *we have* $j^*(\tilde{j}(D)f)=D(j^*(f))$, *where* j^* *is the Ring-homomorphism* $j^*(\mathcal{O}_B)\to\mathcal{O}_X$, *attached to* j.

Proof. We have a morphism of $\mathcal{O}_X$-Modules $j^*\Omega^1(B/k)\to\Omega^1(X/k)$, hence a morphism of $\mathcal{O}_X$-Modules $\tilde{j}:\mathbf{Der}(X/k)\to\mathbf{Hom}_{\mathcal{O}_X}(j^*\Omega^1(B/k),\mathcal{O}_X)$. On the other hand, since $\Omega^1(B/k)$ is generated by global differentials $\omega^1(B/k)$ (i.e. $\Omega^1(B/k)=\omega^1(B/k)\otimes_k\mathcal{O}_B$), we have

$$\mathbf{Hom}_{\mathcal{O}_X}(j^*\Omega^1(B/k),\mathcal{O}_X)\cong\mathrm{Hom}_k(\omega^1(B/k),k)\otimes_k\mathcal{O}_X\cong\mathrm{Der}(B/k)\otimes_k\mathcal{O}_X.$$

Therefore, if we choose a k-basis $(D_i)_{1\leq i\leq n}$ of $\mathrm{Der}(B/k)$, it is also a $\mathcal{O}_X$-basis of $\mathbf{Hom}_{\mathcal{O}_X}(j^*\Omega^1(B/k),\mathcal{O}_X)$. Consider open sets U' of B and U of $j^{-1}(U')$, small enough, an element f of $\mathcal{O}_B(U')$ and a global derivation $D\in\mathrm{Der}(X/k)$. Then there exists a system of elements $(g_i)_{1\leq i\leq n}$ of $\mathcal{O}_X(U)$ such that

$$(\tilde{j}(U)(D|U))f=(D|U)(j^*f)=\sum_{i=1}^{n}g_i j^*(D_i f).$$

Since D is defined everywhere on X, g_i must be defined everywhere on X, hence constants, i.e. $g_i\in k$, $1\leq i\leq n$. Put $D'=\sum_{i=1}^{n}g_iD_i$. Then $\tilde{j}(X)(D)=D'$ and $\tilde{j}(X)(D)f=j^*(D'f)$ for $f\in\mathcal{O}_B(U')$. We denote $\tilde{j}(X)$ by $\tilde{j}$. It is now easy to see that $\tilde{j}$ is a morphism of p-Lie algebras. q.e.d.

Remark. For the simplicity of notation, we sometimes abbreviate $\mathrm{Lie}(G/k)(k)$ as $\mathrm{Lie}(G)$, if there is no fear of confusion.

Here, we shall quote from SGAD, VII, 7.4 some results which we need for the further developement.

Lemma 4. *Let* G *be an infinitesimal finite group-scheme of height* ≤ 1 *over* k.
1) *Then there exists a p-Lie algebra* L *of finite dimension over* k *such that* G *is isomorphic to* $\mathcal{G}_p(L)=\mathrm{Spec}^*(U_p(L))=\mathrm{Spec}(U_p(L)^*)$ *where* $U_p(L)^*$ *is the linear dual of the restricted enveloping algebra* $U_p(L)$ *of* L, *provided with a suitable structure of bigebra.*
2) $\mathrm{Lie}(\mathcal{G}_p(L))\cong L$, *an isomorphism of p-Lie algebras.*

3) *For an exact sequence of p-Lie algebras* $0\to L_1\to L_2\to L_3\to 0$ (*i.e.* L_1 *is normal in* L_2 *such that* $L_2/L_1\simeq L_3$), *there exists an exact sequence of infinitesimal finite group-schemes of height* $\leqq 1$,

$$0 \longrightarrow \mathcal{G}_p(L_1) \longrightarrow \mathcal{G}_p(L_2) \longrightarrow \mathcal{G}_p(L_3) \longrightarrow 0\,.$$

Conversely, if $0\to G_1\to G_2\to G_3\to 0$ *is an exact sequence of infinitesimal finite group-schemes of height* $\leqq 1$, *a sequence of p-Lie algebras*

$$0 \longrightarrow \mathrm{Lie}\,(G_1) \longrightarrow \mathrm{Lie}\,(G_2) \longrightarrow \mathrm{Lie}\,(G_3) \longrightarrow 0$$

is exact.

4. Now we shall prove our main theorems.
The first one is:

Theorem 2. *Let the situation be as in n° 3 of § 2. We put the following assumption (E) on X;*

(*E*) *There exist an immersion j of X into an abelian variety B over k and a homomorphism $\pi : B\to A$ such that a diagram*

$$\begin{array}{ccc} X & \xrightarrow{\ j\ } & B \\ & {\scriptstyle p_X}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & A & \end{array}$$

is commutative.

Then X is an abelian variety and p_X is an isogeny.

Proof. First, note the following points.
1) j is proper, hence is a closed immersion, (cf. EGA, II. 5. 4. 3.).
2) Since the underlying topological spaces of X and A are homeomorphic, there exists a unique point e' of X over the neutral point e of A. e', e are k-rational. Then we can assume that $j(e')$ is the neutral point of B.

By virtue of the assumption (E), there exists an injection of p-Lie algebras $\tilde{j} : \mathrm{Der}\,(X/k)\to\mathrm{Lie}\,(B)$ such that the following diagram is commutative,

$$\begin{array}{ccccccc} & & & & \mathrm{Lie}\,(B) & \longrightarrow & \mathrm{Lie}\,(A) \\ & & & & \uparrow{\scriptstyle \tilde{j}} & & \| \\ 0 & \longrightarrow & \mathrm{Lie}\,(G) & \xrightarrow{\ \tilde{i}\ } & \mathrm{Der}\,(X/k) & \xrightarrow{\ \tilde{p}_X\ } & \mathrm{Lie}\,(A) \\ & & & & \uparrow & & \\ & & & & 0 & & \end{array}$$

Since the composite morphism of p-Lie algebras $\mathrm{Lie}(G) \xrightarrow{\tilde{j}\cdot\tilde{i}} \mathrm{Lie}(B)$ is injective, Lie(G) is commutative because Lie(B) is commutative. Hence G is commutative because G is infinitesimal and of height $\leqq 1$. This case was done in the previous paper [4].

Remark. We have only to assume that there exists a monomorphism from X to an abelian veriety B.

The second one is:

Theorem 3. *Let Y be a complete variety over k and let X be a connected, reduced Galois covering of Y with group G which is infinitesimal and with k as its field of definition. Then the Albanese variety of Y is obtained from the Albanese variety of X as a quotient by an infinitesimal group-scheme over k, where the Albanese varieties are understood in the sense of Serre,* [10].

Proof. Since G is infinitesimal, G is annihilated by a power of the Frobenius endomorphism of G. The argument by induction on the height of G permits us to assume that G is infinitesimal and of height$\leqq 1$. If G is so, let B (resp. A) be the Albanese variety of X (resp. Y) and let j (resp. j') be the canonical morphism of X (resp. Y) into B (resp. A). The situation is summarized in the commutativity of a diagram,

$$\begin{array}{ccccc} G\underset{k}{\times}X & \rightrightarrows & X & \xrightarrow{p_X} & Y \\ & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} \\ & & B & \xrightarrow{p'} & A \end{array}.$$

Then, by the argument analogous to the one of n° 2, § 2, we know that there exists a commutative diagram of $\mathcal{O}_X$-Modules,

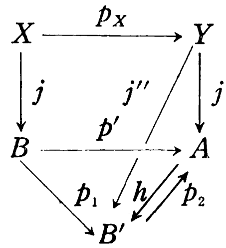
$$\begin{array}{ccccc} 0 \to \mathrm{Lie}(G)\underset{k}{\otimes}\mathcal{O}_X & \xrightarrow{\tilde{i}} & \mathbf{Der}(X/k) & \xrightarrow{\tilde{p}_X} & p_X^*\mathbf{Der}(Y/k) \\ & & \downarrow{\scriptstyle \tilde{j}} & & \downarrow{\scriptstyle p_X^*(\tilde{j}')} \\ & & j^*\mathbf{Der}(B/k) & \xrightarrow[j^*(\tilde{p}')]{} & p_X^*j'^*\mathbf{Der}(A/k) \cong j^*p'^*\mathbf{Der}(A/k) \end{array}$$

where the upper sequence is exact. Taking the global sections on X, we have a commutative diagram of k-vector spaces,

$$
\begin{array}{ccccccc}
0 \longrightarrow \mathrm{Lie}(G) & \xrightarrow{\tilde{i}} & \mathrm{Der}(X/k) & \xrightarrow{\tilde{p}_X} & \Gamma(X, p_X^*\mathbf{Der}(Y/k)) & & \\
& & \downarrow \tilde{j} & & \downarrow \tilde{j}' & & \\
& & \mathrm{Der}(B/k) & \xrightarrow{\tilde{p}'} & \Gamma(X, p_X^* j'^* \mathbf{Der}(A/k)) & \cong & \mathrm{Der}(A/k) \\
& & \wr\downarrow & & & & \wr\downarrow \\
& & \mathrm{Lie}(B) & & & & \mathrm{Lie}(A)
\end{array}
$$

where the upper sequence is exact and where $\tilde{i}$, $\tilde{j}$, $\tilde{p}'$ are morphisms of p–Lie algebras over k. Let $\mathrm{Lie}(G)$ operate on B through $\alpha=\tilde{j}\cdot\tilde{i}$ and let B' be the quotient $B/\mathfrak{n}$ where $\mathfrak{n}$ is the image of $\mathrm{Lie}(G)$ in $\mathrm{Lie}(B)$ which is an ideal of $\mathrm{Lie}(B)$ because $\mathrm{Lie}(B)$ is commutative. We assert that B' is k–isomorphic to A. Let b be a point of B and let U be an affine open set, small enough, of b. Let x be a point of X such that $j(x)=b$. Take an affine open set U' of x in X such that $U'\subset j^{-1}(U)$. For an arbitrary element t of $\mathcal{O}_{B'}(U)$ and for an arbitrary derivation D of $\mathrm{Der}(X/k)$, by virtue of Lemma 3, we have $D(j^*t)=j^*(\tilde{j}(D)t)=0$ where j^* is the Ring-homomorphism $j^*(\mathcal{O}_B)\to\mathcal{O}_X$ at tachedto j. Hence $j^*(t)$ belongs to $\mathcal{O}_Y(U')$. This means that the morphism j induces a morphism j'' of Y into B', the map of the underlying topological spaces being unchanged.

On the other hand, since $\tilde{p}'\cdot\tilde{j}\cdot\tilde{i}=\tilde{j}'\cdot\tilde{p}_X\cdot\tilde{i}=0$, p' factors to the composite of homomorphisms $p'=p_2\cdot p_1$. By virtue of the universality of A, j'' factors to the composite of morphisms, $j''=h\cdot j'$. We shall summarize the situation in a diagram,



where $p'\cdot j=j'\cdot p_X$, $p_1\cdot j=j''\cdot p_X$, $j'=p_2\cdot j''$, $j''=h\cdot j'$ and $p'=p_2\cdot p_1$. Then, since $p_2\cdot h\cdot j'\cdot p_X=p_2\cdot j''\cdot p_X=p_2\cdot p_1\cdot j=p'\cdot j=j'\cdot p_X$ and since p_X is faithfully flat and A is generated by $j'(Y)$, we have $p_2\cdot h=id_A$. Also, since $h\cdot p_2\cdot p_1\cdot j=h\cdot p'\cdot j=h\cdot j'\cdot p_X=j''\cdot p_X=p_1\cdot j$ and since p_1 is faithfully flat and B is generated by $j(X)$, we have $h\cdot p_2=id_{B'}$. Thus h and p_2 define an isomorphism of A and B'. q.e.d.

Corollary. *Let X be a Galois covering of an elliptic curve A with group G which is a finite group-scheme over k and with k as the field of definition. Suppose X is connected and reduced. Then X is an elliptic curve.*

Proof. We may identify A with its Jacobian variety. Suppose G is infinitesimal and of height $\leqq 1$. Then the assumption (E) of Theorem 2 is verified, taking as B the Jacobian variety J of X. Also, Theorem 3 shows that the Albanese variety of X is isogenous to A. Hence the genus of X is 1, i.e. X is an elliptic curve. By induction on the height of G, the assertion holds if G is infinitesimal. From what we have observed in § 1, the assertion is now trivial. q.e.d.

The last one is:

Theorem 4. *Let X be a Galois covering of an abelian variety A with group G which is a finite group-scheme over k and with k as the field of definition. Suppose X is connected and non-singular (i.e. lisse)over k. Then X is an abelian variety.*

Proof. The argument of § 1 shows that we can assume G is infinitesimal. Let H be the kernel of the Frobenius endomorphism F_G of G. Then H is infinitesimal and of height $\leqq 1$, hence is determined by its p-Lie algebra Lie(H). Let X' be the quotient of X by H, i.e. $X'=X/H$. Then X' is connected and non-singular over k by Theorem 2 of [12]. Since the height of G is finite, the induction on the height of G allows us to assume that G is infinitesimal and of height $\leqq 1$. Put $B=A^{p^{-1}}$. Then X divides the Frobenius endomorphism F_B of B, $F_B: B \xrightarrow{h} X \xrightarrow{p_X} A$.

First we shall prove:

Lemma 5. *Let the situation be as above. Then h is faithfully flat.*

Proof. Let $X'=X^{p^{-1}}$. X' is also connected and non-singular over k. The Frobenius endomorphism of X' is divided by

B, $F_{X'}: X' \overset{h'}{\to} B \overset{h}{\to} X$. In order to show that h is faithfully flat, we have only to prove that $F_{X'}$ is faithfully flat and that h' is faithfully flat. For the faithful flatness of $F_{X'}$, it follows from the existence of p-base of the local ring at each point of X'. For the faithful flatness of h', first we shall construct a p-semi-linear morphism of p-Lie algebras $\mathrm{Der}(X'/k) \to \mathrm{Der}(X/k)$. Since X is complete, the dimension of k-vector space $\mathrm{Der}(X/k)$ is finite. Let n be the dimension of X. Then we can find an open covering $\mathfrak{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ of X enough fine so that for any global derivation and for any λ of Λ, there exists a system of regular functions $\{f_i^{(\lambda)}\}_{1 \leq i \leq n}$ on U_λ such that $D | U_\lambda = \sum_{i=1}^{n} f_i^{(\lambda)} \frac{\partial}{\partial t_i^{(\lambda)}}$, where $t_1^{(\lambda)}, \cdots, t_n^{(\lambda)}$ are local parameters of U_λ and where $f_i^{(\lambda)} = \sum_{j=1}^{n} f_j^{(\mu)} \frac{\partial t_i^{(\lambda)}}{\partial t_j^{(\mu)}}$, $i = 1, \cdots, n$, on $U_\lambda \cap U_\mu$. Take the same open covering $\mathfrak{U}$ on X' and put $T_i^{(\lambda)} = (t_i^{(\lambda)})^{p^{-1}}$, $i = 1, \cdots, n$, $\lambda \in \Lambda$. Then $T_i^{(\lambda)}$, $i = 1, \cdots, n$, are local parameters of X' on U_λ. Then we have $\left(\frac{\partial T_i^{(\lambda)}}{\partial T_j^{(\mu)}}\right)^p = \frac{\partial t_i^{(\lambda)}}{\partial t_j^{(\mu)}}$. For a global derivation D of $\mathrm{Der}(X/k)$ expressed as above, we define a section D'_{U_λ} of $\mathbf{Der}(X'/k)$ on U_λ by $D'_{U_\lambda} = \sum_{i=1}^{n} (f_i^{(\lambda)})^{p^{-1}} \frac{\partial}{\partial T_i^{(\lambda)}}$. Then it is easy to see that $\{D'_{U_\lambda}\}$ defines a global derivation D' of $\mathrm{Der}(X'/k)$. This correspondence $D \to D'$ is bijective and, in fact, is a p^{-1}-semi-linear morphism of p-Lie algebras. On the other hand, the action of G on X is determined by the injective morphism of p-Lie algebras $\tilde{i}: \mathrm{Lie}(G) \to \mathrm{Der}(X/k)$. Let G' be an infinitesimal group of height $\leqq 1$ constructed by taking as the structure constants of p-Lie algebra $\mathrm{Lie}(G')$ the p-th roots of structure constants of $\mathrm{Lie}(G)$. Then it is easy to see that $\tilde{i}$ can be lifted to an injective morphism of p-Lie algebras $\tilde{i}': \mathrm{Lie}(G') \to \mathrm{Der}(X'/k)$ which determines the action of G' on X'. The quotient of X' by G' is evidently B. From this argument we know that h' is faithfully flat. Hence h is faithfully flat.

Now we shall return to the proof of Theorem 4.(*) Consider

(*) The essential line of the following proof was communicated by Prof. J.-P. Serre in a letter dated April 18, 1967. The author wishes to express his gratitude to Prof. J.-P. Serre.

a sequence of $\mathcal{O}_B$-Modules

$$\mathbf{Der}(B/k) \xrightarrow{\tilde{h}} h^*\mathbf{Der}(X/k) \xrightarrow{h^*(\tilde{p}_X)} F_B^*\mathbf{Der}(A/k) = h^*p_X^*\mathbf{Der}(A/k).$$

The existence of p-base at every point of B shows that the above sequence is exact. However, in n° 3 of §2, we have seen that the sequence of $\mathcal{O}_X$-Modules

$$0 \longrightarrow \mathrm{Lie}(G)\underset{k}{\otimes}\mathcal{O}_X \longrightarrow \mathbf{Der}(X/k) \xrightarrow{\tilde{p}_X} p_X^*\mathbf{Der}(A/k)$$

is exact. By virtue of Lemma 5, we have an exact sequence of $\mathcal{O}_B$-Modules

$$0 \longrightarrow \mathrm{Lie}(G)\underset{k}{\otimes}\mathcal{O}_B \longrightarrow h^*\mathbf{Der}(X/k) \xrightarrow{h^*(\tilde{p}_X)} F_B^*\mathbf{Der}(A/k).$$

Therefore the image of $\tilde{h}$ is $\mathrm{Lie}(G)\underset{k}{\otimes}\mathcal{O}_B$. Since B is proper over k and since $\mathcal{O}_B$-Module $\mathbf{Der}(B/k)$ is trivial (i.e. $\mathbf{Der}(B/k) \cong \mathrm{Der}(B/k)\underset{k}{\otimes}\mathcal{O}_B$), the morphism $\mathbf{Der}(B/k)\to\mathrm{Lie}(G)\underset{k}{\otimes}\mathcal{O}_B$ is determined by a morphism of p-Lie algebras $\mathrm{Der}(B/k)\to\mathrm{Lie}(G)$. In particular, $\tilde{h}$ is determined by a morphism of p-Lie algebras of global derivations $\tilde{h}: \mathrm{Der}(B/k)\to\mathrm{Der}(X/k)$. The kernel $\mathfrak{n}$ of $\tilde{h}$ is an ideal of $\mathrm{Der}(B/k)$. Consider $\mathfrak{n}$ as an ideal of $\mathrm{Lie}(B)$. Then, it is easily verified that the quotient $B/\mathfrak{n}$ of B by $\mathfrak{n}$ coincides with X. Hence X is an abelian variety. q.e.d.

Part II.

0. In this part, we shall show that if A is an abelian variety, a covariant functor of categories $G\in(C_f^c(k)) \rightsquigarrow \mathrm{E}(G, A)\in(\mathbf{Ab})$ is isomorphic to a covariant functor $G\in(C_f^c(k)) \rightsquigarrow \mathrm{Ext}^1(A, G)\in(\mathbf{Ab})$, (for the notation, see Oort's book [6] and [4]).

1. First, we shall prove:

Lemma 1. *Let X be a Galois covering of an abelian variety A with a group G which is isomorphic to α_p or μ_p and with the field of definition k which is algebraically closed and of positive characteristic p. Then we have:*

(1) *If X is reduced, X is an abelian variety.*

(2) *If X is not reduced, X is isomorphic to a direct product $G \times_k X_{\text{red}}$, and $X_{\text{red}} \cong A$.*

Proof. Suppose G is α_p. Another case is proved analogously. Embed α_p into G_a. Then X is embedded into a principal fibre space P of base A and with group G_a,

$$\begin{array}{ccccc} \alpha_p \times_k X & \rightrightarrows & X & \longrightarrow & A \\ \downarrow & & \downarrow j & & \downarrow \\ G_a \times_k P & \rightrightarrows & P & \longrightarrow & A \end{array}$$

where j is a closed immersion, (cf. [4]). Let i be a canonical closed immersion from X_{red} into X. Then $j \cdot i$ is an isomorphism of X_{red} onto a closed, complete subvariety of codimension 1 of P which is an abelian variety, (cf. Theorem of Appendix of [4]). If $X = X_{\text{red}}$, X is an abelian variety. If $X \neq X_{\text{red}}$, it is not difficult to see that $X_{\text{red}} = A$. Hence, $X \xrightarrow{\sim} \alpha_p \times_k X_{\text{red}}$. q.e.d.

Corollary. *Let X be a Galois covering of an abelian variety A with a group which is infinitesimal and commutative. Then, X can be endowed with a structure of a commutative group-scheme over k.*

Proof. Since G is commutative, G has a composition series, $G_0 = (e) \to G_1 \to G_2 \to \cdots \to G_n \to G_{n+1} = G$, where each quotient is isomorphic to α_p or μ_p. We proceed by induction on the length $n+1$ of G. If $n=0$, our assertion is nothing but Lemma 1. Suppose our assertion holds for a group-scheme G of length $\leqq n$. Let X' be a Galois covering of A with a group G/G_n, induced from X by a morphism $G \to G/G_n$. G/G_n is isomorphic to α_p or μ_p. By virtue of Lemma 1, X' is an abelian variety or isomorphic to a product $G/G_n \times_k A$. For the former case, X is a Galois covering of an abelian variety X' with a group G_n of length $\leqq n$. Hence, by the assumption of induction, X is a commutative group-scheme over k. For the latter case, X comes from a Galois covering X'' of A with a group G_n by extending the group G_n to G, i.e. $X \xrightarrow{\sim} G \times X''/G_n$, (cf. [4]). X'' is also a commutative group-scheme over

k. Then it is easy to see that $G\times X''/G_n \cong X$ is a commutative group-scheme over k. q.e.d.

Now, consider a general case. Let X be a Galois covering of an abelian variety A with a commutative group-scheme G over k. Then G is decomposed to a direct product $G_{\mathrm{red}}\times G_{\mathrm{inf}}$, where G_{inf} is the maximal infinitesimal subgroup of G. Restrict the group G to G_{red} to give a Galois covering $Y=X/G_{\mathrm{inf}}$,

$$\begin{array}{ccccc} G\underset{k}{\times} X & \rightrightarrows & X & \longrightarrow & A \\ \downarrow & & \downarrow & & \| \\ G_{\mathrm{red}}\underset{k}{\times} Y & \rightrightarrows & Y & \longrightarrow & A. \end{array}$$

If Y is connected, it is immediate to see that Y is an abelian variety. In general, if Y is not connected, take a connected component Y_0 of Y and denote by H the subgroup of G_{red} formed by elements g such that $gY_0=Y_0$. Other component of Y is of the form gY_0 where g is an element of a classe of $G'=G_{\mathrm{red}}/H$. It is evident that if we restrict the group G_{red} to G', Y is changed to a direct product $G'\underset{k}{\times} A$. Therefore, there exists a Galois covering Y' of A with a group H such that $Y\xrightarrow{\sim} G_{\mathrm{red}}\times Y'/H$. It is easy to see that Y' is k-isomorphic to Y_0. Since Y_0 is connected, Y' is an abelian variety. Let G'' be a subgroup $H\times G_{\mathrm{inf}}$ of G. As $X/G''\cong Y/H\cong G'\times A$, there exists a Galois covering Y'' of A with group G'' such that $X\xrightarrow{\sim} G\times Y''/G''$. If Y'' is a commutative group-scheme over k, then X is so. On the other hand, $Y''/G_{\mathrm{inf}} \cong Y_0 \cong Y'$. Thus we are reduced to the situation of Corollary to Lemma 1.

Theorem. *Let X be a Galois covering of an abelian variety A with group G which is finite and commutative and with k as a field of definition. Then we can regard X as an element of* $\mathrm{Ext}^1(A, G)$.

2. In the previous paper ([4], Theorem of Part I), we proved that a covariant functor $G\in C_f^c(k) \rightsquigarrow \mathrm{E}(G, Y)\in(\mathbf{Ab})$ is pro-representable if Y is an irreducible variety. Denote by $F^c(Y)$ a profinite group $\in \mathrm{Pro}(C_f^c(k))$ which pro-represents the above functor and which is determined up to isomorphism. For an abelian variety A over k, the above Theorem shows us that $F^c(A)$ is isomorphic to a profinite group of $\mathrm{Pro}(C_f^c(k))$ which pro-represents a functor $G \in (C_f^c(k)) \rightsquigarrow \mathrm{Ext}(A, G) \in (\mathbf{Ab})$. Thus we know $F^c(A) \cong \pi_1(A)$, $\pi_1(A)$ being defined by F. Oort in his book [6].

3. We know that the category $C_f^c(k)$ of finite commutative group-schemes oves k which is algebraically closed and of positive characteristic is decomposed to a direct product of four categories, $\mathcal{A}_{rr}$, $\mathcal{A}_{rl}$, $\mathcal{A}_{lr}$, and $\mathcal{A}_{ll}$. Then any object G of $C_f^c(k)$ is decomposed to a direct product, $G = G_{rr} \times G_{rl} \times G_{lr} \times G_{ll}$, corresponding to the decomposition of categories. Here, $G_{rr} \times G_{rl} \cong G_{\mathrm{red}}$ and $G_{lr} \times G_{ll} \cong G_{\mathrm{inf}}$. Analogously, for an abelian variety A over k, we have $\mathrm{Ext}(A, G) = \mathrm{Ext}(A, G_{rr}) \times \mathrm{Ext}(A, G_{rl}) \times \mathrm{Ext}(A, G_{lr}) \times \mathrm{Ext}(A, G_{ll})$ and $\pi_1(A) = \pi_1^{rr}(A) \times \pi_1^{rl}(A) \times \pi_1^{lr}(A) \times \pi_1^{ll}(A)$, where $\pi_1^i(A)$ is a profinite group in $\mathrm{Pro}(\mathcal{A}_i)$ which pro-represents a functor $\mathrm{Ext}(A, *)$ from $(\mathcal{A}_i)$ to $(\mathbf{Ab})$, $i = rr, rl, lr, ll$.

Finally, we shall give :

Proposition. *Let A be an abelian variety over k. Then we have* :

(1) $\pi_1^{rr}(A) = \prod_{\substack{l:\,\mathrm{prime} \\ l \neq p}} (\mathbf{Z}_l)^{2\dim(A)}$.

(2) $\pi_1^{rl}(A) = (\mathbf{Z}_p)^s$, *where s is an integer such that p^s is the number of points of the kernel of the multiplication by p on A.*

(3) $\pi_1^{lr}(A) = (K_\infty)^s$, (*cf.* [6]).

(4) $\pi_1^{ll}(A) = F(\hat{A}/(\hat{G}_m)^s)$ *where $F(\hat{A}/(\hat{G}_m)^s)$ is the profinite group of* $\mathrm{Pro}(\mathcal{A}_{ll})$ *which pro-represents a functor* $G \in (\mathcal{A}_{ll}) \rightsquigarrow \mathrm{Ext}_{\mathfrak{D}}(\hat{A}/(\hat{G}_m)^s, G) \in (\mathbf{Ab})$, *and where $\hat{A}$ and $\hat{G}_m$ are commutative formal groups attached to A and G_m at their neutral points,* (*cf.* [5]).

We omit the proof.

Kyoto University.

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