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Note on direct summands of modules

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Some years ago the following question was raised by H. Matsumura and solved affirmatively by H. Toda:

"Let N be a subgroup of a finitely generated abelian group M. Suppose that M and $N\oplus M/N$ are isomorphic, is N a direct summand of M?"

A simple proof runs as follows. Let n be an arbitrary integer, M/nM is isomorphic to $N/nN\oplus((M/N)/n(M/N))$. If we denote the order of a finite group G by Card (G), then

$$\begin{aligned}\text{Card}\,(N/nN) &= \text{Card}\,(M/nM)-\text{Card}\,((M/N)/n(M/N))\\ &= \text{Card}\,(M/nM)-\text{Card}\,(M/(N+nM))\\ &= \text{Card}\,(nM/(N+nM))\\ &= \text{Card}\,(N/(N\cap nM))\end{aligned}$$

Therefore $nN=N\cap nM$ for an arbitrary integer, that is, N is a *pure subgroup* of M. By a well known theorem N is a direct summand of M.

In this note we show that this property holds for more general class of modules. The notion of pure subgroups may be generalized to that of modules (Exercise 24, Chap. 1, §2 [1]). We will prove in this context the analogy of the classical theorem that a pure subgroup N of an abelian group M is a direct summand of M provided that M/N is finitely generated.

First we list more or less well known lemmas without proofs. Throughout this note R is a Noetherian commutative ring and A an R-algebra which is of finite type as an R-module. When R

is local, $\hat{A}$ means $\varprojlim_n (A\otimes_R R/\mathfrak{m}^n)$ where $\mathfrak{m}$ is the maximal ideal of R. Let M be an A-module then $\hat{M}=\varprojlim_n (M\otimes_R R/\mathfrak{m}^n)$.

Lemma 1: *Let M be an right A-module of finite type and N an left A-module of finite type. If R is local, there is a natural isomorphism*

$$(M\otimes_A N)^\wedge \xrightarrow{\sim} \hat{M}\otimes_{\hat{A}}\hat{N}$$

Lemma 2: *Let M and N be left (or right) A-modules of finite type. Let R' be a flat commutative R-algebra, then there exists a natural isomorphism*

$$\mathrm{Hom}_A(M, N)\otimes_R R' \xrightarrow{\sim} \mathrm{Hom}_{A\otimes_R R'}(M\otimes_R R', N\otimes_R R').$$

Lemma 3: *Let A and B be rings (not necessarily commutative). Let E be a left A-module and F a (A, B)-bimodule. Let G be right B-module then there is a natural homomorphism;*

$$\sigma: \mathrm{Hom}_B(F, G)\otimes_A E \to \mathrm{Hom}_B(\mathrm{Hom}_A(E, F), G)$$

defined by $\sigma(u\otimes x)(v)=u(v(x))$ where $x\in E$, $u\in \mathrm{Hom}_B(F, G)$ and $v\in \mathrm{Hom}_A(E, F)$.
Moreover, when G is an injective B-module and E of finite presentation, σ is an isomorphism.

Theorem 1: *Let R be a commutative noetherian ring and A an R-algebra which is of finite type as an R-module. Let M be an left (or right) A-module of finite type and N a submodule of M. If M and $N\oplus M/N$ are isomorphic. Then N is a direct summand of M.*

Proof: The problem is certainly local, so we may assume that R is a local ring. Let $\mathfrak{m}$ be the maximal ideal of R. Let L be an arbitrary left A-module of finite type, $L/\mathfrak{m}^n L$ is viewed as an R-module of finite length if $n\geqq 1$. Since M and $N\oplus M/N$ are isomorphic, $\mathrm{Hom}_A(M, L/\mathfrak{m}^n L)$ and $\mathrm{Hom}_A(N, L/\mathfrak{m}^n L)\oplus \mathrm{Hom}_A(M/N, L/\mathfrak{m}^n L)$ are isomorphic. Since $\mathfrak{m}^n$ kills these three modules, they are of finite length as R-modules. Hence $\mathrm{length}_R(\mathrm{Hom}_A(M, L/\mathfrak{m}^n L)) = \mathrm{length}_R(\mathrm{Hom}_A(N, L/\mathfrak{m}^n L)) + \mathrm{length}_R(\mathrm{Hom}_A(M/N, L/\mathfrak{m}^n L))$. This equality shows that the sequence

$$0\to \mathrm{Hom}_R(M/N, L/\mathfrak{m}^n L)\to \mathrm{Hom}_R(M, L/\mathfrak{m}^n L)\to \mathrm{Hom}_R(N, L/\mathfrak{m}^n L)\to 0$$

obtained from the canonical exact sequence $0\to N\to M\to M/N\to 0$ is exact. By lemma 2 (note that $\hat{R}$ is faithfully flat over R)

$$\begin{aligned}\varprojlim_n \mathrm{Hom}_A(M, L/\mathfrak{m}^n L) &\cong \varprojlim_n \mathrm{Hom}_{A/\mathfrak{m}^n A}(M/\mathfrak{m}^n M, L/\mathfrak{m}^n L)\\ &\cong \mathrm{Hom}_{\hat{A}}(\hat{M}, \hat{L})\\ &\cong \hat{R}\otimes_R \mathrm{Hom}_A(M, L)\end{aligned}$$

where every isomorphism is natural one. And the projective limit of exact sequences of modules of finite length is again exact. Therefore

$$0\to \hat{R}\otimes_R \mathrm{Hom}\,(M/N, L)\to \hat{R}\otimes_R \mathrm{Hom}\,(M, L)\to \hat{R}\otimes_R \mathrm{Hom}\,(N, L)\to 0$$

is exact. Since the completion $\hat{R}$ of R is faithfully flat over R,

$$0\to \mathrm{Hom}_A(M/N, L)\to \mathrm{Hom}_A(M, L)\to \mathrm{Hom}_A(N, L)\to 0$$

is exact. Setting $L=N$ we see that the identity map from N to N factors through the canonical injection from N to M. Hence N is a direct summand of M. q.e.d.

In order to prove the corollary, we need:

Lemma 4: *Let* $\mathfrak{m}$ *be one of maximal ideals of* R *and* $A_{\mathfrak{m}}=A\otimes_R R_{\mathfrak{m}}$. *Let* L' *be an* $A_{\mathfrak{m}}$*-module which is of finite length as an* $R_{\mathfrak{m}}$*-module. Then there exists an* A*-module* L *which is of finite length as an* R*-module such that* $L\otimes_R R_{\mathfrak{m}}$ *is isomorphic to* L'.

Proof: First of all there exists an A-module of finite type L_1 such that $(L_1)_{\mathfrak{m}}=L_1\otimes_R R_{\mathfrak{m}}\cong L'$. Since $\mathrm{Ass}_{R_{\mathfrak{m}}}(L')=\{\mathfrak{m}R_{\mathfrak{m}}\}$, $\mathrm{Ass}_R(L_1)$ contains $\mathfrak{m}$. By modifying the arguments of Prop. 4, 1, Chap. 4. [2], we can show that there exists an A-submodule L of L_1 such that $\mathrm{Ass}_R(L)=\{\mathfrak{m}\}$ and $\mathrm{Ass}_R(L_1/L)=\mathrm{Ass}_R(L_1)-\{\mathfrak{m}\}$. Then obviously $L_{\mathfrak{m}}=(L_1)_{\mathfrak{m}}\cong L'$.

Corollary: *Let* L, M *and* N *be (left)* A*-modules of finite type, an exact sequence*

$$0\to L\to M\to N\to 0$$

splits if and only if the canonical sequence obtained by applying

$\mathrm{Hom}_A(\ ,P)$

$$0 \to \mathrm{Hom}_A(N, P) \to \mathrm{Hom}_A(M, P) \to \mathrm{Hom}_A(L, P) \to 0$$

is exact for every A-module P which is of finite length as an R-module.

The proof is clear.

Remark: If M is not of finite type, Theorem 1 is obviously false.

Let M be a left A-module and M' a submodule of M. Let j be the natural injection from M' to M. According to Bourbaki, we call M' a *pure submodule of* M if $1 \otimes j : N \otimes_R M' \to N \otimes_R M$ is an injection for an arbitrary right A-module N. (cf. Exercise 24, Chap. 1, §1. [1]). Tensor products and inductive limits commute, so it is enough to check the injectivity for an A-module of finite type. It is also clear that this notion coincides with classical one for abelian groups.

Theorem 2: *Let R be a commutative noetherian ring and A an R-algebra which is of finite type as an R-module. Then an pure submodule of an A-module M of finite type is a direct summand of M.*

Proof: Let F be an arbitrary left A-module and G an injective R-module. We have an commutative diagram:

$$\begin{array}{ccccc} 0 \to \mathrm{Hom}_R(F, G) \otimes_A M' & \longrightarrow & \mathrm{Hom}_R(F, G) \otimes_A M & \to & \mathrm{Hom}_R(F, G) \otimes_A (M/M') \to 0 \\ \downarrow \sigma & & \downarrow \sigma & & \downarrow \sigma \\ 0 \to \mathrm{Hom}_R(\mathrm{Hom}_A(M', F), G) & \to & \mathrm{Hom}_R(\mathrm{Hom}_A(M, F), G) & \to & \mathrm{Hom}_R(\mathrm{Hom}_A(M/M', F), G) \to 0 \end{array}$$

where the upper horizontal sequence is exact by the definition of pureness and three σ's are isomorphisms by lemma 3. Therefore

$$(*) \quad 0 \to \mathrm{Hom}_R(\mathrm{Hom}_A(M', F), G) \to \mathrm{Hom}_R(\mathrm{Hom}_A(M, F), G) \to \to \mathrm{Hom}_R(\mathrm{Hom}_A(M/M', F), G) \to 0$$

is exact. Let E be the cokernel of $\mathrm{Hom}_A(M, F)\to\mathrm{Hom}_A(M', F)$. Suppose $E\neq 0$. Let G be the injective envelope of E as an R-module. (*) implies that $\mathrm{Hom}_R(E, G)=0$ but $\mathrm{Hom}_R(E, G)\neq 0$ since this contains the injection from E to G. This is a contradiction. Hence

$$0\to\mathrm{Hom}_A(M/M', F)\to\mathrm{Hom}_A(M, F)\to\mathrm{Hom}_A(M', F)\to 0$$

is exact. Hence M' is a direct summand of M. q.e.d.

Corollary 1: *Let M' be a submodule of a left A-module M of finite type M' is pure if and only if $1\otimes j: N\otimes M'\to N\otimes M$ is an injection for an arbitrary A-module N which is of finite length as an R-module.*

The Proof is clear by lemma 1 and theorem 2.

Corollary 2: *Let R' be a faithfully flat commutative R-algebra. Let $A'=A\otimes_R R'$. Let M be an A-module of finite type. Then M' is a direct summand of M if and only if $M'\otimes_R R'$ is a direct summand of $M\otimes_R R'$.*

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REFERENCES

[1] N. Bourbaki: Algèbre commutative; Chap. 1 and 2. Hermann.
[2] N. Bourbaki: Algèbre commutative; Chap. 3 and 4. Hermann.