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On the characters ν^* and τ^* of singularities

By

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In [1], the numerical characters ν^* and τ^* of singularities were introduced and some basic theorems were proven in regards to the effect of permissible monoidal transformations on those characters under the assumption that no *inseparable* residue field extensions occur in the monoidal transformations. They are, of course, satisfactory in the study of singularities of algebraic schemes in which the residue fields have characteristic zero, and played a role of vital importance in the resolution of singularities in characteristic zero. Inseparable residue field extensions are, however, inevitable in the monoidal transformations of algebraic schemes over fields of positive characteristics or over the ring of integers. As was done in [1], if only separable residue field extensions are involved, many of the theorems about the behavior of ν^* and τ^* can be reduced to the case of trivial residue field extensions. Namely, we replace the given scheme by suitable local etale coverings. This approach fails completely if an inseparable residue field extension is involved, and, as is done in this paper, an essentially different approach must be taken. At one crucial point, I make use of Hasse differentiations. This was inspired by [2], Proof of Lemma 7.1, p. 486. In a subsequent paper, the theorems of this paper will play important roles in proving the resolution of singularities of an arbitrary excellent schemes of dimension 2 by means of quadratic and permissible monoidal transformations.

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Throughout this paper, Z will denote a *regular* scheme, X a closed subscheme of Z and x a point of X. To analyse the singularity of X at x, we extract a certain system of numerical characters of the ideal of X in Z at x. J being this ideal, we have defined $\nu^*(J)$ and $\tau^*(J)$, which are also denoted by $\nu_x^*(X/Z)$ and $\tau_x^*(X/Z)$ respectively. (cf [1], Ch. III, §1, Def. 1, p. 205; §1, Def. 2, p. 207; §1, Def. 4, p. 209; §4, Def. 6, p. 221) The last integer $\tau^{(t)}(J)$ of the system $\tau^*(J)$, where $t=t(J)$, will be simply denoted by $\tau(J)$ or by $\tau_x(X/Z)$. (cf [1], Ch. III, §4, Def. 6, p. 221.)

Let R (resp. M, resp. k) be the local ring (resp. maximal ideal, resp. residue field) of Z at x. Both $\nu^*(J)$ and $\tau(J)$ are extracted from the graded (or homogeneous) ideal $gr_M(J, R)$ in the graded k-algebra $gr_M(R)$. (cf [1], Ch. II, §1 and §2, pp. 180–197.) Let $k \to K$ be any field extension, $gr_M(R)_K = gr_M(R) \otimes_k K$ and $gr_M(J, R)_K = gr_M(J, R) \otimes_k K$, which is a graded ideal in $gr_M(R)_K$. In the same way as $\nu^*(J)$ and $\tau^*(J)$ are extracted from $gr_M(J, R)$, we do $\nu^*(J)_K$ and $\tau^*(J)_K$ from $gr_M(J, R)_K$. Namely, let $(\varphi_1, \cdots, \varphi_m)$ be a *standard base of the graded ideal* $gr_M(J, R)_K$, i.e., a minimal base consisting of homogeneous elements φ_j of monotone non-decreasing degrees for $j=1, 2, \cdots, m$. Then $\nu^*(J)_K = (deg\, \varphi_1, \cdots, deg\, \varphi_m, \infty, \infty, \cdots)$ and $\tau^{(i)}(J)_K = rank_K T^{(i)}(J)_K$, where

(1, 0, 1) $(\mu_1, \cdots, \mu_t)$ *being the system of distinct integers, in the increasing order, which appear in* $\nu^*(J)_K$, $T^{(i)}(J)_K$ *is the smallest* K*-submodule* T *of* $gr_M^1(R)_K$ *such that*

$$gr_M^{\mu}(J, R)_K = gr_M^{\mu}(R)_K \cap (gr_M(J, R)_K \cap K[T])\, gr_M(R)_K$$

for all integers $\mu \leqq \mu_i$. $(1 \leqq i \leq t.)$ *Also we denote the integers* t *and* μ_i *by* $t(J)_K$ *and* $\mu^{(i)}(J)_K$, $1 \leq i \leq t$, *respectively.* When the suffix K is omitted, it should be understood that $K=k=R/M$. (cf [1], Ch. III, §4, Lemma 10, p. 221.) Then let $\tau^*(J)_K = (\tau^{(1)}(J)_K, \cdots, \tau^{(t)}(J)_K)$ with $t=t(J)_K$. We write $\tau_x^*(X/Z)_K$ for $\tau^*(J)_K$.

It is easy to see that a standard base of $gr_M(J, R)$ is a standard base of $gr_M(J, R)_K$, so that always $\nu^*(J)_K = \nu^*(J)$, $\mu^{(i)}(J)_K = \mu^{(i)}(J)$ and $t(J)_K = t(J)$. Moreover, if $k \to K$ is separable, $\tau^*(J)_K$

$=\tau^*(J)$. (cf [1], Ch. III, § 4, Lemma 12, p. 223). In general, $\tau^*(J)_K$ can change depending upon K but always

$$(1, 0, 2) \qquad \tau^{(i)}(J)_K \geq \tau^{(i)}(J)_{K'} \quad \textit{if} \quad k \to K \to K'.$$

This can be easily deduced from (1, 0, 1). Moreover, it follows immediately that

(1 0, 3) *$\tau^{(i)}(J)_{\bar{k}} = \min_{k\to K} \{\tau^{(i)}(J)_K\}$, where $\bar{k}$ is the algebraic closure of k. We shall write $\tilde{\tau}^{(i)}(J)$ (resp. $\tilde{\tau}^*(J)$) for $\tau^{(i)}(J)_{\bar{k}}$ (resp. $\tau^*(J)_{\bar{k}}$).* As before, $\tilde{\tau}(J)$ denotes the last integer $\tilde{\tau}^{(t)}(J)$ of $\tilde{\tau}^*(J)$, $\tilde{\tau}^*_x(X/Z) = \tilde{\tau}^*(J)$ and $\tilde{\tau}_x(X/Z) = \tilde{\tau}(J)$.

In this paper, I propose to generalize Theorems 3 and 4 of [1], Ch. III, §5, pp. 233–234. Let Y be a closed subscheme of X such that the monoidal transformation $\pi: X' \to X$ with center Y is *permissible*, i. e.,

(1, 0, 4) *Y is irreducible and regular, and X is normally flat along Y.* (cf [1], Ch. 0, § 4, Def. 1, p. 135.)

This permissibility is a special case of [1], Ch. I, § 1, Def. 6, p. 167. The condition (1, 0, 4) at the point x can be expressed in terms of the local ring R, the ideal J and the prime ideal P of Y in R. For this, see [1], Ch. III, § 5, Def. 8, p. 226. Now, let $p: Z' \to Z$ be the monoidal transformation with center Y. Then X' can be canonically imbedded in Z' and identified with the *strict transform* of X in Z' by p. (cf [1], Ch. I, § 1, preceeding paragraphs of Def. 6, p. 167.) The goal of this paper is to prove:

Theorem (1, A). *Let the assumptions be the same as above; above all the regularity of Z and the permissibility of π. Let x' be any point of X' such that $\pi(x') = x$. Then:*

(1, A, 1) *Always $\nu^*_x(X/Z) \geq \nu^*_{x'}(X'/Z')$.*

(1, A, 2) *If the equality holds in (1, A, 1), then $\tau^{(i)}_x(X/Z)_K \leq \tau^{(i)}_{x'}(X'/Z')_K$*

for all i and for all field extensions $k \to k' \to K$, where $k \to k'$ is the canonical homomorphism of the residue fields at the points x and

x'; *in particular,*

$$\tilde{\tau}^{(i)}_{x}(X/Z) \leq \tilde{\tau}^{(i)}_{x'}(X'/Z')$$

for all i.

Remark (1, A*). We shall later see that, in the situation of (1, A, 2), if the strict inequality holds for one i then the same does for $i+1$. Intuitively speaking, (1, A) says that a *permissible* monoidal transformation does not make singularities any *worse* in terms of the numerical characters $(\nu^*, -\tau^*)$ in lexicographical ordering.

Theorem (1, B). *Let* $\{\pi_\alpha : X_{\alpha+1} \to X_\alpha\}$, $\alpha \geq 0$, *be an infinite sequence of permissible monoidal transformations, and* $\{p_\alpha : Z_{\alpha+1} \to Z_\alpha\}$ *the sequence of monoidal transformations with the same centers, where* $Z_0 = Z$ *and* $X_0 = X$. *Let* $\{x_\alpha\}$ *be an infinite sequence of points* $x_\alpha \in X_\alpha$ *with* $x_0 = x$ *and* $\pi_\alpha(x_{\alpha+1}) = x_\alpha$ *for all* $\alpha \geq 0$. *Let* k_α *be the residue field of* Z_α *at* x_α, *and let* $(\lim_\alpha k_\alpha) \to K$ *be a field extension. Then there exists an integer* $\bar{\alpha}$ *such that*

$$\begin{aligned}(\nu^*_{x_\alpha}(X_\alpha/Z_\alpha), &\quad \tau^*_{x_\alpha}(X_\alpha/Z_\alpha)_K) \\ = (\nu^*_{x_\beta}(X_\beta/Z_\beta), &\quad \tau^*_{x_\beta}(X_\beta/Z_\beta)_K)\end{aligned}$$

for all α *and* $\beta \geq \bar{\alpha}$. *The same holds if* $\tau^*(\ \ )_K$ *is replaced by* $\tilde{\tau}^*(\ )$.

Again intuitively speaking, any singularity can not be *improved* indefinitely in the sense of $(\nu^*, -\tau^*)$, so that the problem of resolution of singularities is reduced to find a finite succession of permissible monoidal transformations which actually *improve* given singularities.

Theorems (1, A) and (1, B) correspond to Theorems 3 and 4, respectively, of [1], pp. 233–234, and the essential point here is that the residue field extension $k \to k'$ is not assumed to be *separable* algebraic. (cf Proofs of Ths. 3 and 4 of [1] loc. cit.)

The proofs of (1, A) and (1, B) will be given after several lemmas below.

Lemma (1, 1). *Let* $k[\xi, \eta]$ *be a polynomial ring with* $s+r$ *indeterminates* $\xi = (\xi_1, \cdots, \xi_s)$ *and* $\eta = (\eta_1, \cdots, \eta_r)$, *where* k *is a field.*

Let g_i be a form of degree ν_i in $k[\eta]$, $1\leq i\leq m-1$, and g^* a polynomial of degree ν in $k[\xi,\eta]$ such that $g^*\in(\eta)^{\mu}k[\xi,\eta]$ with an integer $\mu\geq 0$. Let R be a localization of $k[\xi,\eta]$ by a prime ideal containing all the η_j, $1\leq j\leq r$, and Q the maximal ideal of R. If there exists $a_i\in R$, $1\leq i\leq m-1$, such that

$$(1,1,1) \qquad g^*-\sum_{i=1}^{m-1}a_ig_i\in Q^{\mu+1}$$

then there exist $b_i\in k[\xi,\eta]$, $1\leq i\leq m-1$, such that deg $b_i\leq\nu-\nu_i$, $b_i\in(\eta)^{\mu-\nu_i}k[\xi,\eta]$ for all i, and

$$(1,1,2) \qquad g^*-\sum_{i=1}^{m-1}b_ig_i\in Q^{\mu+1}$$

Moreover, if $\mu=\nu$, then we can choose such b_i so that b_i is a form of degree $\mu-\nu_i$ in $k[\eta]$ for all i, and

$$(1,1,3) \qquad g^*=\sum_{i=1}^{m-1}b_ig_i\,.$$

Proof. Both R and $R/(\eta)R$ are regular. Hence η can be extended to a regular system of parameters of R, say (η,ξ'). We shall first prove that the a_i in $(1,1,1)$ can be so modified that $a_i\in Q^{\mu-\nu_i}$ for all i. Let $\rho_i=\nu_Q(a_i)$, the Q-adic order of a_i, and let $\rho=min_i(\nu_i+\rho_i)$. Let $\tilde{k}=R/Q$, and consider the graded $\tilde{k}$-algebra $gr_Q(R)$. Let A_i be the image of a_i in $gr_Q^{\rho-\nu_i}(R)$, and G_i the initial form of g_i in $gr_Q(R)$. If $a_i\notin Q^{\mu-\nu_i}$ for at least one i, then $\rho<\mu$ and $(1,1,1)$ implies that

$$\sum_{i=1}^{m-1}A_iG_i=0$$

Since $gr_Q(R)=[gr_{(\xi',\eta)}(k[\xi',\eta])]\otimes_k\tilde{k}$ canonically, we can find a finite system $\{u_j\}$, $u_j\in R$, and forms b_{ij} in $k[\xi',\eta]$ of degree $\rho-\nu_i$ such that the images U_j of u_j in $\tilde{k}$ are k-linearly independent and that if B_{ij} is the initial form of b_{ij} in $gr_{(\xi',\eta)}(k[\xi',\eta])$, then

$$A_i=\sum_j B_{ij}U_j\,.$$

By the independency of U_j, we have

$$\sum_{i=1}^{m-1}B_{ij}G_i=0 \quad \text{for all} \quad j\,.$$

Since g_i and b_{ij} are forms in $k[\xi', \eta]$, this implies

$$\sum_{i=1}^{m-1} (\sum_j b_{ij}u_j)g_i=0 .$$

Let $a_i'=a_i-\sum_j b_{ij}u_j$, $1\leq i\leq m-1$. Then we have $a_i'\in Q^{(\rho+1)-\nu_i}$ for all i, and (1, 1, 1) holds if the a_i are replaced by these a_i'. By repeating this process (at most $\mu-\rho$ times), we can achieve the desired situation. From now on, we assume that $a_i\in Q^{\mu-\nu_i}$ for all i, $1\leq i\leq m-1$. Now, let $\bar{R}=R/(\xi')R$. We have a monomorphism $k[\eta]\rightarrow\bar{R}$ and the image of η is a regular system of parameters of $\bar{R}$. We have $\tilde{k}=\bar{R}/\bar{Q}$, where $\bar{Q}=(\eta)\bar{R}=$ the maximal ideal of $\bar{R}$. Let us choose a set of elements $\{\omega_\lambda\}$, $\omega_\lambda\in R$, such that if $\tilde{\omega}_\lambda$ is the image of ω_λ in $\tilde{k}$, then $\{\tilde{\omega}_\lambda\}$ form a free base of $\tilde{k}$ as k-module. We can choose $\{\omega_\lambda\}$ so that if ω is any monomial in ξ, then the image $\tilde{\omega}$ of ω in $\tilde{k}$ is a k-linear combination of those $\tilde{\omega}_\lambda\in\{\tilde{\omega}_\lambda\}$ such that ω_λ are monomials in ξ of degrees $\leq deg\,\omega$. We have $gr_{\bar{Q}}(\bar{R})=gr_{(\eta)}(k[\eta])\otimes_k\tilde{k}$, where $(\eta)=(\eta)k[\eta]$. Let $\tilde{g}^*$ be the image of g^* into $gr_{\bar{Q}}^{\nu}(\bar{R})$. We can write uniquely

$$\tilde{g}^*=\sum_\lambda \tilde{\omega}_\lambda\tilde{g}_\lambda^* \tag{1, 1, 4}$$

where $\tilde{g}_\lambda^*\in gr_{(\eta)}^{\mu}(k[\eta])$. Let $\tilde{a}_i$ be the image of a_i into $gr_{\bar{Q}}^{\mu-\nu_i}(\bar{R})$, and $\tilde{g}_i$ that of g_i into $gr^{\nu_i}_{(\eta)}k[\eta]$ for all i. Then, by (1, 1, 1), we get

$$\tilde{g}^*=\sum_{i=1}^{m-1} \tilde{a}_i\tilde{g}_i \tag{1, 1, 5}$$

We can write $\tilde{a}_i=\sum_\lambda \tilde{a}_{i\lambda}\tilde{\omega}_\lambda$ with $\tilde{a}_{i\lambda}\in gr_{(\eta)}(k[\eta])$. By comparing (1, 1, 4) and (1, 1, 5), we get

$$\tilde{g}_\lambda^*=\sum_{i=1}^{m-1} \tilde{a}_{i\lambda}\tilde{g}_i \tag{1, 1, 6}$$

for every λ which appears in (1, 1, 4). Since $deg\,g^*=\nu$, g^* is a linear combination of monomials in ξ if degress $\leq\nu-\mu$ with coefficients in $k[\eta]$. Hence, by the selection of $\{\omega_\lambda\}$, those ω_λ which actually appear in (1, 1, 4) are monomials in ξ of degrees $\leq\nu-\mu$. Now let $a_{i\lambda}\in k[\eta]$ be the form whose initial form in $gr_{(\eta)}(k[\eta])$ is equal to $\tilde{a}_{i\lambda}$. Let Δ be the set of those λ for which ω_λ are monomials in ξ of degrees $\leq\nu-\mu$. Define

$$(1,1,7) \qquad b_i = \sum_{\lambda\in\Delta} \omega_\lambda a_{i\lambda}$$

for each i, $1\leq i\leq m-1$. These b_i have the required properties in Lemma (1, 1). For instance, (1, 1, 2) can be shown as follows. Let $h=g^*-\sum_{i=1}^{m-1} b_i g_i$. Clearly $h\in(\eta)^\mu k[\xi, \eta]$, so that $h\in(\eta)^\mu R$. On the other hand, by (1, 1, 4)-(1, 1, 7), the image of h in $\bar{R}$ is contained in $\bar{Q}^{\mu+1}$. This means that $h\in(\eta)^{\mu+1}R+(\xi')R$. Since (ξ', η) is a regular system of parameters of R, we get

$$h\in(\eta)^\mu R\cap(\eta)^{\mu+1}R+(g')R\subset(\eta)^\mu(\eta, \xi')R.$$

In particular, $h\in Q^{\mu+1}$. Moreover, if $\mu=\nu$, then g^* is a form in $k[\eta]$. In (1, 1, 7), only $\omega_\lambda=1$ appears, and b_i is a form of degree $\mu-\nu_i$ in $k[\eta]$. Thus h is a form of degree μ in $k[\eta]$. Since η extends to a regular system of parameters of R, $h\in Q^{\mu+1}$ then implies $h=0$. Thus we get (1, 1, 3). Q.E.D.

Let A be a commutative ring. A Hasse differentiation is a sequence of endomorphisms of A as additive group (not as ring), say $\{d_\nu\}$ for $\nu\in Z_0$, such that

(1) $d_0(x)=x$,

(2) $d_\nu(xy)=\sum_{\alpha+\beta=\nu} d_\alpha(x)d_\beta(y)$, and

(3) $d_\alpha(d_\beta(x))=\binom{\alpha+\beta}{\alpha}d_{\alpha+\beta}(x)$

for all $x, y\in A$ and $\alpha, \beta\in Z_0$.

Remark (1, 2, 1). Let A be either a polynomial ring $k[x]$ or a power series ring $k[[x]]$, where k is a ring and $x=(x_1,\cdots, x_s)$ a system of indeterminates. Then there exists a natural system of Hasse differentiations

$$(d^{(1)}, d^{(2)},\cdots, d^{(s)})$$

where $d^{(i)}=\{d^{(i)}_\nu\}$, $\nu\in Z_0$, such that, t being an indeterminate over A,

$$g\in A\to\sum_\nu d^{(i)}_\nu(g)t^\nu\in A[[t]]$$

in the unique homomorphism of k-algebras φ_i: $A\to A[[t]]$ with $\varphi_i(x_j)=x_j$ for all $j\neq i$ and $\varphi_i(x_i)=x_i+t$.

Remark (1, 2, 2), Let $d=\{d_\nu\}$, $\nu\in Z_0$, be a Hasse differentiation of a ring A. Let S be a multiplicative subset of A and $A'=S^{-1}A$, the localization. Then d is uniquely extended to a Hasse differentiation of A'.

Remark (1, 2, 3). Let d and A be the same as in (1, 2, 2). Let Q be an ideal in A. Then we have

$$d_\alpha(Q^\beta)\subset Q^{\beta-\alpha}$$

for all α, $\beta\geq 0$, where $Q^{\beta-\alpha}=A$ if $\beta<\alpha$. (The proof is done by the property (2) of Hasse differentiation and by induction on the pairs (β, α). It follows that d induces a unique differentiation in the Q-adic completion of A).

If e is a k-module with a ring k, then $k[e]$ will denote the symmetric tensor k-algebra of the module e. For instance, if k is a field and e is a k-module of rank N, then $k[e]$ can be identified with the polynomial ring of N indeterminates $(x_1,\cdots,x_N)$ over k, where $(x_1,\cdots,x_N)$ is a free base of the k-module e. If $x=x_1$ for instance, then $k[x^{-1}e]$ will denote the polynomial ring $k[x_2/x_1,\cdots, x_N/x_1]$. Note that $k[e]$ has a natural structure of graded k-algebrs, while $k[x^{-1}e]$ has only a natural structure of k-algebra (without any natural grading). $k[x^{-1}e]$ is the homogeneous part of degree 0 in the localization $k[e][x^{-1}]$ of $k[e]$.

Lemma (1, 3). *Let e be a finite k-module and e' a k-submodule of e, where k is a field. Let $(h_1, h_2,\cdots, h_m)$ be a system of forms in $k[e]$, and $\nu_i=\deg h_i$, $1\leq i\leq m$. Let $x_0\in e-e'$, R a localization of $k[x_0^{-1}e]$ and Q the maximal ideal of R. Suppose*

(0) $\nu_m>0$,

(1) $h_i\in k[e']$ *for* $1\leq i\leq m-1$,

(2) $x_0^{-1}x'\in Q$ *for all* $x'\in e'$, *and*

(3) *there exist* $a_i\in R$, $1\leq i\leq m-1$, *such that*

$$g_m-\sum_{i=1}^{m-1}a_ig_i\in Q^{\nu_m}$$

where $g_i=h_ix_0^{-\nu_i}\in R$, $1\leq i\leq m$.

Then there exist forms b_i of degrees $\nu_m-\nu_i$ in $k[e]$, $1\leq i\leq m-1$, and a k-module e'' with $e\supset e''\supset e'$ such that

(a) $x_0^{-1}x''\in Q$ *for all* $x''\in e''$, *and*

(b) $h_m-\sum_{i=1}^{m-1} b_ih_i\in k[e'']$.

Proof. Let $e''=\{x\in e\,|\,x_0^{-1}x\in Q\}$. Then, by (2), we have $e\supset e''\supset e'$. Without any loss of generality, we may assume $e''=e'$. Let us pick a free base $(y_1,\cdots,y_r)$ of e'. Then $(y_1,\cdots,y_r,x_0)$ extends to a free base of e, say $(y_1,\cdots,y_r,x_0,x_1,\cdots,x_s)$. Identify $k[e]$ with the polynomial ring $k[x,y]$ with $x=(x_0,\cdots,x_s)$ and $y=(y_1,\cdots,y_r)$. In this sense, if $h\in h[e]$, $\#(h)$ will denote the number of those terms of h whose degrees in y are less than ν_m. Given h_m, consider various forms b_i of degrees $\nu_m-\nu_i$ in $k[e]$, $1\leq i\leq m-1$, and let q be the smallest integer obtainable as $q=\#(h_m-\sum_{i=1}^{m-1} b_ih_i)$. If $q=0$, then with such b_i, the assertion (b) is established. Suppose $q>0$. This will lead to a contradiction. To begin with, we may replace h_m with such a difference $h_m-\sum_{i=1}^{m-1} b_ih_i$ as above (and accordingly, a_i by $a_i-b_ix_0^{\nu_i-\nu_m}$ for all i), and assume that $q=\#(h_m)$. Let $\eta_i=y_i/x_0$, $1\leq i\leq r$, and $\xi_j=x_j/x_0$, $1\leq j\leq s$. We shall speak of grading in $k[\xi,\eta]$ in terms of $\xi=(\xi_1,\cdots,\xi_s)$ and $\eta=(\eta_1,\cdots,\eta_r)$. Let μ be the largest integer such that $g_m\in(\eta)^\mu k[\xi,\eta]$. We have $\mu<\nu_m$, because $\mu=\nu_m$ implies $q=0$. Let g' be the sum of those terms of g_m whose degrees in η are equal to μ. Then the assumption (3) implies that

$$g'-\sum_{i=1}^{m-1} a_ig_i\in Q^{\mu+1}.$$

If $g'\in k[\eta]$ and hence g' is homogeneous, then the last part of Lemma (1, 1) implies that there exist forms b_i' degrees $\mu-\nu_i$ in $k[\eta]$ such that

$$g'=\sum_{i=1}^{m-1} b_i'g_i.$$

Then, with $b_i=b_i'x_0^{\nu_m-\nu_i}$ for $1\leq i\leq m-1$, we get $\#(h_m-\sum_{i=1}^{m-1} b_ih_i)<q$, which contradicts the minimality of q. Thus $g'\notin k[\eta]$. Pick a term of g' of the maximal degree ($=\deg g'$). Write it as $c\xi^\alpha\eta^{\beta'}$

where $c\in k$, $\alpha=(\alpha_1,\cdots,\alpha_s)$ and$\beta'=(\beta_1',\cdots,\beta')._r$ $(c\neq 0)$ We then have at least one $\alpha_{i'}>0$. Pick one such i', and define $\alpha'=(\alpha_1',\cdots,\alpha_s')$ by $\alpha_j'=\alpha_j$ for $j\neq i'$ and $\alpha_{i'}'=\alpha_{i'}-1$. Let $(d^{(1)},\cdots,d^{(s)},\delta^{(1)},\cdots,\delta^{(r)})$ be the Hasse differentiations of $k[\xi,\eta]$ corresponding to the indeterminates (ξ,η) in the sense of Remark (1, 2, 1). Let $d=d_{\alpha_1'}^{(1)}\cdots d_{\alpha_s'}^{(s)}$. The extensions of those differentiations to R will be denoted by the same symbols. (cf Remark (1, 2, 2).) The assumption (3) implies, by Remark (1, 2, 3),

(1, 3, 1) $$d(g_m)-\sum_{i=1}^{m-1} d(a_i)g_i\in Q^{\nu_m-|\alpha'|}$$

where $|\alpha'|=\alpha_1'+\cdots+\alpha_s'$. Clearly d does not affect degrees in η. Hence $d(g')$ is the sum of those terms of $d(g_m)$ whose degrees in η are equal to μ. Moreover, $|\alpha'|=|\alpha|-1$, $|\alpha|+|\beta|\leq\nu_m$ and $|\beta|=\mu$. Hence $\nu_m-|\alpha'|\geq\mu+1$. Therefore (1, 3, 1) implies

$$d(g')-\sum_{i=1}^{m-1} d(a_i)g_i\in Q^{\mu+1}$$

By the selection of d, $d(g')$ has degree $=\mu+1$ and has a non-zero term $c\xi_{i'}\eta^\beta$ of degree $\mu+1$. By Lemma (1, 1), there exist $b_i'\in k[\xi,\eta]$, $1\leq i\leq m-1$, such that $\deg b_i'\leq\mu+1-\nu_i$, $b_i'\in(\eta)^{\mu-\nu_i}$ $k[\xi,\eta]$ for all i, and

(1, 3, 2) $$d(g')-\sum_{i=1}^{m-1} b_i'g_i\in Q^{\mu+1}.$$

Let g'' denote this difference.

I claim that $\xi_{i'}\eta^\beta$ has a non-zero coefficient in g'' for some β. Suppose this is not the case. Then let b_i'' be the sum of those terms of b_i' which are divisible by $\xi_{i'}$. Then $\sum_{i=1}^{m-1} b_i''g_i$ is the sum of those terms of $d(g')$ which are divisible by $\xi_{i'}$. Let $b_i^*=b_i''\xi^{\alpha'}x_0^{\nu_m-\nu_i}$, which is a form of degree $\nu_m-\nu_i$ in $k[x,y]$. $\sum_{i=1}^{m-1} b_i^*h_i$ is then equal to the sum of those terms of h_m which are divisible by $\xi^\alpha x_0^{\nu_m-\mu}$ (and have degree μ in y by the selection of μ). This sum is not zero, and we get the contradiction $\#(h_m-\sum_{i=1}^{m-1} b_i^*h_i)<q$. We concluce that $\xi_{i'}\eta^\beta$ has a non-zero coefficient in g''. Let $\delta=\delta_{\beta_1}^{(1)}\cdots\delta_{\beta_r}^{(r)}$. We have $|\beta|=\mu$ and (1, 3, 2) implies

(1, 3, 3) $$\delta(g'')\in Q.$$

(cf Remark (1, 2, 3).) We have $deg\, g'' \leq \mu+1$ and $deg\, \delta(g'') \leq 1$. But $\xi_{i'}$ has a non-zero coefficient in $\delta(g'')$. Hence $\delta(g'')$ has degree 1 and is not contained in $k[\eta]$. Therefore (1, 3, 3) implies that the element $x''=x_0\delta(g'')\in e$ has the property (a) of Lemma (1, 3). By the assumption on e' in the beginning of the proof, we then get $x''\in e'$. This is absurd because $\delta(g'')\notin k[\eta]$. Q.E.D.

Let $x\in X\subset Z$ be the same as those in the beginning of this paper. Let R(resp. M, resp. k) be the local ring (resp. maximal ideal, resp. residue field) of Z at x. By assumption,

(1, 4, 1) *R is regular.*

Let J be the ideal of X in R. Let Y be a closed subscheme of X which contains x. Let P be the ideal of Y in R. We assume that Y is regular at x, i.e.,

(1, 4, 2) *R/P is regular.*

Let $p: Z'\to Z$ be the monoidal transformation of Z with center Y, and X' the strict transform of X in Z' by p, so that p induces the monoidal transformation $\pi: X'\to X$ with center Y. Let us pick any point $x'\in X'$ such that $\pi(x')=x$ and let R' (resp. M', resp. k') be the local ring (resp. maximal ideal, resp. residue field) of Z' at x'. Let J' be the ideal of X' in R. According to the terminology of [1], Chap. III, §2, pp. 209–217 (esp. Definition 5),

(1, 4, 3) *R' is a monoidal transform of R with center P, and J' is the strict transform of J in R' (with respect to the center P).*

We want to show that if $p: Z'\to Z$ is *permissible* for X at x (or, P is a *permissible center for J* in the sense of [1], Ch. III, §5, Definition 8, p. 226), then $\nu^*_x(X/Z)\geq\nu^*_{x'}(X'/Z')$. As is shown below, this question is related to a certain question about $T(J)$ defined by: $T(J)=T^{(t)}(J)$ with $t=t(J)$, i.e.,

(1, 4, 4) *$T(J)$ is the smallest k-submodule of $gr^1_M(R)$ such that*

$$gr_M(J, R)=(gr_M(J, R)\cap k[T(J)])gr_M(R).$$

(cf [1], Chap. II, §2, p. 189, for the definition of $gr_M(J, R)$, and Chap. III, §4, Definition 6 and Lemma 10, p. 221.) We have

$rank_k\ T(J)=\tau(J)=\tau_x(X/Z)$. The question is about the inclusion relation between $T(J)$ and $E(R'/R)$, where

(1, 4, 5) *$E(R'/R)$ is the image of $PM'\cap R$ into $gr^1_M(R)$.*

The ideal $PM'\cap R$ has an obvious geometric significance. Namely, let D be any regular subscheme of Z through x. Then the strict transform of D in Z' by p goes through x' if and only if the ideal I_D of D in R is contained in $PM'\cap R$. Moreover, there exists such D (not unique) with $E(R'/R)=$ the image of I_D into $gr^1_M(R)$.

We want to prove that: *If P is a permissible center for J* (loc. cit.) *and if $\nu^*(J)\leq\nu^*(J')$, then*

(1, 5, 1) $T(J)\subset E(R'/R)$,

(1, 5, 2) *$\nu^*(J)=\nu^*(J')$, and*

(1, 5, 3) *$\tau^{(i)}(J)_K\leq\tau^{(i)}(J')_K$ for all i and for all field extensions $k\to k'\to K$.*

Theorem (1, A) will easily follow from these. But, to prove Theorem (1, B), we need more detailed results than these.

Let us pick $(f_1,\cdots,f_p)$, $f_i\in J$ and $p\geq 0$, such that

(1, 6, 1) *it extends to a standard base of J.*

(cf [1], Ch. III, §1, Def. 3, p. 208; it means that the associated system of initial forms in $gr_M(R)$ extends to a standard base of $gr_M(J, R)$ in the sense of the paragraph above (1, 0, 1).)

(1, 6, 2) *$\nu_P(f_i)=\nu_M(f_i)$ for all i, $1\leq i\leq p$, and*

(1, 6, 3) *if h_i is the initial form of f_i in $gr_M(R)$, then $h_i\in k[E(R'/R)]$ for all i, $1\leq i\leq p$.*

There exists an element $x_0\in P$ with $(x_0)R'=PR'$. Pick any such x_0. Let $\nu_i=\nu_p(f_i)$ and $g_i=x_0^{-\nu_i}f_i$, $1\leq i\leq p$. Let us choose a regular system of parameters (y, x, z) of R, where $y=(y_1,\cdots,y_r)$, $x=(x_0, x_1,\cdots,x_s)$ and $z=(z_1,\cdots,z_t)$, such that

(1, 6, 4) *the initial forms of the y_j in $gr_M(R)$ form a free base of the k-module $E(R'/R)$, and*

(1, 6, 5) $\qquad P=(y, x)R\,.$

Here x_0 may be the one picked above. Let $A=R[x_0^{-1}P]$ $=R[x_0^{-1}x_1,\cdots, x_0^{-1}x_s,\ x_0^{-1}y_1,\cdots, x_0^{-1}y_r]$. Then there exists a prime ideal N in A with $R'=A_N$. Let $y_i'=x_0^{-1}y_i$, $1\leq i\leq r$. Then (y', x_0, z) extends to a regular system of parameters (y', x_0, x', z) of R', where $x'=(x_1',\cdots, x_{s'}')$ with suitable $x_j'\in R'$ and $s'\leq s$.

Let e (resp. $\bar{x}_i$, resp. $\bar{y}_j$) be the image of P (resp. x_i, resp. y_j) into $gr_M^1(R)$. Then we have a canonical isomorphism $\varphi: A/MA \rightarrow k[e/\bar{x}_0]$, induced by the natural homomorphisms $R\rightarrow k$ and $P/x_0 \rightarrow e/\bar{x}_0$. Let $\bar{N}=N/MA$. (Recall $R'=A_N$.) Then, by localization, φ extends to a homomorphism

(1, 6, 6) $\qquad \Psi: R'\rightarrow S=k[e/\bar{x}_0]_{\varphi(\bar{N})}$

which is surjective and whose kernel is $MR'=(x_0, z)R'$. Let $(\bar{y}', \bar{x}_0', \bar{x}', \bar{z}')$ be the initial forms (all linear) of (y', x_0, x', z) in $gr_{M'}(R')$, so that $gr_{M'}(R')=k'[\bar{y}', \bar{x}_0', \bar{x}', \bar{z}']$. Ψ induces a homomorphism of graded k'-algebras

(1, 6, 6*) $\qquad \Psi^*: gr_{M'}(R')\rightarrow gr_Q(S)$

with $Q=\varphi(\bar{N})S$, which is surjective and whose kernel is $(\bar{x}_0', \bar{z}')$ $gr_{M'}(R')$. As $\bar{y}/\bar{x}_0=(\bar{y}_1/\bar{x}_0,\cdots,\bar{y}_r/\bar{x}_0)$ extends to a regular system of parameters of S, we obtain an inclusion $k[\bar{y}/\bar{x}_0]\rightarrow gr_Q(S)$ which maps each $\bar{y}_i/\bar{x}_0$ to its initial form. In this sense, we view

(1, 6, 6**) $\qquad k[\bar{y}/\bar{x}_0]\subset gr_Q(S)$

The natural inclusion $k\subset k'$ and the substitution of variables define a k-homomorphism

(1, 6, 7) $\qquad \lambda: k[\bar{y}]\rightarrow k'[\bar{y}']$

such that $\lambda(\bar{y}_j)=\bar{y}_j'$, $1\leq j\leq r$. Clearly, λ is related to ψ^* by

(1, 6, 7*) $\psi^*(\lambda(h))=h/\bar{x}_0^{\nu}$ *for every form* h *of degree* ν *in* $k[\bar{y}]$.

Let us remark

(1, 6, 8) *If* $g_i'=g_i-g_i''$ *with* $g_i''\in MR'$ *such that* $\nu_{M'}(g_i')\geq\nu_i$, *and if* h_i' *is the image of* g_i' *into* $gr_{M'}^{\nu_i}(R')$, *then*

$$h_i'=\lambda(h_i)\ mod\ (\bar{x}_0', \bar{z}')gr_{M'}(R')\,.$$

In fact, (1, 6, 3) implies that $f_i=f_i'+f_i''$ with some $f_i'\in(y)^{\nu_i}R$ and $\nu_M(f_i'')>\nu_i$. Then $x_0^{-\nu_i}f_i''\in MR'$, so that $\psi(g_i')=\psi(g_i)=\psi(x_0^{-\nu_i}f_i')$ $=h_i/\bar{x}_0^{\nu_i}=\psi^*(\lambda(h_i))$. Hence $\psi^*(h_i')=\psi^*(\lambda(h_i))$. The conclusion of (1, 6, 8) follows.

(1, 6, 9) *Let $\nu>0$ and δ_i' forms of degrees $\nu-\nu_i$ in $k'[\bar{y}']$, $1\leq i<p$, such that*

$$\sum_{i=1}^{p-1}\delta_i'\lambda(h_i)=0\,.$$

Then we can choose $d_i'\in(y')^{\nu-\nu_i}R'$ such that $\sum_{i=1}^{p-1}d_i'g_i\in MR'$ and $\delta_i'=$ the image of d_i' into $gr_{M'}^{\nu-\nu_i}(R')$. In fact, pick a free base $\{\omega_n\}$ of the k-module k'. Write $\delta_i'=\sum_n\lambda(\varepsilon_{in})\omega_n$ with $\varepsilon_{in}\in k[\bar{y}]$. Then the assumption of (1, 6, 9) implies

$$\sum_{i=1}^{p-1}\varepsilon_{in}h_i=0 \tag{1, 6, 9, 1}$$

for all n. Pick any $e_{in}\in(y)^{\nu-\nu_i}R$ such that $\varepsilon_{in}=$ the image of e_{in} into $gr_M^{\nu-\nu_i}(R)$. Also pick any $u_n\in R'$ such that $\omega_n=$ the image of u_n into k'. Let $d_i'=\sum_n(e_{in}/x_0^{\nu-\nu_i})u_n$. Then it is clear that $\delta_i'=$ the image of d_i' into $gr_{M'}^{\nu-\nu_i}(R')$ for all i. Moreover, $\sum_{i=1}^{p-1}d_i'g_i$ $=\sum_n u_nx_0^{-\nu}(\sum_{i=1}^{p-1}e_{in}f_i)$. For every n, $\sum_{i=1}^{p-1}e_{in}f_i\in M^{\nu+1}\cap P^{\nu}=MP^{\nu}$ so that $x_0^{-\nu}\sum_{i=1}^{p-1}e_{in}f_i\in MR'$. Hence $\sum_{i=1}^{p-1}d_i'g_i\in MR'$, as is claimed in (1, 6, 9).

If $\nu^{(1)}(J')\geq\nu^{(1)}(J)$, then we have $\nu_{M'}(g_1)=\nu_1$ (cf [1], Chap. III, §3, Lemma 8, p. 217) so that $\nu^{(1)}(J')=\nu^{(1)}(J)$ and g_1 extends to a standard base of J'. This fact generalizes itself to the following

Lemma (1, 7). *Assume* (1, 4, 1)-(1, 4, 5) *and* (1, 6, 1)-(1, 6, 3). *If $(\nu^{(1)}(J),\cdots,\nu^{(p)}(J))\leq(\nu^{(1)}(J'),\cdots,\nu^{(p)}(J'))$ in the lexicographical ordering, then there exist $g_i''\in(g_1,\cdots,g_{i-1})R'\cap MR'$, $1\leq i\leq p$, such that $(g_i',\cdots,g_p')$ with $g_i'=g_i-g_i''$ extends to a standard base of J'. Moreover, we have $\nu^{(i)}(J)=\nu^{(i)}(J')$ for all $i\leq p$,*

Proof. After (1, 6, 8), the proof of (1, 7) is reduced to find $g_i''\in(g_1,\cdots,g_{i-1})R'\cap MR'$ such that $\nu_{M'}(g_i')\geq\nu_i$ with $g_i'=g_i-g_i''$ for all $i\leq p$. In fact, since $(h_1,\cdots,h_p)$ is a minimal base of the ideal which it generates in $gr_M(R)$, (1, 6, 8) implies that $\nu_{M'}(g_i')=\nu_i$ for all i and that the initial forms of the g_i' in $gr_{M'}(R')$ form a minimal base of the ideal which they generate. Now, the existence of such g_i'' will be shown by induction on p. This is already done for $p=1$. Thus, assume $p>1$ and that we have found g_i'' for all $i<p$. We look for $c_j\in R'$, $1\leq j\leq p-1$, such that $g_p'' = \sum_{i=1}^{p-1} c_j g_j'\in MR'$ and $\nu_{M'}(g_p-g_p'')\geq \nu_p$. Suppose such $\{c_j\}$ does not exist. Then pick one $\{c_j^*\}$ for which $\sum_{j=1}^{p-1} c_j^* g_j'\in MR'$ and, if $g^*=g_p-\sum_{j=1}^{p-1} c_j^* g_j'$, the number $\mu=\nu_{M'}(g^*)$ is the maximum for various such $\{c_j^*\}$. So $\nu_{M'}(g^*)=\mu<\nu_p$. We have $\psi(g^*)=\psi(g_p) =\psi^*(\lambda(h_p))$. Since $deg\, h_P=\nu_p>\mu$, the initial form of g^* in $gr_{M'}(R')$, say h^*, belongs to the ideal $(\bar{x}_0'.\ \bar{z}')\, gr_{M'}(R')$. By induction assumption, $(g_1',\cdots,g_{p-1}')$ extends to a standard base of J' and $\nu^{(i)}(J) =\nu^{(i)}(J')$ for all $i\leq p-1$. Hence, the inequality assumption of (1, 7) means $\nu^{(p)}(J)\leq\nu^{(p)}(J')$. It follows that, if h_i' is the initial form of g_i' in $gr_{M'}(R')$ for $1\leq i\leq p-1$, $h^*\in(h_1',\cdots,h_{p-1}')\, gr_{M'}(R')$. Namely, there exist forms δ_i of degrees $\mu-\nu_i$ in $gr_{M'}(R')$ such that $h^*=\sum_{i=1}^{p-1}\delta_i h_i'$. By (1, 6, 8), $h_i'-\lambda(h_i)\in(\bar{x}_0',\ \bar{z}')\, gr_{M'}(R')$ where $\lambda(h_i)\in k'[\bar{y}']$. Let us write

$$\delta_i=\sum_A \delta_{iA}'\bar{x}'^A+\delta_i''$$

where $\delta_{iA}'\in k'[\bar{y}']$ for every $A=(a_1,\cdots,a_{s'})$ with non-negative integers a_j and $\delta_i''\in(\bar{x}_0',\ \bar{z}')\, gr_{M'}(R')$. We have $h^*\in(\bar{x}_0',\ \bar{z}')\, gr_{M}'(R')$. Therefore we get $\sum_{i=1}^{p-1}(\sum_A \delta_{iA}'\bar{x}'^A)\lambda(h_i)=0$, so that $\sum_{i=1}^{p-1}\delta_{iA}'\lambda(h_i) =0$ for all A. Hence, by (1, 6, 9), there exist $d_{iA}'\in(y')^{\mu-\nu_i-|A|}R'$ such that $\delta_{iA}'=$ the image of d_{iA}' into $gr_{M'}{}^{\nu-\nu_i-|A|}(R')$ and $\sum_{i=1}^{p-1} d_{iA}' g_i'\in MR'$ for all A. Let $c_j'=\sum_A d_{iA}'x'^A+d_j''$, where d_j'' is any element of $MR'=(x_0,\ z)R'$ such that $\nu_{M'}(d_j'')\geq\mu-\nu_j$ and δ_j'' is the image of d_j'' in $gr_{M'}{}^{\mu-\nu_j}(R')$. Then we get $\sum_{j=1}^{p-1} c_j' g_j'\in MR'$ and $\nu_{M'}(g^*-\sum_{j=1}^{p-1} c_j' g_j')>\mu$. This contradicts the maximality of μ. Q.E.D.

Lemma (1, 8). *Let the assumptions be the same as in Lemma* (1, 7). *Assume $\nu^{(i)}(J)=\nu^{(i)}(J')$ for all $i\leq p$. Let $f\in J$ be such that $\nu_P(f)\geq\nu$, where $0<\nu\leq Min\,(\nu^{(p+1)}(J),\ \nu^{(p+1)}(J'))$. Let $g=x_0^{-\nu}f$. Suppose :*

(1, 8, 1) *there exists $(a_1',\cdots,a_p')$ with $a_j'\in R'$ such that $g-\sum_{j=1}^{p} a_j' g_j\in M'^{\nu}$.*

Then :

(1, 8, 2) *there exists* $(b_1',\cdots,b_p')$ *with* $b_j'\in P^{\nu-\nu_j}$ *such that the image of* $f-\sum_{j=1}^{p}b_j'f_j$ *into* $gr_M^{\nu}(R)$ *is contained in* $k[E(R'/R)]$.

Proof. If $\nu_M(f)>\nu$, then (1, 8, 2) is trivially true. Hence we assume $\nu_M(f)=\nu$. Let h be the initial form of f in $gr_M(R)$. The assumption $f\in P^{\nu}$ implies that $h\in k[e]=k[\bar{x},\bar{y}]$. (See (1,6,1)-(1,6,6).) Since $Ker(\psi)=MR'$ (cf (1, 6, 6)) and $R/(MR'\cap R)=k$, we get $\psi(g)=h/\bar{x}_0^{\nu}$. Similarly, $\psi(g_i)=h_i/\bar{x}_0^{\nu_i}$, $1\leq i\leq p$. Let $a_j=\psi(a_j')$. Then (1, 8, 1) implies

$$h/\bar{x}_0^{\nu}-\sum_{j=1}^{p}a_j(h_j/\bar{x}_0^{\nu_j})\in Q^{\nu}$$

where $Q=\psi(M')=$ the maximal ideal of S. (cf (1, 6, 6) and (1, 6, 6*).) Apply Lemma (1, 3) to $\{e, E(R'/R), (h_1,\cdots,h_p,h), \nu, \bar{x}_0, S, Q, (h_1/\bar{x}_0^{\nu_1},\cdots,h_p/\bar{x}_0^{\nu_p}, h/\bar{x}_0^{\nu})\}$ (which play role of $\{e, e', (h_1,\cdots,h_m), \nu_m, x_0, R, Q, (g_1,\cdots,g_m)\}$ in Lemma (1, 3)). We then find forms b_j of degrees $\nu-\nu_j$ in $k[e]$ such that

$$h-\sum_{j=1}^{p}b_jh_j\in k[e'']$$

where $e''=\{\bar{w}\in e\,|\,\bar{w}/\bar{x}_0\in Q\}$. In view of the definition of ψ of (1, 6, 6), if $\bar{w}\in e$ is the image of $w\in P$, then $w/x_0\in M'$ is equivalent to $\bar{w}/\bar{x}_0\in Q$. (Note that $\psi(w/x_0)=\bar{w}/\bar{x}_0$.) It follows that $e''=E(R'/R)$. Now, pick $b_j'\in P^{\nu-\nu_j}$, for each j, such that b_j is the image of b_j' into $gr_M^{\nu-\nu_j}(R)$. Then the image of $f-\sum_{j=1}^{p}b_j'f_j$ in $gr_M^{\nu}(R)$ is equal to $h-\sum_{j=1}^{p}b_jh_j$, which belongs to $k[E(R'/R)]$. Q.E.D.

Lemma (1, 9). *Assume* (1, 4, 1)-(1, 4, 5), *that there exists a system* $(f_1,\cdots,f_p)$ *having the properties* (1, 6, 1) *and* (1, 6, 2), *and that* $(\nu^{(1)}(J),\cdots,\nu^{(p)}(J))\leq(\nu^{(1)}(J'),\cdots,\nu^{(p)}(J'))$ *in the lexicographical ordering. Then there exist* $b'_{ij}\in P^{\nu_i-\nu_j}$ *for* $1\leq j\leq i-1$ *and* $1\leq i\leq p$, *such that* $(f_i',\cdots,f_p')$ *with* $f_i'=f_i-\sum_{j=1}^{i-1}b'_{ij}f_j$ *has the properties* (1, 6, 1)-(1, 6, 3). *Moreover, it follows that* $\nu^{(j)}(J)=\nu^{(j)}(J')$ *for all* $j\leq p$.

Proof. I shall prove the assertion by induction on p. If $p=0$, it is trivially true. Assume $p\geq 1$ and that $(f_1,\cdots,f_{p-1})$ has the properties (1, 6, 1)–(1, 6, 3). By Lemma (1, 7), $\nu^{(j)}(J)=\nu^{(j)}(J')$ for all $j\leq p-1$. Apply Lemma (1, 8) to $f=f_p$ and $\nu=\nu_p$ (p playing the role of $p+1$ in (1, 8)). Here, thanks to Lemma (1, 7) applied to $(g_1,\cdots,g_{p-1})$, the inequality $\nu^{(p)}(J)\leq\nu^{(p)}(J')$ implies the assumption (1, 8, 1) of Lemma (1, 8). Thus, as is in (1, 8, 2), there exists $(b_1',\cdots,b_{p-1}')$ with $b_j'\in P^{\nu_p-\nu_j}$ such that if $f_p'=f_p-\sum_{j=1}^{p-1}b_j'f_j$, then the image of f_p' in $gr_M^{\nu_p}(R)$ is in $k[E(R'/E)]$. Since $(f_1,\cdots,f_p)$ extends to a standard base of J, $\nu_M(f_p')=\nu_p$ and $(f_1,\cdots,f_{p-1},f_p')$ extends to a standard base of J, that satisfies (1, 6, 1)–(1, 6, 3). Q.E.D.

Let $G=k[\eta]$, a polynomial ring over a field k which is naturally graded. Denote by G_μ its homogeneous part of degree μ. Let $\eta=(\eta_1,\cdots,\eta_r)$. We have defined *the E-function* $A^r: G\to Z_0^r$ *with respect to* $(k;\eta)$, and *the E-set* $E^r(H)$ *of any homogeneous ideal* H in G *with respect to* $(k;\eta)$. (See [1], Ch. III, §7, p. 245). Here Z_0^r denote the set of r-tuples of non-negative integers, and for a form φ in G, $A^r(\varphi)$ is the largest $A\in Z_0^r$ in the lexicographical ordering such that the monomial η^A appears with non-zero coefficient in φ. $E^r(H)$ is the set of all $A^r(\varphi)$ with forms $\varphi\in H$. If $(\varphi_1,\varphi_2,\cdots,\varphi_m)$ is any system of forms in G, then we ask whether there exists $A\in E^r((\varphi_1,\cdots,\varphi_{m-1})G)$ such that η^A has non-zero coefficient in φ_m. If the answer is negative, then we say that φ_m is *normalized by* $(\varphi_1,\cdots,\varphi_{m-1})$ *with respect to* $(k;\eta)$. We say that $(\varphi_1,\cdots,\varphi_m)$ is *a normalized standard base* of H *with respect to* $(k;\eta)$, if it is a standard base of H in the sense of the paragraph preceeding (1, 0, 1) and if, at the same time φ_i is normalized by $(\varphi_1,\cdots,\varphi_{i-1})$ with respect to $(k;\eta)$ for all $i\leq m$. (Compare these terminologies with those of [1], Ch. III, §7, Def. 9, p. 248.)

Lemma (1, 10). *Let $G=k[\eta]$ and $H\subset G$ be the same as above. Let $(\varphi_1,\cdots,\varphi_m)$ be a normalized standard base of H with respect to $(k;\eta)$. Let T be any k-submodule of G_1. Then*

(1, 10, 1) $$H=(H\cap k[T])G$$

if and onty if $\varphi_i\in k[T]$ for all i.

Proof. The if-part is trivial. To prove the converse, let us choose a free base $\xi=(\xi_1,\cdots,\xi_\tau)$ of T such that

(1, 10, 2) *if* $A(j)=A^r(\bar{\zeta}_j)$ *for* $1\leq j\leq\tau$, *then* $A(1)>A(2)>\cdots>A(\tau)$, *and*

(1, 10, 3) *for each pair* (i, j), $i<j$, *the coefficient of* $\eta^{A(j)}$ (*resp.* $\eta^{A(i)}$) *in the linear form* $\bar{\zeta}_i\in k[\eta]$ *is* 0 (*resp.* 1).

Let $c(j)$ be the index with $\eta_{c(j)}=\eta^{A(j)}$ for each j. Let $\xi=(\xi_1,\cdots,\xi_{r-\tau})$ be the complementary system of $(\eta_{c(1)},\cdots,\eta_{c(\tau)})$ in η, so that we have a sequence of indices $a(1)<a(2)<\cdots<a(r-\tau)$ with $\xi_q=\eta_{a(q)}$ for all q. Let $\bar{\eta}$ be the system obtained by replacing $\eta_{c(p)}$ by $\bar{\zeta}_p$ in the system η for $1\leq p\leq\tau$. Clearly $k[\eta]=k[\bar{\eta}]$. Let us write ζ_p for $\eta_{c(p)}$. By (1, 10, 3), we get

(1, 10, 4) *for every* p, $\bar{\zeta}_p=\zeta_p+\Sigma_q h_{pq}\xi_q, h_{pq}\in k$, *where* q *runs through those* q *with* $c(p)<a(q)$. *In particular*, $A^r(\bar{\zeta}_p)=A^r(\zeta_p)$. Since $\bar{\eta}_i=\eta_i+\sum_{j>i}f_{ij}\eta_j$ with $f_{ij}\in k$ for all i, we see that

(1, 10, 5) *if* $\bar{A}^r$ *denotes the E-function with respect to* $(k;\bar{\eta})$ (*in the same way as* A^r *is to* $(k;\eta)$), *then* $A^r(\varphi)=\bar{A}^r(\varphi)$ *for all forms* $\varphi\in G$. Now, suppose $\varphi_p\notin k[T]$ for at least one p. Let p denote the smallest index with this property. Let E' be the E-set of the ideal $(\varphi_1,\cdots,\varphi_{p-1})G$ with respect to $(k;\eta)$ and E_0' be the subset of E' which consists of those $A^r(\varphi)$ with forms $\varphi\in(\varphi_1,\cdots,\varphi_{p-1})\,k[T]$. G being a free $k[T]$-module generated by the monomials in ξ, it is easy to prove

(1, 10, 6) *every* $A\in E'$ *is of the form* $A'+B$ *with* $A'\in E_0'$ *and* $B\in Z_0^r$. *This* B *may be so chosen that* η^B *is a monomial in* ξ. Let us write $\varphi_p=\varphi'+\psi$ with $\varphi'\in k[T]$ and $\psi\in(\xi)G$. By (1, 10, 1), we have $\psi\in(\varphi_1,\cdots,\varphi_{p-1})G$. Since $\varphi_p\notin k[T]$, $\psi\neq 0$. Let $A=A^r(\psi)$, which is in E'. Let us write φ' as a form in $\bar{\zeta}$, say

$$\varphi'=\sum_D g_D\bar{\zeta}^D,\ g_D\in k \quad\text{and}\quad D\in Z_0^\tau,$$

where $|D|=deg\,\varphi_p=deg\,\varphi'$. For each D, we shall write D^* for the unique element of Z_0^r with $\bar{\zeta}^D=\bar{\eta}^{D*}$. Following (1, 10, 4), we can write $\bar{\zeta}=\zeta+h\xi$. Since $A\in E'$ and, by assumption φ_p is normalized by $(\varphi_1,\cdots,\varphi_{p-1})$ with respect to $(k;\eta)$, the equality $\varphi_p=\varphi'+\psi$ implies that, φ' and ψ being viewed as forms in η, the term of η^A in ψ must cancel with the same in φ'. This term in φ' should arise from the binomial expansion of $(\zeta+h\xi)^D$ for at least one D. Thus there exists at least one D such that $g_D \neq 0$ and $D^*-A' \in Z_0^r$, where A' is the element of E_0' such that $\eta^{A-A'}$ is a monomial in ξ. (cf (1, 10, 6).) Since $\psi \in (\xi)G$, $D^* \neq A'$ and hence $D^* \neq A$. But $D^*-A' \in Z_0^r$ implies $D^* \in E'$. By (1, 10, 4), we must have $D^* > A$. This implies that the term of η^{D^*} in φ' can only cancel with the same in φ_p. Namely, η^{D^*} has non-zero coefficient in φ_p. But this contradicts the assumption that φ_p is normalized by $(\varphi_1, \cdots, \varphi_{p-1})$ with respect to $(k; \eta)$. Q.E.D.

Lemma (1, 11). *Let $G=k[\eta]$ be the same as in Lemma* (1, 10). *Let $(\psi_1, \cdots, \psi_n)$ be a system of forms in G, and $\nu_j = deg\, \psi_j$ for $1 \leq j \leq n$. Then there exist forms c_j of degrees $\nu_n - \nu_j$ in G for $1 \leq j \leq n-1$, such that $\psi_n - \sum_{j=1}^{n-1} c_j \psi_j$ is normalized by $(\psi_1, \cdots, \psi_{n-1})$ with respect to $(k; \eta)$.* (cf [1], Ch, III, § 7, Lemma 17, p. 246).

Proof. Assume that ψ_n is not normalized by $(\psi_1, \cdots, \psi_{n-1})$ with respect to $(k; \eta)$. Let $\bar{A}=\bar{A}(\psi_n)$ be the largest element in Z_0^r such that $|\bar{A}|=\nu_n$, that $\bar{A} \in E^r((\psi_1, \cdots, \psi_{n-1})G)$ and that $\eta^{\bar{A}}$ appears with a non-zero coefficient in ψ_n. Then the existence of those c_j will be proven by induction on $\bar{A}$. Since $\bar{A} \in E^r((\psi_1, \cdots, \psi_{n-1})G)$, there exists a form φ of degree $\nu_n \in (\psi_1, \cdots, \psi_{n-1})G$ such that $A^r(\varphi)=\bar{A}$. Then clearly we can find a non-zero element $\alpha \in k$ such that the coefficient of $\eta^{\bar{A}}$ in $\psi_n - \alpha\varphi$ is zero. We then have $\bar{A}(\psi_n - \alpha\varphi) < \bar{A}$. Q.E.D.

Back to the preceeding situation, we can improve Lemma (1, 9) as follows.

Lemma (1, 12). *Let the assumptions be the same as in Lemma* (1, 9), *and pick any $(f_1^*, \cdots, f_p^*)$ having the properties* (1, 6, 1) *and* (1, 6, 2). *Then we can find $b_{ij}^* \in P^{\nu_i - \nu_j}$, $1 \leq j \leq i-1$ and $1 \leq i \leq p$, such that $(f_1, \cdots, f_p)$ with $f_i = f_i^* - \sum_{j=1}^{i-1} b_{ij}^* f_j^*$ has the properties* (1, 6, 1)-(1, 6, 3) *and also the following*:

(1, 12, 1) *$\overline{w}$ being an arbitrary but fixed free base of the k-module $T^{(q)}(J)$, where q is the integer with $\mu^{(q)}(J)=\nu^{(p)}(J)$, the initial forms h_i of f_i in $gr_M(R)$ belong to $k[\overline{w}]$ and the system $(h_1, \cdots, h_p)$ is a* *normalized standard base of the ideal* $(h_1,\cdots,h_p)k[\bar{w}]$ *with respect to* $(k;\bar{w})$.

Proof. By (1, 9), we shall assume from the beginning that $(f_1^*,\cdots,f_p^*)$ has the properties (1, 6, 1)-(1, 6, 3). The lemma will be proven by induction on p. It is trivially true for $p=0$. Assume $p>0$ and that $(f_1^*,\cdots,f_{p-1}^*)$ has the property (1, 12, 1). Thus we let $f_i=f_i^*$ for all $i\leq p-1$. Let us extend $\bar{w}$ to a free base $\bar{v}$ of e (= the image of P into $gr_M^1(R)$). Then, by (1, 11), there exist forms c_j of degrees $\nu_p-\nu_j$ in $k[\bar{v}]$, $1\leq j\leq p-1$, such that if h_p^* is the initial form of f_p^* in $gr_M(R)$, then $h_p^*-\sum_{j=1}^{p-1}c_jh_j$ is normalized by $(h_1,\cdots,h_{p-1})$ with respect to $(k;\bar{v})$. Call it h_p. By (1, 10) and by (1, 0, 1) with $K=k$, we get $h_p\in k[\bar{w}]$. It is then easy to prove that h_p is normalized by $(h_1,\cdots,h_{p-1})$ with respect to $(k;\bar{w})$, too. (cf [1], Ch. III, § 7, Remark 2, p. 245.) Pick any $b_{pj}^*\in P^{\nu_p-\nu_j}$ such that c_j is the initial form of b_{pj}^* in $gr_M(R)$. Let $f_p=f_p^*-\sum_{j=1}^{p-1}b_{pj}^*f_j$. Then h_p is the initial form of f_p in $gr_M(R)$ and $(f_1,\cdots,f_p)$ has the property (1, 12, 1). Clearly it has the properties (1, 6, 1)-(1, 6, 2). We have only to check (1, 6, 3), or, that $h_p\in k[E(R'/R)]$. We have $(h_1,\cdots,h_{p-1},h_p^*)k[e]=(h_1,\cdots,h_p)k[e]$, $h_i\in k[E(R'/R)]$ for all $i\leq p-1$ and $h_p^*\in k[E(R'/R)]$. Moreover, $(h_1,\cdots,h_p)$ is a normalized standard base of $(h_1,\cdots,h_p)k[e]$ with respect to $(k;\bar{v})$. Thus $h_p\in k[E(R'/R)]$ follows from (1, 10) applied to $H=(h_1,\cdots,h_p)k[e]$ and $T=E(R'/R)$. Q.E.D.

Lemma (1, 13). *In addition to the assumptions of* (1, 9), *we assume that* p *is such that* $\nu^{(p)}(J)<\nu^{(p+1)}(J)$. *Let* q *be the integer such that* $\mu^{(q)}(J)=\nu^{(p)}(J)$. *Then we have*

$$T^{(q)}(J)\subset E(R'/R).$$

Proof. By (1, 12), we have $(f_1,\cdots,f_p)$ having (1, 6, 1)-(1, 6, 3) and (1, 12, 1). By the assumption on p, (1, 12, 1) implies that $T^{(q)}(J)$ is the smallest k-submodule of $gr_M^1(R)$ generating all the initial forms h_j of f_j. But, by (1, 6, 3), $h_j\in k[E(R'/R)]$ for all j. Hence $T^{(q)}(J)\subset E(R'/R)$.

Lemma (1, 14). *Let the assumptions be the same as in* (1, 13). *Let* $k\to k'\to K$ *be any field extension, where* $k\to k'$ *is the canonical* *homomorphism of the residue fields of R and R'. Let $T=T^{(q)}(J)_K$ and $T'=T^{(q)}(J')_K$. Then we have*

(1, 14, 1) *$\tau^{(a)}(J)_K \leq \tau^{(a)}(J')_K$ for all $a \leq q$, and*

(1, 14, 2) *if $\tau^{(q)}(J)_K=\tau^{(q)}(J')_K$ and if $\nu^{(p)}(J')<\nu^{(p+1)}(J')$, there exists an isomorphism of graded K-algebras*

$$\sigma: K[T] \to K[T']$$

such that, for all $\mu \leq \mu^{(q)}(J)=\mu^{(q)}(J')$,

$$gr^{\mu}_{M'}(J', R')_K = gr^{\mu}_{M'}(R')_K \cap \sigma\{gr_M(J, R)_K \cap K[T]\}\, gr_{M'}(R')_K.$$

Proof. By (1, 12), we have a system $(f_1, \cdots, f_p)$ having (1, 6, 1)–(1, 6, 3) and (1, 12, 1). (Here we fix a free base $\bar{w}$ of $T^{(q)}(J)$. We shall follow the notation of (1, 6, 4)–(1, 6, 8), under the following additional assumptions.

(1, 14, 3) *$y=(y_1, \cdots, y_r)$ is so chosen that, if $\tau=rank_k\,(T^{(q)}(J))$, then $\bar{w}=(\bar{y}_1, \cdots, \bar{y}_\tau)$, and*

(1, 14, 4) *we choose once for all the $g_i'' \in MR'$ of* (1, 6, 8) *so that $(g_1', \cdots, g_p')$ extends to a standard base of $gr_{M'}(J', R')$.*

Here (1, 14, 3) is possible by (1, 13) and (1, 14, 4) by (1, 7). Let ρ be the endomorphism of the graded K-algebra $gr_{M'}(R')_K$ such that $\rho(\bar{x}_0')=\rho(\bar{z}_j')=0$ for all j, $\rho(\bar{y}_i')=\bar{y}_i'$ and $\rho(\bar{x}_i')=\bar{x}_i'$ for all $i \geq 1$. Then, by (1, 6, 8), we have $\rho(h_a')=\lambda(h_a)$ for all a. If $\bar{T}=\rho(T')$, then, by definition of $T^{(q)}(J')$, we must have $H=(H \cap K[\bar{T}])K[\bar{y}', \bar{x}']$ where $H=(\lambda(h_1), \cdots, \lambda(h_p))K[\bar{y}', \bar{x}']$. By (1, 12, 1) and (1, 14, 3), $(h_1, \cdots, h_p)$ is a normalized standard base of $(h)k[\bar{y}]$ with respect to $(k; \bar{y})$, and hence, as is easily shown $(\lambda(h_1), \cdots, \lambda(h_p))$ is a normalized standard base of H with respect to $(K; \bar{y}', \bar{x}')$. It follows, by (1, 10) that $\lambda(h_i) \in K[\bar{T}]$ for all i. Moreover, by the same reason, we get $\lambda(h_i) \in K[\lambda_K(T)]$ for all i, where $\lambda_K: K[\bar{y}] \to K[\bar{y}']$ is the base extension of λ. By the definition of $T=T^{(q)}(J)_K$, $\lambda_K(T)$ must be the smallest K-submodule of $K[\bar{y}']$ which contains all the $\lambda(h_i)$. Therefore $\lambda_K(T) \subset \bar{T}$. Hence $\tau^{(q)}(J)_K = rank_K\, T = rank_K\, \lambda_K(T) \leq rank_K(\bar{T}) \leq rank_K\, T' = \tau^{(q)}(J')_K$, which proves (1, 14, 1) for $a=q$ (and similarly for all $a \leq q$ because the assumptions for the given integer q include the same for smaller integers). If the equality holds, we must have $\lambda_K(T)=\bar{T}$ and $Ker(\rho)\cap T'=(0)$. Hence ρ induces an isomorphism $\sigma_0: T\rightarrow T'$ such that $\rho\sigma_0=\lambda_K|T$. Let $\sigma: K[T]\rightarrow K[T']$ be the unique isomorphism of K-algebras which induces σ_0. I claim that this σ has the property of (1, 14, 2). (The proof of this is quite similar to that of [1], Ch. III, §5, Lemma 16, p. 232.) In fact, since ρ induces an isomorphism in T', we must find forms γ_{ij} of degrees $\nu_i-\nu_j$ in $Ker(\rho)$ such that

$$h_i''=h_i'-\sum_{j=1}^{i-1}\gamma_{ij}h_j'\in K[T']$$

for all i. Then $\rho(h_i'')=\rho(h_i')=\lambda_K(h_i)=\rho\sigma(h_i)$ for all i. Since ρ induces an isomorphism in $K[T']$, we must have $h_i''=\sigma(h_i)$ for all i. In other words, $(\sigma(h_1),\cdots,\sigma(h_p))$ generates the same ideal in $gr_{M'}(R')_K$ as $(h_1',\cdots,h_p')$ does. (1, 14,2) follows immediately. Q.E.D.

Lemma (1, 15). *Let the assumptions be the same as in Lemma* (1, 14), *excluding* (1, 14, 2). *Let $k\rightarrow k'\rightarrow K\rightarrow L$ be any field extensions. We have*:

(1, 15, 1) *If $\tau^{(q)}(J)_K=\tau^{(q)}(J')_K$, then $\tau^{(a)}(J)_K=\tau^{(a)}(J')_K$ for all $a<q$.*

(1, 15, 2) *If $\tau^{(q)}(J)_K=\tau^{(q)}(J')_K$, then $\tau^{(q)}(J)_L=\tau^{(q)}(J')_L$, provided $\nu^{(p)}(J')<\nu^{(p+1)}(J')$.*

Proof. In the proof of (1, 14), it was shown (near its end) that $(\sigma(h_1),\cdots,\sigma(h_p))$ generates the same ideal in $gr_{M'}(R')_K$ as $(h_1',\cdots,h_p')$, provided $\tau^{(q)}(J)_K=\tau^{(q)}(J')_K$. It follows that the equality of (1, 14, 2) holds for all $\mu\leq\mu^{(q-1)}(J')$. This implies (1, 15, 1) for all $a<q$. The above fact about $\sigma(h_j)$ and h_j' remains the same after the base extension $K\rightarrow L$, which shows (1, 15, 2). Q.E.D.

We are now ready to prove the theorems stated early in this paper.

Proof of Th. (1, A). Under the assumptions of (1, A), we may follow the notation (and the assumption) of (1, 4, 1)-(1, 4, 5). The permissibility of the monoidal transformation $\pi: X'\rightarrow X$ implies that P is a permissible center for J. (cf [1], Ch. III, §5, Def. 8, p. 226.) This means that we have a standard base $(f_1,\cdots,f_m)$ of J having the properties (1, 6, 1) and (1, 6, 2) for $p=m$. If $\nu_x^*(X/Z)\leq\nu_{x'}^*(X'/Z')$, then, by (1, 9), we get $\nu_x^*(X/Z)=\nu_{x'}^*(X'/Z')$. Thus (1, A, 1) of the theorem follows. Moreover, in this case, the assumptions of (1, 13) are all satisfied and therefore, by (1, 14), we get $\tau_x^{(i)}(X/Z)_K\leq\tau_{x'}^{(i)}(X'/Z')_K$ for all i and all field extension $k\to k'\to K$. We thus have (1, A, 2) except for the assertion on $\tilde{\tau}^{(i)}$. But this is only a special case of the above inequalities, because $\tau^{(i)}(\ )_K$ remains unchanged if K is replaced by any separable extension. Q.E.D.

Proof of Th. (1, B). Let R_α (resp. M_α, resp. k_α) be the local ring (resp. maximal ideal, resp. residue field) of Z_α at x_α, and let J_α be the ideal of X_α in R_α. Let P_α be the prime ideal of the center of p_α in R_α. With these notations, the rest of the proof of (1, B) is quite similar to the proof of [1], Ch. III, §5, Th. 4, p. 234. Namely, let $\bar{\nu}_i=\lim\limits_\alpha \nu^{(i)}(J_\alpha)$ and $\bar{\mu}_j=\lim\limits_\alpha \mu^{(j)}(J_\alpha)$. These limits exist by Th. (1, A). For each j, let $b(j)$ be the largest integer such that $\bar{\nu}_{b(j)}=\bar{\mu}_j$. Let $\alpha(j)$ be the smallest integer such that, for all $\alpha\geq\alpha(j)$, we have $\bar{\nu}_i=\nu^{(i)}(J_\alpha)$ for all $i\leq b(j)$ and $\tau^{(a)}(J_\alpha)_K=\tau^{(a)}(J_{\alpha+1})_K$ for all $a\leq j$. Such $\alpha(j)$ exists by the existence of $(\bar{\nu}_i, \bar{\mu}_j, b(j))$ and by (1, 14, 1). Clearly the sequence $\{\alpha(j)\}$ is monotone non-decreasing. Let $T_\alpha^{(j)}=T^{(j)}(J_\alpha)_K$ by the given homomorphism $k_\alpha\to K$. Then by (1, 14, 2), we find an isomorphism

$$\sigma_\alpha: K[T_\alpha^{(j)}]\to K[T_{\alpha+1}^{(j)}]$$

such that, if $G(\beta)=gr_{M_\beta}(R_\beta)_K$ and $H(\beta)=gr_{M_\beta}(J_\beta, R_\beta)_K$, then

$$H(\alpha+1)_\mu=G(\alpha+1)_\mu\cap\sigma_\alpha\{H(\alpha)\cap K[T_\alpha^{(j)}]\}G(\alpha+1)$$

for all $\mu\leq\bar{\mu}_j$, where (α, j) is any pair such that $\alpha(j+1)>\alpha\geq\alpha(j)$. Let us modify $\{\sigma_\alpha\}$ as follows: If $\alpha(j+1)>\alpha+1>\alpha\geq\alpha(j)$, then $\bar{\sigma}_\alpha=\sigma_\alpha$; and if $\alpha(j'+1)>\alpha(j')=\alpha(j+1)=\alpha+1>\alpha\geq\alpha(j)$, then

$$\bar{\sigma}_\alpha: K[T_\alpha^{(j)}]\to K[T_{\alpha+1}^{(j')}]$$

which is the composition of σ_α and the canonical inclusion $K[T_{\alpha+1}^{(j)}]$

$\subset K[T_{\alpha+1}^{(j')}]$. Let $K[T]$ be the limit ring of the inductive system $\{\sigma_\alpha\}$, which is a polynomial ring over K. The number of variables in $K[T]$ is bounded by the dimension of R. For each β, let $j(\beta)$ is the largest integer such that $\beta \geq \alpha(j(\beta))$, provided it exists at all. Clearly $j(\beta)$ exists for every $\beta \gg 0$. For every β for which $j(\beta)$ exists, let $\bar{H}(\beta)$ be the ideal in $K[T]$ which is generated by the canonical images of

$$H(\beta)_\mu \cap K[T_\beta^{(j(\beta))}]$$

for all $\mu \leq \bar{\mu}_{j(\beta)}$. Then we get $\bar{H}(\beta) \subset \bar{H}(\beta+1)$ for all $\beta \gg 0$. Since $K[T]$ is noetherian, $\{\bar{H}(\beta)\}$ should be stationary. It follows that there should be only a finite number of $\bar{\mu}_j$. For each j, there can be only a finite number of indices i with $\bar{\nu}_i = \bar{\mu}_j$. In fact, the number is bounded by the rank of the homogeneous part of degree $\bar{\mu}_j$ of any of $gr_{M_\alpha}(R_\alpha)$. Thus the lengths of the integer-valued portions of the $\nu^*(J_\alpha)$ for various α are bounded. It follows immediately that there exists $\bar{\alpha}' \geq 0$ such that

$$\nu^*(J_\alpha) = \nu^*(J_\beta)$$

for all α, $\beta \geq \bar{\alpha}'$. Now, (1, B) is immediate from (1, A, 2). Q.E.D.

Note that (1, 5, 1) is an immediate consequence of (1, 13). Because of its special importance, we reclaim (1, 5, 1) as follows.

Theorem (1, *C*). *Let the assumptions be the same as those of* (1, A). *If* $\nu_x^*(X/Z) = \nu_{x'}^*(X'/Z')$, *then*

$$T(J) \subset E(R'/R)$$

where R (*resp.* R') *is the local ring of* Z *at* x (*resp.* Z' *at* x') *and* J *the ideal of* X *in* R. *Here* $T(J)$ *is defined by* (1, 4, 4) *and* $E(R'/R)$ *by* (1, 4, 5).

The importance of this thcorem is seen even in the following special situations.

Corollary (1, *C*, 1). *Suppose* $\tau_x(X/Z) = dim_x Z$ (*i. e.*, $rank_k\ T(J) = dim\ R$). *Then, if the permissible center* Y *contains* x, Y *must be the closure of the point* x *and* $\nu_x^*(X/Z) > \nu_{x''}^*(X'/Z')$ *for all points*

x'' of X' with $\pi(x'')=x$.

Corollary (1, C, 2). *Suppose $\tau_x(X/Z)+1=dim_x Z$. If $\nu_x^*(X/Z) = \nu_{x'}^*(X'/Z')$ and if Y contains x, then Y must be the closure of the point x and the residue field extension $k \to k'$ is trivial, i.e., $k=k'$. Moreover, in this case $\tau_x^{(i)}(X/Z) \leq \tau_{x'}^{(i)}(X'/Z')$ for all i.*

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