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On the Riemann's relation on symmetric open Riemann surfaces

By

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§1. Introduction.

Let W be an arbitrary Riemann surface and $\{F_n\}$ its exhaustion by regular regions, then there exists on W a canonical homology basis $\{A_i, B_i\}$ of A-type with respect to $\{F_n\}$ such that $A_1, B_1, \cdots, A_{k(n)}, B_{k(n)}$ form a canonical homology basis of F_n mod ∂F_n (Ahlfors (3)). We say that such a basis belongs to $C.H.B.(F_n)_A$ and we denote it by $\{A_i, B_i\} \in C.H.B.(F_n)_A$. Let Γ_h be the Hilbert space of square integrable harmonic differentials defined on W and Γ_1, Γ_2 be its subspaces.

Definition 1 (Accola (1)). *We say that the special bilinear relation holds between ω_1 and ω_2 if we have*

$$(1, 1) \quad (\omega_1, \omega_2{}^*) = \lim_{n\to\infty} \sum_{k=1}^{k(n)} \left(\int_{A_k} \omega_1 \int_{B_k} \bar{\omega}_2 - \int_{A_k} \bar{\omega}_2 \int_{B_k} \omega_1 \right) \quad (a\ \textit{finite sum})$$

for ω_2 and ω_1 (ω_1 with only a finite number of non vanishing periods) where $\omega_2{}^*$ denotes the conjugate differential of ω_2. In the same way we say that the special bilinear relations hold between Γ_1 and Γ_2 if the special bilinear relations hold between all $\omega_1 \in \Gamma_1$ and $\omega_2 \in \Gamma_2$.

Definition 2 (Accola (1)). *We say that the bilinear relations in Accola's sense hold between Γ_1 and ω_2 with respect to $\{F_n\}$ and $\{A_i, B_i\} \in C.H.B.(F_n)_A$ if we have*

$$(1, 2) \qquad (\omega_1, \omega_2{}^*) = \lim_{n\to\infty} \sum_{k=1}^{k(n)} \left(\int_{A_k} \omega_1 \int_{B_k} \bar{\omega}_2 - \int_{A_k} \bar{\omega}_2 \int_{B_k} \omega_1 \right)$$

for ω_2 *and any* $\omega_1 \in \Gamma_1$. *In the analogous way we say that the bilinear relations in Accola's sense hold between* Γ_1 *and* Γ_2 *with respect to* $\{F_n\}$ *and* $\{A_i, B_i\}$ *if we have* (1, 2) *for* Γ_1 *and all* $\omega_2 \in \Gamma_2$.

Definition 3 (Kusunoki (6)). *The exhaustion* $\{F_n\}$ *and* $\{A_i, B_i\}$ $\in C.H.B.(F_n)_A$ *are same as those given in Definition* 2. *If, for given differentials* ω_1 *and* ω_2, *there exists a regular exhaustion* $\{W_\mu\}$ *such that*

$$(1, 3)\qquad \begin{aligned} &1)\quad \{A_i, B_i\} \in C.H.B.(W_\mu)_A \text{ and} \\ &2)\quad (\omega_1, \omega_2^*) = \lim_{\mu\to\infty} \sum_{k=1}^{k(\mu)} \left(\int_{A_k} \omega_1 \int_{B_k} \overline{\omega}_2 - \int_{A_k} \overline{\omega}_2 \int_{B_k} \omega_1 \right), \end{aligned}$$

where $k(\mu)$ *is the genus of* W_μ, *then we say that the bilinear relation in Kusunoki's sense holds between*ω_1 *and* ω_2.

The validity of the bilinear relation in Kusunoki's sense has been considered mainly in case that $\{F_n\}$ is canonical and $\omega_1, \omega_2 \in \Gamma_{hse}$ where Γ_{hse} denotes the class of semiexact harmonic differentials with finite norm (Accola (1), Kobori and Sainouchi (5) and Kusunoki (6)). On the other hand, in case $\{F_n\}$ is not canonical, it was considered by Pfluger (10) and Kobori and Sainouchi (5). The main purpose of this paper is to extend the Theorem of Pfluger (10) to the abstract symmetric Riemann surfaces that belong to the class O_{KD}. In § 2, we shall consider the special bilinear relations on general open Riemann surfaces. In § 3, for the special choice of canonical homology basis the bilinear relations in Kusunoki's sense are discussed on symmetric open Riemann surfaces. In § 4, we shall give a condition that assures the validity of the bilinear relations in Accola's sense on symmetric surfaces. In § 5, on symmetric surfaces we discuss the general period relations in Sainouchi's sense. In this paper we shall use the same notations and terminologies as in Ahlfors and Sario (4).

§ 2. Special bilinear relation.

Lemma 2, 1. *Let* Γ_1 *be a subspace in* Γ_{hse}, *then there exists a subspace* Γ_2 *of* Γ_h *with the property that the special bilinear relations hold between* Γ_1 *and* Γ_2, *but not between* Γ_1 *and any subspace con-*

taining Γ_2, *that is, if* $\omega_2 \in \Gamma_2$ *then there exists a* $\omega_1 \in \Gamma_1$ *such that the special bilinear relation does not hold between* ω_1 *and* ω_2. *We call such a space* Γ_2 *the maximal space associated with* Γ_1 *and denote it by* $M(\Gamma_1)$.

Proof. Let ω_α be a differential with a finite number of non vanishing periods and $\omega_\alpha \in \Gamma_1$. The set of ω_2 such that the special bilinear relation between ω_α and ω_2 holds is obviously a linear closed space and we denote it by Γ_α. Then $M(\Gamma_1)$ is equal to the space $\bigcap_\alpha \Gamma_\alpha$ where ω_α ranges over that class.

Theorem 2, 1 (Matsui (8)). (i) $M(\Gamma_{hse}) = \Gamma_{h0}$, (ii) $M(\Gamma_{h0}) = Cl(\Gamma_{h0} + \Gamma_{he})$, (iii) $M(\Gamma_s \cap \Gamma_{hse}) = \Gamma_{h0} \dot{+} \Gamma_{he} \cap \Gamma_{he}^*$.

Proof. For example we prove the first relation. From the assumption we have for any $\omega_1 \in \Gamma_{hse}$ and any $\omega_2 \in M(\Gamma_{hse})$

$$\begin{aligned}(\omega_1, \omega_2^*) &= \sum_k \left(\int_{A_k} \omega_1 \int_{B_k} \overline{\omega}_2 - \int_{A_k} \overline{\omega}_2 \int_{B_k} \omega_1 \right) + (\omega_1 - T\omega_1, \omega_2^*) \\ &= \sum_k \left(\int_{A_k} \omega_1 \int_{B_k} \overline{\omega}_2 - \int_{A_k} \overline{\omega}_2 \int_{B_k} \omega_1 \right),\end{aligned}$$

where $T\omega_1 = \sum_k b_k \sigma(A_k) - a_k \sigma(B_k)$ (a finite sum), $a_k = \int_{A_k} \omega_1$, $b_k = \int_{B_k} \omega_1$ $\{A_i, B_i\} \in C.H.B.(F_n)_A$, and $\sigma(C)$ denotes the reproducing differential associated with cycle C. On the other hand from the definition $\omega_1 - T\omega_1 \in \Gamma_{he}$. Therefore we have $\omega_2^* \in (\Gamma_{he})^\perp$, hence $M(\Gamma_{hse}) \subset \Gamma_{h0}$. Conversely, for $\omega_2 \in M(\Gamma_{hse})$, we have $(\omega_1 - T\omega_1, \omega_2^*) = 0$ because $\omega_1 - T\omega_1 \in \Gamma_{he}$. Therefore we get $M(\Gamma_{hse}) = \Gamma_{h0}$.

Corollary 2, 1. *The validity of the special bilinear relations between* $\Gamma_1 = \Gamma_{h0}$ *and* $\Gamma_2 = \Gamma_{hse}$, $\Gamma_1 = \Gamma_{hse}$ *and* $\Gamma_2 = \Gamma_{hse}$ *or* $\Gamma_1 = \Gamma_s \cap \Gamma_{hse}$ *and* $\Gamma_2 = \Gamma_{hse}$, *is equivalent to* $\Gamma_{hm} = \Gamma_{he} \cap \Gamma_{h0}$, $\Gamma_{hse} \cap \Gamma_{he}^* = \phi$ *or* $\Gamma_{he} \cap \Gamma_{hse}^* \subset \Gamma_{he}^*$, *respectively* (Accola (1), Mori (9)).

§ 3. Bilinear relations in Kusunoki's sense.

At first we construct a class of symmetric open Riemann surfaces and a cononical homology basis for which we shall investigate the bilinear relations.

Given an open Riemann surface W, its Kerékjártó-Stoilöw compactification will be denoted by W^*, and the ideal boundary of W (in Kerékjártó-Stoilöw's sense) by $\beta(W)$. For a subset V on W, we denote by V^* (resp $\bar{V}$) the closure of V in W^* (resp W) and $V^* \cap \beta(W)$ by $\beta(V)$. For a subset α on $\beta(W)$, the intersection of a neighbourhood of α in W^* with W is called a neighbourhood of α (in W). When α has a neighbourhood Ω such that $\beta(\Omega)=\alpha$, we say that α is isolated in $\beta(W)$. Let F be a canonical region and σ a component of ∂F, then σ is the relative boundary of an end S (i.e. a component of $W-F$). We say that each point on $\beta(S)$ is the derivation of σ, or σ has arbitrary points of $\beta(S)$ as derivation.

In this paper we partition $\beta(W)$ into two disjoint sets α and β such that α is closed on W^* and not empty.

Let $\{F_n\}$ be a regular canonical exhaustion such that there exists a sequence $\{S_n\}$ of the neighbourhoods of α where S_n consists of a finite number of ends (i.e. components of $W-F_n$) $S_n \supset \bar{S}_{n+1}$ and $\bigcap_{n=1}^{\infty} \beta(S_n)=\alpha$, then we put

F_0: a parametric disk,

$$(3,1)\qquad \begin{aligned} &\partial F_n = \Gamma_n = \Gamma_n(\alpha) \cup \sigma_n, \quad \Gamma_n(\alpha) = \bigcup_{i=1}^{\alpha(n)} \Gamma_n^i(\alpha), \quad \sigma_n = \bigcup_{i=\alpha(n)+1}^{p(n)} \sigma_n^i \\ &\bar{F}_{n+1} - F_n = \Big(\bigcup_{i=1}^{\alpha(n)} F_n^i\Big) \cup \Big(\bigcup_{i=\alpha(n)+1}^{p(n)} G_n^i\Big), \end{aligned}$$

where F_n^i $(i=1, 2, \cdots, \alpha(n))$, G_n^i $(i=\alpha(n)+1, \cdots, p(n))$ denote the components of $F_{n+1}-F_n$, each $\Gamma_n^i(\alpha)$ (or σ_n^i) denotes the inner boundary of F_n^i (or G_n^i) and $\Gamma_n^i(\alpha)$ has at least one point of α as derivation. Clearly $p(0)=\alpha(0)=1$.

Let V_h be a parametric disk in F_n^i, and $\partial V_h = C_h$ where $h=p(n-1)+1, \cdots, p(n-1)+\alpha(n)$. We cut F_n along all C_h $(p(i-1)< h \leqslant p(i-1)+\alpha(i)$, $0 \leqslant i \leqslant n-1$ where $p(-1)=0)$ and put

$$(3,2)\qquad \begin{aligned} &R_n = F_n - \cup V_h, \quad \partial R_n = \partial F_n \cup (\cup C_h) = \Gamma_n \cup \Delta_n, \\ &\Gamma_n = \partial F_n, \quad \Delta_n = \cup C_h. \end{aligned}$$

Similarly we cut W along all C_h $(p(i-1)<h \leqslant p(i-1)+\alpha(i)$, $0 \leqslant i \leqslant \infty)$ and put

$$(3,3)\qquad R = W - \cup V_h = S(I_\alpha, W), \quad I_\alpha = \lim_{n\to\infty} \Delta_n.$$

Now we take two copies of R_n, say R_n and R'_n, and put

(3, 4) $$\partial R_n = (\cup C_h) \cup \Gamma_n, \quad \partial R'_n = (\cup C'_h) \cup \Gamma'_n,$$

and adjoin them along C_h and $-C'_h$ for all h. Thus constructed surface will be denoted by $\hat{R}_n$. In the same way we take two copies of R, say R and R', and adjoin them along C_h and $-C'_h$ for all h, and denote a resulting surface by $\hat{R}=\hat{S}(I_\alpha, W)$. It is clear that $\{\hat{R}_n\}$ is a regular exhaustion of $\hat{R}$, but not canonical one and $\partial\hat{R}_n=\Gamma_n \cup \Gamma'_n$.

Remark 3, 1. It is not essential that each C_h is a parametric circle. For example C_h may be an analytic arc in F^i_j.

The involutory mapping j of $\hat{R}$ on itself has the following property: If $p \to h(p)$ is a parametric mapping with domain V, then $p \to \bar{h}(j(p))$ is a parametric mapping with domain $j(V)$. With every differential ω on $\hat{R}$ we associate a differential $\omega^{\sim}$ as follows: If $\omega=a(z)dx+b(z)dy$ in terms of the variable $z=h(p)$ in V, then $\omega^{\sim}=a(\bar{z})dx-b(\bar{z})dy$ in terms of $z=\bar{h}(j(p))$ in $j(V)$. By means of mapping j we get the unique decomposition of a differential ω such that

$$\omega = \omega_s + \omega_a ,$$

where $\omega_a=-\omega_a^{\sim}$ and $\omega_s=\omega_s^{\sim}$. We note the following facts:

(i) For $\omega_i=\omega_{is}+\omega_{ia}$ $(i=1, 2)$ we get

(3, 5) $$(\omega_1, \omega_2{}^*)_{\hat{R}} = 2(\omega_{1s}, \omega_{2a}^*)_R + 2(\omega_{1a}, \omega_{2s}^*)_R ,$$

(ii) When C is a dividing curve on W and $C \subset R=S(I_\alpha, W)$, we get for $\omega=\omega_a+\omega_s \in \Gamma_{hse}(\hat{R})$

(3, 6) $$\int_C \omega_a = 0 , \quad \int_C \omega_s = \int_{j(C)} \omega_s .$$

Next we construct the special canonical homology basis on $\hat{R}$.

(I) *The case that W is planar.* We put

$$C_{h+1} = -B_h \quad \text{for} \quad p(i-1)<h\leqslant p(i-1)+\alpha(i) , \qquad 0\leqslant i\leqslant n-1 .$$

Let P_h be a point on B_h, P_0 a point on C_1, then we join P_0 with P_h by an anlytic curve A_h^+ $(\partial A_h^+=P_h-P_0)$ in the interior of R_n so that

$$A_h^+ \cap A_i^+ = P_0 \qquad \text{for all } h \text{ and } i, \ (h \neq i)$$

Next we denote the analytic curve $-j(A_h^+)$ on R_n' by A_h^- and put

$$A_h = A_h^+ \cup A_h^-, \ B_h = -C_{h+1}. \tag{3, 7}$$

By the same way as in Ahlfors (3) we can prove that $\{A_h, B_h\}$ is a $C.H.B.(\hat{R}_n)_A$. Simply we denote this basis by $\{A_i, B_i\}$ where $A_1, B_1, \cdots, B_{m(n)}$ is a canonical homology basis on $\hat{R}_n$ mod $\partial\hat{R}_n$.

Lemma 3, 1. *Let $\omega = \omega_a + \omega_s$ be a differential in $\Gamma_{hse}(\hat{R}) \cap \Gamma_{hse}^*(\hat{R})$ for $\hat{R} = \hat{S}(I_\alpha, W)$ (W is planar), then we get*

$$\omega_a = du, \quad \text{and} \quad \omega_s^* = dv,$$

where u, v are harmonic functions on R (not on $\hat{R}$) with the properties that $u =$ constant on B_h, $v =$ constant on B_h $(h \geqq 1)$ and $u = v = 0$ on C_1.

Proof. Since W is planar, this is clear from (3, 6) and the fact that $\omega_a = 0$ along I_α.

Lemma 3, 2. *For $\omega_1, \omega_2 \in \Gamma_{hse}(\hat{R}) \cap \Gamma_{hse}^*(\hat{R})$ where $\hat{R} = \hat{S}(I_\alpha, W)$ (W is planar), the following relation holds:*

$$(\omega_1, \omega_2^*)_{\hat{R}} = \lim_{n\to\infty} \Big\{ \sum_{h=1}^{m(n)} \Big(\int_{A_h} \omega_1 \int_{B_h} \bar{\omega}_2 - \int_{A_h} \bar{\omega}_2 \int_{B_h} \omega_1 \Big) + 2\int_{\Gamma_n} (u_1 d\bar{v}_2{}^* - \bar{u}_2 dv_1{}^*) \Big\}, \tag{3, 8}$$

where u_i, v_i are harmonic functions that correspond to ω_i in Lemma 3, 1.

Proof. From (3, 5) and Lemma 3, 1 we have for $\omega_i = \omega_{ia} + \omega_{is} = du_i - dv_i^*$ on R $(i = 1, 2)$

$$(\omega_1, \omega_2^*)_{\hat{R}} = \lim_{n\to\infty} \{2(du_1, dv_2)_{R_n} - 2(dv_1, du_2)_{R_n}\}. \tag{3, 9}$$

From the Green formula we have

$$(du_1, dv_2)_{R_n} = \int_{\Gamma_n} u_1 d\bar{v}_2{}^* - \sum_{h=1}^{m(n)} \int_{B_h} u_1 d\bar{v}_2{}^*.$$

On the other hand we have

$$\int_{B_h} u_1 d\bar{v}_2{}^* = p_{h1}\int_{B_h} d\bar{v}_2{}^*, \quad p_{h1} = \int_{A_h^+}\omega_{1a} = \frac{1}{2}\int_{A_h}\omega_1, \quad \text{and}$$

$$\int_{B_h}\bar{\omega}_2 = -\int_{B_h} d\bar{v}_2{}^*,$$

where p_{h1} denotes the constant value of u_1 on B_h ($h \geqslant 1$). Therefore we get

$$(3,10) \qquad 2(du_1, dv_2)_{R_n} = 2\int_{\Gamma_n} u_1 d\bar{v}_2{}^* + \sum_{h=1}^{m(n)}\int_{A_h}\omega_1\int_{B_h}\bar{\omega}_2\,.$$

Similarly,

$$(3,11) \qquad 2(dv_1, du_2)_{R_n} = 2\int_{\Gamma_n} \bar{u}_2 dv_1{}^* + \sum_{h=1}^{m(n)}\int_{A_h}\bar{\omega}_2\int_{B_h}\omega_1\,.$$

Putting (3, 10), (3, 11) into (3, 9), we get (3, 8). q.e.d.

(II) *The case that W is not planar.* We put

$$R \cap F_n^i = \Omega_h \ (h = p(n-1)+1, \cdots, p(n-1)+\alpha(n))\,,$$
$$R \cap G_n^i = G_h \ (h = p(n-1)+\alpha(n)+1, \cdots, p(n))\,,$$

and suppose the normal forms of the bordered surface Ω_h and G_h are respectively

$$(3,12) \qquad \begin{array}{l} (\prod_{i=1}^{p} a_i b_i a_i^{-1} b_i^{-1}) 1 C_h 1^{-1} (\prod_{i=1}^{q} 1_i \gamma_i 1_i^{-1})\,, \\ \prod_{i=1}^{p} a_i b_i a_i^{-1} b_i^{-1}\,, \end{array}$$

where $\gamma_1, \cdots, \gamma_q$ are sides that correspond to ∂F_n^i, and p is the genus of Ω_h (or G_h). We say that $a_1, a_2, \cdots, a_p$ ($b_1, b_2, \cdots, b_p$) are A-cycles (B-cycles) in Ω_h ($p(n-1)<h\leqslant p(n-1)+\alpha(n)$) or in G_h ($p(n-1)+\alpha(n)<h\leqslant p(n)$), and denote A, B-cycles in Ω_h or G_h by $\{A_i', B_i'\}_h$. Next we put

$$A_i'' = -j(A_i')\,, \quad B_i'' = j(B_i')\,,$$

and we say A_i'' (B_i''), $i=1, 2, \cdots, p$ are A-cycles (B-cycles) in $j(\Omega_h)$ ($p(n-1)+1\leqslant h\leqslant p(n-1)+\alpha(n)$) or $j(G_h)$ ($p(n-1)+\alpha(n)+1\leqslant h\leqslant p(n)$), and denote A, B-cycles in $j(\Omega_h)$ or $j(G_h)$ by $\{A_i'', B_i''\}_h$. We say that the set of all A (resp B)-cycles in all Ω_h, G_h (or $j(\Omega_h)$, $j(G_h)$) constitute the A (resp B)-cycles in R (or R').

Now we put

$$B_h = -C_{h+1} \qquad (h \geqslant 1)\,,$$

where C_h is a component of $\partial\Omega_h=\partial(R\cap F_n^i)$ and $n\geqq 1$, and we join Ω_h with $\Omega_1=R\cap \bar{F}_0^1$ by the sequence of domains $\{E_k\}$ such that

(i) $\Omega_1=E_1, E_2, \cdots, E_n=\Omega_h$ where each E_k is $R\cap \bar{F}_k^i$ with some i and

(ii) $\partial E_k\cap\Gamma_k$ coincides with a component of $\Gamma_k\cap(-\partial E_{k-1})$ for all k.

We take a point $P_h\in B_h$ and a point $P_0\in C_1$, then we can easily find a simple arc s_h joining P_h and P_0 such that s_h does not intersect any A, B-cycles in E_i $(i=1, 2, \cdots, n)$, $s_h\subset D=$ the interior of $\bigcup_{k=1}^{n} E_k$ (except endpoints), and $s_h\cap s_k=P_0$ for $h\neq k$. We put

$$s_h = A_h^+, \quad -j(A_h^+) = A_h^- \quad \text{where} \quad \partial A_h^+ = P_h-P_0, \quad \text{and}$$
$$A_h = A_h^+\cup A_h^- .$$

By the same way as in Ahlfors (3) we can prove that $\{A_h, B_h, \{A_i', B_i'\}_h, \{A_i'', B_i''\}_h\}$ is a $C.H.B.(\hat{R}_n)_A$. Hereafter simply we denote it by $\{A_i, B_i\}$ where $A_1, B_1, \cdots, B_{m(n)}$ is a canonical homology basis on $\hat{R}_n$ mod $\partial\hat{R}_n$.

Let ω be a differential in $\Gamma_{hse}(\hat{R})\cap\Gamma_{hse}^*(\hat{R})$ which has a finite number of non vanishing periods along A, B-cycles in R or R', and put

$$(3, 13) \qquad IT\omega = \sum{}' \{b_i\sigma(A_i)-a_i\sigma(B_i)\} \quad \text{(a finite sum)},$$

where A_i, B_i range over only A, B-cycles in R and R', and $b_i=\int_{B_i}\omega$, $a_i=\int_{A_i}\omega$.

Lemma 3, 3. *Let ω be a differential in $\Gamma_{hse}(\hat{R})\cap\Gamma_{hse}^*(\hat{R})$ which has a finite number of non vanishing periods along A, B-cycles in R and R', then we get*

$$\omega_a' = du, \quad \text{and} \quad \omega_s'^* = dv,$$

where $\omega-IT\omega=\omega'=\omega_a'+\omega_s'$ and u, v are harmonic functions on R with the properties that $u=const$ on B_h, $v=const$ on B_h and $u=v=0$ on C_1.

The proof can be carried out by the same way as in Lemma 3, 1.

Lemma 3, 4. *Let ω_1, ω_2 be the differentials as in Lemma* 3, 3, *then the following relation holds*:

$$(\omega_1, \omega_2{}^*) = \lim_{n\to\infty} \{\sum_{k=1}^{m(n)} (\int_{A_k} \omega_1 \int_{B_k} \bar{\omega}_2 - \int_{A_k} \bar{\omega}_2 \int_{B_k} \omega_1) + 2\int_{\Gamma_n} (u_1 d\bar{v}_2{}^* - \bar{u}_2 dv_1{}^*), \tag{3, 14}$$

where u_i, v_i are functions corresponding to ω_i in Lemma 3, 3.

Proof. For ω_1, ω_2 we have

$$(\omega_1, \omega_2{}^*) = \sum_k{}' (\int_{A_k} \omega_1 \int_{B_k} \bar{\omega}_2 - \int_{A_k} \bar{\omega}_2 \int_{B_k} \omega_1) + (\omega_1 - IT\omega_1, \omega_2{}^* - (IT\omega_2)^*).$$

Decomposing $\omega_i - IT\omega_i = \omega_i' = \omega_{ia}' + \omega_{is}'$, we can prove easily (3, 14) by the analogous way as in Lemma 3, 2. q.e.d.

From Lemma 3, 2 or Lemma 3, 4 we know that if we could get the exhaustion $\{\hat{W}_n\}$ such that

$$\int_{\Gamma_n} u_1 d\bar{v}_2{}^* \to 0 \quad \text{and} \quad \int_{\Gamma_n} \bar{u}_2 dv_1{}^* \to 0,$$

where $\Gamma_n = \partial W_n - \partial W_n \cap I_\alpha$, then the bilinear relation between ω_1 and ω_2 would hold on $\hat{R} = \hat{S}(I_\alpha, W)$.

To find such an exhaustion, we consider in a neighbourhood of α some families of curves on $R = S(I_\alpha, W)$ and near β a certain graph of $R = S(I_\alpha, W)$.

We put for $m > n > n_0$ (n_0: a fixed number)

$$\begin{cases} (R_m - R_n) \cup \Gamma_m = R_{mn} \cup G_{mn}, \ R_{mn} = \bigcup_{i=1}^{\alpha(n)} R_{mn}^i, \ G_{mn} = \bigcup_{i=\alpha(n)+1}^{p(n)} G_{mn}^i, \\ \partial R_{mn}^i \cap \Gamma_m(\alpha) = \bigcup_j \Gamma_m^{ij}(\alpha) = \Gamma_m^i(\alpha), \ (-\partial R_{mn}^i) \cap \Gamma_n(\alpha) = \Gamma_n^i(\alpha), \\ \Omega = \lim_{m\to\infty} R_{mn_0}, \end{cases} \tag{3, 15}$$

where each $\Gamma_n^i(\alpha)$, $\Gamma_m^{ij}(\alpha)$ is a closed curve which has, considering on W, at least one point of α as derivation.

(A) *The families of curves in a neighbourhood of α.*

Define the families of curves $\boldsymbol{C}(\alpha)$, $\boldsymbol{C}_{mn}$, $\boldsymbol{C}_{mn}^i$, $\boldsymbol{C}_n$, $\boldsymbol{L}_{mn}$, $\boldsymbol{L}_{mn}^i$, $\boldsymbol{L}_{mn}^{ij}$ on Ω as follows (Marden and Rodin (7)):

$\boldsymbol{C}(\alpha) = \{\gamma : \gamma$ is a countable union of closed curves in $\Omega^* - \beta(\Omega)$ and separates Γ_{n_0} and $\alpha\}$,

$\boldsymbol{C}_n = \{\gamma : \gamma \in \boldsymbol{C}(\alpha)$ and $\gamma \subset (\Omega - R_n)^* - \Gamma_n\}$,

$C^i_{mn} = \{\gamma : \gamma$ is a finite number of closed curves in $R^{i*}_{mn} - \beta(R^i_{mn})$ and separates $\Gamma^i_n(\alpha)$ and $\Gamma^i_m(\alpha) = \bigcup_j \Gamma^{ij}_m(\alpha)\}$,

$C_{mn} = \{\gamma : \gamma = \bigcup_i \gamma_i,\ \gamma_i \in C^i_{mn}\}$

$L_{mn} = \{\gamma' : \gamma'$ is an arc in $R^*_{mn} = \bigcup_i R^{i*}_{mn}$ connecting $\Gamma_n(\alpha)$ to $\Gamma_m(\alpha)\}$,

$L^i_{mn} = \{\gamma' : \gamma'$ is an arc in R^{i*}_{mn} connecting $\Gamma^i_n(\alpha)$ to $\Gamma^i_m(\alpha)\}$,

$L^{ij}_{mn} = \{\gamma' : \gamma'$ is an arc which connnects $\Gamma^i_n(\alpha)$ to $\Gamma^{ij}_m(\alpha)$ and lies in the interior of R^i_{mn} except endpoints$\}$.

Now we put

$$\mathrm{N}_{mn} = \max_{i,j} \lambda(L^{ij}_{mn})\lambda(C^i_{mn}), \quad \mathrm{M}_{mn} = \sum_i \lambda(C^i_{mn}), \tag{3,16}$$

where λ expresses the extremal length.

Definition 3, 1. *We say that $R = S(I_\alpha, W)$ has K-property in the neighbourhood of α if R satisfies the conditions*

$$\lambda(C(\alpha)) = 0, \quad \textit{and} \quad \overline{\lim_{m\to\infty}} \mathrm{N}_{mn} < \mathrm{N} < \infty. \tag{3,17}$$

Lemma 3, 5. (i) *If* $\lambda(C(\alpha)) = 0$, $\lambda(C_n) = 0$ (Kusunoki (6)).

(ii) *If* $\lambda(C_n) = 0$, *then* $\lim_{m\to\infty} \lambda(C_{mn}) = 0$ (Suita (12)).

(iii) $\lambda(C^i_{mn})\lambda(L^i_{mn}) = 1$, *and so* $\lambda(L_{mn}) \cdot \mathrm{M}_{mn} = 1$ (Marden and Rodin (4)).

(B) *A graph of $R = S(I_\alpha, W)$ near β.*

Let D^i_n $(i = \alpha(n)+1, \cdots, p(n),)$ be annuli each of which has an inner boundary σ^i_n, and we assume $D^i_n \cap D^k_m = \phi$ for $i \neq j$ or $n \neq k$ $(n, k > n_0)$ and $D^i_n \cap I = \phi$ for all n, i $(n > n_0)$. We put $D_n = \bigcup_{i=\alpha(n)+1}^{p(n)} D^i_n$ and denote the harmonic modulus of D^i_n and D_n by ν^i_n and ν_n, respectively. Also we denote by $U_0 + iV_0$ the function which maps $D = \bigcup_{m=n_0}^{\infty} D_n$ onto the strip domain $0 < U_0 < T = \sum_{n=n_0}^{\infty} \nu_n$, $0 < V_0 < 2\pi$. We call this strip domain the graph of R near β associated with $\{D_n\}$.

Definition 3, 2. *We say that R has S-property near β if R satisfies the condition*

$$\sum_{n=n_0}^{\infty} \min_i \nu^i_n = \infty, \tag{3,18}$$

and that R has A-property near β if

(3, 19) $\min_i \nu_n^i \geqq E > 0$ *for any n (E: a constant).*

Definition 3, 3. *We write $R \in S_0(I_\alpha, W)$ when R has K-property in the neighbourhood of α and S-property near β. When $R \in S_0(I_\alpha, W)$ we write $\hat{R} \in \hat{S}_0(I_\alpha, W)$.*

Lemma 3, 6. *For $\hat{R} \in \hat{S}_0(I_\alpha, W)$ (W is planar and α is isolated), the bilinear relation in Kusunoki's sense holds between ω_1 and ω_2 ($\omega_i \in \Gamma_{hse}(\hat{R}) \cap \Gamma_{hse}^*(\hat{R})$, $i=1, 2$).*

Note that $\{\hat{R}_n\}$ is not canonical.

Proof. From the assumption, we can suppose $\partial R_{mn} \cap \sigma_m = \phi$ for $n > n_0$.

(i) *Evaluation of the integrals along $\Gamma_n(\alpha)$ in* (3, 8).

We put

$$\max_{p \in \Gamma_n(\alpha)} (|u_1(p)|, |u_2(p)|) = t_n(u_1, u_2)$$
$$du_j = dU_j + idV_j, \quad dv_j = dU'_j + idV'_j, \quad f_j dz = dU_j + idU_j^*,$$
$$f'_j dz = dU'_j + idU'^*_j, \quad g_j dz = dV_j + idV_j^*, \quad g'_j dz = dV'_j + idV'^*_j$$

where U_j, U'_j, V_j, V'_j ($j=1, 2$) are real harmonic functions on R. Then we get

$$\text{(3, 20)} \quad \begin{aligned} &|du_j^*|, |du_j^*|, |du_j|, |dv_j| \leqq \rho |dz| \qquad (j=1, 2), \\ &D_K(\rho) = \int_K\int \rho^2 dxdy \leqq 8\{D_K(u_1) + D_K(u_2) + D_K(v_1) + D_K(v_2)\}, \end{aligned}$$

where $\rho = |f_1| + |f_2| + |f'_1| + |f'_2| + |g_1| + |g_2| + |g'_1| + |g'_2|$, and K is a compact domain in R. Moreover from (ii) in Lemma 3, 5, for fixed n there exists m such that

$$\text{(3, 21)} \qquad \lambda(C_{mn}) \geqq \left\{\frac{1}{\sqrt{\mathrm{M}_{mn}}}\right\}^2 > \{t_n(u_1, u_2)\}^2.$$

To make the notations simple, with respect to above m, n we put

$$S = R_{mn}^i, \quad \cup S = \bigcup_{i=1}^{\alpha(n)} R_{mn}^i, \quad \partial S \cap \Gamma_m = \Gamma' = \cup \Gamma^j,$$
$$(-\partial S) \cap \Gamma_n = \Gamma \quad \text{where } \Gamma \text{ is a closed curve},$$
$$C = C_{mn}^i, \quad L = L_{mn}^i, \quad L_j = L_{mn}^{ij}, \quad t_n(u_1, u_2) = t(u_1, u_2),$$
$$\mathrm{M} = \mathrm{M}_{mn} = \sum_i \lambda(C_{mn}^i), \quad \mathrm{N} = \mathrm{N}_{mn}.$$

Then there exists $\gamma_s = \bigcup_k \gamma_k \in \boldsymbol{C}$ (γ_k: a component) and $\gamma_j' \in \boldsymbol{L}_j$ such that

$$(3,22)\qquad \int_{\gamma_s} |du_i|,\ \int_{\gamma_s} |dv_i^*| < \int_{\gamma_s} \rho |dz| < \sqrt{2\lambda(\boldsymbol{C}) D_S(\rho)}\,,$$
$$\int_{\gamma_j'} |du_i| < \int_{\gamma_j'} \rho |dz| < \sqrt{2\lambda(\boldsymbol{L}_j) D_S(\rho)}\,.$$

From the definitions of $\boldsymbol{L}_j$ and $\boldsymbol{C}$, γ_s intersects γ_j' at least once at a point q that lies on a component γ_k of γ_s. Therefore we get

$$|u_i(q)| < t(u_1, u_2) + \int_{\gamma_j'} |du_i|, \qquad (i = 1, 2)\,.$$

Consequently we obtain for any point p on γ_k

$$|u_i(p)| < t(u_1, u_2) + \int_{\gamma_j'} |du_i| + \int_{\gamma_k} |du_i|, \qquad (i = 1, 2)\,.$$

From (3, 21) we can get for each S

$$(3,23)\qquad \int_{\gamma_s} |u_1 d\bar{v}_2^*|,\ \int_{\gamma_s} |\bar{u}_2 dv_1^*| < \left\{\frac{1}{\sqrt{\mathrm{M}}} + \sqrt{2 \max_j \lambda(\boldsymbol{L}_j) D_S(\rho)} + \sqrt{2\lambda(\boldsymbol{C}) D_S(\rho)}\right\} \int_{\gamma_s} \rho |dz|\,.$$

Summing up the relation (3, 23) for all S, in the former notations we have

$$(3,24)\qquad \int_{\gamma_{mn}} |u_1 d\bar{v}_2^*|,\ \int_{\gamma_{mn}} |\bar{u}_2 dv_1^*| < \sqrt{D_{R_{mn}}(\rho)}\,\{2\sqrt{\mathrm{M}_{mn}\lambda(\boldsymbol{L}_{mn})} + 2\sqrt{\mathrm{N}_{mn} D_{R_{mn}}(\rho)} + 4\sqrt{\mathrm{M}_{mn} D_{R_{mn}}(\rho)}\}\,,$$

where γ_{mn} denotes $\cup \gamma_s$ ($\gamma_{mn} \in \boldsymbol{C}_{mn}$). Let m_0 be an integer such that (3, 21) is fulfilled with $n = n_0$ and $m = m_0$, then ther exists $\gamma_{m_0 n_0}$ which satisfies (3, 24). Next, let m_1 be an integer such that (3, 21) is fulfilled with $n = n_1 > m_0$ and $m = m_1$, then then there exists $\gamma_{m_1 n_1}$ which satisfies (3, 24). By this way, we can get from (3, 20) a sequence of the level curves $\{\gamma_{m_\mu n_\mu}\} = \{\gamma_\mu\}$ such that

$$\int_{\gamma_\mu} |u_1 dv_2^*| + \int_{\gamma_\mu} |\bar{u}_2 dv_1^*| \to 0\,.$$

(ii) *Evaluation of the integrals along σ_n in* (3, 8).

We denote the level curve $U_0=t$ by $\theta_t=\bigcup_{i>\alpha(n)}^{p(n)}\theta_t^i$ and $\sum_{k=n_0}^{n-1}\nu_k\leqslant t<\sum_{k=n_0}^{n}\nu_k$. From the fact $\int_{\theta_t^i}dv_k^*=0$, we have

$$\left|\int_{\theta_t}u_1d\bar v_2-\int_{\theta_t}\bar u_2dv_1^*\right|\leqslant\sum_{i=\alpha(n)+1}^{p(n)}\left\{\int_{\theta_t^i}|du_1|\int_{\theta_t^i}|dv_2^*|+\int_{\theta_t^i}|d\bar u_2|\int_{\theta_t^i}|dv_1^*|\right\}$$

$$\leqslant L(t)=2\sum\{(\int_{\theta_t^i}|du_1|^2+(\int_{\theta_t^i}|du_2|)^2+(\int_{\theta_t^i}|dv_2^*|^2+(\int_{\theta_t^i}|dv_2^*|^2)\}.$$

We set $du_k=a_kdU_0+b_kdV_0$, $dv_k=a_k'dU_0+b_k'dV_0$ $(k=1,2)$, then by the successive applications of Schwarz inequality we obtain

$$L(t)\leqslant 4\pi\Lambda_0(t)\int_0^{2\pi}\{|b_1|^2+|b_2|^2+|b_1'|^2+|b_2'|^2\}dV_0,$$

where $\Lambda_0(t)=\max_i\int_{\theta_t^i}dV_0=\max_i\frac{\nu_n}{\nu_n^t}$. Hence again applying the Schwarz inequality, we get

$$\int_0^T\frac{L(t)}{\Lambda_0(t)}dt\leqslant\|du_1\|_D^2+\|du_2\|_D^2+\|dv_1\|_D^2+\|dv_2\|_D^2.$$

Consequently, under the condition $\int_0^T\frac{dt}{\Lambda_0(t)}=\sum_i\min_i\nu_n^i=\infty$, we have

$$\varliminf_{t\to T}L(t)=0,$$

and so there exists a sequence of the level curve $\{\theta_\mu:U_0=t_\mu'\}$ such that

$$L(t_\mu')\to 0\quad\text{as}\quad\mu\to\infty.$$

From the results of (i) and (ii) we can get an exhaustion $\{\hat W_\mu\}$ such that

$$\int_{\Gamma_\mu}|u_1d\bar v_2^*|+\int_{\Gamma_\mu}|\bar u_2dv|\to 0\quad\text{as}\quad\mu\to\infty,$$

where $\partial W_\mu-\partial W_\mu\cap I_\alpha=\gamma_\mu\cup\theta_\mu=\Gamma_\mu$, q.e.d.

In the analogous way as in Lemma 3, 6, we have from Lemma 3, 4 the following

Lemma 3, 7. *For* $\hat R\in\hat S_0(I_\alpha,W)$ *(*W *is not planar and* α *is isolated), the bilinear relation in Kusunoki's sense holds between* ω_1 *and* ω_2 *where* $\omega_i\in\Gamma_{hse}(\hat R)\cap\Gamma_{hse}^*(\hat R)$ $(i=1,2)$ *and* ω_i *have a finite* *number of non vanishing periods along A, B-cycles in R and R'.*

Theorem 3, 1. *For $\hat{R}\in\hat{S}_0(I_\alpha, W)$ (α is closed), the bilinear relation in Kusunoki's sense holds between ω_1 and ω_2 where $\omega_i\in\Gamma_{hse}(\hat{R})\cap\Gamma_{hse}^*(\hat{R})$ ($i=1, 2$) and ω_i have a finite number of non vanishing periods along A, B-cycles in R and R'.*

Note that besides non vanishing periods along A, B-cycles in R and R', ω_i may have an infinite number of non vanishing periods.

Proof. As in the proof of Lemma 3, 6 or Lemma 3, 7, we take the sequence of the level curves $\{\theta_\mu\}$ where $\theta_\mu=\bigcup_i\theta_\mu^i$. Let m_μ, n_μ be the integers such that

(i) the inner boundary of D_μ^k is $\sigma_{m_\mu}^k\in\Gamma_{m_\mu}$, and

(ii) (3, 21) is fulfilled with $m=m_\mu$ and $n=n_\mu$.

To make the notations simple, with respect to above m_μ, n_μ we put

$$m_\mu=m\,,\quad n_\mu=n\,,\quad \theta_\mu^i=\theta_i\,,\quad R_{m_\mu n_\mu}^i=S\quad\text{where}\quad \partial S\cap D_\mu^k\neq\phi\,,$$

then there exists the curves γ_s which satisfies (3, 23). We put $\theta_s=\bigcup\theta_i$ (θ_i is a curve on S such that $\beta(\gamma_s)\cap\beta(\theta_i)=\phi$, where $\beta(c)$ denotes all the points of $\beta(W)$ that are the derivation of c).

Thus we get an exhaustion $\{\hat{W}_\mu\}$ such that

$$\int_{\Gamma_\mu}|u_1d\bar{v}_2^*|+\int_{\Gamma_\mu}|\bar{u}_2dv_1^*|\to 0\quad\text{as}\quad \mu\to\infty\,,$$

where $\bigcup_s\{\gamma_s\cup\theta_s\}=\Gamma_\mu=\partial W_\mu-\partial W_\mu\cap I_\alpha$. q.e.d.

Corollary 3, 1, 1. *For $\hat{R}\in\hat{S}_0(I_\alpha, W)$ (W is of finite genus and α is closed) the bilinear relation in Kusunoki's sense holds between ω_1 and ω_2 where $\omega_i\in\Gamma_{hse}(\hat{R})\cap\Gamma_{hse}^*(\hat{R})$ ($i=1, 2$). Then $\hat{R}\in O_{KD}$ (from Corollary* 2, 1).

Corollary 3, 1, 2. *When W is z-plane, $\beta=\phi$ and I_α is a set of slits on real positive axis, Theorem* 3, 1 *reduces to the Theorem of Pfluger* (10).

Remark 3, 2. Theorem 3, 1 is true for ω_1 and ω_2 with an infinite number of non vanishing periods along A, B-cycles in R and R', if $\|IT_n\omega_1\|$ and $\|IT_n\omega_2\|$ are uniformly bounded, where $IT_n\omega=\sum_{k=1}^{n}{}' b_k\sigma(A_k)-a_k\sigma(B_k)$, and A_i, B_i range over only A, B-cycles in R and R'. From the Corollary 4 in (8) we can construct an example of surface $\hat{R}$ such that for any differentials $\omega \in \Gamma_{hse}(\hat{R})$, $\|IT_n\omega\|$ are bounded.

§ 4. Bilinear relations in Accola's sense.

Suppose $D_n'^i$ is an annulus in R_n whose outer boundary is $\Gamma_n^i(\alpha)$ ($\Gamma_n^i(\alpha)$: Cf (3, 1)) and $D_n'^i \cap D_m'^j = \phi$ for $n \neq m$ or $i \neq j$. We put

$D_n' = \bigcup_{i=1}^{\alpha(n)} D_n'^i$

$C_n'^i = \{\gamma : \gamma$ is a colsed curve in $D_n'^i$ which separates the inner boundary and outer boundary$\}$,

$C_n' = \{\gamma : \gamma = \bigcup_{i=1}^{\alpha(n)} \gamma_i,\ \gamma_i \in C_n'^i\}$

$L_n'^i = \{\gamma : \gamma$ is an arc that connects Γ_0 to $\Gamma_n^i(\alpha)$ and lies in the interior of R_n except endpoints$\}$,

$\{A_i, B_i\}$: the same $C.H.B.(\hat{R}_n)$ as in Theorem 3, 1,

and we set

$$X_n = \sum_{i=1}^{\alpha(n)} \lambda(C_n'^i),\qquad Y_n = \max_i \lambda(L_n'^i) X_n, \tag{4, 1}$$

Definition 4, 1. *We say that $R = S(I_\alpha, W)$ (α is closed) has A'-property in the neighbourhood of α if R satisfies the conditions*:

$$X_n \to 0 \quad as \quad n \to \infty, \quad and \quad \overline{\lim}\, Y_n < Y < \infty. \tag{4, 2}$$

Definition 4, 2. *We say that $R = S(I_\alpha, W)$ (α is closed) belongs to $S_1(I_\alpha, W)$ if R has A'-property in the neighbourhood of α and R has A-property near β. When $R \in S_1(I_\alpha, W)$, we write $\hat{R} \in \hat{S}_1(I_\alpha, W)$.*

Note $\hat{S}_1(I_\alpha, W) \subset \hat{S}_0(I_\alpha, W)$.

Theorem 4, 1. *If $\hat{R} \in \hat{S}_1(I_\alpha, W)$ (α is closed), then the bilinear relation in Accola's sense holds with respect to $\{\hat{R}_n\}$ and $\{A_i, B_i\}$ between ω_1 and ω_2 where $\omega_i \in \Gamma_{hes}^*(\hat{R}) \cap \Gamma_{hse}(\hat{R})$ ($i = 1, 2$) with only a finite number of non vanishing periods along A, B-cycles in R and R'.*

Proof. The proof is omitted since it is simpler than that of Theorem 3, 1. Note $\{\hat{R}_n\}$ is not canonical.

Remark 4, 1. It is easily seen that α satisfying the conditions stated in Theorem 4, 1 consists of only a finite number of ideal boundary points on $\beta(W)$.

§5. Period relation in Sainouchi's sense on $\hat{R}=\hat{S}(I_\alpha, W)$.

We assume that $\{\hat{R}_n\}$ is an exhaustion and $\partial\hat{R}_n \cap \partial R_n = \Gamma_n(\alpha)\cup\sigma_n = \Gamma_n$ as in (3. 1) or (3, 15), where $\Gamma_n(\alpha)=\bigcup_{i=1}^{\alpha(n)}\Gamma_n^i(\alpha)$, $\sigma_n = \bigcup_{i>\alpha(n)}^{p(n)}\sigma_n^i$. For $\omega_1, \omega_2\in\Gamma_{hse}(\hat{R})\cap\Gamma_{hse}^*(\hat{R})$ with a finite number of non vanishing periods along A, B-cycles in R and R', we obtain as in Lemma 3, 4

$$(5, 1) \qquad (\omega_1, \omega)_2^* = \lim_{n\to\infty}\Big[\sum_{i=1}^{m(n)}\Big(\int_{A_i}\omega_1\int_{B_i}\bar{\omega}_2-\int_{A_i}\bar{\omega}_2\int_{B_i}\omega_1\Big) + 2\int_{\Gamma_n}(u_1 d\bar{v}_2^* - \bar{u}_2 dv_1^*),$$

where u_k, v_k $(k=1, 2)$ are harmonic functions on R, $\omega_k - IT\omega_k = \omega_k' = \omega_{ka}'+\omega_{ks}'$, $\omega_{ka}'=du_k$ in R and $\omega_{ks}'^*=dv_k$ in R. Note that dv_k^* $(k=1,2)$ have in general a non vanishing period along a dividing cycle on R (not dividing on $\hat{R}$). Let D_n^h $(h=1, 2,\cdots, p(n))$ be annuili each of which includes a contour $\Gamma_n^h(\alpha)$ $(1\leqslant h\leqslant \alpha(n))$ or σ_n^h $(\alpha(n)<h\leqslant p(n))$, and we assume that $D_n^h\cap I_\alpha=\phi$ for all n, h, and $D_n^h\cap D_m^j=\phi$ for $h\neq j$ or $n\neq m$. We put $D_n=\bigcup_{h=1}^{p(n)}D_n^h$, and denote the harmonic modulus of D_n^i and D_n by ν_n^i and ν_n, respectively. Also we denote by $U_0(p)+iV_0(p)$ the function which maps $D=\bigcup_{n=1}^{\infty}D_n$ onto the strip domain $0<U_0<R=\sum_{n=1}^{\infty}\nu_n$, $0<V_0<2\pi$. By the same way as in Sainouchi (11) we get the following

Theorem 5, 1. *If $R\in S(I_\alpha, W)$ satisfies the conditions*

$$(5, 2) \qquad \sum_{n=1}^{\infty}\min_i \nu_n^i = \infty\ ,$$

then for $\{A_i, B_i\}$ and ω_1, ω_2 as in Lemma 3, 4 *there exists a regular exhaustion $\{\hat{W}_\mu\}$ such that*

$$(\omega_1, \omega_2^*) = \lim_{\mu\to\infty}\Big\{\sum_{i=1}^{m(\mu)}\Big(\int_{A_i}\omega_1\int_{B_i}\bar{\omega}_2-\int_{A_i}\bar{\omega}_2\int_{B_i}\omega_1\Big) + \sum_{i=1}^{\alpha(\mu)}\Big(\frac{d_\mu'^i}{\theta_\mu^i}\int_{\Gamma_\mu^i}u_1 dV_0 - \frac{d_\mu^i}{\theta_\mu^i}\int_{\Gamma_\mu^i}\bar{u}_2 dV_0\Big)\Big\}\ ,$$

where $m(\mu)$ *denotes the genus of* $\hat{W}_\mu$, $W\cap\partial\hat{W}_\mu=\Gamma_\mu(\alpha)\cup\sigma_\mu$, $\Gamma_\mu(\alpha)$ $=\cup\Gamma_\mu^t$, $\theta_\mu^t=\int_{\Gamma_\mu^t}dV_0$, $d_\mu^t=\int_{\Gamma_\mu^t}dv_1^*$ *and* $d_\mu'^t=\int_{\Gamma_\mu^t}d\bar{v}_2^*$.

Proof. From (5, 1) we can prove this Theorem quite similarly as in Sainouchi (11) except that $\{\hat{W}_\mu\}$ is not canonical, and so the proof is omitted.

We say that $\hat{R}=\hat{S}(I_\alpha, W)$ belongs to $\hat{S}_2(I_\alpha, W)$ if R satisfies (5, 2). Then $W\in O_{KD}$ (Sainouchi (11)).

In the present step it is not sure that there exists a surface of class $\hat{S}_2(I_\alpha, W)$ (or a surface of class $\hat{S}(I_\alpha, W\in O_{KD})$) which is not of class O_{KD}.

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