J. Math. Kyoto Univ.
7-2 (1967) 205-220

A note on theorems of Bott and Samelson

By

Hisao NAKAGAWA*) **)

(Communicated by Prof. H. Toda, Sept. 9, 1967)

Introduction.

Let there be given an n ($\geqq 2$)-dimensional complete and connected Riemannian manifold M of class C^{∞}. Throughout this paper, let a geodesic be parametrized by arc length, unless otherwise stated. A *geodesic loop* is a geodesic segment for which the initial and the final points coincide. Let γ be a geodesic loop parametrized by arc length s $(0 \leqq s \leqq 2l)$ such that $\gamma(0)=\gamma(2l)$, where self-intersections are permitted. γ is said to be *fundamental*, if there is no parameter s such that $\gamma(s)=\gamma(0)$ for $0<s<2l$. Throughout this paper, we mean by a *geodesic loop* a fundamental geodesic loop. A *closed geodesic* is by definition a geodesic loop whose initial tangent vector coincides with the final tangent vector. In connection with the study of homological properties of compact irreducible symmetric spaces of rank one, Bott [3][1] has studied the homological structure of Riemannian manifolds M having the following properties:

(a) there exists a point p such that all geodesics starting from p are closed,

(b) all of these closed geodesics passing through the point p are of the same length,

(c) all of these closed geodesics passing through the point p are simple and of the same index λ.

Making use of the Morse theory of the loop space and the

---

*) This work was supported by the Sakko-kai Foundation.

**) The helpful suggestions of the referee are grateful acknowledged.

1) Numbers in brackets refer to the Bibliography at the end of this paper.

Leray spectral sequences of the two kinds, he has obtained the following important Theorem A concerning the Riemannian manifold with the properties mentioned above:

Theorem A. *If λ is positive, then the integral cohomology ring $H^*(M, Z)$ is a truncated polynomial ring generated by an element, and if λ is equal to* 0, *then the universal covering manifold of M is a homological sphere.*

Studying the Riemannian manifold M which satisfies only the conditions (a) and (b), Samelson [**15**] has determined the cohomological structure of M by a different method from that followed by Bott, that is, by constructing a map of a real projective space PR^n onto M, which sends projective lines onto the geodesics passing through p. Summing up his results, we can state

Theorem B. *If M is simply connected, then $H^*(M, Z)$ is a truncated polynomial ring generated by an element, and if M is not simply connected, then $H^*(M, Z)$ is isomorphic to $H^*(PR^n, Z)$.*

On the other hand, Varga [**18**] has recently proved

Theorem C. *If an even dimensional simply connected homogeneous Riemannian manifold M satisfies the conditions* (*a*) *and* (*b*), *then M is homeomorphic to a symmetric space of rank one.*

In another paper [**13**], Riemannian manifolds with certain foliated structures are studied and we can show that such Riemanian manifolds with foliated structure satisfy the following two conditions:

(d) there exists a point p such that all geodesics starting from p are geodesic loops,

(e) all of these geodesic loops passing through the point p are of the same length.

The condition (d) is slightly weaker than the condition (a), but we can not make clear whether there exists any essential difference between them or not. Hatsuse and Takagi [**7**] have recently investigated a two dimensional W^*-manifold which is one of examples of Riemanian manifolds with the conditions (d) and (e). Thus, it might be interesting to study the homological structure of the Riemannian manifold satisfying the conditions (d) and (e). The purpose of this paper is to prove the following

Main Theorem. *Let M be an n ($\geqq 2$)-dimensional complete and connected Riemannian manifold satisfying the conditions (d) and (e).*

(1) *If M is simply connected, then $H^*(M, Z)$ is a truncated polynomial ring generated by an element.*
(2) *If M is not simply connected, then M has the same homology group as a real projective space PR^n and the universal covering manifold of M is homeomorphic to a sphere S^n.*

The Main Theorem will be proved by a similar device to, but slightly simpler than that followed by Bott, i.e., by making use of the Morse theory of the path space and the Leray-Serre spectral sequence of the fibre space.

In §1, we investigate the fundamental properties of a Riemannian manifold M satisfying the conditions (d) and (e), and prove some lemmas for later use. In §2 we shall prove the Main Theorem for case (1), i.e., for simply connected M (Theorem 2.2). In §3 we shall prove the Main Theorem for case (2), i.e., for non-simply connected case (Theorem 3.4). In the last proof of §3, we study the orientability of M in the case that M is not simply connected and obtain Theorem 3.5.

§1. Preparations.

Throughout this paper, M denotes an n ($\geqq 2$)-dimensional complete and connected Riemannian manifold satisfying the following two properties:

(d) there exists a point p such that all geodesics starting from p are geodesic loops,
(e) all of these geodesic loops passing through the point p are of the same length $2l$.

The fixed point p in the condition (d) is called the *basic point* of M and $2l$ in the condition (e) is called the *loop length* at p. We know many examples of such manifolds; for instance, a symmetric space of rank one, a real projective space and a complete Riemannian manifold satisfying the $W_{0,0}$-condition.[2] However, as remarked in the introduction, we do not know whether there exists any example showing that there exists at least one geodesic loop which is not a closed geodesic or not.

First of all, we prove the following

Lemma 1.1. *M is simply connected or a fundamental group* $\pi_1(M)$ *is of order* 2.

Proof. It is easily seen that arbitrary two geodesic loops in the loop space $\Omega(p, p)$ are (p, p)-homotopic, for all geodesics issuing from p are geodesic loops. Then the product $\gamma_2 * \gamma_1$ of any two geodesic loops γ_1 and γ_2 is null-homotopic, i.e., $\gamma_2 * \gamma_1 \sim e_p$, because we have

$$\gamma_2 * \gamma_1 \sim \gamma_1^{-1} * \gamma_1 \sim e_p,$$

where e_p denotes the constant curve at p and γ_1^{-1} is the inverse geodesic loop of γ_1 defined by $\gamma_1^{-1}(s) = \gamma_1(2l - s)$. For an arbitrary geodesic segment γ in $\Omega(p, p)$, γ is expressed as a product of geodesic loops $\gamma_1, \gamma_2, \cdots, \gamma_k$ at p as follows:

$$\gamma = \gamma_k * \gamma_{k-1} * \cdots * \gamma_1.$$

When k is even, this implies that γ is null-homotopic, and when k is odd, it is easily seen that γ is homotopic to the geodesic loop γ_1. Hence γ is null-homotopic, or homotopic to a geodesic loop.

As a consequence of the facts proved above, we see that the fundamental group $\pi_1(M, p)$ at p is equal to 0, if a geodesic loop is null-homotopic, and that the group $\pi_1(M, p)$ is of order 2, if there exists at least one geodesic loop which is not null-homotopic. Thus Lemma 1.1 is proved completely.

We have easily

Lemma 1.2. *For an arbitrary geodesic loop* γ, $\gamma(2l)$ *is a conjugate point of* $\gamma(0)$ *along* γ *and with multiplicity* $n-1$.

Lemma 1.2 shows that all geodesics starting from p have conjugate points. Therefore M is compact.

2) Ōtsuki [14].

Taking an arbitrary point x in M, we consider the path space $\Omega(p, x)$. The geodesic segment σ in $\Omega(p, x)$ is said to be *of order* k, if there exist k real numbers $s_1, s_2, \cdots, s_k$ such that $0 < s_1 < s_2 < \cdots < s_k < L(\sigma)$ and

$$\sigma(s_1) = \sigma(s_2) = \cdots = \sigma(s_k) = p ,$$

where $L(\sigma)$ denotes the arc length of σ.

We denote by γ_μ a geodesic loop of index μ, and by $\operatorname{Ind}\gamma$ the index of a geodesic segment γ. Thus we have

Lemma 1. 3. *For an arbitrary geodesic loop γ_μ, there is a point q of γ_μ different from p such that, for an arbitrarily given integer k, there exist in $\Omega(p, q)$ two and only two geodesic segments σ_1 and σ_2 of order k, whose indices satisfy*

$$(1.1) \qquad \operatorname{Ind}\sigma_1 \geqq k(n-1) , \qquad \operatorname{Ind}\sigma_2 \geqq k(n-1)+\mu .$$

Proof. Let $\gamma_\mu(s_0)$ be a minimal point of p along γ_μ. We choose a point $q_1 = \gamma_\mu(s_1)$ for a parameter s_1 satisfying $0 < s_1 < s_0$, and define two geodesic segments γ_1 and γ_2 joining p and q_1 by $\gamma_1(s) = \gamma_\mu(s)$ for $0 \leqq s \leqq s_1$, and $\gamma_2(s) = \gamma_\mu(2l-s)$ for $0 \leqq s \leqq 2l - s_1$, respectively. Then we easily see that $\operatorname{Ind}\gamma_1 = 0$ and $\operatorname{Ind}\gamma_2 = \mu$ hold.

The index $\operatorname{Ind}\sigma$ of a geodesic segment σ of order k in $\Omega(p, q_1)$ satisfies the inequality $\operatorname{Ind}\sigma \geqq k(n-1)$, if σ has the property $\sigma(s) = \gamma_1(s-2kl)$ for $s \geqq 2kl$, and $\operatorname{Ind}\sigma$ satisfies the inequality $\operatorname{Ind}\sigma \geqq k(n-1)+\mu$, if σ has the property $\sigma(s) = \gamma_2(s-2kl)$ for $s \geqq 2kl$. Thus, there exist necessarily at least two geodesic segments in the path space $\Omega(p, q_1)$ satisfying (1. 1).

Let U' be a spherical neighbourhood centered at p and with radius $\delta = d(p, C(p))$, where $C(p)$ is a cut locus of p. For any point x in the intersection of γ_1 and U', geodesic segments joining p and x different from γ_1 are not minimal because of the property of the cut locus, and hence they intersect the complement of U'. This shows that their lengths are not less than δ and not greater than $2l - \delta$.

We now define a map f of $[\delta, 2l-\delta] \times S_p(1)$ into M by $f(s, u) = \exp_p(su)$, where $S_p(1)$ is the unit sphere centered at the origin 0 of the tangent space $T(M)_p$. Since f is continuous and

$[\delta, 2l-\delta]\times S_p(1)$ is compact, the image $\mathrm{Img}\, f$ is closed. Taking account of the condition (e) that all geodesic loops at the basic point p are of the same length $2l$, we see that p is not contained in the image $\mathrm{Img}\, f$. Thus p has a neighbourhood U'' such that $U''\cap \mathrm{Img}\, f=\phi$. If we take a point $q=\gamma_\mu(s_2)$ arbitrarily in $U''-\{p\}$, then any geodesic segment σ joining p and q which is a part of a geodesic loop satisfies the inequality

$$L(\sigma)<\delta \qquad \text{or} \qquad L(\sigma)>2l-\delta\,.$$

When $L(\sigma)<\delta$, σ is minimal and therefore σ coincides with γ_1. When $L(\sigma)>2l-\delta$, the length of a complement of σ in the geodesic loop is less than δ and hence σ coincides with γ_2. Consequently there exist two and only two geodesic segments σ_1 and σ_2 of order k in $\Omega(p, q)$ such that σ_1 and σ_2 satisfy the inequalities (1.1).

Thus Lemma 1.3 is proved completely.

Remark 1.1. Taking account of Lemma 1.3, we show that if each point in M is basic then all geodesics in M are simply closed.

For a geodesic loop γ_μ of index μ, we fix a point $q=\gamma_\mu(s_2)$ satisfying the properties of Lemma 1.3. We have to show now that the path space $\Omega(p, q)$ is non-degenerate, since we shall develop discussions in the sense of the Morse theory in $\Omega(p, q)$. We suppose that there exists a geodesic segment σ of order k in $\Omega(p, q)$ at which the Hessian of the energy function is degenerate. Then there is a non-zero Jacobi field $X(s)$ along σ such that $X(0)=0$ and $X(L(\sigma))=0$. From the property of Jacobi fields, $X(s)$ is orthogonal to the tangent vector $\sigma'(s)$ of σ at $\sigma(s)$. On the other hand, we see by virtue of Lemma 1.2 that there exist $n-1$ linearly independent Jacobi fields $X^1(s), X^2(s), \cdots, X^{n-1}(s)$ along σ such that

$$X^1(2jl)=X^2(2jl)=\cdots=X^{n-1}(2jl)=0 \quad \text{for} \quad j=0, 1, 2, \cdots.$$

Thus each of Jocobi fields $X^1(s), X^2(s), \cdots, X^{n-1}(s)$ is orthogonal to $\sigma'(s)$. Consequently $X(s)$ is expressed as a linear combination of $X^1(s), X^2(s), \cdots, X^{n-1}(s)$. Thus we get $X(2kl)=0$ and hence we see that q is a conjugate point of p along a geodesic segment σ, defined by $\sigma_1(s)=\sigma(s)$ for $2kl\leqq s\leqq L(\sigma)$. This geodesic segment σ_1 is a part of a geodesic loop. Therefore the Hessian of the energy function at σ_1 must be degenerate. This is a contradiction. Thus $\Omega(p, q)$ should be non-degenerate.

By choosing the point q fixed, there exist two and only two geodesic segments σ_1 and σ_2 of an arbitrary order k in the path space $\Omega(p, q)$ such that the index $\mathrm{Ind}\,\sigma_1$ is greater than or equal to $k(n-1)$ and the index $\mathrm{Ind}\,\sigma_2$ is greater than or equal to $k(n-1)+\mu$. Accordingly, the index $\mathrm{Ind}\,\sigma$ of any geodesic segment σ in $\Omega(p, q)$ satisfies one of the following relations:

$$\mathrm{Ind}\,\sigma=0,\ \mathrm{Ind}\,\sigma=\mu,\ \mathrm{Ind}\,\sigma\geqq n-1,\ \mathrm{Ind}\,\sigma\geqq n-1+\mu,\ \mathrm{Ind}\,\sigma\geqq 2(n-1),\ \mathrm{Ind}\,\sigma\geqq 2(n-1)+\mu,\cdots,$$

and for each of the relations above there exists a unique geodesic segment satisfying that relation. We denote the indices of all geodesic segments in $\Omega(p, q)$ by $0, \mu_1, \mu_2,\cdots$ and suppose that they are arranged in such a way that

$$0\leqq\mu_1\leqq\mu_2\leqq\cdots,$$

where μ_2 is greater than or equal to $n-1$.

By means of the Morse theory of the path space, since $\Omega(p, q)$ is non-degenerate, it has the same homotopy type as that of a CW-complex Λ obtained by attaching a μ_1-cell e^{μ_1}, a μ_2-cell $e^{\mu_2},\cdots$ to $\Omega^r(p, q)$, where $\Omega^r(p, q)$ is a certain set of piecewise differentiable curves belonging to $\Omega(p, q)$. As is well known, a cell decomposition of Λ is given by $\{e^0, e^{\mu_1}, e^{\mu_2},\cdots\}$. Now we shall prove

Lemma 1. 4. *If there is a geodesic loop at p of index* 0, *then M is not simply connected.*

Proof. First we consider the case $n\geqq 3$. Let γ_0 be a geodesic loop of index 0. The cell decomposition of the CW-complex Λ is given by $\{e^0, e^0, e^{\mu_2}, e^{\mu_3},\cdots\}$, and μ_2 is not less than 2. For a 0-chain group and a 1-chain group of Λ, we have

$$C_0(\Lambda)=Ze^0+Ze^0,\qquad C_1(\Lambda)=0,$$

which implies that the 0-dimensional homology group $H_0(\Lambda)$ of Λ is given by

$$H_0(\Lambda) = Z + Z .$$

Since $\Omega(p,q)$ and Λ are of the same homotopy type and the homology group is homotopy invariant, we get

$$H_0(\Omega(p, q)) = Z + Z .$$

This means that M is not simply connected. In fact, suppose that M is simply connected, then $\Omega(p, q)$ is pathwise connected and hence it is connected. Thus we get $H_0(\Omega(p, q)) = Z$. This is a contradiction.

In the case $n=2$, by the same device as that followed by Hatsuse and Takagi [7] in a two dimensional W^*-manifold (see Remark 1.3 below for the definition of a W^*-manifold), we can prove that M is not orientable. M is therefore not simply connected. Thus Lemma 1.4 is proved completely.

Remark 1.2. If M is simply connected, then each geodesic loop at p is of positive index.

Remark 1.3. In Lemma 1.4, we have assumed the existence of a geodesic loop at p of index 0. However, Hatsuse and Takagi [7] have introduced the notion of a W^*-manifold, which is an $n(\geqq 2)$-dimensional complete and connected Riemannian manifold satisfying the property that the first conjugate locus of p consists of a single point p and all geodesic loops at p are of the same length. It is easily seen that a Riemannian manifold is a W^*-manifold if and only if it satisfies the conditions (d), (e) and the condition that all geodesic loops at p are of index 0.

They have proved that any W^*-manifold is not simply connected. In their proof for the case $n=2$, the following properties of W^*-manifolds play essential roles: a W^*-manifold satisfies the conditions (d) and (e), and that there is at least one geodesic loop at p of index 0.

§ 2. Proof of Main Theorem: the case (1).

In this section, we assume that M denotes an $n(\geqq 2)$-dimensional complete, simply connected and connected Riemannian manifold satisfying the conditions (d) and (e). Denote by $E(p, M)$ the space of all piecewise differentiable curves having the fixed point p as their initial points. Then, as is well known, the set $(E(p, M), M, \Omega)$ is a fibre space in the sense of Serre, where Ω denotes the fibre. We shall prove

Proposition 2. 1. *If M is simply connected, then the index of each geodesic loop at p is positive and less than n.*

Proof. Since M is simply connected, the indices of all geodesic loops at p are positive, as a consequence of Lemma 1. 4.

Suppose that there is a geodesic loop γ_λ at p of index $\lambda \geqq n$. For a fixed point q on γ_λ given by Lemma 1. 3, the non-degenerate path space $\Omega(p, q)$ has the geodesic segments of index $0, \lambda_1, \lambda_2, \cdots$ such that

$$n \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots .$$

Thus $\Omega(p, q)$ has the same homotopy type as a CW-complex Λ which has a cell decomposition $\{e^0, e^{\lambda_1}, e^{\lambda_2}, \cdots\}$. Therefore Λ is $(n-1)$-connected and so is $\Omega(p, q)$. It is well known that $\pi_j(M) \approx \pi_{j-1}(\Omega) \approx \pi_{j-1}(\Omega(p, q))$ for $j \geqq 1$. Consequently we obtain that M is n-connected, and in particular, we get $\pi_n(M) = H_n(M) = 0$ by means of Hurewicz isomorphism theorem, which contradicts to $H_n(M) = Z$. Consequently, there is no geodesic loop γ_λ of index $\geqq n$.

We state the following

Theorem 2. 2. *If M is simply connected, then there is a positive integer λ such that the integral cohomology ring $H^*(M, Z)$ is a truncated polynomial ring generated by an element of dimension $\lambda+1$. Thus*

$$H_{k(\lambda+1)}(M) \approx H^{k(\lambda+1)}(M) = Z \qquad \text{for} \quad k = 0, 1, 2, \cdots, m$$

and the other (co)homology groups are zero, where n is necessarily equal to $m(\lambda+1)$.

Proof. The case $n=2$ is obvious, and we may restrict ourselves to the case $n\geqq 3$. From Proposition 2.1, there is a geodesic loop γ_λ of index λ such that $0<\lambda\leqq n-1$. For a fixed point $q=\gamma_\lambda(s_2)$ in Lemma 1.3, the indices of geodesic segments contained in $\Omega(p, q)$ are listed up as follows:

$$0, \lambda_1, \lambda_2, \lambda_3, \cdots,$$

where $\lambda_1=\lambda\leqq n-1$ and $n\leqq\lambda_2\leqq\lambda_3\leqq\cdots$. Consequently, a cell decomposition of a CW-complex Λ, which has the same homotopy type as $\Omega(p, q)$, is given by $\{e^0, e^{\lambda_1}, e^{\lambda_2}, \cdots\}$.

First we consider the case $\lambda=n-1$. In this case it is easily seen that Λ is $(n-2)$-connected. It follows that M is $(n-1)$-connected as in the proof of Proposition 2.1. Obviously $H_n(M)=H^n(M)=Z$. Then the theorem is proved for this case.

Next we consider the case $\lambda<n-1$. Then we have easily

$$H^j(\Omega) = H^j(\Lambda) = \begin{cases} Z & \text{for } j = 0, \lambda \\ 0 & \text{for } j \neq 0, \lambda \text{ and } j < n. \end{cases}$$

Now the theorem is a direct consequence of the following

Lemma 2.3. *If, in a simply connected and connected topological space M, the j-th cohomology group $H^j(\Omega)$ of a path space Ω of M satisfies the above condition for $0<\lambda<n-1$, then there is an element u of $H^{\lambda+1}(M)$ such that the cup-product with the element u induces an isomorphism $H^j(M)\approx H^{j+\lambda+1}(M)$ for $0\leqq j\leqq n-(\lambda+1)$.*

Proof. Consider the cohomology spectral sequence $\{E_r^{p,q}\}$ associated with the fibre space $(E(p, M), M, \Omega)$. (For the details see [16].) By the fundamental theorem of [16] we get

$$E_2^{p,q} = H^p(M, H^q(\Omega)) = H^p(M)\otimes H^q(\Omega)+\text{Tor}\,(H^{p+1}(M), H^q(\Omega)) \quad \text{for } q < n.$$

By means of the assumption, we have $E_2^{p,q}=0$ if $q\neq 0, \lambda$ and $q<n$, and the same is true for $E_r^{p,q}$, $r\geqq 2$. Then d_r: $E_r^{p,q}\to E_r^{p+r,q-r+1}$ is trivial if $r\neq\lambda+1$, $(p, q)\neq(0, n)$ and $p+q\leqq n$. From the fundamental property $H(E_r^{p,q})=E_{r+1}^{p,q}$ of the spectral sequence it follows that $E_2^{p,q}=E_{\lambda+1}^{p,q}$ and $E_{\lambda+2}^{p,q}=E_\infty^{p,q}$ if $(p, q)\neq(0, n)$ and

$p+q\leqq n$. Since the total space is contractible, we have $E_\infty^{p,q}=0$ for $(p, q)\neq(0, 0)$. Consider the homomorphisms

$$d_{\lambda+1}^{j,\lambda} = d_{\lambda+1}\colon E_{\lambda+1}^{j,\lambda} \to E_{\lambda+1}^{j+\lambda+1,0}.$$

Since each element of $E_{\lambda+1}^{j+\lambda+1,0}$ is a $d_{\lambda+1}$-cycle, we have $E_{\lambda+1}^{j+\lambda+1,0}/\operatorname{Img} d_{\lambda+1}^{j,\lambda}=E_{\lambda+2}^{j+\lambda+1,0}=0$ for $0<j+\lambda+1\leqq n$, that is, $d_{\lambda+1}^{j,\lambda}$ is an epimorphism for $0<j+\lambda+1\leqq n$. We have also, for $j+\lambda+1\leqq n$, $\operatorname{Ker} d_{\lambda+1}^{j,\lambda}/d_{\lambda+1}E_{\lambda+1}^{j-\lambda-1,2\lambda}=H(E_{\lambda+1}^{j,\lambda})=E_{\lambda+2}^{j,\lambda}=0$. Here $E_{\lambda+1}^{j-\lambda-1,2\lambda}=0$ since $2\lambda\neq0$, λ and $j-\lambda-1<0$ if $2\lambda\geqq n$. Thus we get $\operatorname{Ker} d_{\lambda+1}^{j,\lambda}=0$, and $d_{\lambda+1}^{j,\lambda}$ is a monomorphism for $j+\lambda+1\leqq n$. Consequently we have obtained an isomorphism

$$d_{\lambda+1}\colon E_2^{j,\lambda} = E_{\lambda+1}^{j,\lambda} \approx E_{\lambda+1}^{j+\lambda+1,0} = E_2^{j+\lambda+1,0} \qquad \text{for} \quad 0<j+\lambda+1 \leqq n.$$

Let v be a generator of $H^\lambda(\Omega)=Z$. Using the identification $E_2^{p,q}=H^p(M)\otimes H^q(\Omega)+\operatorname{Tor}(H^{p+1}(M), H^q(\Omega))$, we see that the formula $d_{\lambda+1}(x\otimes v)=\phi^*(x)\otimes 1$ defines an isomorphism

$$\phi^*\colon H^j(M) \approx H^{j+\lambda+1}(M) \qquad \text{for } 0 \leqq j \leqq n-(\lambda+1).$$

Put $u=\phi^*(1)$ in $H^{\lambda+1}(M)$. By means of the derivativity of $d_{\lambda+1}$, we have, for $x\in H^j(M)$, $d_{\lambda+1}(x\otimes v)=d_{\lambda+1}((x\otimes 1)(1\otimes v))=(-1)^{\lambda+1}(x\otimes 1)d_{\lambda+1}(1\otimes v)=(-1)^{\lambda+1}(x\cup u)\otimes 1$. Thus we obtain $(-1)^{\lambda+1}\phi^*(x)=x\cup u$ for $0\leqq j\leqq n-(\lambda+1)$, and the lemma is proved.

This completes the proof of Theorem 2.2.

Remark 2.1. When $\lambda=n-1$, it follows from Wang exact sequence that $H_j(\Omega)=Z$ for $j=0, n-1, 2(n-1), \cdots$ and $H_j(\Omega)=0$ for other j. Thus we get $\lambda_i=i(n-1)$ for $i=1, 2, 3, \cdots$. When $\lambda<n-1$, we get $n=m(\lambda+1)$ and further computations in the spectral sequence show that $H_j(\Omega)=Z$ for $j\geqq0$ and $j\equiv0, \lambda(\operatorname{mod} n+\lambda-1)$ and $H_j(\Omega)=0$ for the other values of j. Thus we have $\lambda_{2i}=i(n+\lambda-1)$ for $i=1, 2, 3, \cdots$ and $\lambda_{2i+1}=\lambda+i(n+\lambda-1)$ for $i=0, 1, 2, \cdots$.

Remark 2.2. We assume that n is greater than or equal to 3. If there is a geodesic loop γ_λ of positive index λ in $\Omega(p, q)$, then all geodesic loops at p are of the same index λ. In fact, if we suppose that there exists a geodesic loop γ_μ of index $\mu\neq\lambda$, then we get $H_{\mu+1}(M)=Z$ by developing a similar discussion concerning γ_μ as those developed in the proof of Theorem 2. 2. This contradicts Theorem 2. 2.

Remark 2. 3. According to the cohomology theory (Adams [1], Adem [2] and Milnor [10]), if the integral cohomology ring is a truncated polynomial ring generated by a unique element of dimension $\lambda+1$, then λ is necessarily equal to 1, 3, 7 or $n-1$, where n should be equal to 16 in the case $\lambda=7$. In particular, when n is odd, λ must be equal to $n-1$.

When $n(\geqq 5)$ is odd, M is a homological sphere and hence a homotopical sphere, because of $\lambda=n-1$. Taking account of Smale's Theorem on the generalized Poincaré conjecture, we see that M is homeomorphic to a sphere, if $n(\geqq 5)$ is odd.

Remark 2. 4. As examples of simply connected Riemannian manifolds satisfying the conditions (d) and (e), we have symmetric spaces of rank one, that is,

(1) the sphere S^n $(n\geqq 2)$,

(2) the complex projective space PC^n $(n=2m\geqq 4)$,

(3) the quaternion projective space PQ^n $(n=4m\geqq 8)$,

(4) the Cayley projective space PCa^n $(n=16)$.

These spaces have a truncated polynomial ring generated by a unique element as their integral cohomology ring.

Eells and Kuiper [5] have recently constructed compact simply connected manifolds which have the same integral cohomology ring as the quaternion or the Cayley projective space, but, whose homotopy type are different from the corresponding projective space.

§ 3. Proof of Main Theorem: the case (2).

Let M be an $n(\geqq 2)$-dimensional complete and connected Riemannian manifold satisfying the conditions (d) and (e) mentioned in the section 1.

First of all, we shall prove

Lemma 3. 1. *If M is not simply connected, then there is no geodesic loop at p of positive index.*

Proof. By virtue of Lemma 1. 1, the fundamental group

$\pi_1(M)$ of M is of order 2. Let $\tilde{M}$ be an universal covering manifold of M and $\pi^{-1}(p)=\{\tilde{p}_1, \tilde{p}_2\}$ the inverse image of p under the covering map π. Since all geodesic loops at p in M are (p, p)-homotopic to each other and none of them is null-homotopic, the locus of the final points of their lifts starting from $\tilde{p}_1$ (from $\tilde{p}_2$) in $\tilde{M}$ consists of the single point $\tilde{p}_2$ (of the single point $\tilde{p}_1$). Thus all geodesics starting from $\tilde{p}_1$ are geodesic loops at $\tilde{p}_1$ and have the same length $4l$. Consequently $\tilde{M}$ satisfies the conditions (d) and (e) such that two points $\tilde{p}_1$ and $\tilde{p}_2$ are basic, and the loop length at $\tilde{p}_1$ or $\tilde{p}_2$ is equal to $4l$.

Suppose that there is a geodesic loop γ_λ at p of positive index λ. Let $\tilde{\gamma}_\lambda$ be the lift of γ_λ starting from $\tilde{p}_1$. Then an index of the extended geodesic loop of $\tilde{\gamma}_\lambda$ at $\tilde{p}_1$ is not less than $n-1+\lambda$, that is, not less than n. This contradicts Proposition 2. 1, i.e., the fact that there is no geodesic loop at $\tilde{p}_1$ in $\tilde{M}$, whose index is greater than or equal to n. Thus all geodesic loops at p in M must be of index 0. This completes the proof.

Combining Lemmas 1. 4 and 3. 1 together, we get

Corollary 3. 2. *If there is a geodesic loop of index* 0, *then all geodesic loops at* p *are of index* 0.

Remark 3. 1. Taking account of Lemma 1. 4 and Corollary 3. 2, we see that M is simply connected if and only if all geodesic loops at p in M are of positive index. That is to say, the fundamental group of M is of order 2 if and only if all geodesic loops at p in M are of index 0. By means of Remark 2. 2, if there is a geodesic loop at p of positive index λ, then all geodesic loops at p are of the same index λ, that is, an index of M at p in the sense of Bott is equal to λ. Thus a theorem due to Samelson is a generalization of a theorem due to Bott.

As mentioned in the section 1, a W^*-manifold introduced by Hatsuse and Takagi is by definition a complete and connected Riemannian manifold satisfying the conditions (d), (e) and the condition that all geodesic loops at p are of index 0. Hatsuse and Takagi [7] has shown that a two-dimensional W^*-manifold is homeomorphic to a real projective space PR^2. We have now the following Corollary 3.3 from Lemma 3.1 or Corollary 3.2.

Corollary 3.3. *If M is not simply connected, then M is a W^*-manifold.*

If there is a geodesic loop at p of index 0, *then M is a W^*-manifold.*

In particular, if M is not simply connected for $n=2$, then M is homeomorphie to PR^2.

Next we shall prove the latter half of Main Theorem. We verify the following

Theorem 3.4. *If M is not simply connected, then M has the same homology group as PR^n and $\tilde{M}$ is homeomorphic to a sphere S^n.*

Proof. First we shall prove that $\tilde{M}$ is homeomorphic to S^n. Let $\{\tilde{p}_1, \tilde{p}_2\}$ be the inverse image of p under the covering map π. Because the universal covering manifold $\tilde{M}$ satisfies the conditions (d) and (e), the indices of all extended geodesic loops at $\tilde{p}_1$ in $\tilde{M}$ are less than or equal to $n-1$, as a consequence of Proposition 2.1. By virtue of Lemma 3.1, all geodesic loops at p in M are of index 0, and therefore the lifts of them starting from $\tilde{p}_1$ are of index 0. Thus the final point of any lift is a first conjugate point of $\tilde{p}_1$ along the lift. As it is already seen that the locus of the final points of all lifts consists of a single point $\tilde{p}_2$, we can take $Q(\tilde{p}_1)=\tilde{p}_2$, where $Q(\tilde{p}_1)$ denotes the locus of first conjugate points of $\tilde{p}_1$ along all geodesics issuing from $\tilde{p}_1$. Similarly we get $Q(\tilde{p}_2)=\tilde{p}_1$. Since the projection of a minimal geodesic segment joining $\tilde{p}_1$ and $\tilde{p}_2$ in $\tilde{M}$ is a geodesic loop at p in M, the length of the minimal geodesic segment is equal to $2l$, and is equal to that of all lifts of geodesic loops at p. Hence the cut locus $C(\tilde{p}_1)$ of $\tilde{p}_1$ consists of $\tilde{p}_2$ and $C(\tilde{p}_2)$ consists of $\tilde{p}_1$. Thus $\tilde{M}$ is homeomorphic to S^n.

Let f be a homeomorphism of S^n onto $\tilde{M}$. The sphere may be regarded as an universal covering manifold of M with the covering map $p=\pi\circ f$ and the fundamental group $\pi_1(M)$ is of order 2. Making use of the spectral sequence of such covering manifold and the homology groups of the cyclic group of order 2, we can precisely calculate the homology group of M (cf. *Hu* [8]). We have now the following results:

$$\begin{aligned} H_j(M) &= 0 && \text{for } j\text{; even}, && 2 \leqq j \leqq n\,, \\ H_j(M) &= Z_2 && \text{for } j\text{; odd}, && 1 \leqq j \leqq n-1\,, \\ H_n(M) &= Z && \text{if } n \text{ is odd.} \end{aligned}$$

We shall sketch briefly the proof of the relations mentioned above. By the covering space theorem of the homotopy theory, we get

$$\pi_j(M) = 0 \qquad \text{for } 2 \leqq j \leqq n-1\,.$$

Taking account of the above homotopical structure of M, we have in the spectral sequence of the covering manifolds

$$H_j(M) = H_j(Z_2) \qquad \text{for } 1 \leqq j \leqq n-1\,,$$

where $H_j(Z_2)$ denotes the j-th homology group of the cyclic group Z_2 of order 2. On the other hand, taking account of the assertion given by the complete computation of the homology groups $H_j(Z_2)$, we have

$$\begin{aligned} H_j(Z_2) &= 0 && \text{for } j\text{; even}, && j \geqq 2\,, \\ H_j(Z_2) &= Z_2 && \text{for } j\text{; odd}, && j \geqq 1\,. \end{aligned}$$

After determining the orientability of M, the n-th homology group $H_n(M)$ of M is given. Thus the homology groups of M are obtained.

Concerning the orientability of M, we see [8] that the following theorem holds, because the sphere S^n is the universal covering manifold of M and the fundamental group is the cyclic group of order 2.

We can state

Theorem 3.5. *If M is not simply connected, then M is orientable for odd n, and M is not orientable for even n.*

Remark 3.2. As a consequence of Theorem 3.4, we can prove the fact that the integral cohomology ring $H^*(M, Z)$ is isomorphic to $H^*(PR^n, Z)$, and moreover it is obvious that the third assertion of Corollary 3. 3 holds.

Tokyo University of Agriculture and Technology

BIBLIOGRAPHY

[1] J. F. Adams, On the non-existence of elements of Hopf invariant one, Ann. of Math., **72** (1960), 20-104.

[2] J. Adem, Relations on iterated reduced powers, Proc. Nat. Acad. Sci. U.S.A., **39** (1953), 636-638.

[3] R. Bott, On manifolds all of whose geodesics are closed, Ann. of Math., **60** (1954), 375-382.

[4] E. Cartan, Sur certaines forms riemanniennes remarquables des géométries a groupe fondamental simple, Ann. Sci. Ecole Norm. Sup., **44** (1927), 347-467.

[5] J. Eells and N. Kuiper, Closed manifolds which admit non-degenerate functions with three critical points, Nederl. Acad. Wetensch. Proc. Ser. A, **64** (1961), 411-417.

[6] L. W. Green, Auf Wiedersehensflächen, Ann. of Math., **78** (1963), 289-299.

[7] K. Hatsuse and R. Takagi, On W^*-manifolds, to appear.

[8] S. T. Hu, Homotopy theory, New York, 1959.

[9] W. Klingenberg, Contributions to Riemannian geometry in the large, Ann. of Math., **69** (1959), 654-665.

[10] J. W. Milnor, Some consequences of a theorem of Bott, Ann. of Math., **68** (1958), 444-449.

[11] ——————, Morse theory, Ann. of Math. Studies, No. 51, (1963).

[12] M. Morse, The calculus of variations in the large, New York, 1934.

[13] H. Nakagawa, On Riemannian manifolds with certain foliated conditions, to appear.

[14] T. Ōtsuki, On focal elements and the spheres, Tôhoku Math. J., **17** (1965), 285-304.

[15] H. Samelson, On manifolds with many closed geodesics, Portugaliae Math., **22** (1963), 193-196.

[16] J. P. Serre, Homologie singuliere des espaces fibrés, Ann. of Math., **54** (1951), 425-505.

[17] S. Smale, Generalized Poincaré conjecture in dimesions greater than four, Ann. of Math., **74** (1961), 391-406.

[18] H. O. Singh Varma, Homogeneous manifolds all of whose geodesics are closed, Indag. Math., **48** (1965), 813-819.