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Theory of Finsler spaces and differential geometry of tangent bundles

By

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The purpose of the present paper is to show the close relation between the theory of Finsler spaces and differential geometry of tangent bundles, and to develope the former as a theory of special linear connections on the tangent bundles. In a previous paper [11], a special linear connection Γ' on the tangent bundle was derived from a given Finsler connection, and called the linear connection of Finsler type. It was also shown that, by symmetrization of Γ', the linear connection given by K. Yano and A.J. Ledger [16] was obtained as a specially simple case. Further, it was known that the tangent bundle was thought of as a Riemann manifold with a Riemann metric which was defined by lifting a given Finsler metric on the base manifold to the tangent bundle [13], [15].

In the present paper, we are specially concerned with the difference between the linear connection of Finsler type Γ' and the Riemann connection $\bar{\Gamma}$ with respect to the lifted Riemann metric. The difference will be exactly formulated by defining the *strain tensor of the Finsler connection* under consideration, by means of which the parallel displacement of tangent vectors with respect to Γ' will be compared with that with respect to $\bar{\Gamma}$. These considerations lead us to the concept of *normal Finsler connections*, which seem natural from the standpoint of the theory of linear connections on the tangent bundles.

It appears from the theory of Finsler spaces that the last section is an appendix, because some theorems given in that place will be concerned only with the problems arising from Riemann metrics and not from Finsler metrics. Those theorems, however, will give interesting results as for sectional curvatures of the lifted Riemann metrics.

§1. Fibre bundles

Let $F(M)$ (M, π, G) be the bundle of frames over a differentiable n-manifold M, where $\pi: F(M)\rightarrow M$ is the natural projection, and $G=GL(n, R)$ is the structural group. Throughout the present paper, we shall denote by the letter z a point of $F(M)$, and by (x^i, z^i_a) a canonical coordinate of z, corresponding to a local coordinate (x^i) on the base manifold M, namely,

$$z = (z_a), \quad a = 1, \cdots, n; \quad z_a = z^i_a \frac{\partial}{\partial x^i} \in M_x ,$$

where we use the notation M_x to denote the tangent vector space to M at a point x.

Next, let $T(M)$ (M, τ, F, G) be the bundle of tangent vectors to M, where $\tau: T(M)\rightarrow M$ is the natural projection, and F is the standard fibre, that is, a real vector n-space. In the following, we shall denote by the letter y a point of $T(M)$, and by (x^i, y^i) a canonical coordinate of y, corresponding to a local coordinate (x^i) on M, namely, $y=y^i(\partial/\partial x^i)$. Further, let (e_a), $a=1, \cdots, n$, be a base of F, which is considered to be fixed throughout the paper. Then, the operation of $g\in G$ on F is defined, referring to the base (e_a), such that

$$g = (g^a_b): f = f^a e_a \rightarrow gf = g^a_b f^b e_a .$$

By the projection τ of the tangent bundle $T(M)$, the induced bundle $\tau^{-1}F(M)$ is obtained from the frame bundle $F(M)$, which will be denoted by $Q(T(M), \pi_1, G)$. The structural group of Q is $G=GL(n, R)$, too, and the total space Q is given by

$$Q = \{(y, z)\in T(M)\times F(M) \mid \tau(y) = \pi(z)\} .$$

Thus, a point $q=(y, z)\in Q$ is a pair of a tangent vector y and a tangent frame z to M at the same point $x=\tau(y)=\pi(z)\in M$. The principal bundle Q will be called the *Finsler bundle* of M under consideration, which will play an important role in the theory of Finsler spaces. The mapping π_1 is the natural projection $Q \to T(M)$ such that $\pi_1(y, z) = y$. A right translation R_g of Q by $g \in G$ is naturally induced from a right translation $\underline{R}_g$ of $F(M)$ such that

$$R_g(y, z) = (y, \underline{R}_g(z)), \qquad \text{namely,} \quad (y, z)g = (y, zg)\,.$$

We shall denote by $q = (y, z)$ a point of Q and by (x^i, y^i, z^i_a) a canonical coordinate of q, where (x^i, y^i) and (x^i, z^i_a) are canonical coordinates of y and z respectively.

In order to consider the differential geometry of the tangent bundle $T(M)$, we shall be concerned with the frame bundle $FT(M)(T(M), \pi', G')$ over $T(M)$, where $\pi' : FT(M) \to T(M)$ is the natural projection, and $G' = GL(2n, R)$ is the structural group.

Let F' be a real vector $2n$-space, and then F' may be identified with the direct sum $F \oplus F$, and we have the fixed base (e'_α), $\alpha = 1, \cdots, 2n$, which is obtained from the above fixed base (e_a) of F as follows [11]:

$$e'_a = (e_a, 0), \qquad a = 1, \cdots, n\,,$$
$$e'_{(a)} = (0, e_a), \qquad (a) = n + a.$$

Therefore, referring to the base (e'_α), we can obtain the operation of $g' \in G'$ on F' as well as that of $g \in G$ on F. Let $\rho : F \oplus F \to F'$ be the identification, and then we see, for $f = f^a e_a \in F$,

$$\rho(f, 0) = f^a e'_a, \quad \rho(0, f) = f^a e'_{(a)}\,.$$

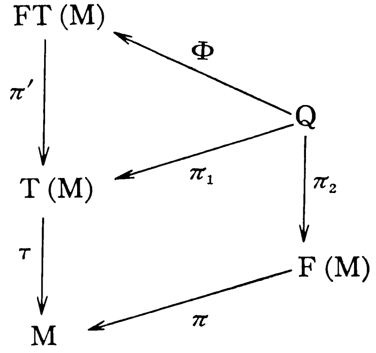
Further, we shall define a natural homomorphism $\varphi : G \to G'$ [11] by

$$g = (g^a_b) \in G$$
$$\to \varphi(g) = \begin{pmatrix} g^a_b & 0 \\ 0 & g^a_b \end{pmatrix} \in G'\,.$$

It is easy to verify that

$$(1.1) \qquad \rho(gf, 0) = \varphi(g)\rho(f, 0),$$
$$\rho(0, gf) = \varphi(g)\rho(0, f)\,.$$

Finally, we shall illustrate fibre bundles and mappings used



often in the following. The mapping π_2 is called the induced mapping such that $\pi_2(y, z)=z$, and the mapping Φ is a bundle homomorphism, which will be defined in §5 by a non-linear connection H on $T(M)$, together with the above homomorphism $\varphi: G\to G'$.

§2. Non-linear connections

A frame $z\in\pi^{-1}(x)$ at a point $x\in M$ is interpreted as an admissible mapping from the standard fibre F of $T(M)$ to the fibre $\tau^{-1}(x)$ over x. Since the mapping is a homeomorphism, we denote by z^{-1} the inverse mapping of z. Further, by means of the fixed base (e_a) of F, a global coordinate (f^a) on F is introduced such that $f=f^a e_a$. Hence, $f\in F$ gives a tangent vector field $j(f)$ on F:

$$j(f) = f^a \frac{\partial}{\partial f^a}, \qquad f = f^a e_a.$$

We shall call $j(f)$ a *parallel vector field*, corresponding to $f\in F$.

Now, since a tangent vector X to the manifold M at x is a point of the fibre $\tau^{-1}(x)$ over x, $z^{-1}X$ is a vector of F. Corresponding to $z^{-1}X$, the parallel vector field $j(z^{-1}X)$ on F is obtained. Then, given a point $y\in\tau^{-1}(x)$, we obtain the tangent vector $j(z^{-1}X)_{z^{-1}y}$ to F at the point $z^{-1}y$, and thus

$$X^v = dz(j(z^{-1}X)_{z^{-1}y}), \tag{2.1}$$

where dz is the differential of the mapping z. X^v is a tangent vector to $T(M)$ at the point y and obviously vertical. It will further be easily verified that X^v depends only on y and not on the frame z used. X^v as thus obtained is called the *vertical lift* of X to y [11]. If we put $X=X^i(\partial/\partial x^i)$ in terms of a local coordinate (x^i), then X^v is expressed by

$$X^v = X^i \frac{\partial}{\partial y^i}, \tag{2.2}$$

referring to the canonical coordinate (x^i, y^i) on $T(M)$.

By means of the notion of the vertical lift, we can introduce a special vertical vector field $\mathfrak{v}$ on $T(M)$, which will play an important role in future. That is, since a point y of $T(M)$ is regarded as a tangent vector to M at $x=\tau(y)$, the vertical lift y^v of y to the point y is defined as above, which will be denoted by $\mathfrak{y}$. Referring to a canonical coordinate (x^i, y^i), $\mathfrak{y}$ is expressed by $y^i(\partial/\partial y^i)_y$. It is remarked that the tangent vector field $\mathfrak{y}$ as thus defined was introduced without a tangent vector field on M, and, in this sense, we refer it the *intrinsic vector field* on $T(M)$.

Further, we consider an integral curve of the intrinsic vector field, which will be called an *intrinsic curve* on $T(M)$. The differential equations of an intrinsic curve are given by

$$\frac{dx^i}{dt} = 0, \quad \frac{dy^i}{dt} = y^i . \tag{2.3}$$

It is seen from (2. 3) that the curve has been already treated in [8].

Next, we shall be concerned with a concept of non-linear connection on M, or in $T(M)$, which is one of the basic concepts in the present paper.

Definition. A *non-linear connection* H in the tangent bundle $T(M)$ is a distribution $y \in T(M) \to H_y$ such that a tangent vector space $T(M)_y$ to $T(M)$ at any point y is expressed

$$T(M)_y = H_y \oplus T(M)^v_y \qquad \text{(direct sum)},$$

where $T(M)^v_y$ is the vertical subspace of $T(M)_y$.

The modern definition of a non-linear connection as above was given by W. Barthel [1], who developed the theory of holonomy groups of nonlinear connections. He imposed further two conditions, namely, homogeneity and differentiability. The homogeneous condition is not required for our present purpose, though it has been often used in our previous papers. On the other hand, the differentiable condition is, of course, necessary in our discussion, but it will be used without mention.

H_y is called the horizontal subspace of $T(M)_y$, and $X \in H_y$ is horizontal. If a linear connection Γ is given in the frame bundle $F(M)$, the associated linear connection H is obtained in $T(M)$ [12, p. 43]. The associated connection is a specially simple case of non-linear connection. With respect to a non-linear connection H, the lifting operator $l_y: M_x \to H_y$, $x = \tau(y)$, is introduced such that $\tau l_y = 1$. If we refer to a local coordinate (x^i) of x, and put $X = X^i(\partial/\partial x^i)$, the lift $l_y X$ of X to a point y is expressed

$$(2.4) \qquad l_y X = X^i\left(\frac{\partial}{\partial x^i} - F_i^j(x, y)\frac{\partial}{\partial y^j}\right),$$

where $F_i^j(x, y)$ are functions of x^k and y^k, and called the *connection parameters* of H. In a case of an associated connection with a linear connection in $F(M)$, the connection parameters are reduced to the form $y^k F_k{}^j{}_i(x)$.

The notion of a lift of a curve C on M to $T(M)$ will be introduced with respect to a given non-linear connection H. A lift of C is a horizontal curve covering C. Let $x^i(t)$ be a curve C, and then a lift of C is given by the differential equation

$$(2.5) \qquad \frac{dy^i}{dt} + F_j^i(x(t), y(t))\frac{dx^j}{dt} = 0.$$

A point $y(t)$ of the lift is called to be obtained from its starting point $y(0)$ by the *parallel displacement* along the curve C. This is nothing but the notion of an usual parallel displacement of a tangent vector $y(0)$ to M at the point $x(0)$.

In a similar way to the case of the intrinsic vector field $\mathfrak{v}$, we obtain the horizontal vector field $\mathfrak{h}$, that is, the lift of a tangent vector $y \in M_x$ to the point $y \in T(M)$. $\mathfrak{h}$ will be called the *H-intrinsic vector field* on $T(M)$ with respect to the non-linear connection H. In terms of a local coordinate (x^i, y^i) on $T(M)$, $\mathfrak{h}$ is expressed

$$(2.6) \qquad \mathfrak{h} = y^i\left(\frac{\partial}{\partial x^i} - F_i^j(x, y)\frac{\partial}{\partial y^j}\right).$$

An integral curve of the H-intrinsic vector field $\mathfrak{h}$ is called the *H-intrinsic curve*, which is given by the differential equations:

$$(2.7) \qquad \frac{dx^i}{dt} = y^i, \quad \frac{dy^i}{dt} + F_j^i(x(t), y(t))y^j(t) = 0.$$

A projection of an H-intrinsic curve on the base manifold M is called a *path* with respect to the non-linear connection H. It will be easy to show that a path is characterized by the property that the tangent vector of the curve is parallel along the curve. The differential equation of a path is given by

$$\frac{d^2x^i}{dt^2} + F^i_j\Big(x(t), \frac{dx}{dt}\Big)\frac{dx^j}{dt} = 0 . \tag{2.8}$$

§ 3. Flat connections in Finsler subbundles

We shall be concerned with the Finsler bundle $Q(T(M), \pi_1, G)$ of the manifold M. Given a point $q \in Q$, we define a mapping

$$L_q : G \to Q, \quad g \in G \to qg \in Q ,$$

and hence, corresponding to an element A of the Lie algebra $L(G)$ of G, a vertical vector field $F(A)$ is obtained such that $F(A)_q = L_q(A)$ at q, where by the same letter L_q we denoted the differential of the above mapping L_q. $F(A)$ is called the fundamental vector field on Q, corresponding to $A \in L(G)$. It is well known [12] that, given a base $(A_1, \cdots, A_r)$, $r = n^2$, of $L(G)$, the set of fundamental vector fields $(F(A_1), \cdots, F(A_r))$ spans a vertical subspace Q''_q of a tangent vector space Q_q to Q at any point q. The similar fact holds good in a case of the frame bundle $F(M)$.

Lemma 1. *Let $F(A)$ and $\underline{F}(A)$ be fundamental vector fields on the Finsler bundle Q and the frame bundle $F(M)$ respectively, corresponding to an element A of the Lie algebra $L(G)$ of $G = GL(n, R)$. Then, the induced mapping $\pi_2 : Q \to F(M)$, $q = (y, z) \in Q \to z \in F(M)$, carries $F(A)$ to $\underline{F}(A)$.*

Proof. $\underline{F}(A)$ is defined by a mapping

$$L_z : G \to F(M), \quad g \in G \to zg \in F(M),$$

such that $\underline{F}(A)_z = L_z(A)$. The proof will be easily obtained by the fact that $\pi_2 L_q = L_z$, $q = (y, z)$.

Lemma 2. *A tangent vector $X \in Q_q$ vanishes, if and only if*

$$\pi_1(X) = 0, \quad \pi_2(X) = 0 ,$$

where π_1 ; $Q \to T(M)$ is the projection, and $\pi_2 : Q \to F(M)$ is the induced mapping.

Proof. The condition is clearly necessary. Conversely, if it is satisfied, then X is vertical by means of $\pi_1(X)=0$, which implies that there exists an element $A\in L(G)$ such that $X=F(A)_q$. As a consequence of Lemma 1, $\pi_2(X)=0=\underline{F}(A)_z$, $q=(y, z)$, and hence $A=0$.

Now, we shall introduce following three subspaces of a tangent vector space Q_q to Q at q:

$$Q_q^v = \{X\in Q_q \mid \pi_1(X) = 0\}, \quad Q_q^{iv} = \{X\in Q_q \mid \pi_2(X) = 0\},$$
$$Q_q^{*v} = \{X\in Q_q \mid \tau\pi_1(X) = 0\} .$$

Q_q^v is well known and called the vertical subspace, while Q_q^{iv} will be called the *induced-vertical subspace* [7, p. 146], and Q_q^{*v} the *quasi-vertical subspace.*

Proposition 1.

(1) $\quad Q_q^{*v} = Q_q^v \oplus Q_q^{iv} \quad$ (*direct sum*),

(2) $\quad R_g Q_q^{iv} = Q_{qg}^{iv}$,

where R_g *is a right translation of* Q *by* $g\in G$.

(3) *Given* $x\in M$ *and* $z\in\pi^{-1}(x)$, *we define a mapping*

$$s_z\colon \tau^{-1}(x) \to Q, \quad y\in\tau^{-1}(x) \to (y, z)\in Q ,$$

and then $s_z(T(M_y^v))=Q_q^{iv}$, $q=(y, z)$, *where* $T(M)_y^v$ *is the vertical subspace of the tangent vector space* $T(M)_y$ *to* $T(M)$ *at* y.

Proof. We shall first prove (3). Given a tangent vertical vector $X\in T(M)_y^v$, we see $\pi_2 s_z(X)=0$ by virtue of $\pi_2 s_z=$constant, which implies that $s_z(X)\in Q_q^{iv}$. Consequently we have $s_z(T(M)_y^v)\subset Q_q^{iv}$. To prove the reverse inclusion, if we take $X\in Q_q^{iv}$, it follows that $\tau\pi_1(X)=\pi\pi_2(X)=0$, which implies that $\pi_1(X)\in T(M)_y^v$. Further, it follows from $\pi_1 s_z=$identity and $\pi_2 s_z=$constant that $\pi_1(X-s_z\pi_1(X))=0$, $\pi_2(X-s_z\pi_1(X))=0$. Therefore we have $X=s_z\pi_1(X)$ from Lemma 2, and hence $s_z(T(M)_y^v)\supset Q_q^{iv}$. Thus we proved (3).

Next, we shall show (1). Given $X\in Q_q^{*v}$, it follows from $\tau\pi_1(X)=0$ that $\pi_1(X)\in T(M)_y^v$, and from (3) $s_z\pi_1(X)\in Q_q^{iv}$. Since $\pi_1(X-s_z\pi_1(X))=0$, we obtain $X=s_z\pi_1(X)+Y$, where Y is vertical. Therefore we have $Q_q^{*v}=Q_q^v+Q_q^{iv}$. The fact that $Q_q^v\cap Q_q^{iv}=0$ is a direct result of Lemma 2. Finally (2) will be easily verified from $\pi_2 R_g = \underline{R}_g \pi_2$, where $\underline{R}_g$ is a right translation of $F(M)$ by $g \in G$.

Now, given a point $x \in M$, a subbundle $Q(x)$ of Q can be introduced, whose base space is the fibre $\tau^{-1}(x)$ over x. $Q(x)$ is called the *Finsler subbundle* over x. A point of $Q(x)$ is a pair (y, z) of a tangent vector y and a frame z at the fixed point x. It is obvious that a tangent vector space $Q(x)_q$ to $Q(x)$ at $q=(y, z)$ is nothing but the quasi-vertical subspace Q_q^{*v} of Q_q, because $Q_q^{*v} = \{X \in Q_q \mid \pi_1(X) \text{ is vertical}\}$. Then, we observe that (1) and (2) of Proposition 1 mean that the distribution $Q^{iv}: q \in Q(x) \to Q_q^{iv}$ on $Q(x)$ is a connection in $Q(x)$. This connection on the fibre $\tau^{-1}(x) = M_x$ is called the *flat connection* in the Finsler subbundle $Q(x)$ over x. On the other hand, (3) of Proposition 1 means that the mapping s_z is the lifting operator to the point (y, z) with respect to the flat connection. The base space of $Q(x)$ is homeomorphic to the vector space F (the standard fibre of $T(M)$), and (y^i) as often used in the preceding section is thought of as a global coordinate on the base space $\tau^{-1}(x)$ of $Q(x)$. Then, the lift $s_z X$ of a tangent vector $X = X^i(\partial/\partial y^i)_y$ to (y, z) is immediately expressed by $s_z(X) = X^i(\partial/\partial y^i)_{(y,z)}$ in terms of the coordinate (y^i, z^i) on $Q(x)$. Accordingly, it is seen that the connection parameters of the flat connection vanish identically in terms of the coordinate (y^i). Therefore, the covariant differentiation with respect to the flat connection is nothing but the partial differentiation by y^i. In the classical theory of Finsler spaces, this fact has been well known and often used in order to derive new tensors from a given scalar or tensor (see, for example, [14]).

§ 4. Finsler metrics

It is well known that a tensor field T of type V on a differentiable manifold M is thought of as a mapping from the frame bundle $F(M)$ to a vector space V such that $T\underline{R}_g = g^{-1}T$ is satisfied [5, p. 66], where $\underline{R}_g$ is a right translation of $F(M)$ by $g \in G$, and g^{-1} in the right-hand side is the operation of the representation of g^{-1} on V. On the other hand, in classical theory, components of a tensor field T on a Finsler space M are generally functions not only of x^i but also of y^i, and hence T cannot be regarded as a mapping from $F(M)$ to some vector space. In previous papers, the present author has been shown that, from the modern standpoint, a tensor field T on a Finsler space M is regarded as a mapping from the Finsler bundle Q to some vector space such that $TR_g = g^{-1}T$. Thus, for an example, a tensor field of (1, 1)-type T is a mapping $Q \to F \otimes F^*$ (F^* is the dual space of the real vector n-space F), so that T is expressed

$$T(q) = T^i_j(x, y) z^{-1a}_{\ i} z^j_b e_a \otimes e^b \qquad \text{at} \quad q = (x^i, y^i, z^i_a)\,,$$

where $z^{-1a}_{\ i}$ are elements of the inverse matrix of the matrix (z^i_a), and (e^b) is the dual base of the fixed base (e_a) of F. Coefficients $T^i_j(x, y)$ are functions of x^k and y^k, which are classical components of T. $y = \pi_1(q)$ is called the element of support of $T(q)$, following E. Cartan [4].

As an example of Finsler tensor fields, let us remember the characteristic field γ on Q, which is defined by $\gamma(q) = z^{-1}y$, $q = (y, z)$. The characteristic field γ is a Finsler tensor field of (1, 0)-type, and has played an important role in our previous papers. We may say that γ is the element of support itself, because the classical components of γ are y^i.

Definition. *Finsler metric function* L is a positive-valued scalar (namely, a Finsler tensor field of (0, 0)-type) and further supposed to be positively homogeneous of degree 1. Thus L satisfies

$$LR_g = L, \qquad\qquad Lh_r = r \cdot L\,,$$

where h_r, $r > 0$, is the homogeneous mapping [1] such that $h_r(y, z) = (ry, z)$, which is used in order to define the homogeneity with respect to y [7].

Then, there exists an unique positive-valued function $\underline{L}$ on $T(M)$ such that $\underline{L}\pi_1 = L$. In fact $\underline{L}$ is given by Ls_z, where s_z is used in Proposition 1. Then, $(\underline{L}(y))^{1/2}$ is called the *absolute length* of a tangent vector y to M. The arc-length of a curve $t \to x(t)$ on M is by definition the integral of the absolute length of the tangent vector $x'(t)$ to the curve.

Let us denote by Δ_0 the covariant differential operator with respect to the flat connection as defined in §3. Then, from the function L, a tensor field of $(0, 2)$-type G is defined by $G=\frac{1}{2}\Delta_0^2(L^2)$. We now have to impose upon L the regular condition:

(1) $G(f, f)\geqq 0, \qquad f\in F,$

(2) $G(f, f) = 0, \qquad$ if and only if $f=0$.

The tensor field of $(0, 2)$-type G as thus obtained is called the *Finsler metric tensor,* constructed from the Finsler metric function L. In the following, components of G are written $g_{ij}(x, y)$ as usual, and hence we obtain $G(q)=g_{ij}(x, y)z_a^i z_b^j e^a\otimes e^b$, where $q=(x^i, y^i, z_a^i)$.

Let X and Y be Finsler vector fields, that is, tensor fields of $(1, 0)$-type, and then we have $X(q)$, $Y(q)\in F$, $q\in Q$, and hence a real number $G(X, Y)_q=G(q)(X(q), Y(q))$, which is called the scalar product of X, Y with respect to the element of support $y=\pi_1(q)$. In particular, the non-negative number $(G(X, X)_q)^{1/2}$ is called the *relative length* of X with respect to y.

Now, we shall deal with a non-linear connection H in $T(M)$ and a Finsler metric L. In §2, the notions of parallel displacement and path have been introduced.

Definition. A non-linear connection H is said to be *metrical* with respect to a given Finsler metric L, when the following two conditions are satisfied:

(1) A parallel displacement of a tangent vector to M preserves its absolute length.

(2) Any path coincides with an extremal of the variation problem with respect to the Lagrangean $\underline{L}$, provided that the path parameter is taken as the arc-length s of the path.

It follows from the homogeneity of L that the condition (1) is expressed by $d(g_{ij}(x, y)y^i y^j)=0$, where $dy^i=-F_j^i(x, y)dx^j$ from (2.5), and hence we obtain

$$F_{0j} = \gamma_{00j}\,, \tag{4.1}$$

where we put $F_{ij}=g_{ik}F_j^k$ and γ_{ijk} are Christoffel's symbols constructed from $g_{ij}(x, y)$ with respect to x^k. Further, following the usual manner, index 0 means the contraction by the element of support y^k. Next, an extremal in the condition (2) is given by

$$\frac{d^2x^i}{ds^2}+\gamma_j{}^i{}_k\left(x,\frac{dx}{ds}\right)\frac{dx^j}{ds}\frac{dx^k}{ds}=0,\qquad (\gamma_j{}^i{}_k=g^{il}\gamma_{jlk}),$$

and hence (2) is expressed

$$F_{i0}=\gamma_{0i0}\,. \tag{4.2}$$

We sum up the above results for the later use.

Proposition 2. *A non-linear connection H is metrical with respect to a Finsler metric L, if and only if the connection parameters* F^i_j *satisfy* (4.1) *and* (4.2), *where* γ_{ijk} *are Christoffel's symbols constructed from the components of the Finsler metric tensor* $g_{ij}(x, y)$ *with respect to x.*

Given a Finsler metric L, there exists really a metrical non-linear connection H. In fact. if we put $G^i=\frac{1}{2}\gamma_0{}^i{}_0$, the non-linear connection H whose connection parameters are given by

$$F^i_j(x,y)=\frac{\partial G^i}{\partial y^j} \tag{4.3}$$

is then metrical, as is well known [4]. The connection H as thus obtained will be called *Berwald's non-linear connection* [2, p. 45].

§5. Finsler decomposition of tensor fields

Assume that a non-linear connection H be given in the tangent bundle $T(M)$, and we shall define a bundle homomorphism from the Finsler bundle Q to the frame bundle $FT(M)$ of the tangent bundle $T(M)$ as follows. The homomorphism $\varphi: G\to G'$ between the structural groups has been defined in §1. If we take a point $q=(y, z)\in Q$, $z=\tau_2(q)$ is a frame, that is, a set of linearly independent n vectors $z=(z_a)$ at $x=\tau(y)=\pi(z)\in M$, and hence their vertical lifts $z^v=(z^v_a)$ to the point $y\in T(M)$ span the vertical subspace $T(M)$, while their horizontal lifts $z^h=(z^h_a)$ to y with respect to H span the horizontal subspace H_y. Therefore the set (z^h, z^v) is considered as a frame at y, that is, a point of $FT(M)$. Put $\Phi(q)=(z^h, z^v)$, and then the mapping $\Phi: Q\to FT(M)$ is defined.

It is obvious that $\pi'\Phi=\pi_1$, and Φ is compatible with the homomorphism $\varphi: G\rightarrow G'$, namely $\Phi R_g=R'_{\varphi(g)}\Phi$, where $R'_{\varphi(g)}$ is a right translation of $FT(M)$ by $\varphi(g)\in G'$. Consequently the mapping Φ as above defined, together with φ, is a bundle homomorphism [12, p. 20], [11, p. 258 and the final remark].

In the following, we shall show that *a tensor field on* $T(M)$ *is interpreted as a set of Finsler tensor fields*, by making use of the above bundle homomorphism Φ. In order to show that, for example, let us consider a tensor field of (0, 2)-type T on $T(M)$. Then, T is regarded as a mapping $FT(M)\rightarrow F'^*\otimes F'^*$ and satisfies $TR'_{g'}=g'^{-1}T$, where $R'_{g'}$ is a right translation of $FT(M)$ by $g'\in G'$. Now, we shall introduce a mapping $T_{11}: Q\rightarrow F^*\otimes F^*$ such that, for $f_1, f_2\in F$,

$$T_{11}(f_1, f_2) = T(\rho(f_1, 0), \rho(f_2, 0))\Phi ,$$

where $\rho: F\oplus F\rightarrow F'$ is the identification as defined in §1. In order to see that T_{11} is really a Finsler tensor field, it is enough to verify that $T_{11}R_g=g^{-1}T_{11}$, $g\in G$. In fact, it follows from (1.1) that

$$\begin{aligned} T_{11}R_g(f_1, f_2)_q &= T(\rho(f_1, 0), \rho(f_2, 0))_{\Phi(q)\varphi(g)} \\ &= (\varphi(g)^{-1}T)(\rho(f_1, 0), \rho(f_2, 0))_{\Phi(q)} = T(\varphi(g)\rho(f_1, 0), \varphi(g)\rho(f_2, 0))_{\Phi(q)} \\ &= T(\rho(gf_1, 0), \rho(gf_2, 0))_{\Phi(q)} = T_{11}(gf_1, gf_2)_q = (g^{-1}T_{11})(f_1, f_2)_q . \end{aligned}$$

Therefore, we obtain a Finsler tensor field of (0, 2)-type T_{11}. Further, three Finsler tensor fields T_{12}, T_{21} and T_{22} of the same type will be introduced as follows:

$$\begin{aligned} T_{12}(f_1, f_2) &= T(\rho(f_1, 0), \rho(0, f_2))\Phi , \\ T_{21}(f_1, f_2) &= T(\rho(0, f_1), \rho(f_2, 0))\Phi , \\ T_{22}(f_1, f_2) &= T(\rho(0, f_1), \rho(0, f_2))\Phi . \end{aligned}$$

Definition. The set of four Finsler tensors $(T_{11}, T_{12}, T_{21}, T_{22})$ as above defined is called the *Finsler decomposition* of the tensor field of (0, 2)-type T on the tangent bundle $T(M)$.

We shall deal with components of T and its Finsler decomposition. Given a point $q=(y, z)$ of Q, T is expressed

$$T\Phi(q) = T_{\alpha\beta}e'^{\alpha}\otimes e'^{\beta}$$

and then we obtain, for an example,

$$T_{21}(f_1, f_2)_q = T_{\alpha\beta}e'^{\alpha}\otimes e'^{\beta}(f_1^a e'_{(a)}, f_2^b e'_b) = T_{(a)b}f_1^a f_2^b .$$

Thus, components of T_{21}, referring to the frame z, are equal to $T_{(a)b}$, where $(a)=n+1, \cdots, 2n$, and $b=1, \cdots, n$. Therefore, if $T_{\alpha\beta}$, $\alpha, \beta=1, \cdots, 2n$, are components of T in terms of the frame $\Phi(q)$, $q=(y, z)$, components of the Finsler decomposition of T are given by

$$\begin{array}{ll} T_{11}: T_{ab}, & T_{12}: T_{a(b)}, \\ T_{21}: T_{(a)b}, & T_{22}: T_{(a)(b)}, \\ a, b = 1, \cdots, n; & (a), (b) = n+1, \cdots, 2n . \end{array}$$

In general, it may be said that *components of the Finsler decomposition of T, referring to the frame z, are given by the classification of components of T, referring to the frame $\Phi(q)$, $q=(y, z)$, based on ranges of indices.*

On the other hand, if we consider components of the Finsler decomposition with respect to canonical coordinates, the circumstances will become complicated. In fact, let $T_{\lambda\mu}$ be components of T with respect to a coordinate (x^i, y^i), and then we have

$$T = T_{\lambda\mu}z'^{\lambda}_{\alpha}z'^{\mu}_{\beta}e'^{\alpha}\otimes e'^{\beta} ,$$

where we put $z'=\Phi(q)$. Then it follows immediately that

$$T_{21}(f_1, f_2)_q = T_{\lambda\mu}z'^{\lambda}_{(a)}z'^{\mu}_{b}f_1^a f_2^b .$$

Since $z'_{(a)}$ is defined as the vertical lift of z_a, we obtain

$$z'_{(a)} = z^i_a\frac{\partial}{\partial y^i}, \qquad \text{hence } z'_{(a)} = (0, z^i_a) .$$

On the other hand, z'_b is defined as the horizontal lift of z_b, we obtain

$$z'_b = z^i_b\left(\frac{\partial}{\partial x^i} - F^j_i\frac{\partial}{\partial y^j}\right), \qquad \text{hence } z'_b = (z^i_b, -z^j_b F^i_j) .$$

Consequently, we see that

$$T_{21}(f_1, f_2)_q = (T_{(i)j} - T_{(i)(k)}F^k_j)z^i_a z^j_b f_1^a f_2^b .$$

Since, referring to the frame z, the tensor T_{21} should be expressed $T_{21}(f_1, f_2)_q = (T_{21})_{ij} z^i_a z^j_b f^a_1 f^b_2$, it follows then from the above that

$$(T_{21})_{ij} = T_{(i)j} - T_{(i)(k)} F^k_j ,$$

which are components of T_{21}, referring to the coordinate (x^i). It will be easily seen that, in general, components of the Finsler decomposition *having at least one of the index* 1 (for example, T_{21}, T_{12}, T_{11}) are expressed by components of the tensor, *together with the connection parameters* F^i_j of the non-linear connection H used.

The Finsler decomposition of a tensor field on the tangent bundle $T(M)$ will be defined for a tensor field of any type in the similar way. As an another example, we consider a tensor field of (1, 1)-type T, which is a mapping $FT(M) \to F' \otimes F'^*$. Now we shall define mappings

$$\rho_1 : F' \to F, \qquad f'^{\alpha} e'_{\alpha} \in F' \to f'^{a} e_a \in F ,$$
$$\rho_2 : F' \to F, \qquad f'^{\alpha} e'_{\alpha} \in F' \to f'^{(a)} e_a \in F .$$

That is, the first half of f' is chosen by ρ_1 and the latter half by ρ_2, with respect to components of f'. Then we put, for $f \in F$,

$$T^1_1(f) = \rho_1(T(\rho(f, 0))\Phi), \qquad T^2_1(f) = \rho_2(T(\rho(f, 0))\Phi) ,$$
$$T^1_2(f) = \rho_1(T(\rho(0, f))\Phi). \qquad T^2_2(f) = \rho_2(T(\rho(0, f))\Phi) .$$

It will be easy to show that these T^1_1, T^2_1, T^1_2 and T^2_2 are Finsler tensor fields of (1, 1)-type, and thus the Finsler decomposition $(T^1_1, T^2_1, T^1_2, T^2_2)$ of T is constructed.

We now return to the consideration of a Finsler metric tensor G. In a previous paper [11], we obtained a Riemann metric tensor $\bar{G}$ on $T(M)$ from G, which was called the *lifted Riemann metric* of G. Since this concept is one of the basic concepts in the theory of tangent bundles, we shall again give the way to define it. Let X be a tangent vector to $T(M)$ at y, and then we have

$$X = hX + vX, \qquad hX \in H_y, \quad vX \in T(M)^v_y .$$

If we put $X_1 = \tau X$, we have $hX = l_y X_1$, and hence it will be natural that the length of hX should be defined to be the relative length $(G(X_1, X_1)_y)^{1/2}$ of X_1 with respect to the Finsler metric G. On the other hand, if a point $x \in M$ is regarded to be fixed, the Finsler metric $G = g_{ij}(x, y) z^i_a z^j_b e^a \otimes e^b$ is considered as a Riemann metric on the fibre $\tau^{-1}(x)$, and g_{ij} are components of the metric in terms of the global coordinate (y^i) on $\tau^{-1}(x)$, because z^i_a are components of the frame z^v (the vertical lift of z) in terms of (y^i). Since vX is tangent to $\tau^{-1}(x)$, the Riemann length $|vX|$ of vX is obtained with respect to the above Riemann metric. Put $vX = X^i(\partial/\partial y^i)$, and then $|vX|^2 = g_{ij}(x, y) X^i X^j$. Besides, vX is regarded as the vertical lift of $X_2 = X^i(\partial/\partial x^i) \in M_x$, and $g_{ij}(x, y) X^i X^j = G(X_2, X_2)$. Then, we define the Riemann length $(\bar{G}(X, X)_y)^{1/2}$ of X with respect to $\bar{G}$ as the Pythagorean sum of the above lengths of X_1 and X_2, that is,

$$\bar{G}(X, X) = G(X_1, X_1) + G(X_2, X_2).$$

by means of which the lifted Riemann metric $\bar{G}$ is defined.

The Finsler decomposition $(\bar{G}_{11}, \bar{G}_{12}, \bar{G}_{21}, \bar{G}_{22})$ of the lifted Riemann metric tensor $\bar{G}$ has been given in [11, (4.2)] as follows:

$$\bar{G}_{11} = \bar{G}_{22} = G, \qquad \bar{G}_{12} = \bar{G}_{21} = 0.$$

§ 6. Finsler connections

A definition of a Finsler connection has been given in previous papers [6], ···, [11], from the viewpoint of fibre bundles. In this section, however, we shall give an alternative characterization of it.

Definition. A *Finsler connection* (Γ, H) on a differentiable manifold M is a pair of a non-linear connection H in the tangent bundle $T(M)$ and a connection Γ in the Finsler bundle Q of M.

By making use of the lifting operator l_q with respect to Γ, we define two subspaces of a tangent vector space Q_q to Q at q as follows:

$$\Gamma^h_q = l_q H_y, \quad \Gamma^v_q = l_q T(M)^v_y, \qquad q = (y, z),$$

and then a pair (Γ^h, Γ^v) of distributions is obtained on Q. It will be easy to show that (Γ^h, Γ^v) is a Finsler connection in a sense of previous papers. Conversely, let (Γ^h, Γ^v) be a Finsler connection in a sense of previous papers, and then we put

$$H_y = \pi_1 \Gamma_q^h, \quad \Gamma_q = \Gamma_q^h \oplus \Gamma_q^v \quad \text{(direct sum)}, \qquad q = (y, z),$$

and then the pair (Γ, H) is a Finsler connection in a sense of the above definition. Therefore those definitions of Finsler connections are equivalent each other.

The theory of Finsler connections based on the above definition has been developed in detail in previous papers, and we shall then describe in outline for our present purpose.

For a vector $f \in F$, two tangent vector fields $B^h(f)$ and $B^v(f)$ on the Finsler bundle Q are obtained by the rule

$$B^h(f)_q = l_q(zf)^h, \quad B^v(f)_q = l_q(zf)^v, \qquad q = (y, z),$$

where $(\ )^h$ and $(\ )^v$ denote the horizontal and vertical lifts respectively with respect to H, while l_q the lift to q with respect to Γ. $B^h(f)$ and $B^v(f)$ are called the h- and v-basic vector fields respectively. Corresponding to the fixed base (e_a) of F, the set $(B^h(e_a))$ spans the h-horizontal subspace Γ^h, while the set $(B^v(e_a))$ does the v-horizontal subspace Γ^v. In terms of a canonical coordinate (x^i, y^i, z_a^i) on Q, those are expressed

$$B^h(e_a) = z_a^i\left(\frac{\partial}{\partial x^i} - F_i^k\frac{\partial}{\partial y^k} - z_b^k F_k{}^j{}_i \frac{\partial}{\partial z_b^j}\right),$$

$$B^v(e_a) = z_a^i\left(\frac{\partial}{\partial y^i} - z_b^k C_k{}^j{}_i \frac{\partial}{\partial z_b^j}\right),$$

where F_i^k are connection parameters of H and $(F_k{}^j{}_i, C_k{}^j{}_i)$ are that of Γ. If $x \in M$ is fixed, $C_k{}^j{}_i$ are connection parameters of the connection Γ^v, that is, the restriction of Γ to the Finsler subbundle $Q(x)$ over x as defined in § 3.

Let $F(A)$ be a fundamental vector field on Q, corresponding to an element A of the Lie algebra $L(G)$ of the structural group G of Q. Then, the structural equations of the Finsler connection (Γ, H) are given by

$$\begin{aligned}
[B^h(f_1), B^h(f_2)] &= F(R^2(f_1, f_2)) + B^h(T(f_1, f_2)) + B^v(R^1(f_1, f_2)),\\
[B^h(f_1), B^v(f_2)] &= F(P^2(f_1, f_2)) + B^h(C(f_1, f_2)) + B^v(P^1(f_1, f_2)),\\
[B^v(f_1), B^v(f_2)] &= F(S^2(f_1, f_2)) \qquad\qquad\qquad\quad + B^v(S^1(f_1, f_2)),
\end{aligned}$$

for $f_1, f_2 \in F$. From these equations, the curvature tensors and torsion tensors are obtained. First, R^2, P^2 and S^2 are called the *h*-, *hv*- and *v*-curvature tensors respectively, and components of R^2, P^2 and S^2 are $R_{j\cdot kl}^{\ i}$, $P_{j\cdot kl}^{\ i}$ and $S_{j\cdot kl}^{\ i}$ respectively, which are given by

$$\begin{aligned} R_{j\cdot kl}^{\ i} &= \frac{\delta F_j{}^i{}_k}{\delta x^l} - \frac{\delta F_j{}^i{}_l}{\delta x^k} + C_j{}^i{}_h\left(\frac{\delta F_k^h}{\delta x^l} - \frac{\delta F_l^h}{\delta x^k}\right) + F_j{}^h{}_k F_h{}^i{}_l - F_j{}^h{}_l F_h{}^i{}_k, \\ P_{j\cdot kl}^{\ i} &= \frac{\partial F_j{}^i{}_k}{\partial y^l} - \frac{\delta C_j{}^i{}_l}{\delta x^k} + C_j{}^i{}_h \frac{\partial F_k^h}{\partial y^l} + F_j{}^h{}_k C_h{}^i{}_l - C_j{}^h{}_l F_h{}^i{}_k, \\ S_{j\cdot kl}^{\ i} &= \frac{\partial C_j{}^i{}_k}{\partial y^l} - \frac{\partial C_j{}^i{}_l}{\partial y^k} + C_j{}^h{}_k C_h{}^i{}_l - C_j{}^h{}_l C_h{}^i{}_k, \end{aligned}$$

where we used the differential operators

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \frac{\partial}{\partial y^j} F_i^j .$$

Next, T and C are called the $h(h)$-and $h(hv)$-torsion tensors, and those components are given by $T_j{}^i{}_k = F_j{}^i{}_k - F_k{}^i{}_j$ and $C_j{}^i{}_k$. Finally, R^1, P^1 and S^1 are called the $v(h)$-, $v(hv)$-and $v(v)$-torsion tensors, and those components are given by

$$R_j{}^i{}_k = \frac{\delta F_j^i}{\delta x^k} - \frac{\delta F_k^i}{\delta x^j}, \quad P_j{}^i{}_k = \frac{\partial F_j^i}{\partial y^k} - F_k{}^i{}_j, \quad S_j{}^i{}_k = C_j{}^i{}_k - C_k{}^i{}_j .$$

Two kinds of covariant derivative $\Delta^h K$ and $\Delta^v K$ of a Finsler tensor field K are obtained with respect to Γ^h and Γ^v respectively. For example, take a Finsler vector field K, and then the components $K^i{}_{|j}$ and $K^i|_j$ of $\Delta^h K$ and $\Delta^v K$ respectively are given by

$$K^i{}_{|j} = \frac{\delta K^i}{\delta x^j} + K^k F_k{}^i{}_j, \quad K^i|_j = \frac{\partial K^i}{\partial y^j} + K^k C_k{}^i{}_j .$$

Now, assume that a Finsler metric tensor G be given, and then a Finsler connection (Γ, H) is said *metrical* with respect to G when $\Delta^h G = \Delta^v G = 0$. The following proposition is well known, and so we show it without proof [4].

Proposition 3. *There exists uniquely a Finsler connection* (Γ, H) *satisfying the following four conditions*:

(1) *Metrical.*

(2) *The* $h(h)$*-torsion* $T=0$.

(3) *The* $v(v)$*-torsion* $S^1=0$.

(4) *The condition* F: $F_j^i(x, y)=y^k F_k{}^i{}_j(x, y)$ *holds good.*

A geometrical meaning of (4) has been shown in [7], from the standpoint of fibre bundles. The Finsler connection as thus determined from the given Finsler metric G is called *Cartan's connection* [10]. It is well known that the connection parameters of Cartan's connection are given by

$$C_{ijk} = g_{jl}C_i{}^l{}_k = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}, \quad F_j^i = \gamma_j{}^i{}_0 - C_j{}^i{}_k \gamma_0{}^k{}_0,$$
$$F_{ijk} = g_{jl}F_i{}^l{}_k = \gamma_{ijk} - C_{ijl}F_k^l - C_{jkl}F_i^l + C_{kil}F_j^l,$$

where $\gamma_j{}^i{}_k$ are Christoffel's symbols constructed from components of G with respect to x^k.

In the following, we shall enumerate some important properties of Cartan's connection for the later use.

(5) The non-linear connection H is Berwald's one, and hence metrical.

(6) $y^i{}_{|j}=0$, and $y^i|_j=\delta_j^i$ (Kronecker's deltas). The former identities mean the above condition (4), while the latter does $y^k C_k{}^i{}_j=0$.

(7) C_{ijk} coincide with the Christoffel's symbols constructed from $g_{ij}(x, y)$ with respect to y^k. We said in §5 that $g_{ij}(x, y)$ are regarded as components of the Riemann metric on the fibre $\tau^{-1}(x)$, and hence C_{ijk} are connection parameters of the Riemann connection on $\tau^{-1}(x)$.

(8) The $v(h)$-torsion R^1 and the $v(hv)$-torsion P^1 are obtained from the h-curvature R^2 and the hv-curvature P^2 respectively by contraction by the element of support:

$$R_j{}^i{}_k = y^l R_l{}^i{}_{jk}, \quad P_j{}^i{}_k = y^l P_l{}^i{}_{jk},$$

while $y^l S_l{}^i{}_{jk}=0$.

(9) The $v(hv)$-torsion P^1 is given by

$$P_{ijk} = g_{il}P_j{}^l{}_k = C_{ijk|l}y^l,$$

which implies that P_{ijk} are completely symmetric and $P_{0jk}=0$.

§ 7. Strain tensors of Finsler connections

By means of the bundle homomorphism $\Phi: Q \to FT(M)$ as defined in § 5, a connection Γ' in $FT(M)$ is derived from the connection Γ in Q, where (Γ, H) is a given Finsler connection in Q. Γ' was called the *linear connection of Finsler type* [11]. In [11], we were concerned with the connection in a great detail, and the following two facts have to be noticed here for the later use.

Proposition 4. [11, (4.4)] *If a Finsler connection (Γ, H) is metrical with respect to a Finsler metric G, the linear connection of Finsler type Γ' derived from (Γ, H) is metrical with respect to the lifted Riemann metric $\bar{G}$ of G. The converse is true, too.*

Proposition 5. [11, (2.16), (2.17)] *The Finsler decomposition of the torsion tensor T' of the linear connection of Finsler type Γ' is given by torsion tensors of the Finsler connection (Γ, H):*

$$T'^{1}_{1\,1} = T, \quad T'^{2}_{1\,1} = R^{1}, \quad T'^{1}_{1\,2} = C,$$
$$T'^{2}_{1\,2} = P^{1}, \quad T'^{1}_{2\,2} = 0, \quad T'^{2}_{2\,2} = S^{1}.$$

The Finsler decomposition of the curvature tensor R' of Γ' is given by curvature tensors of (Γ, H):

$$R'^{1}_{111} = R'^{2}_{211} = R^{2}, \qquad R'^{1}_{211} = R'^{2}_{111} = 0,$$
$$R'^{1}_{112} = R'^{2}_{212} = P^{2}, \qquad R'^{1}_{212} = R'^{2}_{112} = 0,$$
$$R'^{1}_{122} = R'^{2}_{222} = S^{2}, \qquad R'^{1}_{222} = R'^{2}_{122} = 0.$$

We have to agree entirely with H. Busemann, who said [3] that the term "Finsler space" evokes in most mathematician the picture of an impenetrable forest whose entire vegetation consists of tensors. Indeed, according to our general theory of Finsler connections, there are three curvature tensors and five torsion tensors. Further, we have three Ricci's identities and eleven Bianchi's identities. But, from the standpoint of the differential geometry of tangent bundles, we observe from Proposition 5 that those five torsion tensors are nothing but the Finsler decomposition of the *only one* torsion tensor T' and those three curvature tensors are that of the *only one* curvature tensor R', and so on.

Now, we shall have the Riemann connection $\bar{\Gamma}$ with respect to the lifted Riemsnn metric $\bar{G}$ of the Finsler metric G. It will be clear that $\bar{\Gamma}$ does not generally coincide with the linear connection of Finsler type Γ', even if the original Finsler connection (Γ, H) is metrical and so is Γ'. We shall first show

Theorem 1. *Let $\bar{\Gamma}$ be the Riemann connection in $FT(M)$ with respect to the lifted Riemann metric $\bar{G}$ of a Finsler metric G, and let Γ' be a linear connection of Finsler type in $FT(M)$ derived from a Finsler connection (Γ, H). If Γ' coincides with $\bar{\Gamma}$, then the metric G is Riemannian.*

Proof. The Riemann connection $\bar{\Gamma}$ is uniquely determined from $\bar{G}$ by the condition:

(1) Metrical with respect to $\bar{G}$.

(2) The torsion T' vanishes.

Therefore, if $\Gamma'=\bar{\Gamma}$, it follows from Proposition 4 that (Γ, H) is metrical. If follows further from $T'=0$ and Proposition 5 that $C=0$. Thus we see

$$\Delta'' G = 0: \quad \frac{\partial g_{ij}}{\partial y^k} - C_{ijk} - C_{jik} = 0,$$

which implies that $g_{ij}(x, y)$ are functions of x alone.

If G is Riemannian, we obtain the linear Finsler connection (Γ, H) [11] which is essentially equivalent to the Riemann connection with respect to G. Then we obtain the linear connection of Finsler type Γ' derived from the above (Γ, H). In this simplest case, it may be expected that Γ' coincides with the Riemannian $\bar{\Gamma}$. But, this is not so, as will be shown later.

In order to formulate exactly the difference between connections $\bar{\Gamma}$ and Γ', we shall define a tensor K expressing the difference as follows:

Definition. Let $\bar{\Gamma}$ and Γ' be connections in $FT(M)$ as mentioned in Theorem 1. The *strain tensor* K of the Finsler connection (Γ, H) is a tensor field of (1, 2)-type on $T(M)$, that is, a mapping

K: $FT(M)\to F'\otimes F'^*\otimes F'^*$ such that

$$K(f') = \omega'(\bar{B}(f')), \qquad f'\in F',$$

where ω' is the connection form of Γ' [11, (2.3)], and $\bar{B}(f')$ is the basic vector field with respect to $\bar{\Gamma}$, corresponding to $f'\in F'$.

By making use of the strain tensor K, the basic vector field $B'(f')$ with respect to Γ' is expressed by

$$\bar{B}(f') = B'(f')+F(K(f')), \tag{7.1}$$

where $F(K(f'))$ is the fundamental vector on $FT(M)$, corresponding to the element $K(f')$ of the Lie algebra $L(G')$ of the structural group G' of $FT(M)$. Further, the connection form ϖ of the connection $\bar{\Gamma}$ is given by

$$\varpi = \omega'-K(\theta), \tag{7.2}$$

where θ is the basic form on $FT(M)$, namely, $\theta_{z'}=z'^{-1}\pi'$ at $z'\in FT(M)$.

Since the Riemann connection $\bar{\Gamma}$ is metrical, we are, in future, concerned only with a metrical Finsler connection (Γ, H) and so Γ' is supposed to be metrical. We shall first treat the curvature tensor $\bar{R}$ of the Riemann connection $\bar{\Gamma}$, which is given by the structural equation of the connection:

$$[\bar{B}(f'_1), \bar{B}(f'_2)] = F(\bar{R}(f'_1, f'_2)), \quad f'_1, f'_2\in F'.$$

Making use of (7.1), the left-hand side of the above is written

$$\begin{aligned}
&= [B'(f'_1), B'(f'_2)]+[B'(f'_1), F(K(f'_2))] \\
&\qquad -[B'(f'_2), F(K(f'_1))]+[F(K(f'_1)), F(K(f'_2))] \\
&= F(R'(f'_1, f'_2))+B'(T'(f'_1, f'_2))-B'(K(f'_2)f'_1)+F(B'(f'_1)K(f'_2)) \\
&\qquad +B'(K(f'_1)f'_2)-F(B'(f'_2)K(f'_1))+F([K(f'_1), K(f'_2)]) \\
&\qquad +F(F(K(f'_1))K(f'_2))-F(F(K(f'_2))K(f'_1)).
\end{aligned}$$

Comparing first the horizontal parts of the above, we obtain

$$T'(f'_1, f'_2) = K(f'_1, f'_2)-K(f'_2, f'_1), \tag{7.3}$$

where we put $K(f'_2)f'_1=K(f'_1, f'_2)$. Next, comparing the vertical parts, and according to (7.3) and the formula

$$F'(A)K(f') = K(Af') - [A, K(f')], \quad A \in L(G'), \quad f' \in F',$$

we obtain

$$\bar{R}(f'_1, f'_2) = R'(f'_1, f'_2) + \Delta' K(f'_2, f'_1) - \Delta' K(f'_1, f'_2) - [K(f'_1), K(f'_2)] - K(T'(f'_1, f'_2)), \tag{7.4}$$

where Δ' is the covariant differential operator with respect to Γ'.

Let T'_* and K_* be pure covariant tensors obtained from T' and K by the metric tensor $\bar{G}$ respectively, namely,

$$T'_*(f'_1, f'_2, f'_3) = \bar{G}(f'_2, T'(f'_1, f'_3)),$$
$$f'_1, f'_2, f'_3 \in F',$$
$$K_*(f'_1, f'_2, f'_3) = \bar{G}(f'_2, K(f'_1, f'_3)).$$

Since both of $\bar{\Gamma}$ and Γ' are metrical with respect to $\bar{G}$, it follows from (7.1) that

$$\bar{B}(f')\bar{G} = 0 = B'(f')\bar{G} + F(K(f'))\bar{G} = F(K(f'))\bar{G},$$

which implies that

$$K_*(f'_1, f'_2, f'_3) + K_*(f'_2, f'_1, f'_3) = 0. \tag{7.5}$$

Further it follows from (7.3) and (7.5) that

$$2K_*(f'_1, f'_2, f'_3) = T'_*(f'_1, f'_2, f'_3) - T'_*(f'_2, f'_3, f'_1) + T'_*(f'_3, f'_1, f'_2), \tag{7.6}$$

which is the equation giving the strain tensor K.

Since K_* is skew-symmetric with respect to the first two indices, the Finsler decomposition of K_* is completely known by its part $(K_{111}, K_{112}, K_{121}, K_{122}, K_{221}, K_{222})$ (we omit the sign $*$). It follows from (7.6) and Proposition 5 that those are given by

$$\begin{aligned} 2(K_{111})_{abc} &= T_{abc} - T_{bca} + T_{cab}, \\ 2(K_{112})_{abc} &= R_{cab} + C_{abc} - C_{bac}, \\ 2(K_{121})_{abc} &= C_{cab} + C_{acb} - R_{bca}, \\ 2(K_{122})_{abc} &= P_{cab} + P_{bac}, \\ 2(K_{221})_{abc} &= P_{acb} - P_{bca}, \\ 2(K_{222})_{abc} &= S_{abc} - S_{bca} + S_{cab}, \end{aligned} \tag{7.7}$$

where we used the letters

$$T_{abc} = g_{bd}T_a{}^d{}_c, \quad C_{abc} = g_{bd}C_a{}^d{}_c,$$
$$R_{abc} = g_{ad}R_b{}^d{}_c, \quad P_{abc} = g_{ad}P_b{}^d{}_c, \quad S_{abc} = g_{ad}S_b{}^d{}_c.$$

Finally, we shall pay attention to the simplest case where the original metric G is Riemannian and the Finsler connection (Γ, H) is linear, that is, G and (Γ, H) are Riemannian in essential. It then follows from [11, Proposition 1] that non-zero components of (7.7) are

$$2(K_{121})_{abc} = -y^d R_{dbca}, \quad 2(K_{112})_{abc} = y^d R_{dcab},$$

where R_{abcd} are components of the curvature tensor constructed from the Riemann metric G. Therefore, *the strain tensor K does not vanish in general, even if G and (Γ, H) are Riemannian.*

§ 8. Normal Finsler connections

In this section, we shall treat one of the essential problems of the theory of Finsler spaces, that is, to find the most natural connection. It may be admitted that one of the most essential conditions satisfied by the connection is to be metrical with respect to a given Finsler metric G. Therefore, we shall consider only a *metrical Finsler connection* in the following.

Next, we shall pay attention to the parallel displacement of tangent vectors to $T(M)$ with respect to the linear connection of Finsler type Γ' derived from the metrical Finsler connection (Γ, H) under consideration, and compare it with the parallel displacement with respect to the Riemann connection $\bar{\Gamma}$ as above treated.

$$\text{Finsler metric } G \begin{array}{l} \nearrow \ (\Gamma, H) \text{ in } Q \ \rightarrow \ \Gamma' \text{ in } FT(M), \\ \searrow \ \bar{G} \text{ on } T(M) \ \rightarrow \ \bar{\Gamma} \text{ in } FT(M), \end{array}$$

(Γ, H) …… metrical with respect to G,
Γ' …… derived from (Γ, H) by Φ,
$\bar{G}$ …… the lifted Riemann metric of G,
$\bar{\Gamma}$ …… Riemannian with respect to $\bar{G}$.

Let C: $[0, 1] \rightarrow T(M)$, be a differentiable curve on $T(M)$ and let C' and $\bar{C}$ be lifts of C to $FT(M)$ with respect to the connection Γ' and $\bar{\Gamma}$ respectively, where their starting points coincide each other with z'_0. Then, $C'(t)$ and $\bar{C}(t)$ are frames at the point $C(t)\in T(M)$ obtained from z'_0 by parallel displacements along C with respect to Γ' and $\bar{\Gamma}$ respectively.

Generally speaking [5, p. 59], if δ is a curve on $FT(M)$ which covers C and issues from z'_0, the lift $\bar{C}$ is given from δ by modification by the suitable right translation as follows:

$$\bar{C}(t) = \delta(t)g'(t), \qquad g'(t)\in G',$$

where the curve $t\to g'(t)$ on G' satisfies the differential equation

$$\frac{dg'}{dt}+\varpi\left(\frac{dz'}{dt}\right)g'(t) = 0\,,$$

with the initial condition $g'(0)=e'$ (the unit of G'), where $t\to z'(t)$ expresses the curve δ.

Now, take $\delta=C'$, and it follows from (7.2) that $g'(t)$ satisfies

$$(8.1)\qquad \frac{dg'}{dt}-K\left(z'^{-1}\frac{dy}{dt}\right)g'(t) = 0\,,$$

where $t\to y(t)$ is the original curve C on $T(M)$ and $t\to z'(t)$ is the lift C'. Summing up the result, we have

Proposition 4. *In the notation of Theorem* 1, *let* $\bar{C}(t)$ *and* $C'(t)$ *be lifts of a curve* $C(t)$ *on* $T(M)$ *to* $FT(M)$ *with respect to the connections* $\bar{\Gamma}$ *and* Γ' *respectively. Then we have* $\bar{C}(t)=C'(t)g'(t)$, *where* $t\to g'(t)$ *is the curve on the structural group* G' *of* $FT(M)$, *satisfying* (8.1), *where* $t\to y(t)$ *is the curve* C *and* $t\to z'(t)$ *is the curve* C'.

The above curve $t\to g'(t)$ on G' is called the *strain of parallel displacements* with respect to the Finsler connection (Γ, H) under consideration.

We now restrict our discussion to the case where C is horizontal with respect to the non-linear connection H of the Finsler connection (Γ, H). Then, in general, C' is written in the form ΦC^*, where C^* is a lift of C to the Finsler bundle Q with respect to Γ. It then follows from the definition of the bundle homomorphism Φ that

$$z'^{-1}\frac{dy}{dt} = \Phi(q)^{-1}\frac{dy}{dt} = \rho(f(t), 0)\,,$$

where $t \to f(t)$ is a curve on F. Thus, (8. 1) is written in the form

$$\frac{dg'}{dt} - K(\rho(f(t), 0))g'(t) = 0\,. \tag{8. 1h}$$

On the other hand, if C is supposed to be vertical, we obtain similarly

$$\frac{dg'}{dt} - K(\rho(0, f(t))g'(t) = 0\,. \tag{8. 1v}$$

Since $g'(0) = e'$ is supposed, we obtain $(dg'/dt)_{t=0} = A$ which is an element of the Lie algebra $L(G')$. When C is horizontal, it follows from (8. 1h) that $A = K(\rho(f(0), 0))$ and hence, for $f_1 \in F$, we obtain

$$\begin{aligned} A(\rho(f_1, 0)) &= K(\rho(f_1, 0), \rho(f(0), 0)) \\ &= (K_1{}^1{}_1(f_1, f(0)), K_1{}^2{}_1(f_1, f(0)))\,, \end{aligned}$$

where we made use of the Finsler decomposition of the strain tensor K. Further we have

$$A(\rho(0, f_1)) = (K_2{}^1{}_1(f_1, f(0)), K_2{}^2{}_1(f_1, f(0)))\,.$$

Consequently we obtain, for the case of a horizontal curve C, $A = A_h$:

$$A_h = \begin{pmatrix} K_1{}^1{}_1 f(0) & K_1{}^2{}_1 f(0) \\ K_2{}^1{}_1 f(0) & K_2{}^2{}_1 f(0) \end{pmatrix}. \tag{8. 2h}$$

When C is vertical, we similarly obtain from (8. 1v) $A = A_v$:

$$A_v = \begin{pmatrix} K_1{}^1{}_2 f(0) & K_1{}^2{}_2 f(0) \\ K_2{}^1{}_2 f(0) & K_2{}^2{}_2 f(0) \end{pmatrix}. \tag{8.2 v}$$

In order to compare (8. 2) with the simplest case where G and (Γ, H) are Riemannian, we shall find $A_h = A_h^0$ and $A_v = A_v^0$ in the latter case, and then obtain

$$A_h^0 = \begin{pmatrix} 0 & -R_c{}^b{}_a f(0)^c \\ R_{ac}{}^b f(0)^c & 0 \end{pmatrix}, \quad A_v^0 = \begin{pmatrix} R_{ca}{}^b f(0)^c & 0 \\ 0 & 0 \end{pmatrix},$$

where $R_c{}^b{}_a = y^d R_{d\cdot ca}^{\;b}$ and $R_{ca\cdot}^{\;\;b} = g^{bd} g_{ce} R_a{}^e{}_d$.

Here, let us remember the definition of the homomorphism $\varphi: G \to G'$, that is, $\varphi(g) = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$, $g \in G$, and hence an element of the Lie algebra of $\varphi(G)$ is of the form $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$. Therefore, it seems natural to pay attention to the places denoted by $*$ of A_h and A_v, and to compare them with those of A_h^0 and A_v^0 respectively. For the purpose to find natural and simple Finsler connections, we are led to the following definition.

Definition. A Finsler connection is said to be *normal of the first kind*, if the following two conditions are satisfied:

(1) Metrical.

(2) $K_1{}^1{}_1 = K_2{}^2{}_1 = K_2{}^2{}_2 = 0$, where K's are components of the Finsler decomposition of the strain tensor K.

Then, as for a normal Finsler connection of the first kind, we shall show

Theorem 2. *The connection parameters F_j^i, $F_j{}^i{}_k$ and $C_j{}^i{}_k$ of a normal Finsler connection of the first kind are given by*

(1) $$C_{ijk} = g_{ijk},$$

where $C_{ijk} = g_{jl} C_i{}^l{}_k$ and $g_{ijk} = \dfrac{1}{2}\dfrac{\partial g_{ij}}{\partial y^k}$.

(2) $F_{jk} = g_{jl} F_k^l$ *must satisfy the following differential equations:*

$$\frac{\partial F_{jk}}{\partial y^i} - \frac{\partial F_{ji}}{\partial y^k} + \frac{\partial F_{ki}}{\partial y^j} - \frac{\partial F_{ik}}{\partial y^j} - 2(g_i{}^l{}_j F_{lk} - g_k{}^l{}_j F_{li}) + \gamma_{jik} - \gamma_{jki} = 0,$$

where $g_i{}^l{}_j = g^{lh} g_{ihj}$ and γ_{ijk} are Christoffel's symbols constructed from $g_{ij}(x, y)$ with respect to x^k.

(3) $F_{ijk} = g_{jl} F_i{}^l{}_k$ *are given by*

$$F_{ijk} = \frac{1}{2}\left(\frac{\partial F_{jk}}{\partial y^i} - \frac{\partial F_{ik}}{\partial y^j} + \gamma_{ijk} - \gamma_{jik}\right) - g_i{}^l{}_j F_{lk}.$$

Therefore the connection is uniquely determined, if F_{ij} are given such as to satisfy (2).

Proof. It follows from (7.7) that the conditions for a Finsler connection (Γ, H) to be normal of the first kind are given by

$$\frac{\partial g_{ij}}{\partial x^k} - 2g_i{}^l{}_j F_{lk} = F_{ijk} + F_{jik}, \tag{8.3}$$

$$2g_{ijk} = C_{ijk} + C_{jik}, \tag{8.4}$$

$$T_{ijk} - T_{jki} + T_{kij} = 0, \tag{8.5}$$

$$P_{ijk} = P_{kji}, \tag{8.6}$$

$$S_{ijk} - S_{jki} + S_{kij} = 0. \tag{8.7}$$

It is easy to show that (8.5) and (8.7) give $T_{ijk} = S_{ijk} = 0$, namely, $F_j{}^i{}_k = F_k{}^i{}_j$ and $C_j{}^i{}_k = C_k{}^i{}_j$. Then (1) is a direct result from (8.4). It follows from the definition of $P_j{}^i{}_k$ that

$$P_{ijk} = \frac{\partial F_{ij}}{\partial y^k} - 2g_i{}^l{}_k F_{lj} - F_{kij},$$

and hence (8.6) is written in the form

$$F_{kij} - F_{ikj} = \frac{\partial F_{ij}}{\partial y^k} - \frac{\partial F_{kj}}{\partial y^i}.$$

Combining the above with (8.3), we immediately obtain (3), and then (2) means $F_{ijk} = F_{kji}$. Thus the proof is complete.

In order to consider (2) in Theorem 2, let us suppose as usual that $F_{ij}(x, y)$ be homogeneous of degree 1 with respect to y [7]. Then, the contraction of (2) by y^k gives

$$3F_{ji} - F_{ij} = \frac{\partial F_{j0}}{\partial y^i} + \frac{\partial F_{0i}}{\partial y^j} - \frac{\partial F_{i0}}{\partial y^j} + \gamma_{ji0} - \gamma_{j0i} - 2g_i{}^l{}_j F_{l0}.$$

Therefore, F_{ij} will be uniquely determined if F_{i0} and F_{0i} are given. It is noticed that, if the non-linear connection H is supposed to be metrical, then F_{i0} and F_{0i} are already determined by Proposition 2.

Theorem 3. *A normal Finsler connection* (Γ, H) *of the first kind is Cartan's connection, provided that the non-linear connection* H *be metrical and* $F_j^i(x, y)$ *are homogeneous of degree* 1 *with respect to* y.

Proof. As already observed, the connection satisfying all of the conditions as mentioned above are uniquely determined. On the other hand, Cartan's connection satisfies those conditions, and hence the proof is complete.

Consequently, we may say that Cartan's connection seems to be quite natural from our standpoint.

In the above discussion, we considered general horizontal and vertical curves on $T(M)$, and $(dg'/dt)_{t=0}\in L(G')$. We shall now be concerned with an H-intrinsic curve (horizontal) and an intrinsic curve (vertical). As for an H-intrinsic curve which has in § 2, $f(t)$ in (8. 1h) is equal to $y^a(t)e_a$, and hence (8. 1h) is then been defined written down

$$(8.8)\qquad \begin{aligned} &\frac{dg'^a_\alpha}{dt}-(K_1{}^1{}_1)_b{}^a{}_0 g'^b_\alpha-(K_2{}^1{}_1)_b{}^a{}_0 g'^{(b)}_\alpha=0\,,\\ &\frac{dg'^{(a)}_\alpha}{dt}-(K_1{}^2{}_1)_b{}^a{}_0 g'^b_\alpha-(K_2{}^2{}_1)_b{}^a{}_0 g'^{(b)}_\alpha=0\,. \end{aligned}\qquad \alpha=1,\cdots,2n\,,$$

In particular, as for Cartan's connection (Γ, H), (8. 8) becomes quite simple:

$$\frac{dg'^a_\alpha}{dt}-\frac{1}{2}R_{0\cdot 0b}^{\ a}g'^{(b)}_\alpha=0,\quad \frac{dg'^{(a)}_\alpha}{dt}+\frac{1}{2}R_{0\cdot 0b}^{\ a}g'^b_\alpha=0\,,$$

from which we have

$$\begin{aligned} &\frac{d}{dt}(g'^a_b-g'^{(a)}_{(b)})-\frac{1}{2}R_{0\cdot 0c}^{\ a}(g'^{(c)}_b+g'^{\,c}_{(b)})=0\,,\\ &\frac{d}{dt}(g'^{(a)}_b+g'^{\,a}_{(b)})+\frac{1}{2}R_{0\cdot 0c}^{\ a}(g'^c_b-g'^{(c)}_{(b)})=0\,. \end{aligned}$$

Since the curve $t\to g'(t)$ satisfies $g'(0)=e'$ the equations

$$g'^a_b(t)-g'^{(a)}_{(b)}(t)=0,\quad g'^{(a)}_b(t)+g'^{\,a}_{(b)}(t)=0$$

must hold for $t=0$. Therefore, the above differential equations show that these two equations hold for any t.

Next, we shall consider an intrinsic curve as defined in § 2. In this case, $f(t)$ in (8. 1v) is equal to $y^a(t)e_a$, and hence (8. 1v) is then written down

$$\frac{dg'^a_\alpha}{dt} - (K_1{}^1{}_2)_b{}^a{}_0 g'^b_\alpha - (K_2{}^1{}_2)_b{}^a{}_0 g'^{(b)}_\alpha = 0 ,$$

$$(8.9) \qquad \alpha = 1, \cdots, 2n ,$$

$$\frac{dg'^{(a)}_\alpha}{dt} - (K_1{}^2{}_2)_b{}^a{}_0 g'^b_\alpha - (K_2{}^2{}_2)_b{}^a{}_0 g'^{(b)}_\alpha = 0 .$$

In particular, if we consider Cartan's connection, (8.9) becomes solely $dg'/dt=0$, and hence $g'(t)$ is reduced to the point e'.

We sum up these facts as for Cartan's connection in the following.

Theorem 4. *Let* (Γ, H) *be Cartan's connection constructed from a Finsler metric* G. *The strain of parallel displacements* $g'(t)$ *with respect to* (Γ, H) *is of the following form*:

(1) *For an* H-*intrinsic curve*, $g'(t)=\begin{pmatrix} g^a_b & g^a_{(b)} \\ -g^a_{(b)} & g^a_b \end{pmatrix}$, *where* g^a_b *and* $g^a_{(b)}$ *are determined by the differential equations*

$$\frac{dg^a_b}{dt} + \frac{1}{2} R^a_{0\cdot 0c} g^c_{(b)} = 0 , \quad \frac{dg^a_{(b)}}{dt} - \frac{1}{2} R^\alpha_{0\cdot 0c} g^c_b = 0 .$$

with the initial conditions $g^a_b(0)=\delta^a_b$, $g^a_{(b)}(0)=0$.

(2) *For an intrinsic curve*, $g'(t)$ *is reduced to the unit of* G'.

It is observed that the differential equations in the above (1) which give g^a_b and $g^a_{(b)}$ will not become simpler formally, even if G and (Γ, H) are Riemannian. Thus, we may say that Cartan's connection gives the simplest parallel displacements of frames along an H-intrinsic curve and an intrinsic curve.

From the viewpoint of Theorem 4, we now lay down the following definition.

Definition. In the notation of Theorem 4, a Finsler connection is said to be *normal of the second kind* if it is metrical and the strain of parallel displacements $g'(t)$ is of the following form:

(1) For an H-intrinsic curve, $g'(t)=\begin{pmatrix} g^a_b & g^a_{(b)} \\ -g^a_{(b)} & g^a_b \end{pmatrix}$.

(2) For an intrinsic curve, $g'(t)$ is reduced to the unit.

It will be easy to show from (8.8) and (8.9) that the conditions (1) and (2) as above are expressed

$$(8.10)\qquad (K_1{}^1{}_1)_b{}^a{}_0 = (K_2{}^2{}_1)_b{}^a{}_0, \quad (K_2{}^1{}_1)_b{}^a{}_0 = -(K_1{}^2{}_1)_b{}^a{}_0,$$
$$(K_1{}^1{}_2)_b{}^a{}_0 = (K_2{}^1{}_2)_b{}^a{}_0 = (K_1{}^2{}_2)_b{}^a{}_0 = (K_2{}^2{}_2)_b{}^a{}_0 = 0.$$

Theorem 5. *Let $F_j{}^i$, $F_j{}^i{}_k$ and $C_j{}^i{}_k$ be the connection parameters of a normal Finsler connection of the second kind. If $F^i_j(x, y)$ be assumed to be homogeneous of degree* 1 *with respect to y, these connection parmeters are as follows*:

$$(1)\qquad C_{ijk}+C_{jik} = 2g_{ijk}, \quad C_{ij0} = 0.$$
$$(2)\qquad F_{ij}-F_{ji} = \gamma_{0ij}-\gamma_{0ji}, \quad F_{i0} = \gamma_{0i0}, \quad F_{0i} = \gamma_{00i}.$$
$$(3)\qquad F_{ijk} + F_{jik} = \frac{\partial g_{ij}}{\partial x^k} -2g_i{}^l{}_jF_{lk},$$

where the same notation of Theorem 2 *is used.*

Proof. The first equation of (1) and (3) mean solely that the connection is metrical. According to (7.7), the equation (8.10) are written down

$$(8.11)\qquad S_{ij0}-S_{j0i}+S_{0ij} = 0,$$
$$(8.12)\qquad T_{ij0}-T_{j0i}+T_{0ij} = P_{i0j}-P_{j0i},$$
$$(8.13)\qquad R_{0ij}+C_{ij0}-C_{ji0} = 0,$$
$$(8.14)\qquad R_{i0j}-C_{0ji}-C_{j0i} = R_{j0i}-C_{0ij}-C_{i0j},$$
$$(8.15)\qquad P_{0ij}+P_{ji0} = 0.$$

It follows from the definition of S_{ijk} that (8.11) becomes $C_{ij0}=0$, where we used the first of (1). Thus we obtain $R_{0ij}=0$ from (8.13). It follows from the first of (1) and (8.14) that $R_{i0j}=R_{j0i}$. Hence (8.13) and (8.14) become

$$(8.16)\qquad R_{0ij} = 0, \quad R_{i0j} = R_{j0i}.$$

Next, by the same way as in the proof of Theorem 2, we obtain

$$P_{0ij} = \frac{\partial F_{0i}}{\partial y^j} -F_{ji}-F_{j0i}, \quad P_{ji0} = F_{ji}-F_{0ji},$$

where we have to notice that the homogeneous condition on F^i_j was used. Then, (8.15) gives

$$(8.17)\qquad \frac{\partial F_{0j}}{\partial y^i} = \gamma_{0ji}+\gamma_{j0i}.$$

Similarly (8.12) is written

$$\frac{\partial F_{i0}}{\partial y^j} - \frac{\partial F_{j0}}{\partial y^i} - F_{ij} + F_{ji} = \gamma_{ji0} - \gamma_{ij0}. \tag{8.18}$$

Contractions of (8.17) and (8.18) by y^i give the second and the third equations of (2), and then, inserting $F_{i0} = \gamma_{0i0}$ in (8.18), we obtain the first of (2). It now remains to treat (8.16), but it will be shown by direct calculation that (8.16) is an automatical result of the facts as have been already verified.

It follows immediately from (2) in the above theorem and Proposition 2 that

Corollary 1. *The non-linear connection H of a normal Finsler connection of the second kind (Γ, H) is metrical, provided that $F^i_j(x, y)$ be homogeneous of degree* 1 *with respect to y.*

For the purpose to make little the strain of parallel displacements of a Finsler connection (Γ, H), we are finally led to the following definition.

Definition. A Finsler connection is called *normal*, if it is normal of the first kind and of the second kind.

Then, we establish from Theorem 3 and Corollary 1 that

Corollary 2. *A normal Finsler connection is definitely Cartan's one, provided that $F^i_j(x, y)$ of the connection parameters be homogeneous of degree* 1 *with respect to y.*

§ 9. Curvature tensors

It will be rather complicated to find the curvature tensor $\bar{R}$ of the Riemann connection $\bar{\Gamma}$ with respect to the lifted Riemann metric $\bar{G}$ by referring to the local coordinate (x^i, y^i) on the tangent bundle $T(M)$. In this section, we shall do by making use of the frame $\Phi(q)$, $q \in Q$. The curvature tensor $\bar{R}$ has been expressed by the curvature tensor R' and the torsion tensor T' of the linear connection of Finsler type Γ', together with the strain tensor K, in the abstract form (7.4). In that equations, if $f'_1, f'_2 \in F'$ are taken as $\rho(f, 0)$ or $\rho(0, f)$, $f \in F$, we shall obtain the Finsler decomposition of the curvature tensor $\bar{R}$ as follows:

$$
\begin{aligned}
(\bar{R}_{1111})_{abcd} &= R_{abcd} - \tfrac{1}{2}R_{eab}R_{c\ d}^{\ e} + (\tfrac{1}{2}R_{ecb} - g_{ecb})(\tfrac{1}{2}R_{d\ a}^{\ e} - g_{d\ a}^{\ e}) \\
&\qquad - (\tfrac{1}{2}R_{edb} - g_{edb})(\tfrac{1}{2}R_{c\ a}^{\ e} - g_{c\ a}^{\ e}), \\
(\bar{R}_{1112})_{abcd} &= -\tfrac{1}{2}(R_{dca|b} - R_{dcb|a}) - (g_{dca|b} - g_{dcb|a}) - P_{ecd}R_{a\ b}^{\ e}, \\
(\bar{R}_{1122})_{abcd} &= R_{cdab} + (\tfrac{1}{2}R_{dae} - g_{dae})(\tfrac{1}{2}R_{cb\cdot}^{\ \ e} - g_{c\cdot b}^{\ e}) \\
&\qquad - (\tfrac{1}{2}R_{dbe} - g_{dbe})(\tfrac{1}{2}R_{ca\cdot}^{\ \ e} - g_{c\ a}^{\ e}), \\
(\bar{R}_{1212})_{abcd} &= \tfrac{1}{2}R_{bca}|_{d} - g_{bca|d} + P_{abd|c} - P_{abe}P_{c\ d}^{\ e} \\
&\qquad + (\tfrac{1}{2}R_{bea} - g_{bea})g_{c\ d}^{\ e} + \tfrac{1}{2}(\tfrac{1}{2}R_{bce} - g_{bce})R_{da\cdot}^{\ \ e}, \\
(\bar{R}_{2221})_{abcd} &= P_{cda}|_{b} - P_{cdb}|_{a} + \tfrac{1}{2}(P_{a\ c}^{\ e}R_{bde} - P_{b\ c}^{\ e}R_{ade}), \\
(\bar{R}_{222})_{abcd} &= g_{dae}g_{b\ c}^{\ e} - g_{dbe}g_{a\ c}^{\ e} + P_{eda}\ P_{b\ c}^{\ e} - P_{edb}P_{a\ c}^{\ e}.
\end{aligned} \tag{9.1}
$$

Remarks. (i) Since the lifted Riemann metric $\bar{G}$ is determined by the original Finsler metric G and the non-linear connection H, the above $\bar{R}$ does not depend on the choice of Γ of the Finsler connection (Γ, H). In the above formulas, the lifted Riemann metric is given by means of Berwald's non-linear connection H. The symbols in the right-hand members are that of Cartan's connection. (ii) It follows from the meaning of the Finsler decomposition, components of $\bar{R}$ are given by (9.1), referring to the frame $\Phi(q)$, that is, for example,

$$
\bar{R}_{abcd} = (\bar{R}_{1111})_{abcd}, \quad \bar{R}_{a(b)c(d)} = (\bar{R}_{1212})_{abcd},
$$
$$
a, b, c, d = 1, \cdots, n; \quad (b) = n+b; \quad (d) = n+d.
$$

Since the general formulas (9.1) are complicated, we next consider the case where G is Riemannian. In this case, the non-linear connection H used to obtain the lifted Riemann metric $\bar{G}$ is naturally the associated linear connection with the Riemann connection with respect to G. Then, we obtain

$$
\begin{aligned}
&\bar{R}_{abcd} = R_{abcd} - \tfrac{1}{2}R_{0eab}R_{0\cdot cd}^{\ e} + \tfrac{1}{4}(R_{0ebc}R_{0\cdot ad}^{\ e} - R_{0ebd}R_{0\cdot ac}^{\ e}), \\
&\bar{R}_{abc(d)} = \tfrac{1}{2}R_{0dab;c}, \\
&\bar{R}_{ab(c)(d)} = R_{abcd} + \tfrac{1}{4}(R_{0dae}R_{b\cdot 0c}^{\ e} - R_{0dbe}R_{a\cdot 0c}^{\ e}), \\
&\bar{R}_{a(b)c(d)} = \tfrac{1}{2}R_{acbd} + \tfrac{1}{4}R_{0bce}R_{a\cdot 0d}^{\ e}, \\
&\bar{R}_{(a)(b)(c)d} = 0, \quad \bar{R}_{(a)(b)(c)(d)} = 0. \\
&a, b, c, d = 1, \cdots, n. \\
&(a) = n+a, \quad (b) = n+b, \quad (c) = n+c, \quad (d) = n+d.
\end{aligned} \tag{9.2}
$$

Remarks. (i) In (9.2), R_{abcd} are, of course, components of the curvature tensor R of the Riemann connection Γ with respect to the Riemann metric G on the base manifold M. The index 0 means the contraction by y as usual, and the semi-colon means the covariant derivative with respect to the connection Γ. (ii) Members in the right-hand sides of (9.2) are components referring to the frame z, while members in the left-hand sides are components referring to the frame $\Phi(q)$, $q=(y, z)$.

From (9.2), we shall obtain some interesting theorems as for sectional curvatures of the Riemann manifold $T(M)$.

Theorem 6. *Let M be a Riemann manifold with a Riemann metric G. Then the tangent bundle $T(M)$ over M is regarded as a Riemann manifold with the lifted Riemann metric $\bar{G}$ of G. Let $\bar{S}(X, Y)$ be the sectional curvature of a 2-section spanned by tangent vectors X and Y to $T(M)$.*

(1) *If X and X are vertical, then $\bar{S}(X, Y)=0$* [13, Theorem 18].

(2) *If X is the intrinsic vector $\mathfrak{v}$, then $\bar{S}(X, Y)=0$ for any Y.*

Proof. Suppose that X and Y be vertical, and hence there exist $f_1, f_2 \in F$ such that $X=\Phi(q)(0, f_1)$ and $Y=\Phi(q)(0, f_2)$. It then follows from (9.2) that

$$\begin{aligned}\bar{R}(X, Y, X, Y) &= \bar{R}(\rho(0, f_1), \rho(0, f_2), \rho(0, f_1), \rho(0, f_2)) \\ &= \bar{R}_{2222}(f_1, f_2, f_1, f_2) = 0\,,\end{aligned}$$

which proves (1). Next, if $X=\mathfrak{v}$, X is expressed $X=\Phi(q)(0, y)$, where $y=y^a e_a \in F$. Hence we see

$$\begin{aligned}\bar{R}(X, Y, X, Y) &= \bar{R}(\rho(f_1, f_2), \rho(0, y), \rho(f_1, f_2), \rho(0, y)) \\ &= (\bar{R}_{1212})_{abcd} f_1^a y^b f_1^c y^d = 0\,,\end{aligned}$$

where we put $Y=\Phi(q)(f_1, f_2)$, $f_1, f_2 \in F$. Thus, (2) is proved.

It follows from Theorem 6 that, if the Riemann manifold $T(M)$ is of constant curvature, the curvature must vanish identically, and hence we obtain

Corollary 3. *If the tangent bundle $T(M)$ over a Riemann manifold M is considered as a Riemann manifold with the lifted Riemann metric, it is impossible that $T(M)$ is of non-vanishing constant curvature.*

Next, let us compare a sectional curvature of $T(M)$ with one of M, corresponding to each other by lifting tangent vectors. Let X and Y be tangent vectors to M, and we have the vertical lifts X^v and Y^v of X and Y respectively. It follows from Theorem 6 that the sectional curvature $\bar{S}(X^v, Y^v)=0$. On the other hand we shall obtain

Theorem 7. *In the notation of Theorem* 6, *we denote by* $S(X, Y)$ *the sectional curvature of a 2-section spanned by tangent vectors* X *and* Y *to* M. *Then*

(1) $$\bar{S}(X^h, Y^h) \leqq S(X, Y),$$

where X^h *and* Y^h *are horizontal lifts of* X *and* Y *respectively. The equality holds good, if and only if* $R(y, X, Y)=R_{j\cdot kl}^{\ i}y^j X^k Y^l=0$.

(2) $$\bar{S}(X^h, Y^v) \geqq 0,$$

where Y^v *is the vertical lift of* Y. *The equality holds good, if and only if* $R(X, Y, y)=R_{j\cdot kl}^{\ i}X^j Y^k y^l=0$.

Proof. If $X \in M_x$ is expressed $X=zf$, $z \in \pi^{-1}(x)$, $f \in F$, we obtain

$$X^h=\Phi(q)(f, 0), \quad X^v=\Phi(q)(0, f), \quad q=(y, z).$$

It then follows from (9.2) that

$$\bar{S}(X^h, Y^h) = S(X, Y)-\tfrac{3}{4}|R(y, X, Y)|^2,$$
$$\bar{S}(X^h, Y^v) = \tfrac{1}{4}|R(X, Y, y)|^2,$$

where $|\cdots|$ denotes the length with respect to the original Riemann metric G on M. As a consequence of the above equations, we have the theorem.

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