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Intersections of quotient rings of an integral domain

By

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Let D be an integral domain with identity having quotient field K. A domain between D and K is called an *overring* of D, and a *valuation overring* of D is an overring of D which is also a valuation ring. Davis [2], Gilmer and Ohm [6], Goldman [8], and Pendleton [15] have recently considered domains D with the *QR-property*: Each overring of D is a quotient ring of D. A domain with the QR-property is necessarily Prüfer [2; pp. 197-8], [6; p. 99], and a Noetherian domain has the QR-property if and only if it is a Dedekind domain with torsion class group [2; p. 200], [6; p. 100], [8; p. 114].

Davis in [2] and Gilmer in [4] have considered a related property on a domain D, which we shall refer to here as the *QQR-property*: Each overring of D is an intersection of quotient rings of D. Since any quotient ring of D is an intersection of localizations of D (that is, quotient rings of D taken with respect to the complement of prime ideals of D), the QQR-property for D is equivalent to the condition that each overring of D is an intersection of localizations of D. It is well-known that any Dedekind domain has the QQR-property; more generally, any Prüfer domain has the QQR-property [2; p. 197]. Davis in [2; p. 200] raises the question of the validity of the converse; he proves that if each prime ideal of D is of finite rank and if D has the QQR-

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property, then D is Prüfer. In Section 1 we consider more closely the relation between the concepts "D is Prüfer" and "D has the QQR-property". We are able to generalize the positive result of Davis just cited (Theorem 1.4) and to add a second positive result (Corollary 1.7), but we give an example later in Section 4 showing that a domain with the QQR-property need not be Prüfer. Theorem 1.9 shows that the domain D has the QQR-property if and only if D_M has the QQR-property for each maximal ideal M of D. Theorem 1.10 states that if J is a quasi-local domain with the QQR-property which is not integrally closed, then the integral closure $\bar{J}$ of J is the unique minimal overring of J in the sense that each proper overring of J contains $\bar{J}$. Hence Section 2 is devoted to a consideration of domains which admit a unique minimal overring, and Theorem 3.3 of Section 3 gives a characterization of quasi-local domains with the QQR-property.

All domains considered in this paper are assumed to contain an identity. The terminology is that of Zariski-Samuel [17] [18].

1. Prüfer domains and the QQR-property. We establish in this section some consequences of the QQR-property in a domain D. Theorem 1.4 and Corollary 1.7 relate the QQR-property to the property of being Prüfer. We shall have frequent occasion to use the following result from [4], which we quote directly:

Suppose D' is an overring of the Prüfer domain D, and let Ω be the set of prime ideals P of D such that $PD' \subset D'$. Then

(i) *If M is a maximal ideal of D' and if $P = M \cap D$, then $D_P = D'_M$ and $M = PD_P \cap D'$. Therefore D' is Prüfer.*

(ii) *For P a proper prime ideal of D, $P \in \Omega$ if and only if $D_P \supseteq D'$. Further, $D' = \cap_{P\in\Omega} D_P$.*

(iii) *If A' is an ideal of D' and $A = A' \cap D$, then $A' = AD'$.*

(iv) *$\{PD'\}_{P\in\Omega}$ is the set of proper prime ideals of D'.*

Also, we shall make use of the following fact concerning localizations.

If D' is an overring of the domain D and P is a prime ideal of D such that $D' \subseteq D_P$, then $D'_{P'} = D_P$ where $P' = D' \cap PD_P$.

1.1 PROPOSITION. *Let D be an integral domain and let D' be* *an overring of D. If D' is an intersection of quotient rings of D then D' is an intersection of localizations of D.*

Proposition 1.1 is immediate from the easily-proved fact that a quotient ring of D is an intersection of localizations of D. (For example, see [16].)

1.2 COROLLARY. *The following statements concerning the integral domain D are equivalent.*

(1) *D has the QQR-property.*

(2) *Each overring of D is an intersection of localizations of D.*

(3) *Each overring of D which is quasi-local is an intersection of localizations of D.*

Proof. In view of Proposition 1.1 we need only show that (3) implies (1). Thus suppose D' is an overring of D and $\{M_\alpha\}$ is the collection of maximal ideals of D'. By hypothesis, D'_{M_α} is an intersection of localizations of D. Since $D'=\bigcap_\alpha D'_{M_\alpha}$, (1) then follows.

In considering questions concerning overrings of the domain D we find that an important role is played by the valuation overrings of D—that is, valuation rings lying between D and K. Our first results therefore deal with the condition that every valuation overring of D is an intersection of quotient rings of D.

1.3 LEMMA. *Let V be a valuation overring of D which is an intersection of localizations of D: $V=\bigcap_\alpha D_{P_\alpha}$. Then each D_{P_α} is a valuation ring, the set $\{P_\alpha\}$ is linearly ordered under inclusion, and $M=\bigcup_\alpha P_\alpha$ is the center of V on D. Also, $D_M\subseteq V$ and equality holds if and only if D_M is a valuation ring. If A is a nonmaximal proper ideal of V, then $A\subset P_\alpha D_{P_\alpha}$ for some α.*

Proof. Since each D_{P_α} contains V, each D_{P_α} is a valuation ring and $\{P_\alpha D_{P_\alpha}\}$ is chained under inclusion. Because $P_\alpha= P_\alpha D_{P_\alpha}\cap D$ for each α, $\{P_\alpha\}$ is also chained under inclusion. Therefore $M=\bigcup_\alpha P_\alpha$ is prime in D. We next observe that $\bigcup_\alpha P_\alpha D_{P_\alpha}$ is the maximal ideal M' of V. That $\bigcup_\alpha P_\alpha D_{P_\alpha}\subseteq M'$ is clear. And if x is a nonunit of V, $1/x\notin V=\bigcap_\alpha D_{P_\alpha}$ implies $1/x\notin D_{P_\alpha}$ for some

α. Hence $M' \subseteq \bigcup_{\alpha} P_\alpha D_{P_\alpha}$. Therefore the center of V on D is $M' \cap D = (\cup P_\alpha D_{P_\alpha}) \cap D = \bigcup_{\alpha} (P_\alpha D_{P_\alpha} \cap D) = \bigcup_{\alpha} P_\alpha = M$. It follows that $D_M \subseteq V$. If D_M is a valuation ring, then $D_M = V$, for $M \subseteq M'$ implies that $MD_M \subseteq MV \subseteq M'$; MD_M the maximal ideal of D_M. Finally, if A is an ideal of V such that $A \subset M'$, then $A \not\supseteq \bigcup_{\alpha} P_\alpha D_{P_\alpha}$ so $A \not\supseteq P_\alpha D_{P_\alpha}$ for some α. Hence $A \subset P_\alpha D_{P_\alpha}$.

1.4 THEOREM. *If each valuation overring of D is an intersection of quotient rings of D, and if D satisfies the ascending chain condition (a.c.c.) for prime ideals then D is a Prüfer domain.*

Proof. Let M be a maximal ideal of D. There is a valuation overring of D having center M on D. By Lemma 1.3, there is a chain $\{P_\alpha\}$ of prime ideal of D such that $\bigcup_{\alpha} P_\alpha = M$ and each D_{P_α} is a valuation ring. Since D satisfies a.c.c. on prime ideals, some P_α is equal to M, and D_M is a valuation ring. It follows that D is a Prüfer domain.

1.5 THEOREM. *If each valuation overring of D is an intersection of quotient rings of D, then for P a nonmaximal prime of D, D_P is a valuation ring.*

Proof. Let M be a maximal ideal of D such that $P \subset M$ and let V be a valuation overring of D having prime ideals P' and M' such that $P' \cap D = P$, $M' \cap D = M$, and M' is the maximal ideal of V. [13, p. 37]. By Lemma 1.3 there is a set $\{P_\alpha\}$ of prime ideals of D such that $V = \bigcap_{\alpha} D_{P_\alpha}$, and $P' \subset P_\alpha D_{P_\alpha}$ for some α. Therefore, $P = P' \cap D \subseteq P_\alpha D_{P_\alpha} \cap D = P_\alpha$. It follows that $D_{P_\alpha} \subseteq D_P$, and D_P is a valuation ring as we wished to show.

1.6 THEOREM. *If each valuation overring of D is an intersection of quotient rings of D, then $\bar{D}$, the integral closure of D, is a Prüfer domain.*

Proof. If $\bar{D}$ is not a Prüfer domain, then there is a maximal ideal $\bar{M}$ of $\bar{D}$ such that $\bar{D}_{\bar{M}}$ is not a valuation ring. It follows (see, for example, [18, p. 21]) that there is a valuation overring V of $\bar{D}$ such that V has prime ideals $P' \subset M'$ with $P' \cap \bar{D} = M' \cap \bar{D}$

$=\bar{M}$. Hence if $\bar{M}\cap D=M$, then $P'\cap D=M$ also. By hypothesis V is an intersection of quotient rings of D. Thus, there is a set $\{P_\alpha\}$ of prime ideals of D such that $V=\bigcap_\alpha D_{P_\alpha}$, and $P'\subset P_\alpha D_{P_\alpha}$ for some α. This means that $M\subseteq P'\cap D\subseteq P_\alpha D_{P_\alpha}\cap D=P_\alpha$; hence $M=P_\alpha$ and $D_M=D_{P_\alpha}$ is a valuation ring. But $\bar{M}\cap D=M$ implies that $D_M\subseteq\bar{D}_{\bar{M}}$, and by assumption $\bar{D}_{\bar{M}}$ is not a valuation ring. This contradiction establishes Theorem 1.6.

1.7 COROLLARY[2]. *If D has the QQR-property, then $\bar{D}$, the integral closure of D, is Prüfer. Therefore if D is an integrally closed domain, D has the QQR-property if and only if D is a Prüfer domain.*

If J is a domain lying between a domain D and a quotient ring D_N of D, then $D_N=J_N$. Therefore we have

1.8 PROPOSITION. *If D is a domain with the QQR-property and if D' is an overring of D, then D' has the QQR-property.*

Our next result reduces the problem of characterizing domains with the QQR-property to the study of quasi-local domains.

1.9 THEOREM. *The domain D has the QQR-property if and only if D_{M_α} has the QQR-property for each maximal ideal M_α of D.*

Proof. If D has the QQR-property, then by Proposition 1.8 each D_{M_α} has the QQR-property. We assume, conversely, that each D_{M_α} has the QQR-property. By Proposition 1.2, D will have the QQR-property if each quasi-local overring of D is an intersection of localizations of D. Hence, let D' be a quasi-local overring of D with maximal ideal M'. If $P=M'\cap D$ and if M_α is a maximal ideal of D containing P, then $D_{M_\alpha}\subseteq D_P\subseteq D'$. Since D_{M_α} has the QQR-property, D' is an intersection of localizations of D_{M_α}. But each localization of D_{M_α} is a localization of D, hence D' is an intersection of localizations of D as we wished to show.

TERMINOLOGY. We will say that the overring D_1 of the

2) Corollary 1.7 can also be obtained as a special case of Corollary 1 of [2]. A domain D such that each valuation overring of D is an intersection of quotient rings of D need not have the QQR-property. Example 4.2 illustrates this fact.

domain D is the *unique minimal overring* of D if $D \subset D_1$ and if for any overring D_2 of D not equal to D we have $D_1 \subseteq D_2$.

1.10 THEOREM. *Let D be a quasi-local domain with maximal ideal M. If D has the QQR-property and is not a valuation ring, then $\bar{D}$, the integral closure of D, is the unique minimal overring of D.*

Proof. Since D is quasi-local and is not a valuation ring, D is not Prüfer, so that $D \subset \bar{D}$. If D' is an overring of D, there is a set $\{P_\alpha\}$ of prime ideals of D such that $D' = \bigcap_\alpha D_{P_\alpha}$, and if $D \neq D'$, then $M \notin \{P_\alpha\}$ since $D_M = D$. Therefore each P_α is non-maximal, and Theorem 1.5 shows that each D_{P_α} is then a valuation ring. Therefore $D' = \bigcap_\alpha D_{P_\alpha}$ is an integrally closed domain containing D. Hence $\bar{D} \subseteq D'$, and we conclude that $\bar{D}$ is the unique minimal overring of D.

2. Unique minimal overrings. Theorem 1.10 leads us to an investigation of domains which possess a unique minimal overring. From the fact that a domain is the intersection of the localizations taken with respect to its set of maximal ideals, it follows that a domain which has a unique minimal overring is quasi-local. Also, since an integrally closed domain is an intersection of valuation rings, if D is integrally closed and has a unique minimal overring, then D is a valuation ring. Our primary interest lies in the case where D is quasi-local and $\bar{D}$, the integral closure of D, is the unique minimal overring of D.

We consider first the situation when D and D_1 are domains with $D \subset D_1$ such that there are no domains properly between D and D_1.

2.1 LEMMA. *If A and B are distinct ideals of D_1 such that $A \cap D = B \cap D$, then $A \cap B = A \cap D$. Therefore, $A \cap B$ is an ideal of D.*

Proof. We have $A \cap D = (A \cap D) \cap (B \cap D) \subseteq A \cap B$. If $A \cap D \subset A \cap B$, then there is an $x \in (A \cap B) - D$, and since there are no domains properly between D and D_1, we have $D_1 = D[x]$. If $y \in A$, then $y = d_0 + d_1 x + \cdots + d_n x^n$ where $d_0, d_1, \cdots, d_n \in D$. Hence $d_0 = y - d_1 x - \cdots - d_n x^n \in A \cap D = B \cap D$, and $y = d_0 + d_1 x + \cdots + d_n x^n \in B$.

It follows that $A \subseteq B$; similarly $B \subseteq A$, so that $A = B$. Consequently, if A and B are distinct ideals of D_1 such that $A \cap D = B \cap D$, then $A \cap B = A \cap D$.

2.2 COROLLARY. *If P is a prime ideal of D, then there are at most two prime ideals of D_1 lying over P. In particular, if D is quasi-local and D_1 is integral over D, D_1 has at most two maximal ideals.*

2.3 LEMMA. *Assume that D is quasi-local with maximal ideal M. If D_1 is integral over D, then M is the conductor of D in D_1.*

Proof. Let $y \in M$ and assume that $yD_1 \not\subseteq D$. Then there is an $x \in yD_1 - D$; say $y\xi = x$. Since $x \notin D$, since there are no domains properly between D and D_1, and since D_1 is integral over D, we have $D_1 = D + Dx + \cdots + Dx^n$ for some positive integer n. We choose n minimal with this property. There are $d_0, \cdots, d_n \in D$ such that $\xi^n = d_0 + d_1 x + \cdots + d_n x^n$. Hence $x^n = y^n \xi^n = y^n d_0 + \cdots + y^n d_n x^n$ and $x^n(1 - y^n d_n) = y^n d_0 + \cdots + y^n d_{n-1} x^{n-1}$. Since D is quasi-local and $y \in M$, $1 - y^n d_n$ is a unit of D and $x^n \in D + Dx + \cdots + Dx^{n-1}$. It follows that $D_1 = D + Dx + \cdots + Dx^{n-1}$, which contradicts our choice of n. We conclude that M is the conductor of D in D_1.

2.4 THEOREM. *Let D be a quasi-local domain with maximal ideal M and integral closure $\bar{D}$ such that $D \subset \bar{D}$. If $\bar{D}$ is Prüfer and if there are no domains properly between D and $\bar{D}$, then $\bar{D}$ is the unique minimal overring of D.*

Proof. By Lemma 2.3, M is the conductor of D in $\bar{D}$ and by Corollary 2.2, $\bar{D}$ has at most two maximal ideals. For $x \in K - D$ we show that $\bar{D} \subseteq D[x]$. If $x \in \bar{D} - D$, then $\bar{D} = D[x]$ by hypothesis. If $\bar{D}$ is quasi-local with maximal ideal $\bar{M}$, then since $\bar{D}$ is integral over D, $\bar{M}$ is the only prime of $\bar{D}$ lying over M. Hence $M = M\bar{D}$ has radical $\bar{M}$ in $\bar{D}$. For $x \notin \bar{D}$, we have $1/x \in \bar{M}$, since $\bar{D}$ is a valuation ring, so that $(1/x)^n \in M$ for some positive integer n. Thus for $\xi \in \bar{D}$, we have $\xi = \xi(1/x^n)x^n \in D[x]$ and $\bar{D} \subseteq D[x]$.

The remaining case is when $\bar{D}$ has two maximal ideals—say M_1 and M_2. By Lemma 2.1, $M = M_1 \cap M_2$ and since $\bar{D}$ is Prüfer

$\bar{D}_{M_1}$ and $\bar{D}_{M_2}$ are valuation rings. For $x \notin \bar{D}$, we have $x \notin \bar{D}_{M_1}$ or $x \notin \bar{D}_{M_2}$. Assume $x \notin \bar{D}_{M_1}$; then $1/x \in M_1 \bar{D}_{M_1}$. Hence $1/x = u/v$ where $u \in M_1$ and $v \in \bar{D} - M_1$. If $y \in M_2 - M_1$, then $1/x = uy/vy$ and $vy \in \bar{D} - M_1$. Thus $xuy = vy \in \bar{D} - D$. But $uy \in M_1 \cap M_2 \subset D$. Hence $\bar{D} = D[vy] \subseteq D[x]$. We conclude that $\bar{D}$ is the unique minimal overring of D and Theorem 2.4 is established.

We remark that without the condition that $\bar{D}$ is a Prüfer domain the conclusion of Theorem 2.4 need not follow. For example if $F = R(\sqrt{2})(x)$, where R is the field of rational numbers and x is transcendental over R then the power series ring $F[[y]]$ is a rank one valuation ring of the form $F + M$ where M is the maximal ideal of $F[[y]]$. $D = R + M$ is a quasi-local domain with integral closure $\bar{D} = R(\sqrt{2}) + M$. There are no domains properly between D and $\bar{D}$; however $\bar{D}$ is not a unique minimal overring of D because $\bar{D} \nsubseteq D[x]$.

We consider now the question of what domains admit a unique minimal overring. As we have already mentioned, if D is an integrally closed domain which has a unique minimal overring, then D is a valuation ring. If D is not integrally closed and has a unique minimal overring D_1, then $D \subset D_1 \subseteq \bar{D}$ where $\bar{D}$ is the integral closure of D. We show by example in Section 4 that the unique minimal overring D_1 of D need not be quasi-local. However, Corollary 2.2 shows that D_1 has at most two maximal ideals. We consider this case in Proposition 2.5.

2.5 PROPOSITION. *If D_1, the unique minimal overring of D, has two maximal ideals, then $\bar{D}$, the integral closure of D, is an intersection of two valuation rings V_1 and V_2. If N_i is the maximal ideal of V_i, then $N_1 \cap N_2 = M$, the maximal ideal of D.*

Proof. Let M_1 and M_2 be the maximal ideals of D_1. If V_1 and V_2 are valuation overrings of D such that V_i has maximal ideal N_i, where $N_i \cap D_1 = M_i$, and if $x \in (N_1 \cap N_2) - M$, we have $D_1 \subseteq D[x]$. Hence if $\xi \in D_1$, $\xi = d_0 + d_1 x + \cdots + d_n x^n$, where $d_0, \cdots, d_n \in D$. Also $\xi - d_0 \in D_1$ and $d_1 x + \cdots + d_n x^n \in N_1 \cap N_2$; thus $\xi - d_0 \in N_1 \cap N_2 \cap D_1 = M_1 \cap M_2$. By Lemma 2.1, $M_1 \cap M_2 = M \subseteq D$. Therefore the assumption that $x \in (N_1 \cap N_2) - M$ implies that

$\xi - d_0 \in D$ and hence $\xi \in D$. This contradicts the fact that $D \subset D_1$. We conclude that $N_1 \cap N_2 = M$. If a valuation ring V_3 is an overring of D and if N_3, the maximal ideal of V_3, is such that $N_3 \cap D = M$, then $D_1 \subseteq V_3$ and $N_3 \cap D_1$ is either M_1 or M_2—say $N_3 \cap D_1 = M_1$. By the argument just given, $N_3 \cap N_2 = M$. We show that this implies that $V_1 = V_3$. $T = V_1 \cap V_2 \cap V_3$ is a domain such that $T_{N_i \cap T} = V_i$ for each i [1, p. 132]. It follows that $(N_3 \cap T) \cap (N_2 \cap T) = (N_3 \cap N_2) \cap T = M \cap T \subseteq N_1 \cap T$. Because there are no containment relations between $V_1 = T_{N_1 \cap T}$ and $V_2 = T_{N_2 \cap T}$, it follows that $N_2 \cap T \nsubseteq N_1 \cap T$. Hence $N_3 \cap T \subseteq N_1 \cap T$, and because of the symmetry of our argument, $N_3 \cap T = N_1 \cap T$ so that $V_3 = T_{N_3 \cap T} = T_{N_1 \cap T} = V_1$. Therefore V_1 and V_2 are the only valuation rings which are overrings of D and which have center M on D. It follows that $\bar{D} = V_1 \cap V_2$, [18, p. 17], and the proof of Proposition 2.5 is complete.

If D is not integrally closed and if D_1, the unique minimal overring of D, is quasi-local with maximal ideal M_1 then we may have $M_1 = M$, the maximal ideal of D, or $M \subset M_1$. When $M \subset M_1$, Lemma 2.3 shows that M is an ideal of D_1 and by Lemma 2.1 there are no ideals properly between M and M_1. It then follow that $M_1^2 \subseteq M$ [17, p. 237]. We have not been able, in the case where $M \subset M_1$ and D_1 is quasi-local, to determine whether the integral closure of D is Prüfer. But if $M_1 = M$ we show that $\bar{D}$ is a valuation ring.

2.6 PROPOSITION. *Let D_1 be the unique minimal overring of D, and assume that D_1 is quasi-local. If M, the maximal ideal of D, is also the maximal ideal of D_1, then $\bar{D}$, the integral closure of D, is a valuation ring with maximal ideal M.*

Proof. Let V be a valuation ring which is an overring of D such that V has center M on D. If N is the maximal ideal of V, then for $x \in N$ either $x \in M$ or $D_1 \subseteq D[x]$. But if $D_1 \subseteq D[x]$, then for $\xi \in D_1$ we have $\xi = d_0 + d_1 x + \cdots + d_n x^n$ where $d_0, \cdots, d_n \in D$. Hence $\xi - d_0 = d_1 x + \cdots + d_n x^n \in N \cap D_1 = M \subset D$, which means that $\xi \in D$. This contradicts the fact that $D_1 \nsubseteq D$. Therefore V has maximal ideal M and we conclude that $V = \bar{D}$, the integral closure of D [18, p. 17].

We remark that under the hypothesis of Proposition 2.6, D_1 may be properly contained in $\bar{D}$. In fact, $\bar{D}$ need not be a finite ring extension of D_1.

We summarize the results of this section as they apply to domains with the QQR-property.

2.7 THEOREM. *Let D be a quasi-local domain with maximal ideal M having the QQR-property. We suppose that $D \subset \bar{D}$; $\bar{D}$ the integral closure of D. Then $\bar{D}$ is Prüfer with at most two maximal ideals, $\bar{D}$ is the unique minimal overring of D, and M is the conductor of D in $\bar{D}$.*

Proof. By Theorem 1.10, $\bar{D}$ is the unique minimal overring of D. By Corollary 1.7, $\bar{D}$ is Prüfer. Corollary 2.2 shows that $\bar{D}$ has at most two maximal ideals, and Lemma 2.3 states that M is the conductor of D in $\bar{D}$.

3. A characterization of quasi-local domains with the QQR-property. In this section we denote by D a quasi-local domain with maximal ideal M and by $\bar{D}$ the integral closure of D.

TERMINOLOGY. We say that a prime ideal P of an integral domain is *unbranched* if P is the only P-primary ideal. Otherwise we say that P is *branched* [3, p. 252]. We remark that a prime ideal P of a Prüfer domain is branched if and only if P properly contains the union of the chain of primes properly contained in P [3].

3.1 LEMMA. *If $\bar{D}$ is a Prüfer domain with $\{M_\alpha\}$ the set of maximal ideals of $\bar{D}$ and if M is unbranched, then each M_α is unbranched.*

Proof. If some M_α is branched and if for that fixed α we denote by $\{Q_\beta\}$ the set of M_α-primary ideals, then we have $\bigcap_\beta Q_\beta = P_\alpha$, a prime ideal of $\bar{D}$ properly contained in M_α.[3] Since $\bar{D}$ is integral over D, $P_\alpha \cap D = P$ is a prime ideal of D properly

3) Note that this is the only place where the hypothesis that $\bar{D}$ is Prüfer is used; see [14].

contained in M. Hence there is a β such that $Q_\beta \cap D \subset M$, and $Q_\beta \cap D$ is M-primary. Therefore M is branched and Lemma 3.1 is proved.

3.2 LEMMA. *Let $\{M_\beta\}$ be the set of maximal ideals of $\bar{D}$. If M is the conductor of D in $\bar{D}$ and if each M_β is unbranched, then M is unbranched.*

Proof. If Q is an M-primary ideal of D, then since M is an ideal of $\bar{D}$, QM is an ideal of $\bar{D}$. Therefore, $QM = \bigcap_\beta QM\bar{D}_{M_\beta}$. We have $QM \subseteq Q \subseteq M$, and QM and M have the same radical in $\bar{D}$. Thus, $\{M_\beta\}$ is the set of minimal primes of QM in $\bar{D}$ since $\bar{D}$ is integral over D. For any β, $QM\bar{D}_{M_\beta}$ is primary for $M_\beta\bar{D}_{M_\beta}$, and hence, $QM\bar{D}_{M_\beta} = M_\beta\bar{D}_{M_\beta}$ since M_β is unbranched. It follows that $M \subseteq \bigcap_\beta M_\beta\bar{D}_{M_\beta} = QM \subseteq Q$ so that $Q = M$ and M is unbranched as we wished to show.

We can give now a characterization of quasi-local domains with the QQR-property.

3.3 THEOREM. *If D is not a valuation ring, the following are equivalent:*

(*a*) *D has the QQR-property.*

(*b*) *$\bar{D}$ is the unique minimal overring of D and the maximal ideals of $\bar{D}$ are unbranched.*

(*c*) *There are no domains properly between D and $\bar{D}$, $\bar{D}$ is a Prüfer domain, and M is unbranched.*

Proof. (a)→(b): By Theorem 1.10, $\bar{D}$ is the unique minimal overring of D, and by Corollary 1.7, $\bar{D}$ is a Prüfer domain. Hence if M_α is a maximal ideal of $\bar{D}$, $\bar{D}_{M_\alpha}$ is a valuation ring. Since $M_\alpha \cap D = M$, we see that $\bar{D}_{M_\alpha}$ is an overring of D which is not a quotient ring of D. Because D has the QQR-property, there is a set $\{P_\beta\}$ of prime ideals of D such that $\bar{D}_{M_\alpha} = \bigcap_\beta D_{P_\beta}$. Also $D_{P_\beta} = \bar{D}_{\bar{P}_\beta}$ where $\bar{P}_\beta = P_\beta D_{P_\beta} \cap \bar{D}$. Hence by Lemma 1.3, $M_\alpha = \bigcup_\beta \bar{P}_\beta$. Since $\bar{D}_{M_\alpha}$ is not a quotient ring of D, we have $\bar{D}_{\hat{M}_{\hat{\alpha}}} \subset D_{P_\beta} = \bar{D}_{\bar{P}_\beta}$. Thus $\bar{P}_\beta \subset M_\alpha$ for each β, and M_α is the union of a chain of prime ideals of $\bar{D}$ which are properly contained in M_α. Since $\bar{D}$ is a

Prüfer domain, it follows that M_α is unbranched.

(b)→(c): It is clear that there are no domains properly between D and $\bar{D}$. Corollary 2.2 shows that $\bar{D}$ has at most two maximal ideals. If $\bar{D}$ has two maximal ideals then by Proposition 2.5, $\bar{D}$ is a Prüfer domain. If $\bar{D}$ is quasi-local then $M\bar{D}$ has radical $\bar{M}$, the maximal ideal of $\bar{D}$. By Lemma 2.3, $M\bar{D}=M$ and by hypothesis $\bar{M}$ is unbranched; hence $M=\bar{M}$. It follows in this case from Proposition 2.6 that $\bar{D}$ is Prüfer. Since M is the conductor of D in $\bar{D}$, Lemma 3.2 shows that M is unbranched as an ideal of D.

(c)→(a): Theorem 2.4 shows that $\bar{D}$ is the unique minimal overring of D. Let D_1 be an overring of D. If $D=D_1$, then $D_1=D_M$. If $D\subset D_1$, then $\bar{D}\subseteq D_1$. Since $\bar{D}$ is Prüfer, $\bar{D}$ has the QQR-property, and there is a set $\{\bar{P}_\beta\}$ of prime ideals of $\bar{D}$ such that $D_1=\bigcap_\beta \bar{D}_{\bar{P}_\beta}$. If a maximal ideal $M_\alpha\in\{\bar{P}_\beta\}$, then by Lemma 3.1, M_α is unbranched and there is a set $\{\bar{P}_\gamma\}$ of non-maximal prime ideals of $\bar{D}$ such that $\bar{D}_{M_\alpha}=\bigcap_\gamma \bar{D}_{\bar{P}_\gamma}$. Therefore $D_1=\bigcap_\lambda \bar{D}_{\bar{P}_\lambda}$ where each $\bar{P}_\lambda$ is a non-maximal prime of $\bar{D}$. Hence to show that D has the QQR-property it will suffice to show that if $\bar{P}$ is a non-maximal prime of $\bar{D}$ then $\bar{D}_{\bar{P}}=D_P$ where $\bar{P}\cap D=P$. Since $\bar{D}$ is integral over D, P is a non-maximal prime of D and $\bar{D}\subseteq D_P$. Thus $PD_P\cap\bar{D}=N$ is a prime ideal of $\bar{D}$ lying over P and $\bar{D}_N=D_P\subseteq\bar{D}_{\bar{P}}$. Hence $\bar{P}\subseteq N$. It follows that $N=\bar{P}$ and $\bar{D}_{\bar{P}}=D_P$. We conclude that D has the QQR-property.

3.4 COROLLARY. *If J is an integral domain with the QQR-property and if each maximal ideal of J is branched then J is a Prüfer domain.*

Proof. If N is a maximal ideal of J, then J_N has the QQR-property and NJ_N is branched. Hence by Theorem 3.3, J_N is a valuation ring.

3.5 REMARK. Our results in Section 3 show that in the statement of Theorem 2.7 we can actually say that M is the Jacobson radical of $\bar{D}$.

4. Examples. We now consider some examples of domains with the QQR-property which are not Prüfer domains. The examples indicate in some cases negative answers to possible generalizations of results previously obtained.

We denote by G the countable weak direct sum of the additive group of integers, ordered lexicographically. We let k be a field and k_0 be a subfield over which k is algebraic. As in [7, p. 248] we consider $x_1, x_2, \cdots, x_n, \cdots$ elements of an extension field of k which are algebraically independent over k. We define a valuation v on $K=k(x_1, x_2, \cdots)$ as follows. For any nonzero element a of k and any nonnegative integers $r_1, r_2, \cdots, r_n$ we define $v(ax_1^{r_1}x_2^{r_2}\cdots x_n^{r_n}) =(r_1, r_2, \cdots, r_n, 0, \cdots)\in G$; $v(f(x))=$ minimum value of the nonzero monomials occuring in $f(x)$, for any $f(x)\in k[x_1, x_2, \cdots]$; and $v(\xi)=v(f)-v(g)$ for any $\xi=f/g\in k(x_1, x_2, \cdots)$. Let V be the valuation ring associated with v and let M_1 be the maximal ideal of V. We observe that $V=k+M_1$ and that M_1 is an unbranched ideal of V. Consider the domain $D=k_0+M_1$. D is a quasi-local domain with maximal ideal M_1 and integral closure $\bar{D}=V$. Lemma 3.2 shows that M_1 is unbranched as an ideal of D. There is a one-to-one correspondence between domains between D and $\bar{D}$ and fields between k_0 and k.

4.1 EXAMPLE. In the above construction we take k and k_0 to be fields such that $k_0\subset k$ and such that there are no fields properly between k_0 and k. Theorem 3.3 then proves that the domain $D=k_0+M_1$ has the QQR-property.

In [2, p. 200], Davis raised what is, in our terminology, this question: Must a domain D with the QQR-property be Prüfer? Davis showed that the answer is affirmative if each prime ideal of D has finite rank—a result generalized by our Theorem 1.4. And Example 4.1 shows that the answer to Davis' question is negative.

4.2 EXAMPLE. In our construction preceding Example 4.1 we choose for k_0 and k fields such that there exist proper intermediate fields. Then Theorem 1.10 shows that $D=k_0+M_1$ does not have the QQR-property. But D does have the property that each valuation ring containing D which is an overring of D is an intersection of quotient rings of D. To prove this statement we observe that since $\bar{D}=k+M_1$ has unbranched maximal ideal, $\bar{D}$ is an intersection of valuation rings which properly contain $\bar{D}$. Hence it suffices to show that each valuation ring V such that $\bar{D}\subset V\subset K$, is a quotient ring of D. Any such V is of the form $(\bar{D})_P$ for some prime P of $\bar{D}$, $P\subset M_1$. Then if $m\in M_1-P$ and $x\in k$, $xm\in M_1-P$, implying that $x=xm/m\in D_P$. It follows that $k\subseteq D_P$, and hence that $k+M_1=\bar{D}\subseteq D_P$. Therefore $D_P=\bar{D}_{(PD_P\cap\bar{D})}=\bar{D}_P=V$.

Example 4.2 shows that the condition we considered in Theorems 1.4–1.6 is weaker than the QQR-property. In Example 4.3 we construct a quasi-local domain D with the QQR-property such that the integral closure of D is not a valuation ring. This example also shows that the unique minimal overring of a quasi-local domain may not be quasi-local.

4.3 EXAMPLE[4]. Let k denote a prime field and let $x_1, x_2, \cdots, x_n, \cdots$ be elements of an extension field which are algebraically independent over k. We define a valuation v on $k(x_1, x_2, \cdots)$ having value group G as in Example 4.1. If we let $y_i=x_i-1$ for each i, then $k(y_1, y_2, \cdots)=k(x_1, x_2, \cdots)$. Define a valuation w on $k(y_1, y_2, \cdots)$ by $w(f(y_1, \cdots, y_n))=v(f(x_1, \cdots, x_n))$; $w(\xi)=w(f)-w(g)$ where $\xi=f/g\in k(y_1, y_2, \cdots)$. We denote by V and W the valuation rings associated with the valuations v and w respectively. If M_1 and M_2 are the maximal ideals of V and W, we show easily that $V=k+M_1$, $W=k+M_2$, and that M_1 and M_2 are unbranched. Let $D=k+M$, where $M=M_1\cap M_2$, and denote by $\bar{D}$ the integral closure of D. We show that $\bar{D}=V\cap W$. It suffices to observe that each element t of $V\cap W$ is integral over D. Thus there are elements a, b of k such that $t-a\in M_1$, $t-b\in M_2$. And since $t-a\in W$ and $t-b\in V$, it follows that $s=(t-a)(t-b)\in M_1\cap M_2$. Therefore, $t^2-(a+b)t+(ab-s)=0$, and t is integral over D. It follows that $\bar{D}=W\cap V$. Since $V\neq W$, $\bar{D}$ is a Prüfer domain with two maximal

4) Constructions like that of Example 4.3 occur in [5] and [10]. In particular, our proof that $\bar{D}=V\cap W$ comes from [10].

ideals, $M_1 \cap \bar{D}$ and $M_2 \cap \bar{D}$ [12, p. 56]. We observe that $D = k + M$ is quasi-local with maximal ideal M, so that $D \subset \bar{D}$. Because $M = M_1 \cap M_2$, M is an ideal of $\bar{D}$ and is the conductor of D in $\bar{D}$. We now show that there are no domains properly between D and $\bar{D}$. If $\bar{M}_i = M_i \cap \bar{D}$ then $\bar{D}_{\bar{M}_1} = V$ and $\bar{D}_{\bar{M}_2} = W$. Also $\bar{D}/M = \bar{D}/\bar{M}_1 \cap \bar{M}_2 \cong \bar{D}/\bar{M}_1 \oplus \bar{D}/\bar{M}_2$ and $\bar{D}/\bar{M}_i \cong k$. The inclusion map of D into $\bar{D}$ induces a map of $D/M \cong k$ into $\bar{D}/M \cong k \oplus k$ which we will denote by τ. We have $\tau(1) = 1 \oplus 1$ so that $\tau(a) = a \oplus a$ for each $a \in k$. It is easy to see that $k \oplus k$ is a simple ring extension of $\tau(k)$ by any element of $(k \oplus k) - \tau(k)$. Hence for any $d \in \bar{D} - D$, we have $D[d]/M = \bar{D}/M$. Since $M \subset D$, it follows that $D[d] = \bar{D}$ and there are no rings properly between D and $\bar{D}$. Because $\bar{D}$ is a Prüfer domain, Theorem 2.4 implies that $\bar{D}$ is a unique minimal overring of D. Also the valuation rings V and W are the intersection of valuation rings properly containing them. Since $\bar{D}$ is a Prüfer domain, it follows that the maximal ideals of $\bar{D}$ are unbranched. Theorem 3.3 then shows that D has the QQR-property.

We next give an example of a domain with the QQR-property which is neither Prüfer nor quasi-local.

4.4 EXAMPLE. We consider $D = k + M$, the quasi-local domain with the QQR-property which was constructed in Example 4.3. Let T be a valuation ring on $k(x_1, x_2, \cdots)$ of the form $k + M_3$, where M_3 is the maximal ideal of T, and such that there are no containment relations between T, V, and W.

4.5 PROPOSITION. *$D^* = T \cap D$ is a domain with the QQR-property which is neither quasi-local nor Prüfer.*

Proof. By an argument similar to that given in Example 4.3 we show that any valuation ring containing D^* contains either T, V, or W. Therefore the integral closure of D^* is $R = T \cap V \cap W$ and the maximal ideals of D^* are the centers of T, V and W on D^*. Since $D^* \subseteq D$, $M_1 \cap D^* = M_2 \cap D^* = P$. Because there are no containment relations among T, V, and W, R is a Prüfer domain with distinct maximal ideals $M_1 \cap R$, $M_2 \cap R$, and $M_3 \cap R$. Therefore $D^* \subset R$ and D^* is not Prüfer. Let $N = M_3 \cap D^*$. We show that $D^*_N = T$ and $D^*_P = D$. Since $R_{M_3 \cap R} = T$, if $x \in T$, then $x = a/b$

where a, $b \in R$ and $b \notin M_3 \cap R$. We choose $y \in (R \cap M_1 \cap M_2) - M_3$. Then $x = ay/by$, $ay \in T \cap D = D^*$, $by \in D^*$, and $by \notin D^* \cap M_3$. Therefore $x \in D^*_N$ and we conclude that $D^*_N = T$. Since $D^*_P \subseteq D$, it is clear that $P \neq N$ and D^* is not quasi-local. If $x \in D$, we show there exists $y \in D^* - P$ such that $yx \in D^*$. If $x \in T$, then $x \in T \cap D = D^*$ and we take $y = 1$. If $x \notin T$, then $x^{-1} \in M_3$. Let Q be the radical of $x^{-1}T$. Q is a prime ideal of T and $Q \cap D^* \nsubseteq M \cap D^*$. This follows because R is integral over D^* and $Q \cap R$ is a prime ideal of R lying over $Q \cap D^*$. Thus by the "going up" theorem [11, p. 749], if $Q \cap D^* \subseteq M \cap D^*$, then $Q \cap R$ is contained in a prime ideal of R lying over $M \cap D^*$. But $M_1 \cap R$ and $M_2 \cap R$ are the only prime ideals of R lying over $M \cap D^*$ and since $R_{Q \cap R} = T_Q$, $R_{M_1 \cap R} = V$, and $R_{M_2 \cap R} = W$, if either $M_1 \cap R$ or $M_2 \cap R$ contains $Q \cap R$, then either V or W is contained in T_Q. But $x \in V \cap W$ and $x \notin T_Q$ since $x^{-1} \in Q$. Therefore $Q \cap D^*$ is not contained in $M \cap D^*$. We choose $a \in (Q \cap D^*) - M$. There is a positive integer n such that $a^n \in x^{-1}T$. If $y = a^n$, then $y \in D^* - M$ and $yx = a^n x \in T$. Therefore $yx \in D^* = T \cap D$. We conclude that $D^*_{M \cap D^*} = D$. It follows from Theorem 1.9 that D^* has the QQR-property.

4.6 REMARK. If $D_1, D_2, \cdots, D_n$ are quasi-local domains with the QQR-property such that each D_i has quotient field K, then $D = \cap_{i=1}^n D_i$ also has K as its quotient field. For if M_i is the maximal ideal of D_i, then $M = \cap_{i=1}^n M_i$ is the Jacobson radical of $\cap_{i=1}^n (V_i \cap W_i)$ where V_i and W_i are valuation rings of K such that $V_i \cap W_i$ is the integral closure of D_i (we may have $V_i = W_i$ for some i). Since M is an ideal of D, D and $\cap_{i=1}^n (V_i \cap W_i)$ have the same quotient field. Therefore D has quotient field K. The intersection, however, of a finite number of quasi-local domains with the QQR-property need not have the QQR-property. In fact our final example is an integral domain D which is an intersection of two quasi-local domains with the QQR-property each having quotient field K such that the integral closure of D is not Prüfer.

4.7 EXAMPLE. Let F be a field of characteristic zero and let y be transcendental over F. We construct as in Example 4.1 a valuation ring $V = F(y) + M$, where M, the maximal ideal of V, is unbranched. $F(y)$ is an algebraic field extension of dimension 2 over each of the fields $F(y^2)$ and $F(y^2+y)$. Therefore $D_1=F(y^2)+M$ and $D_2=F(y^2+y)+M$ are quasi-local domains with the QQR-property. Since the sum of $F(y)$ and M, as additive subgroups of V, is direct, we have $D=D_1\cap D_2=(F(y^2)\cap F(y^2+y))+M$. But it is not hard to show that $F(y^2)\cap F(y^2+y)=F$ [9, p. 31]. Therefore $D=F+M$ is an integrally closed quasi-local domain which is not Prüfer.

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