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Irreducible representations of the binary modular congruence groups mod p^{λ}

By

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1. Introduction

Let p be an odd prime number and $\boldsymbol{Z}_{\lambda}$ (λ a natural number) be the ring $\boldsymbol{Z}/(p^{\lambda})$. $SL(2, \boldsymbol{Z}_{\lambda})$ are called the binary modular congruence groups mod p^{λ}. In this paper we construct all irreducible representations (on the complex field) of these groups.

Let us consider the additive groups $G=\boldsymbol{Z}_{\lambda}\times\boldsymbol{Z}_{\lambda-k}$ $(0\leq k\leq\lambda)$ and let σ and Δ be integers such that $\sigma\not\equiv 0(p)$ and $\Delta=p^{k}\Delta'$ with $\Delta'\not\equiv 0(p)$. For $u=(u_1, u_2)$ and $v=(v_1, v_2)\in G$, put

$$\langle u, v\rangle = e_{\sigma}\left[\frac{2(u_1v_1+\Delta u_2v_2)}{p^{\lambda}}\right],$$

where $e_{\sigma}[x]=e^{2\pi i\sigma x}$. Then G is self-dual with respect to this function on $G\times G$.

If we restrict the projective representation of $Sp(G)$, the symplectic group of G, defined on $L_2(G)$ (see Weil [3]) to $SL(2, \boldsymbol{Z}_{\lambda})$, we obtain a system of representations R_k $(0\leq k\leq\lambda)$. It is shown that representation R_k $(0\leq k\leq\lambda)$ are subrepresentations of R_0 or R_1 and all irreducible representations of these groups are naturally realized in cartain subspaces invariant under the representations R_k $(0\leq k\leq\lambda)$.

In 1946, H. D. Kloosterman [1] constructed the representations R_0 and R_{λ} by means of transformation formula of theta functions under the modular group and obtained the larger part of irreducible representations of $SL(2, \boldsymbol{Z}_{\lambda})$ by decomposing R_0 into invariant irreducible subspaces. In [2], we gave the construction of the

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representations R_0 and R_1 by means of the projective representation of $Sp(G)$ (they were also constructed by J.A. Shalika). Recently T. Shintani has given the classification of all irreducible representations of those groups by means of the notion of induced representations. Our results as well as Shintani's were reported at the Symposium on theory of group representations and some of its applications held in Kyoto in July, 1967.

2. Construction of system of representations R_k $(0 \leq k \leq \lambda)$

Let $Z_\lambda{}^*$ be the set of all invertible elements in Z_λ and $\left(\frac{\cdot}{p}\right)$ be the Legendre symbol. For $\alpha \in Z$, we define the homomorphism α of G by $u\alpha = (\alpha u_1, \alpha u_2)$.

Let H be the finite dimensional Hilbert space $L_2(G)$ and let $A(\alpha)$ $(\alpha \in Z_\lambda{}^*)$, $B(\beta)$ $(\beta \in Z_\lambda)$ and W be the operators on H defined by

$$A(\alpha)\phi(u) = \left(\frac{\alpha}{p}\right)^k \phi(u\alpha),$$

$$B(\beta)\phi(u) = e_\sigma\left[\frac{\beta(u_1{}^2 + \Delta u_2{}^2)}{p^\lambda}\right]\phi(u),$$

$$W\phi(u) = c\sum_{v \in G}\phi(v)e_\sigma\left[-\frac{2(u_1v_1 + \Delta u_2v_2)}{p^\lambda}\right], \qquad (\phi \in H).$$

Here c is defined by

$$p^{-\lambda+(k/2)}\left(\frac{\Delta'}{p}\right)^{\lambda-k}\left(\frac{\sigma}{p}\right)^k \varepsilon,$$

where

$$\varepsilon = \begin{cases} 1 & \left(k \text{ odd}, \ \left(\frac{-1}{p}\right) = 1\right) \\ -i & \left(k \text{ odd}, \ \left(\frac{-1}{p}\right) = -1\right) \\ (-1)^\lambda & (k \text{ even}). \end{cases}$$

We define elements d_α $(\alpha \in Z_\lambda{}^*)$, ζ_β $(\beta \in Z_\lambda)$ and w of $SL(2, Z_\lambda)$ by

$$d_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad \zeta_\beta = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Any element g of $SL(2, \boldsymbol{Z}_\lambda)$ is expressed by d_α, ζ_β and w:

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \zeta_{\alpha\gamma^{-1}} w d_\gamma \zeta_{\delta\gamma^{-1}}, \qquad \text{if} \quad \gamma \not\equiv 0(p)$$

and

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = w \zeta_{-\gamma\alpha^{-1}} w d_{-\alpha} \zeta_{\beta\alpha^{-1}}, \qquad \text{if} \quad \gamma \equiv 0(p).$$

There exists a representation $T(g)$ of $SL(2, \boldsymbol{Z}_\lambda)$ with representation space H such that

$$T(d_\alpha) = A(\alpha), \quad T(\zeta_\beta) = B(\beta) \qquad \text{and} \quad T(w) = W.$$

Let us denote this representation by $R_k(\sigma, \Delta)$ or simply by R_k. We often suppress the dependence of representations and operators on k, σ and Δ if confusion does not occur. R_0 and R_1 were constructed by means of the projective representations of $Sp(G)$ in [2, §§ 3, 4]. R_k $(2 \leq k \leq \lambda)$ can be constructed analogously; however we shall construct these as subrepresentations of R_0 and R_1.

Let k be 0 or 1 and let l be an integer such that $k+2l \leq \lambda$. Let H_0 be the subspace of the representation space of $R_k(\sigma, \Delta)$ $(k=0, 1)$ with elements ϕ satisfying the following conditions:

1) $\phi(u_1, u_2) = 0 \qquad \text{unless} \quad u_2 \equiv 0(p^l)$;
2) $\phi(u_0, u_2) = \phi(u_1, u_2') \qquad \text{if} \quad u_2 \equiv u_2'(p^{\lambda-k-l})$.

Let us prove that H_0 is invariant under the representations $R_k(\sigma, \Delta)$ and the representation realized on H_0 turns out to be $R_{k+2l}(\sigma, \Delta p^{2l})$. For $\phi \in H_0$, we define function ϕ' on $\boldsymbol{Z}_\lambda \times \boldsymbol{Z}_{\lambda-k-2l}$ by $\phi'(u_1, u_2) = \phi(u_1, p^l u_2)$. We obviously have that $A(\alpha)\phi$, $B(\beta)\phi \in H_0$ and

$$(A(\alpha)\phi)'(u) = \phi'(u\alpha),$$
$$(B(\beta)\phi)'(u) = e_\sigma\left[\frac{u_1^2 + \Delta p^{2l} u_2^2}{p^\lambda}\right]\phi'(u).$$

For the operator W, we need some calculation.

$$W\phi(u) = c \sum_{v \in G} \phi(v) e_\sigma\left[\frac{-2(u_1 v_1 + \Delta u_2 v_2)}{p^\lambda}\right]$$

$$= c\sum\phi(v_1, p^l(a+p^{\lambda-k-2l}b))e_\sigma\Big[-\frac{2\{u_1v_1+\Delta u_2(p^la+p^{\lambda-k-l}b)\}}{p^\lambda}\Big]$$
$$(v_1\in \boldsymbol{Z}_\lambda,\quad a\in \boldsymbol{Z}_{\lambda-k-2l}\quad \text{and}\quad b\in Z_l)$$
$$= c\sum\nolimits_{v_1\in \boldsymbol{Z}_\lambda,\, a\in \boldsymbol{Z}_{\lambda-k-2l}}\phi(v_1, p^la)e_\sigma\Big[-\frac{2(u_1v_1+\Delta p^lu_2a)}{p^\lambda}\Big]$$
$$\times\sum\nolimits_{b\in \boldsymbol{Z}_l}e_\sigma\Big[-\frac{2\Delta' u_2b}{p^l}\Big].$$

So we have shown that $W\phi(u)=0$ if $u_2\not\equiv 0(p^l)$ and that

$$W\phi(u) = cp^l\sum\nolimits_{v_1\in \boldsymbol{Z}_\lambda,\, a\in \boldsymbol{Z}_{\lambda-k-2l}}\phi(v_1, p^la)e_\sigma\Big[-\frac{2(u_1v_1+\Delta p^lu_2a)}{p^\lambda}\Big],$$

if $u_2\equiv 0(p^l)$.

Therefore $W\phi\in H_0$ and

$$(W\phi)'(u) = cp^l\sum\nolimits_{v_1\in \boldsymbol{Z}_\lambda,\, v_2\in \boldsymbol{Z}_{\lambda-k-2l}}\phi'(v_1, v_2)e_\sigma\Big[-\frac{2(u_1v_1+\Delta p^{2l}u_2v_2)}{p^\lambda}\Big].$$

So H_0 is invariant under $R_k(\sigma, \Delta)$ $(k=0, 1)$ and the restriction of the representation $R_k(\sigma, \Delta)$ to H_0 is equivalent to the representation which we called $R_{k+2l}(\sigma, \Delta p^{2l})$.

Let us write, for representation $R_k(\sigma, \Delta)$ $(0\leq k\leq\lambda)$,

$$T(g)\phi(u) = \sum\nolimits_{v\in G}K(g|u, v)\phi(v)\qquad (\phi\in H).$$

Explicit form of $K(g|u, v)$ is determined by the expression of $g\in SL(2, \boldsymbol{Z}_\lambda)$ by d_α, ζ_β and w.

3. Preliminary results for the decomposition of the representations $R_k(\sigma, \Delta)$ $(0\leq k\leq\lambda-1)$.

For $x_1\in \boldsymbol{Z}_\lambda$ and $x_2\in \boldsymbol{Z}_{\lambda-k}$, consider the matrix

$$V=V(x_1, x_2) = \begin{pmatrix} x_1 & -\Delta x_2 \\ x_2 & x_1\end{pmatrix},$$

and let it operate on G by the formula

$$Vu = \begin{pmatrix} x_1 & -\Delta x_2 \\ x_2 & x_1\end{pmatrix}\begin{pmatrix} u_1 \\ u_2\end{pmatrix} = \begin{pmatrix} x_1u_1-\Delta x_2u_2 \\ x_2u_1+x_1u_2\end{pmatrix}.$$

The totality of V which satisfy

$$x_1^2+\Delta x_2^2 \equiv 1(p^\lambda)$$

forms an abelian group C with respect to matrix multiplication. The order of the group C is $p^{\lambda-1}\left(p-\left(\frac{-\Delta}{p}\right)\right)$ [1, p. 351] if $k=0$ and $2p^{\lambda-k}$ if $1\leq k\leq\lambda-1$. $u_1^2+\Delta u_2^2$ is invariant mod p^λ under transformations $V\in C$, so operators of H induced by $V\in C$ commute with operators of the representations.

Let χ be a character of C and H_χ be the subspace of H which is formed by the elements satisfying

$$\phi(Vu) = \chi(V)\phi(u)\,.$$

H_χ is invariant under the representation $R_k(\sigma, \Delta)$ and we denote the restriction of $R_k(\sigma, \Delta)$ to H_χ by $R_k(\sigma, \Delta, \chi)$ or simply by $R_k(\chi)$ and its operators by $T_\chi(g)$.

Let G_l $(k\leq l\leq\lambda)$ be the set of those $u\in G$ which satisfy i) u_1 and $u_2\equiv 0$ $(p^{\lambda-l})$ and ii) u_1 or u_2 is not divisible by $p^{\lambda-l+1}$. Let C_l $(k\leq l\leq\lambda)$ be the subgroups of C of those $V(x_1, x_2)$ which satisfy $x_1\equiv 1$ (p^l) and $x_2\equiv 0$ (p^{l-k}). It is proved that the stationary subgroup of C at $u\in G_l$ is C_l. We call a character of C primitive if its restriction to $C_{\lambda-1}$ is not trivial. The number of the primitive characters is $p^{\lambda-2}(p-1)\left(p-\frac{-\Delta}{p}\right)$ if $k=0$ and $p^{\lambda-1}-p^{\lambda-2}$ if $1\leq k\leq\lambda-1$.

Let $\phi\in H_\chi$ (χ primitive), then $\phi(u)=0$ unless $u\in G_\lambda$. Let θ be a system of representatives of the C-transitive parts of G_λ. Then for $\phi\in H_\chi$

$$T(g)\phi(u) = \textstyle\sum_{v\in\theta}K_\chi(g\,|\,u, v)\phi(v)\,,$$

where

$$K_\chi(g\,|\,u, v) = \textstyle\sum_{V\in C}K(g\,|\,u,\ Vv)\chi(V)\,.$$

4. Irreducibility and equivalence of the representations $R_k(\sigma, \Delta, \chi)$ $(1\leq k\leq\lambda-1)$ corresponding to primitive characters

In this section, we assume $\lambda\geq 2$ and $1\leq k\leq\lambda-1$.

Lemma 1. *Let*

$$u_1^2+\Delta u_2^2 \equiv v_1^2+\Delta v_2^2\ (p^\lambda) \qquad (u, v\in G_\lambda)\,.$$

If $v_1 \not\equiv 0\ (p)$, *there exists a* $V \in C$ *such that* $u = Vv$.

Let G_λ^1 and G_λ^2 denote the subsets of G_λ consisting of elements u which satisfy $u_1 \not\equiv 0\ (p)$ or $u_1 \equiv 0\ (p)$ respectively and put $\theta^i = \theta \cap G_\lambda^i\ (i=1, 2)$. We denote the subspace of H_χ consisting of elements whose carriers are in $G_\lambda^i (i=1, 2)$ by H_χ^i.

Lemma 2. *Let* $u^1, \cdots, u^m$ *be pairwise* C*-inequivalent elements of* G_λ^2 *and* χ *be primitive. Then the linear transformation from* H_χ^1 *to* C^m *defined by*

$$\phi \longrightarrow ((T_\chi(w)\phi)(u^1), \cdots, (T_\chi(w)\phi)(u^m))$$

is onto.

Proof. It is sufficient to prove that the adjoint transformation is one-to-one. Let, for $\phi \in H_\chi^2$ and for all $v \in G_\lambda^1$, $(T_\chi(-w)\phi)(v) = 0$.

Then

$$0 = T_\chi(-w)\phi(v) = \bar{c}\textstyle\sum_{u\in G} e_\sigma\left[\dfrac{2(u_1v_1 + \Delta u_2v_2)}{p^\lambda}\right]\phi(u).$$

Now let $V = V(x_1, x_2) \in C_{\lambda-1}$. Then $x_1 \equiv 1\ (p^{\lambda-1})$ and if $u_1 \equiv 0\ (p)$, we have

$$(Vu)_1 = x_1u_1 - \Delta x_2u_2 \equiv u_1\ (p^{\lambda-1})$$
$$(Vu)_2 = x_2u_1 + x_1u_2 \equiv u_2\ (p^{\lambda-k}).$$

Therefore $e_\sigma\left[\dfrac{2(u_1v_1 + \Delta u_2v_2)}{p^\lambda}\right]$ $(v_1 \equiv 0\ (p))$ is a $C_{\lambda-1}$-invariant function of u defined on G_λ^2. So by primitivity of χ, we have

$$\textstyle\sum_{u\in G} e_\sigma\left[\dfrac{2(u_1v_1 + \Delta u_2v_2)}{p^\lambda}\right]\phi(u) = 0,$$

if $v_1 \equiv 0\ (p)$. Because Fourier transform of ϕ is 0, we have $\phi = 0$. The Lemma is thus proved.

Let S be $\{s = u_1^2 + \Delta u_2^2;\ u \in \theta\}$ and θ_s be $\{u \in \theta;\ u_1^2 + \Delta u_2^2 = s\}$. Let n_s denote the number of elements in θ_s. By Lemma 1 if $s \not\equiv 0\ (p)$, then $n_s = 1$. Also by Lemma 1, we can replace $\theta^1 = \theta \cap G_\lambda^1$ by $\{u_0\alpha, \alpha \in \boldsymbol{Z}_\lambda^*/\{\pm 1\}\}$, where u_0 is an arbitrarily fixed element in G_λ^1. From now on we take this specially chosen θ.

Theorem i) *If* $1 \leq k \leq \lambda - 1$ *and* χ *is primitive,* $R_k(\sigma, \Delta, \chi)$ *is irreducible.*

ii) $R_k(\sigma, \Delta, \chi_1)$ *and* $R_k(\sigma, \Delta, \chi_2)$ (χ_1, χ_2 *primitive*) *are equivalent if and only if* $\chi_1 = \chi_2$ *or* $\chi_1 = \chi_2^{-1}$.

Proof. Let A be a linear transformation from H_{χ_1} to H_{χ_2} commuting with operators of each representation. By commutativity of A with operators corresponding to ζ_β ($\beta \in \boldsymbol{Z}_\lambda$), A can be represented by a matrix-valued functcon on S,

$$S \ni s \rightarrow (a_s(u, v))_{u, v \in \theta_s}.$$

If $s \not\equiv 0\ (p)$, the above matrix is a scalar which we denote by $a(u)$ ($u \in \theta_s$). For $v \in \theta^1$,

$$\sum_{w \in \theta_s} a_s(u, w) K_{\chi_1}(g | w, v) = K_{\chi_1}(g | u, v) a(v) \qquad (u \in \theta_s). \tag{1}$$

In particular, if $u, v \in \theta^1$, we have

$$a(u) K_{\chi_1}(g | u, v) = K_{\chi_2}(g | u, v) a(v). \tag{2}$$

Putting $g = d_\alpha$, we have $a(u\alpha) = a(u)$ if $u \in \theta^1$. So $a(u)$ is independent of $u \in \theta^1$ and we denote it by a.

Now let $\chi_1 = \chi_2 = \chi$. Then for $v \in \theta^1$,

$$\sum_{w \in \theta_s} a_s(u, w) K_\chi(g | w, v) = K_\chi(g | u, v) a \qquad (u \in \theta_s);$$

that is, if $\theta_s = \{w_1, \cdots, w_m\}$ ($m = n_s$), ${}^t(K_\chi(g | w_1, v), \cdots, K_\chi(g | w_m, v)) \in C^m$ is an eigenvector of the matrix $(a_s(u, v))_{u, v \in \theta_s}$ with eigenvalue a. So by Lemma 2 we conclude that $(a_s(u, v))_{u, v \in \theta_s}$ is a diagonal matrix with all diagonal elements equal to a. i) is thus proved.

Let us return to the formula (1) and assume that A is not identically 0. If $a = 0$, $(a_s(u, v))_{u, v \in \theta_s}$ is zero matrix by Lemma 2, which contradicts to the assumption on A, so we have $a \neq 0$. Therefore for $u, v \in \theta^1$ we have by (2)

$$K_{\chi_1}(g | u, v) = K_{\chi_2}(g | u, v). \tag{3}$$

Now $\chi(V) + \chi(V)^{-1}$ ($V = V(x_1, x_2) \in C$) is independent of x_2; hence let $f_\chi(x_1)$ ($x_1 \in \boldsymbol{Z}_\lambda$) be $\chi(V) + \chi(V)^{-1}$ if there exists x_2 such that $V(x_1, x_2) \in C$, and 0 otherwise. Let us define $\tilde{f}_\chi(\alpha)$, $\alpha \in \boldsymbol{Z}_\lambda$ by the formula

$$\tilde{f}_\chi(\alpha) = \sum_{V \in C} e_\sigma\left[\frac{\alpha x_1}{p^\lambda}\right] \chi(V).$$

Then $\tilde{f}_\chi(\alpha)$ is the Fourier transform of $f_\chi(x_1)$:

$$\tilde{f}_\chi(\alpha)=\sum_{x_1\in Z_\lambda} e_\sigma\left[\frac{\alpha x_1}{p^\lambda}\right] f_\chi(x_1)\,.$$

If $\alpha\equiv 0(p)$, $e_\sigma\left[\frac{\alpha x_1}{p^\lambda}\right]$ is $C_{\lambda-1}$-invariant as a function on C, so by primitivity of χ, we have that $\tilde{f}_\chi(\alpha)=0$. On the other hand Formula (3) implies that for $\alpha\not\equiv 0(p)$,

$$\tilde{f}_{\chi_1}(\alpha)=\tilde{f}_{\chi_2}(\alpha)\,.$$

So by the uniqueness of the Fourier transform, we have

$$\chi_1(V)+\chi_1(V)^{-1}=\chi_2(V)+\chi_2(V)^{-1}\,.$$

"Only if" part of ii) is thus proved. "If" part of i) is easy.

5. Description of all irreducible representations

Let first $\lambda\geq 2$. The representation $R_\lambda(\sigma)$ was investigated by Kloosterman [1, pp. 368–375]. It contains two inequivalent irreducible representations $R_\lambda{}^i(\sigma)$ $(i=0,1)$ of dimension $2^{-1}p^{\lambda-2}(p^2-1)$.

We remark that the dimension of the representation space of $R_k(\sigma,\Delta,\chi)$ $(1\leq k\leq\lambda-1$, χ primitive) are equal to $2^{-1}p^{\lambda-2}(p^2-1)$.

Therem i) *Let* $1\leq k, k'\leq\lambda-1$ *and* χ_1,χ_2 *be primitive characters. Then* $R_k(\sigma_1,\Delta_1,\chi_1)$ *and* $R_{k'}(\sigma_2,\Delta_2,\chi_2)$ *are equivalent if and only if* $k=k'$, $\left(\frac{\sigma_1}{p}\right)=\left(\frac{\sigma_2}{p}\right)$, $\left(\frac{\Delta_1}{p}\right)=\left(\frac{\Delta_2}{p}\right)$ *and* $\chi_1=\chi_2$ *or* χ_2^{-1}.

ii) $R_\lambda{}^i(\sigma)$ *and* $R_\lambda{}^i(\sigma')$ $(i,j=0,1)$ *are equivalent if and only if* $\left(\frac{\sigma}{p}\right)=\left(\frac{\sigma'}{p}\right)$ *and* $i=j$.

iii) $R_k(\sigma,\Delta,\chi)$ $(1\leq k\leq\lambda-1$, χ *primitive*) *and* $R_\lambda{}^i(\sigma)$ *are inequivalent.*

Proof. If part of i) and ii) can be shown easily.

The other part of the proof of the theorem is based on the consideration of the spectral properties of operators corresponding to ζ_β $(\beta\in Z_\lambda)$. For this purpose we use the following facts.

Let $p^l\|a$ mean that $p^l|a$ and a is not divisible by p^{l+1}. Let $1\leq k\leq\lambda$. If $p^l\|u_1^2+\Delta u_2^2$ $(u\in\theta,\ l<k)$, then l is even and $p^l\|u_1^2$. If we put $u_1^2+\Delta u_2^2=p^l a$ and $u_1=p^{l/2}u_1'$, we have $a\equiv u_1'^2(p)$.

Now let $1 \leq k \leq \lambda - 1$ and $p^k | u_1^2 + \Delta u_2^2$ $(u \in \theta)$ (then $u_1 \equiv 0$ (p) and $u_2 \not\equiv 0$ (p)) and put $u_1^2 + \Delta u_2^2 = p^k a$. Then if k is odd, $a \equiv \Delta' u_2^2$ (p) and if k is even, $a \equiv u_1'^2 + \Delta' u_2^2$ (p) where $u_1 = p^{k/2} u_1'$.

On the other hand $R_0(\sigma_1, \Delta, \chi)$ and $R_0(\sigma_2, \Delta, \chi)$ are equivalent and Kloosterman [1] proved that $R_0(\sigma, \Delta, \chi)$ (χ primitive) are irreducible representations of dimension $p^\lambda + \left(\frac{-\Delta}{p}\right) p^{\lambda-1}$. He also proved that $R_0(\sigma, \Delta, \chi_1)$ and $R_0(\sigma, \Delta, \chi_2)$ (χ_1, χ_2 primitive) are equivalent if and only if $\chi_1 = \chi_2$ or $\chi_1 = \chi_2^{-1}$. There were obtained $2^{-1} p^{\lambda-2}(p-1)\left(p - \left(\frac{-\Delta}{p}\right)\right)$ irreducible representations and they are inequivalent to those representations described above because the dimensions of the representation spaces are different.

The irreducible representations of $SL(2, \boldsymbol{Z}_\lambda)$ obtained do not degenerate to those of $SL(2, \boldsymbol{Z}_{\lambda-1})$ and the number of them is equal to

$$2^{-1} p^{\lambda-2}(p-1)(p+1) + 2^{-1} p^{\lambda-2}(p-1)(p-1) + 4 \sum_{k=1}^{\lambda-1} (p^{\lambda-k} - p^{\lambda-k-1}) + 4$$
$$= p^\lambda + 3p^{\lambda-1}.$$

On the other hand, the number of the conjugate classes of $SL(2, \boldsymbol{Z}_\lambda)$ is equal to

$$p^\lambda + 4 \sum_{k=0}^{\lambda-1} p^k.$$

Thus all irreducible representations of $SL(2, \boldsymbol{Z}_\lambda)$ are obtained, because those of $SL(2, \boldsymbol{Z}_1)$ were constructed by Kloosterman [1].

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[2] S. Tanaka: On irreducible unitary representations of some special linear groups of the second order, II Osaka J. Math. **3** (1966), 229-242.

[3] A. Weil: Sur certains groupes d'opérateurs unitaires, Acta Math., **111** (1953), 143-211.

Added in proof

The author came to notice that classification of irreducible representations of modular congruence groups was also given by

J. A. Shalike in: Representations of the two by two unimodular group over local fields, I, II, Seminar of representations of Lie groups, Institute for Advanced Study, 1966. His method and result are different from those of the present paper.