J. Math. Kyoto Univ.
7-2 (1967) 113-121

On the Structure of $H^*(BSF; Z_p)$

by

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(Received June 5, 1967)

§ 1. **Statements of Results.** The study of characteristic classes for orthogonal fibre bundles has been very useful in differential topology, differential geometry, and algebraic topology. In recent years, it has become clear that characteristic classes for PL-bundles and spherical fibre spaces will also be useful and should be studied. In this paper, we give a structure theorem for the cohomology modulo an odd prime of the classifying space for oriented spherical fibre spaces.

Let $BSF=BSG$ be the classifying space for oriented spherical fibre spaces (see [10] and [12]). $\boldsymbol{MSF}=\{MSF(n)\}$ be the associated Thom spectrum, and let[2] $\phi: H^*(BSF)\to H^*(\boldsymbol{MSF})$ be the Thom isomorphism. Let $r=2p-2$ throughout this paper. The Wu classes, $q_i\in H^{ir}(BSF)$, are defined by $q_i=\phi^{-1}(\mathcal{P}^i(\phi(1))$. Milnor [10] has shown that $H^*(BSF)$ is isomorphic to a free commutative algebra generated by q_i and βq_i (the Bockstein of q_i) in dimensions $<pr-1$. Gitler and Stasheff [5] have shown that a new element, the first exotic class, e_1, comes in dimension $pr-1$. Stasheff [13] has extended Milnor's computations and shown that q_i and βq_i generate a free commutative subalgebra of $H^*(BSF)$ in dimensions $<2pr$.

Our first theorem is the following.

THEOREM 1. a). *Let* $\theta: Z_p[q_i]\otimes E(\beta q_i)\to H^*(BSF)$ *be the natural map. Then* θ *is a monomorphism.*

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1) Partially supported by the National Science Foundation, and a Fulbright Grant.

2) All cohomology groups, unless otherwise stated, will have coefficient Z_p, p an odd prime.

b). *There exists a homomorphism* Φ, *of Hopf algebras over the mod* p *Seenrod algebra* $\mathfrak{A}$, $\Phi: H^*(BSF)\rightarrow Z_p[q_i]\otimes E(\beta q_i)$ *such that* $\Phi\theta=id$.

Using theorem 1 we prove the mod p analogue of theorem 2.5 in [4], giving a structure theorem for $H^*(BSF)$.

THEOREM 2. *There is a Hopf algebra over* $\mathfrak{A}$, C, *which is* $(pr-2)$-*connected, such that*

$$H^*(BSF)\approx Z_p[q_i]\otimes E(\beta q_i)\otimes C\,,$$

the isomorphism being an isomorphism of Hopf algebras over $\mathfrak{A}$.

We remark that Stasheff [13] has proved theorem 2 in dimensions $<2pr$ and given C explicitly in dimensions $<2pr$, namely C is a free commutative algebra on elements $\{a(e_1)\}$, where $\{a\}$ runs through an additive basis for $\mathfrak{A}/\mathfrak{A}(\mathcal{P}^1)$, in dimensions $\leq pr$. It is not difficult to see that for $p=3$, $(\beta e_1)^3=0$, so C is not a free commutative algebra in general.

Our third theorem concerns the structure of the spectrum ***MSF***. In [4] it was shown that ***MSF*** is of the same mod 2 homotopy type as a wedge of Eilenberg-MacLane spectra.

THEOREM 3. ***MSF*** *is of the same homotopy type as a wedge of Eilenberg-MacLane spectra.*

§ 2. Proof of Theorems 1 and 2.

Since part a) of theorem 1 is an immediate corollary of part b), we restrict our attention to part b). We first prove an easy lemma.

LEMMA 2.1. *There exists an epimorphism of Hopf algebras over* $\mathfrak{A}$, $\rho: H^*(BSO)\rightarrow Z_p[q_i]$.

Proof. We first make $Z_p[q_i]$ into a Hopf algebra over $\mathfrak{A}$ by $\psi(q_i)=\sum_{k=0}^{i} q_k\otimes q_{i-k}$, and defining $\mathcal{P}^t(q_i)$ by the Wu formulae (cf. [10]). $H^*(BSO)=Z_p[P_j]$, $j\geq 1$. Since $q_i=\lambda P_{ir/4}+$decomposable terms with $\lambda\not\equiv 0$ (p) (cf. [7])[1], we may write $H^*(BSO)=Z_p[q_i, P_j]$, $i\geq 1, j\not\equiv 0\,(r/4)$. Define ρ by $\rho(q_i)=q_i$ and $\rho(P_j)=0, j\not\equiv 0\,(r/4)$.

1) $\lambda=(-1)^{\frac{p+1}{2}\cdot\frac{p-1}{2}}$.

Since $\mathfrak{A}(P_j)\subset\operatorname{Ker}\rho$, ρ is a map of algebras over $\mathfrak{A}$. Since $\psi(P_j)=\sum_{0=k}^{j} P_k\otimes P_{j-k}$, if $j\not\equiv 0\,(r/4)$, then either k or $k-j\not\equiv 0\,(r/4)$. Thus ρ is a map of Hopf algebra over $\mathfrak{A}$.

Let $L(n)=S^{2n+1}/Z_p$, $L=\cup L(n)$, the lens spaces. Let $\lambda: L(n)\to CP(n)=S^{2n+1}/S^1$ be the natural map. Let $C_\lambda=CP(n)\bigcup_\lambda C(L(n))$ be the mapping cone of λ and $\pi: C_\lambda\to SL(n)=C_\lambda/CP(n)$ be the natural map. Let $i: BSO\to BSF$ be the natural map. Our main lemma is the following one.

LEMMA 2. 2. *For each positive integer n, there exists maps f_n: $C_\lambda\to BSO$ and h_n: $SL(n)\to BSF$ such that $h_n^*(\beta q_i)\neq 0\in H^{ir+1}(SL(n))$ if $i\leq n(p-1)$ and such that the following diagram is homotopy commutative:*

$$\begin{array}{ccc} C_\lambda & \xrightarrow{f_n} & BSO \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle i} \\ SL(n) & \xrightarrow{h_n} & BSF. \end{array}$$

Furthermore, the maps $\{h_n;\ n=1, 2, \cdots\}$ can be chosen such that $h_n=h|SL(n)$ for a map h: $SL\to BSF$. Thus $h^*(\beta q_i)\neq 0\in H^{ir+1}(SL)$ for all $i\geq 1$.

Proof. Let $j: CP(n)\to C_\lambda$ be the inclusion. In the following diagram

$$\begin{array}{ccccc} \tilde{K}(C_\lambda) & \xrightarrow{j^*} & K(CP(n)) & \xrightarrow{\lambda^*} & K(L(n)) \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} & & \\ \widetilde{KO}(C_\lambda) & \xrightarrow{j^*} & KO(CP(n)), & & \end{array}$$

Ker λ^* is an ideal generated by ξ^p-1, $\xi\in K(CP(n))$ the class of the canonical line bundle over $CP(n)$ (cf. [6]). Consider J: $KO(CP(n))\to J(CP(n))$. There exists a positive integer $e=e(n)$ such that $J((p+1)^e(\Psi_{p+1}-1)r(\xi))=0$ (theorem 1. 3 of [1]). Hence there exists an element $\alpha\in\tilde{K}(C_\lambda)$ such that $j^*(\alpha)=(p+1)^e(\xi^{p+1}-\xi)=(p+1)^e(\Psi_{p+1}-1)\xi$ and $J(j^*(r(\alpha))=0$ since Ψ_{p+1} commutes with r (*cf.* [2]).

Let f_n represent $r(\alpha)$. Then $if_n|CP(n): CP(n)\to BSF$ is homotopic to zero. Hence there exists an h_n: $SL(n)\to BSF$ such that $if_n\simeq h_n\pi$. For $0<i\leq n$, $H^{2i}(C_\lambda;\ Z)$ is mapped by j^* monomor-

phically onto $p\cdot H^{2i}(CP(n)\,;\,Z)$. Let $x=c_1(\xi)\in H^2(CP(n)\,;\,Z)$, then $y_{2i}=j^{*-1}(p\cdot x^i)$ generates $H^{2i}(C_\lambda\,;\,Z)$. Obviously, $y_{2i}\cdot y_{2j}=p\cdot y_{2i+2j}$ $(i,\ j>0)$, and the cup products in $H^*(C_\lambda)$ are trivial. For the total Chern class c, we have $c(\xi^{p+1}-\xi)=(1+(p+1)x)/(1+x)=1+\sum_{i=1}^{n}(-1)^{i-1}(p\cdot x^i)$ and $c(\alpha)=(1+\sum_{i=1}^{n}y_{2i})^{(p+1)^e}\equiv 1+\sum_{i=1}^{n}(-1)^{i-1}y_{2i}\ (p)$. Thus, $P_k(r(\alpha))\equiv \pm y_{4k}\ (p)$. Since $q_i\equiv\lambda P_{ir/4}+$decomposable terms with $\lambda\not\equiv 0\ (p)$, $\pi^*h_n^*(q_i)=f_n^*(q_i)=q_i(r(\alpha))=\lambda P_{ir/4}(r(\alpha))=\pm\lambda y_{ir}\neq 0$ for $i\leq n/(p-1)$, whence $h_n^*(q_i)\neq 0$ and $h_n^*(\beta q_i)=\beta h_n^*(q_i)\neq 0$.

Consider a map h_n satisfying the conditions of the theorem, then, if $n\geq m>0$, $h_m=h_n|SL(m)$ does so. Since $[SL(n), BSF]$ is finite the set of the homotopy classes of such maps h_n's is finite for each n. Then it is an easy arithmetic that there exists a sequence $\{h_n,\ h=1, 2, \cdots\}$ such that $h_n\simeq h_{n+1}|SL(n)$. Using the homotopy extension theorem successively, we have a map h: $SL\to BSF$ such that $h_n=h|SL(n)$ satisfies the conditions with respect to some f_n (which might be not $f_{n+1}|C_\lambda(n)$). This completes the lemma.

The idea of the proof of part b) of theorem 1 is to consider the map $BSO\times SL\times SL\times\cdots\times SL\to BSF$ given by i: $BSO\to BSF$, h: $SL\to BSF$, and multiplication and to show that the image of the induced map on cohomology is $Z_p[q_i]\otimes E(\beta q_i)$. We start with some preliminary considerations.

Let M be a connected, graded, locally finite algebra over Z_p. Let $\underset{N}{\otimes}M=M\otimes\cdots\otimes M$, N-times. The symmetric group $\mathcal{S}(N)$ acts on $\underset{N}{\otimes}M$ (with the usual signs) and let $\mathcal{S}^N(M)\subset\underset{N}{\otimes}M$ denote the set of elements left fixed by the action of $\mathcal{S}(N)$. For any one-to-one map ε: $\{1, \cdots, N\}\to\{1, \cdots, N'\}$, we obtain ε^*: $\underset{N'}{\otimes}M\to\underset{N}{\otimes}M$ and $\mathcal{S}(\varepsilon)$: $\mathcal{S}^{N'}(M)\to\mathcal{S}^N(M)$. $\mathcal{S}(\varepsilon)$ depends only on N and N' and is an isomorphism in dimensions $\leq N\cdot$(connectivity of $M+1$). Let $\mathcal{S}(M)=\lim_{N}\mathcal{S}^N(M)$, an algebra over Z_p. The identity $\underset{2N}{\otimes}M=\underset{N}{\otimes}M\otimes\underset{N}{\otimes}M$ induces an algebra homomorphism $\mathcal{S}^{2N}(M)\to\mathcal{S}^N(M)\otimes\mathcal{S}^N(M)$ and hence a homomorphism ψ: $\mathcal{S}(M)\to\mathcal{S}(M)\otimes\mathcal{S}(M)$. Thus $\mathcal{S}(M)$ is a Hopf algebra over Z_p.

If f: $M\to M'$, we obtain $\underset{N}{\otimes}f$: $\underset{N}{\otimes}M\to\otimes M'$ and $\mathcal{S}(f)$: $\mathcal{S}(M)\to\mathcal{S}(M')$. If M is an algebra over $\mathcal{A}$, then so is $\underset{N}{\otimes}M$ and $\mathcal{S}^N(M)$, and $\mathcal{S}(M)$ is a Hopf algebra over $\mathfrak{A}$. If f is an $\mathfrak{A}$-map, then so is $\mathcal{S}(f)$.

LEMMA 2. 3. *Let $M=M_1 \oplus M_2$ as algebras (with base points identified). Then $\mathcal{S}(M) \xrightarrow{\psi} \mathcal{S}(M)\otimes\mathcal{S}(M)\to\mathcal{S}(M_1)\otimes\mathcal{S}(M_2)$ is an isomorphism of Hopf algebras with inverse $\mathcal{S}(M_1)\otimes\mathcal{S}(M_2)\to\mathcal{S}(M)\otimes\mathcal{S}(M) \xrightarrow{\phi} \mathcal{S}(M)$.*

The proof is left to the reader.

Let $H'(SL)\subset H^*(SL)$ be the submodule consisting of $H^0(SL)$, $H^{ir}(SL)$ and $H^{ir+1}(SL)$. Let π': $H^*(SL)\to H'(SL)$ be the natural projection. Note that π' is a map of algebras over $\mathfrak{A}$. Let $H^{\text{odd}}(SL(n))=H^0(SL(n))+\sum H^{2j+1}(SL(n))$, π'': $H^*(SL(n))\to H^{\text{odd}}(SL(n))$ the natural map. Note that $H^*(SL(n))\approx H^{\text{odd}}(SL(n))\oplus H^*(C_\lambda)$ as algebras.

Because $H^*(BSF)$ is cocommutative and coassociative, the iterated diagonal ψ induces Ψ : $H^*(BSF)\to\mathcal{S}(H^*(BSF))$. We define

$$\Phi: H^*(BSF)\to Z_p[q_i]\otimes\mathcal{S}(H'(SL))$$

to be the following composition.

$$H^*(BSF) \xrightarrow{\psi} H^*(BSF)\otimes H^*(BSF) \xrightarrow{1\otimes\Psi} H^*(BSF)\otimes\mathcal{S}(H^*(BSF))$$
$$\xrightarrow{i^*\otimes\mathcal{S}(h^*)} H^*(BSO)\otimes\mathcal{S}(H^*(SL)) \xrightarrow{\rho\otimes\mathcal{S}(\pi')} Z_p[q_i]\otimes\mathcal{S}(H'(SL)).$$

Φ is a homomorphism of Hopf algebras over $\mathfrak{A}$ by construction and lemma 2. 1. Part b) of theorm 1 follows immediately from the following lemma.

LEMMA 2. 4. a). $\operatorname{Im}\Phi=\operatorname{Im}(\Phi\theta)$.

b). $\Phi\theta$ is a monomorphism.

Proof. Since $\mathcal{S}(H'(SL))\approx\mathcal{S}(H'(SL(n))$ in dimensions $\leq 2n+1$, we may replace SL, h and π' by $SL(n)$, $h_n=h|SL(n)$ and π': $H^*(SL(n))\to H'(SL(n))$ respectively. By the coassociativity of $H^*(BSF)$ and lemmas 2. 2, 2. 3, we have the following commutative diagram.

$$
\begin{array}{ccc}
H^*(BSF) & \xrightarrow{(1\otimes\Psi)\psi} & H^*(BSF)\otimes\mathcal{S}(H^*(BSF)) \\
\downarrow (1\otimes\Psi)\psi & & \downarrow i^*\otimes\mathcal{S}(\pi'' h_n^*) \\
H^*(BSF)\otimes\mathcal{S}(H^*(BSF)) & & H^*(BSO)\otimes\mathcal{S}(H^{\mathrm{odd}}(SL(n)) \\
\downarrow i^*\otimes\mathcal{S}(h_n^*) & & \downarrow \psi\otimes 1 \\
H^*(BSO)\otimes\mathcal{S}(H^*(SL(n))) & & H^*(BSO)\otimes H^*(BSO)\otimes\mathcal{S}(H^{\mathrm{odd}}(SL(n))) \\
\searrow 1\otimes\psi & & \swarrow 1\otimes(\mathcal{S}(f_n^*)\Psi)\otimes 1 \\
 & H^*(BSO)\otimes\mathcal{S}(H^*(SL(n))\otimes\mathcal{S}(H^*(SL(n)) & \\
 & \downarrow 1\otimes\mathcal{S}(f_n^*)\otimes\mathcal{S}(\pi'') & \\
\downarrow id & H^*(BSO)\otimes\mathcal{S}(H^*(C_\lambda))\otimes\mathcal{S}(H^{\mathrm{odd}}(SL(n))) & \\
 & \swarrow (1\otimes\phi) & \\
H^*(BSO)\otimes\mathcal{S}(H^*SL(n))) . & &
\end{array}
$$

Apply $\rho\otimes\mathcal{S}(\pi')$ to the left-side line to obtain Φ. Clearly

$$
\operatorname{Im}\Phi\subset\operatorname{Im}
\left(\begin{array}{l}
H^*(BSO)\otimes\mathcal{S}(H^{\mathrm{odd}}(SL(n))\to H^*(BSO)\otimes\mathcal{S}(H^{\mathrm{odd}}(SL(n))\cap H'(SL(n))) \\
\xrightarrow{\psi\otimes 1} H^*(BSO)\otimes H^*(BSO)\otimes\mathcal{S}(H^{\mathrm{odd}}(SL(n))\cap H'(SL(n))) \\
\to H^*(BSO)\otimes\mathcal{S}(H^*(C_\lambda)\cap H'(SL(n)))\otimes\mathcal{S}(H^{\mathrm{odd}}(SL(n))\cap H'(SL(n))) \\
\to Z_p[q_i]\otimes\mathcal{S}(H'(SL(n)))
\end{array}\right)
$$

$$
\subset\operatorname{Im}
\left(\begin{array}{l}
Z_p[q_i]\otimes\mathcal{S}(H^{\mathrm{odd}}(SL(n)))\to Z_p[q_i]\otimes\mathcal{S}(H^{\mathrm{odd}}(SL(n))\cap H'(SL(n))) \\
\to\cdots\to Z_p[q_i]\otimes\mathcal{S}(H'(SL(n)))
\end{array}\right)
$$

as $H^*(BSO)\to\mathcal{S}(H^*(C_\lambda)\cap H'(SL(n)))$ factors through $Z_p[q_i]$. Finally, $Z_p[q_i]\otimes E(\beta q_i)\xrightarrow{\theta} H^*(BSF)\xrightarrow{(1\otimes\Psi)\psi} H^*(BSF)\otimes\mathcal{S}(H^*(BSF))\xrightarrow{i^*\otimes\mathcal{S}(\pi'' h_n^*)} H^*(BSO)\otimes\mathcal{S}(H^{\mathrm{odd}}(SL(n)))\to Z_p[q_i]\otimes\mathcal{S}(H^{\mathrm{odd}}(SL(n))\cap H'(SL(n)))$ is an epimorphism so $\operatorname{Im}\Phi\subset\operatorname{Im}(\Phi\theta)\subset\operatorname{Im}\Phi$ and part a) is proved. Part b) follows from the fact that the composition $Z_p[q_i]\otimes E(\beta q_i)\xrightarrow{\theta} H^*(BSF)\to Z_p[q_i]\otimes\mathcal{S}(H'(SL(n)))\to Z_p[q_i]\otimes\mathcal{S}(H^{\mathrm{odd}}(SL(n))\cap H'(SL(n)))$ is an isomorphism in dimensions $\leq 2n+1$. This finishes the proof of theorem 1.

We remark that one can give a shorter proof of part a) of theorem 1 using lemma 8.2 of [14], a generalization of the argument used in the proof of theorem 4.3 of [5], and some Hopf algebra arguments. These methods don't seem to give part b) however.

We now turn to the proof of theorem 2. Theorem 2 follows from part b) of theorem 1 using the mod p analogue of the proof of theorem 2.5 of [4]. We give here only an outline. Define $C=H^*(BSF)/(\overline{Z_p[q_i]\otimes E(\beta q_i)})\cdot H^*(BSF)$. Since $H^*(BSF)$ is commutative the ideal is a two-sided ideal over $\mathcal{A}$ and C is a Hopf algebra over $\mathcal{A}$. Let $\gamma: H^*(BSF)\to C$ be the quotient map. By theorem 4.4. of [11], the isomorphism of theorem 2 is given by the composition $H^*(BSF)\xrightarrow{\psi} H^*(BSF)\otimes H^*(BSF)\xrightarrow{\Phi\otimes\gamma} Z_p[q_i]\otimes E(\beta q_i)\otimes C$, and all maps are homomorphisms of Hopf algebras over $\mathcal{A}$.

We remark that the second part of theorem 2.11 of [4] is incerrect as the isomorphism is only as Hopf algebras over $\mathcal{A}'$, the subalgebra of $\mathcal{A}$ generated by $\mathcal{P}^i$.

§3. Proof of Theorem 3.

The proof of theorem 3 will be analogous to that of corollary 2.10 of [4]. It depends on a determination of $H^*(\boldsymbol{MSF})$ as a module over $\mathcal{A}$.

PROPOSITION 3.1. *In $H^*(\boldsymbol{MSF})$, $Q_i(U)=\lambda U\cdot(\beta q_{p(i)}+$decomposable terms), where $\lambda\not\equiv 0$ (p), Q_i is the primitive element in a dimension $r\cdot p(i)+1$, and $p(i)=p^{i-1}+p^{i-2}+\cdots+p+1$.*

Proof. By the Wu formulae (*cf.* [10]), $\mathcal{P}^t q_s=(-1)^t\binom{s(p-1)-1}{t}q_{s+t}+$decomposable terms, and $\mathcal{P}^t\beta q_s=(-1)^t\binom{s(p-1)}{t}\beta q_{s+t}+$decomposable terms. Recall that $Q_0=\beta$ and $Q_i=[\mathcal{P}^{p^{i-1}}, Q_{i-1}]$ (see [8]). Hence

$Q_i(U)=U\cdot(\mathcal{P}^{p^{i-1}}(\beta q_{p(i-1)})-Q_{i-1}(q_{p^{i-1}})+$decomposable terms), by induction on i. Now $\mathcal{P}^{p^{i-1}}(\beta q_{p(i-1)})=0$ for dimensional reasons. $Q_{i-1}(q_{p^{i-1}})$ has terms of the form $\mathcal{P}^{p^t}(\beta q_{p(i)-p^t})$ with $1\leq t\leq i-2$ and a term $\beta\mathcal{P}^1\mathcal{P}^p\cdots\mathcal{P}^{p^{i-2}}(q_{p^{i-1}})$ plus decomposable terms. Since

$$\binom{(p(i)-p^t)(p-1)}{p^t} \equiv \binom{(p-1)p^{i-1}+\cdots+\widehat{(p-1)p^t}+\cdots+(p-1)}{p^t}$$
$$\equiv 0 \qquad (p),$$

$\mathcal{P}^{p^t}(\beta q_{p(i)-p^t})$ is decomposable. Finally, by downward induction on k, we prove that $\mathcal{P}^{p^k}\mathcal{P}^{p^{k+1}}\cdots\mathcal{P}^{p^{i-2}}(q_{p^{i-1}})=q_{p^{i-1}+p^{i-2}+\cdots+p^k}+$decomposable terms. Namely,

$$\begin{aligned}\mathcal{P}^{p^{k-1}}\mathcal{P}^{p^k}\cdots\mathcal{P}^{p^{i-2}}(q_{p^{i-1}}) &= \mathcal{P}^{p^{k-1}}(\lambda q_{p^{i-1}+\cdots+p^k})+\text{decomposable terms}\\ &= -\lambda\binom{(p^{i-1}+\cdots+p^k)(p-1)-1}{p^{k-1}}q_{p^{i-1}+\cdots+p^{k-1}}\\ &\qquad +\text{decomposable terms}\end{aligned}$$

and the coefficient is non-zero mod p. Thus

$\beta\mathcal{P}^1\mathcal{P}^p\cdots\mathcal{P}^{p^{i-2}}(q_{p^{i-1}})=\lambda\beta q_{p(i)}+$decomposable terms, with $\lambda\not\equiv 0$ (p)

and the proposition is proved.

PROPOSITION 3. 2. *Let θ: $\mathcal{A}\to H^*(\boldsymbol{MSF})$ be defined by $\theta(a)=a(U)$. Then*

$$\operatorname{Ker}\theta = \mathcal{A}(\beta).$$

Proof. $\beta(U)=0$, hence $\mathcal{A}(\beta)\subset\operatorname{Ker}\theta$. θ defines a map of coalgebras θ' : $\mathcal{A}/\mathcal{A}(\beta)\to H^*(\boldsymbol{MSF})$. To show θ' is a monomorphism, it is enough to show $P(\mathcal{A}/\mathcal{A}(\beta))\cap\operatorname{Ker}\theta'=0$. An additive base for $P(\mathcal{A}/\mathcal{A}(\beta))$ is given by Q_i, $i>0$, and $\mathcal{P}^{\Delta_j}$, where $\Delta_j=(0,\cdots,1,\cdots)$ in Milnor's notation. $\mathcal{P}^{\Delta_j}$ goes to a non-zero element under the composition $\mathcal{A}/\mathcal{A}(\beta)\to H^*(\boldsymbol{MSF})\to H^*(\boldsymbol{MSO})$ [9] and $\theta'(Q_i)=Q_i(U)\neq 0$ for $i>0$ by proposition 3. 1.

THEOREM 3. 3. *Let M be a connected coalgebra over $\mathcal{A}$. Let $\theta:\mathcal{A}\to M$ be defined by $\theta(a)=a(1)$. Assume that* $\operatorname{Ker}\theta=\mathcal{A}(\beta)$. *Then, as an $\mathcal{A}$-module,*

$$M\approx\sum\mathcal{A}/\mathcal{A}(\beta)\oplus\sum\mathcal{A}.$$

Proof. The proof of this theorem is analogous to, and easier than, the proof of theorem 8. 1 of [3] (see the remarks after the proof of theorem 8. 1 of [3]).

COROLLARY 3. 4. *$H^*(\boldsymbol{MSF})$, as an $\mathcal{A}$-module, is isomorphic* *to a direct sum of copies of $\mathfrak{A}/\mathfrak{A}(\beta)$ and $\mathfrak{A}$.*

Proof. This follows immediately from proposition 3.2 and theorem 3.3.

The mod p version of lemma 4.1 of [4] and corollary 3.4 imply that ***MSF*** is of the same homotopy type mod p as a wedge of Eilenberg-MacLane spectra. Since this is true for all p including 2, theorem 3 follows.

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