J. Math. Kyoto Univ.
7-3 (1967) 315-324

Some remarks on boundary values of harmonic functions with finite Dirichlet integrals

By

Yukio Kusunoki and Shin-ichi Mori

(Received November 27, 1967)

Introduction. In this article we shall show some results concerning the boundary values of an HD-function (a single-valued harmonic function with finite Dirichlet integral) given as the limit of HD-functions.

Let R be an open Riemann surface of hyperbolic type or more generally a Green space. We consider a D-normal compactification R^* of R, a notion introduced by Maeda [6], and the ideal boundary $\Delta = R^* - R$. Several well-known compactifications, for instance, those of Wiener, Royden, Martin and Kuramochi are D-normal. Every HD-function u on R is, by definition, expressed as $u(a) = \int_\Delta f d\omega_a$ with a resolutive function f and the harmonic measure ω. The f is determined except a set of harmonic measure zero and is denoted by $H^{-1}u$. In some compactifications $H^{-1}u$ is given as the limit values of u. We extend the definition of the linear operator H^{-1} to define $H^{-1}u$ for HD-functions u given outside of compact sets on R and prove Theorem 2 which will play a fundamental role in the sequel. From Theorem 3 we shall derive Theorem 4 which is regarded as a generalization of the corresponding theorem in Kusunoki [5] obtained for Kuramochi boundary.

1. In the following we shall denote by R an open Riemann surface of hyperbolic type. First of all we state the following known result (Constantinescu-Cornea [1], Doob [3], Maeda [6]) together with a new simple proof.

LEMMA 1. *Let R^* be any resolutive compactification of R and $\Delta=R^*-R$ the ideal boundary of R. If u is a harmonic function on R which is expressed as*

$$u(a)=H_f(a)=\int_\Delta f d\omega_a, \quad a\in R$$

with a resolutive function f and the harmonic measure ω_a with respect to a, then we have

$$(L.H.M.|u|)(a)=\int_\Delta |f| d\omega_a$$

where L.H.M. v stands for the least harmonic majorant of v.

PROOF. Since the resolutive functions are ω-summable and $|u|$ is dominated by harmonic function $\int_\Delta |f| d\omega_a$, $L.H.M.|u|$ exists and

$$L.H.M.|u|\leqq\int_\Delta |f| d\omega.$$

The opposite inequality is obtained as follows. Since $2\max(u,0)-u=|u|$, $2\max(u,0)\leqq u+L.H.M.|u|$ and therefore

$$2(u\vee 0)\leqq u+L.H.M.|u|$$

where $u\vee 0=L.H.M.\max(u,0)$. While $u=u\vee 0-(-u)\vee 0$, so we have

$$u\vee 0+(-u)\vee 0\leqq L.H.M.|u|$$

which is the required, because the left hand side is equal to

$$H_{\max(f,0)}+H_{\max(-f,0)}=\int_\Delta |f| d\omega.$$

CORORALLY. *$u=H_f$ is non-negative on R if and only if $f\geqq 0$ on Δ ω-almost everywhere (ω-a.e.).*

Indeed, if $u\geqq 0$ on R, then $L.H.M.|u|=u$ hence $\int_\Delta(|f|-f)d\omega=0$ by Lemma 1. The integrand is non-negative, so $f\geqq 0$ on Δ ω-a.e.

The converse is trivial.

From this corollary we know that $u=H_f$ vanishes identically if and only if $f=0$ on Δ ω-*a.e.* Identifying two functions which are mutually equal on Δ ω-*a.e.*, the mapping

$$u=H_f\to f$$

is then one-to-one and positive linear.

2. Suppose that $\{u_n\}$ is a sequence of harmonic functions on R which converges to zero on R. T en under what conditions can we conclude that $L.H.M.|u_n|\to 0$ on R? It is of course true if u_n are non-negative on R, but generally not true if each u_n does not have a definite sign on R. Example I (below) shows that it does not hold if u_n are uniformly bounded. Now we shall prove the following theorem which provides an answer for above question and is useful for our later purpose.

THEOREM 1. *Let $\{u_n\}$ be a sequence of harmonic functions on R which converges to* 0 *on R. If the Dirichlet integrals $\|du_n\|^2 =\int_R du_n\wedge {}^*du_n$ converge to* 0 *for $n\to\infty$, then we have*

$$L.H.M.|u_n|\to 0 \quad \text{on } R.$$

PROOF. By Royden decomposition one can write as

$$|u_n|=v_n+\varphi_n$$

where $v_n\in HD(R)$ and φ_n are Dirichlet potentials on R. Since $|u_n|$ are subharmonic, $v_n=L.H.M.|u_n|\geqq 0$ and

$$\|dv_n\|\leqq\|d|u_n|\|=\|du_n\| \quad \text{(Dirichlet principle)}.$$

While $\|du_n\|\to 0$ and $\|d\varphi_n\|\leqq\|du_n\|$, hence

$$\|dv_n\|\to 0,\quad \|d\varphi_n\|\to 0.$$

Now we show that the limit function of $\{v_n\}$ exists on R and is identically zero. Suppose the contrary, then there exists a point $c\in R$ and a subsequence $\{v_n^*\}$ of $\{v_n\}$ such that the limit of $\{v_n^*(c)\}$ exists (may be $+\infty$) and is different from zero. Let $\{\varphi_n^*\}$ be the corresponding subsequence of $\{\varphi_n\}$. Take again a subsequence $\{\varphi_n^{**}\}$ of $\{\varphi_n^*\}$ such that

$$\|d\varphi_n^{**}-d\varphi_{n+1}^{**}\|<\frac{1}{2^n}, \quad n=1, 2, \cdots,$$

then $\{\varphi_n^{**}\}$ converges quasieverywhere on R to a Dirichlet potential φ and $\|d\varphi_n^{**}-d\varphi\|\to 0$ (Hilfssatz 7.8 [2]). Therefore $\|d\varphi\|=\lim\|d\varphi_n^{**}\|=0$ and φ is zero quasieverywhere on R. Since $u_n\to 0$ on R, the corresponding subsequence $\{v_n^{**}\}$ of $\{v_n^*\}$ converges to 0 except a polar set e. While $\{dv_n\}$ is a Cauchy sequence in norm, hence $\{dv_n^{**}\}$ is so. Then for any fixed point $p_0\in R$ $v_n^{**}(p)-v_n^{**}(p_0)$ $(p\in R)$ converge to 0 uniformly on every compact set on R. Taking the point $p_0\notin e$, we know therefore that v_n^{**} converge to 0 everywhere on R, in particular $\lim v_n^*(c)=\lim v_n^{**}(c)=0$, which is a contradiction.

EXAMPLE I. Let R be a unit disc and

$$u_n(z)=r^n\sin n\theta, \quad z=re^{i\theta}$$

then the harmonic functions u_n converge to 0 on R and $|u_n|\leqq 1$, but $L.H.M.|u_n|$ does not converge to zero.

Indeed,

$$(L.H.M.|u_n|)(re^{i\theta})=\frac{1}{2\pi}\int_0^{2\pi}|\sin n\varphi|\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}d\varphi$$

so, in particular

$$(L.H.M.|u_n|)(0)=\frac{1}{2\pi}\int_0^{2\pi}|\sin n\varphi|d\varphi=\frac{2}{\pi}.$$

Note that $\|du\|^2=n\pi\to\infty$, furthermore

$$\varliminf_{n\to\infty}(L.H.M.|u_n|)(e^{i\theta})=\varliminf_{n\to\infty}|\sin n\theta|=0 \quad \text{for any } \theta.$$

3. A resolutive compactification R^* of R is said to be D-normal (Maeda [6]) if every HD-function u on R can be written as

$$u(a)=H_f(a)=\int_\Delta f\,d\omega_a, \quad a\in R$$

with a resolutive function f on Δ. The f is determined except a set of harmonic measure zero as we noted before. Hence we shall write

$$f=H^{-1}u, \quad u\in HD(R).$$

Several known compactifications, for instance, those of Wiener, Royden, Martin and Kuramochi are all D-normal [6]. In case of Wiener and Royden, $H^{-1}u$ is the continuous extension of u onto Δ. In case of Martin, $H^{-1}u$ is given as the fine limit of u on Δ which exist ω-*a.e.* on Δ (Naim [7]). In case of Kuramochi, $H^{-1}u$ is given as the continuation of u onto Δ except a set of capacity zero, hence of harmonic measure zero (Constantinescu-Cornea [2]).

Next we shall slightly extend the definition of the operator H^{-1}, that is, we define $H^{-1}u$ for HD-functions u defined on the neighborhood of the ideal boundary. Let E be a compact set on R and $u\in HD(R-E)$, then one finds that for a normal exhaustion $\{R_n\}$ of R the limit function

$$U(a)=\lim_{n\to\infty} H_u^{R_n}(a)$$

exists on R, $U\in HD(R)$, moreover U is uniquely determined by u independently on the choice of exhaustions. To see this take a relatively compact subregion E' containing $\bar{E}(=E\cup\partial E)$ and a Dirichlet function $\tilde{u}$ *on* R such that $\tilde{u}=u$ on $R-E'$ (for instance, take E' with smooth boundary and set $\tilde{u}=H_u^{E'}$ on E' and $\tilde{u}=u$ on $R-E'$), then we know that U is nothing else but the harmonic part of Royden decomposition of $\tilde{u}$. Thus we define $H^{-1}u$ by

$$H^{-1}u=H^{-1}U, \quad u\in HD(R-E)$$

This definition clearly coincides with the original one if E is empty. Moreover $\tilde{u}-U$ is a Dirichlet potential and $\tilde{u}=u$ on $R-E'$, hence in cases of the compactifications of Wiener, Royden, Martin and Kuramochi $H^{-1}u$ is given as the extention of u with exactly the same properties as stated before.

Now we shall prove the following fundamental

THEOREM 2. *Let R^* be a D-normal compactification of R and E be a compact set* (*may be empty*) *on R. Suppose $\{u_\nu\}$ is a* *sequence of HD-functions on $R-E$ such that $u_\nu \to u \in HD(R-E)$ and $\|du_\nu - du\|_{R-E} \to 0$ as $\nu \to \infty$, then we have*

$$\varliminf_{\nu\to\infty} |H^{-1}u_\nu(b) - H^{-1}u(b)| = 0$$

for $b \in \Delta = R^* - R$ except a set of harmonic measure zero.

PROOF. Take two relatively compact subregions E_1 and E_2 of R so that $\overline{E} \subset E_1$, $\overline{E}_1 \subset E_2$ and each component of $E_2 - \overline{E}_1$ is conformally equivalent with a ring domain on the complex plane. And define

$$\tilde{u} = \begin{cases} u & \text{on } R - E_1 \\ H_u^{E_1} & \text{on } \overline{E}_1 \end{cases}$$

$$\tilde{u}_\nu = \begin{cases} u_\nu & \text{on } R - E_2 \\ H_{f_\nu}^{E_2 - \overline{E}_1} & \text{on } E_2 - E_1 \text{ with the function } f_\nu \text{ such that } f_\nu = u_\nu \\ & \text{on } \partial E_2 \text{ and } f_\nu = u \text{ on } \partial E_1 \\ H_u^{E_1} & \text{on } \overline{E}_1 \end{cases}$$

Under our hypothesis $u_\nu - u$ (and their derivatives) converge to 0 uniformly on every compact set on $R-E$, hence $\tilde{u}_\nu - \tilde{u}$ converge to 0 on R. Moreover it is proved that

$$\|d\tilde{u}_\nu - d\tilde{u}\| \to 0$$

Because, since $\|d\tilde{u}_\nu - d\tilde{u}\|^2 = \|d\tilde{u}_\nu - d\tilde{u}\|^2_{E_2 - \overline{E}_1} + \|du_\nu - du\|^2_{R - \overline{E}_2}$, it suffices to show $\|d\tilde{u}_\nu - d\tilde{u}\|_{E_2 - \overline{E}_1} \to 0$, which is seen from Lemma 2 below. Now let $U_\nu = \lim_{n\to\infty} H_{\tilde{u}_\nu}^{R_n}$, $U = \lim_{n\to\infty} H_{\tilde{u}}^{R_n}$, then by Dirichlet principle

$$\|dU_\nu - dU\| \leqq \|d\tilde{u}_\nu - d\tilde{u}\|$$

and further $U_\nu \to U$ on R, which is proved as in the proof of Theorem 1. Hence by means of Lemma 1 and Theorem 1 we know

$$\int_\Delta |H^{-1}u_\nu - H^{-1}u| \, d\omega = \int_\Delta |H^{-1}U_\nu - H^{-1}U| \, d\omega$$
$$= L.H.M. |U_\nu - U| \to 0.$$

It follows that by Fatou's theorem

$$0 \geqq \int \varliminf_{\nu\to\infty} |H^{-1}u_\nu - H^{-1}u| \, d\omega \geqq 0,$$

which implies our conclusion.

LEMMA 2. *Let* $S=\{\rho<|z|<1, \rho>0\}$ *and* f_n *be the functions on* ∂S *such that* $f_n(re^{i\theta})$ *and* $\frac{\partial}{\partial\theta}f_n(re^{i\theta})$ $(r=\rho$ *or* $1)$ *are continuous with respect to* θ *and converge uniformly to* 0 *as* $n\to\infty$, *then*

$$\|dH^S_{f_n}\|_S\to 0, \quad n\to\infty$$

PROOF. It suffices to prove the case $f_n(\rho e^{i\theta})\equiv 0$. Consider the functions

$$F_n(re^{i\theta})=\frac{r-\rho}{1-\rho}f_n(e^{i\theta})$$

which belong to class $C^1(\bar{S})$ and possess the same boundary value with f_n, then by Dirchlet principle

$$\begin{aligned}\|dH^S_{f_n}\|^2_S&\leq\|dF_n\|^2_S\\ &=c_1\int_0^{2\pi}f^2_n(e^{i\theta})d\theta+c_2\int_0^{2\pi}\left(\frac{\partial}{\partial\theta}f_n(e^{i\theta})\right)^2d\theta\end{aligned}$$

with $\quad c_1=\frac{1+\rho}{2(1-\rho)}, \quad c_2=\frac{1}{1-\rho^2}\left[\frac{1}{2}(1-\rho)^2-2\rho(1-\rho)+\rho^2\log\frac{1}{\rho}\right].$

Under our hypothesis $\|dF_n\|_S\to 0$ as $n\to\infty$, hence $\|dH^S_{f_n}\|_S\to 0$, q.e.d.

The following example shows that the conclusion of above theorem is false if u_n are uniformly bounded and converge to u on R.

EXAMPLE II. Let $u_n(z)$ and $v_n(z)$ be the harmonic measure on $R=\{|z|<1\}$ with respect to the sets

$$A_n=\bigcup_{m=0}^{n-1}\left\{e^{i\theta};\ \frac{2m}{n}\pi\leq\theta\leq\frac{2m+1}{n}\pi\right\}$$

and $B_n=\partial R-A_n$ respectively. Let $w_n(z)=u_n(z)-v_n(z)$, then $|w_n(z)|\leq 1$ and

$$w_n(z)\to 0 \text{ on } R, \text{ but } \lim_{n\to\infty}|w_n(e^{i\theta})|=1 \text{ almost everywhere on } \partial R.$$

In fact, since $v_n(z)=u_n(ze^{-i\pi/n})$ we have via Poisson's formula

$$|w_n(re^{i\theta})|\leq\frac{1-r^2}{2\pi}\int_{A_n}\frac{\varepsilon_n}{|z-\zeta|^2[|z-\zeta|^2+\varepsilon_n]}d\varphi$$

where $z=re^{i\theta}$, $\zeta=e^{i\varphi}$ and

$$\varepsilon_n=\varepsilon_n(z,\zeta)=2r\left[\left(1-\cos\frac{\pi}{n}\right)\cos(\theta-\varphi)-\sin\frac{\pi}{n}\sin(\theta-\varphi)\right]$$

For each z $\{\varepsilon_n\}$ converges to 0 uniformly with respect to ζ, consequently

$$w_n(z)\to 0, \quad z\in R.$$

While, $|w_n(e^{i\theta})|=1$ except a finite number of points on ∂R, hence

$$\lim_{n\to\infty}|w_n(e^{i\theta})|=1 \ \ a.e..$$

We note that on account of the identity $u_n+v_n\equiv 1$

$$u_n=w_n\vee 0\to\frac{1}{2}, \quad v_n=(-w_n)\vee 0\to\frac{1}{2}.$$

Compare with Example I.

4. Theorem 2 gives us some informations for $H^{-1}u$ when $H^{-1}u_\nu$ possess the known behaviors. As an application of this sort we shall show the following

THEOREM 3. *Under the condition of Theorem 2, if every $H^{-1}u_\nu$ is constant α_ν ω-a.e. on a given subset $\gamma(\subset\Delta)$ with positive harmonic measure, then $H^{-1}u$ is constant ω-a.e. on γ.*

PROOF. Let $f_\nu=H^{-1}u_\nu$, $f=H^{-1}u$ and $f_\nu=\alpha_\nu$ (const.) on γ except a set γ_ν of harmonic measure zero. By Theorem 2 we have

$$\varliminf_{\nu\to\infty}|f_\nu(b)-f(b)|=0, \quad b\in\Delta-\delta,$$

δ being a set of harmonic measure zero. Let $\gamma'=\gamma-(\bigcup_\nu\gamma_\nu\cup\delta)$ and b_0 be a point of γ', then there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$f(b_0)=\lim_{k\to\infty}f_{n_k}(b_0)=\lim_{k\to\infty}\alpha_{n_k}\equiv A.$$

Clearly $u_{n_k}\to u$ on $R-E$ and $\|du_{n_k}-du\|_{R-E}\to 0$, hence again by Theorem 2

$$\varliminf_{k\to\infty}|f_{n_k}(c)-f(c)|=0, \quad c\in\Delta-\delta',$$

δ' being a set of harmonic measure zero. Let q be any point of

$\gamma'-\delta'=\gamma-\delta''$, $\delta''=\bigcup_{\nu}\gamma_\nu\cup\delta\cup\delta'$, then there exist a subsequence $\{f_{\nu_n}\}$ of $\{f_{n_k}\}$ such that

$$f(q)=\lim_{n\to\infty} f_{\nu_n}(q)=\lim_{n\to\infty}\alpha_{\nu_n}.$$

The $\{\alpha_{\nu_n}\}$ is a subsequence of $\{\alpha_{n_k}\}$, thus we know

$$f(q)=A, \quad q\in\gamma-\delta'', \quad \text{q.e.d.}$$

To apply above theorem to more special but interesting case we consider the Kerékjartó-Stoilow compactification $\hat{R}$ of R and set for each point $e\in\hat{R}-R$

$$\varDelta_e=\bigcap_U \overline{U\cap R}$$

where U run over the neighborhoods of e in $\hat{R}$ and the closure is taken on a compactification R^* of R. $\varDelta_e$ are connected, closed and $\varDelta=R^*-R=\bigcup_e \varDelta_e$. The compactification R^* is said to be of type S if $\varDelta_e$ are mutually disjoint. The compactifications of Wiener, Royden, Martin and Kuramochi are all of type S.

Theorem 4. *Let R^* be a D-normal compactification of R of type S and u be a canonical potential which is single-valued, regular outside of a compact set E on R, then $H^{-1}u$ is constant ω-a.e. on each connected component $\varDelta_e$ of $\varDelta$.*

Proof. By definition of canonical potentials and Theorem 3 it is enough to prove the following fact (cf. [4], [5]). Let γ be a Jordan closed curve which divides R into disjoint parts R' and R'', then for the generalized harmonic measure

$$\omega_\gamma=\lim_{n\to\infty} H_{f_n}^{R_n}, \ f_n=1 \text{ on } \partial R_n\cap R',\ f_n=0 \text{ on } \partial R_n\cap R''$$

we have ω-*a.e.*

$$H^{-1}\omega_\gamma=\begin{cases}1 & \text{on } \varDelta'=\varDelta\cap\overline{R'} \\ 0 & \text{on } \varDelta''=\varDelta\cap\overline{R''}\end{cases}$$

where the closure is taken on R^*. To show this let φ be a continuous function on R^* such that φ is identically equal to 1 (resp. 0) in the neighborhood of $\varDelta'$ (resp. $\varDelta''$). Such a function φ exists, for

R^* is of type S and Δ', Δ'' are disjoint. Now since $\omega_a^{R_n}$ converge vaguely to ω_a (cf. [2] p. 87), it follows

$$H_{f_n}^{R_n}(a)=\int_{\partial R_n}\varphi d\omega_a^{R_n}\rightarrow\int_{\Delta}\varphi d\omega_a=H_\varphi(a)$$

Therefore $H_\varphi=\omega_\gamma$. Since $\omega_n=H_{f_n}^{R_n}$ converge uniformly to ω_γ on every compact set on R, we have for $m>n$

$$\|d\omega_n-d\omega_m\|_{R_n}^2\leqq\|d\omega_n\|_{R_n}^2-2\langle d\omega_n,d\omega_m\rangle_{R_n}+\|d\omega_m\|_{R_m}^2$$

$$=\int_{\partial R_n\cap R'}d\omega_n^*-2\int_{\partial R_n\cap R'}d\omega_m^*+\int_{\partial R_m\cap R'}d\omega_m^*$$

$$=\int_\gamma d(\omega_n-\omega_m)^*\rightarrow 0,\quad n\rightarrow\infty.$$

Consequently $\omega_\gamma=H_\varphi\in HD(R)$ hence $H^{-1}\omega_\gamma=\varphi$ ω-*a.e.* on Δ, q.e.d.

Theorem 4 can also be proved in quite analogous way as [5] under the use of normal derivatives in Maeda's sense [6]. So far as the Kuramochi's compactification concerns above result is weaker than Theorem 2 of Kusunoki [5], where the exceptional set is of capacity zero.

REFERENCES

[1] Constantinescu, C. and Cornea, A.: Über das Verhalten der analytischen Abbildungen Riemannscher Flächen auf dem idealen Rand von Martin. Nagoya Math. J. 17 (1960) 1-87.

[2] ———————: Ideale Ränder Riemannscher Flächen, Berlin (1963).

[3] Doob, J. L.: Boundary properties of functions with finite Dirichlet integrals. Ann. Inst. Fourier 12 (1962) 573-621.

[4] Kusunoki, Y.: Theory of Abelian integrals and its applications to conformal mappings. Mem. Col. Sci. Univ. of Kyoto Ser. A. Math. 32 (1959) 235-258.

[5] ——————: Characterizations of canonical differentials. J. Math. Kyoto Univ. 5 (1966) 197-207.

[6] Maeda, F-Y.: Normal derivatives on an ideal boundary. J. Sci. Hiroshima Univ. Ser. A-I 28 (1964) 113-131.

[7] Naim, L.: Sur le rôle de la frontière de R. S. Martin dans la thêorie du potentiel. Ann. Inst. Fourier 7 (1957) 183-281.

Kyoto University and Kobe University