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A limit theorem of branching processes and continuous state branching processes*

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§0. Introduction

In 1951, Feller [3] showed that a class of one-dimensional diffusion processes on $[0, \infty)$ can be obtained as a limit of Galton-Watson branching processes if one changes the scale of time and mass (=size) in an appropriate way. Lamperti [12] determined the class of Markov processes on $[0, \infty)$ which can be obtained as a limit of Galton-Watson branching processes. A main objective of the present paper is to consider a similar problem in more complicated situations. We shall show in §4 below an example of branching processes with particles moving in an n-dimensional space R^n which converge, when we change the scale of time and mass in an appropriate way, to a continuous random motion of mass distributions on R^n. To formulate such a limit process, we shall develop the theory of continuous state branching processes (C. B-processes) in earlier sections.

The concept of C. B-processes was introduced by Jiřina [7] and they were studied in some special cases, by Lamperti [11], Silverstein [16] and Watanabe [17]. The general theory was developed by Jiřina [8] and Motoo [13]. In particular, Motoo

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determined the infinitely divisible laws on the space of measures on a compact space and gave some interesting examples of C. B-processes. In §1 and §2, we shall obtain and extend Motoo's results concerning the formulation and existence of C. B-processes. The method we adopt here is a natural extension of that used in [17]. The theory of infinitesimal generators is added for the purpose of applying it to the limit theorem of §4. The theory is quite parallel to that given in Ikeda-Nagasawa-Watanabe [6] in the case of ordinary branching processes. In §3, we shall consider the case when C. B-processes are diffusion processes. In §4, a typical limit theorem will be given.

§1. Infinitely divisible distribution on the space of measures

Let S be a compact metrizable space, $\mathfrak{S}$ be the set of all non-negative Radon measures[1] on S and $\mathfrak{S}_0$ be the subset of $\mathfrak{S}$ formed of all probability Radon measures on S. Let $\overline{\mathfrak{S}}=\mathfrak{S}\cup\{\Delta\}$, where Δ is an extra-point and let $\widetilde{\mathfrak{S}}=[0,\infty]\times\mathfrak{S}_0$. $\widetilde{\mathfrak{S}}$ is a compact metrizable space by the product topology.[2] Define a mapping ρ; $\tilde{\lambda}=(\bar{\lambda},\lambda_0)\in\widetilde{\mathfrak{S}}\to\lambda=\rho(\bar{\lambda},\lambda_0)\in\overline{\mathfrak{S}}$ by

$$\rho(\bar{\lambda},\lambda_0)=\begin{cases}\bar{\lambda}\cdot\lambda_0, & \text{if } \bar{\lambda}<\infty,\\ \Delta\ , & \text{if } \bar{\lambda}=\infty\end{cases} \tag{1.1}$$

and define the topology of $\overline{\mathfrak{S}}$ as the strongest of all topologies rendering ρ continuous. $\overline{\mathfrak{S}}$ is a compact metrizable space.[3] Let $C^+(S)$ be the set of all *strictly positive* continuous functions on S.[4] It is easy to see that, for each $f\in C^+(S)$, the function $\varphi_f(\lambda)$ defined by

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1) i.e., bounded Borel measures.

2) The topology of $\mathfrak{S}_0$ is that of weak convergence: $\mathfrak{S}_0$ is compact metrizable by this topology.

3) Cf. Bourbaki [1] Chap. 9, p. 44.

4) This notation, which is slightly different from the usual one, is more convenient in future discussions.

$$\varphi_f(\lambda)=\begin{cases} e^{-(\lambda,f)}, & \lambda\in\mathfrak{S}, \\ 0, & \lambda=\Delta, \end{cases} \tag{1.2}$$

where

$$(\lambda, f)=\int_S f(x)\lambda(dx)$$

is a continuous function on $\overline{\mathfrak{S}}$.

Let $\mathfrak{M}\equiv\mathfrak{M}(\mathfrak{S})$ be the set of all substochastic Radon measures[5] on $\mathfrak{S}$. Clearly $\mathfrak{M}(\mathfrak{S})$ can be regarded as the set of all probability Radon measures on $\overline{\mathfrak{S}}$ by the relation

$$P(\{\Delta\})=1-P(\mathfrak{S}), \qquad P\in\mathfrak{M}(\mathfrak{S}).$$

Define the Laplace transform of $P\in\mathfrak{M}(\mathfrak{S})$ by

$$L_P(f)=\int_{\overline{\mathfrak{S}}}\varphi_f(\lambda)P(d\lambda)=\int_{\mathfrak{S}}e^{-(\lambda,f)}P(d\lambda). \tag{1.3}$$

Hence the Laplace transform $L_P(f)$ is a function defined on $C^+(S)$ and it is clear that, if $f_n\to f$ point wise ($f_n, f\in C^+(S)$), then $L_P(f_n)\to L_P(f)$.[6]

Proposition 1.1. *Let $P_i\in\mathfrak{M}(\mathfrak{S})$, $i=1, 2$, and $L_{P_1}(f)=L_{P_2}(f)$ for all $f\in C^+(S)$. Then, $P_1=P_2$.*

This proposition follows at once from the following

Lemma 1.1. *The linear hull of $\{\varphi_f(\lambda); f\in C^+(S)\}$ is dense in $C_0(\mathfrak{S})$ where $C_0(\mathfrak{S})=\{F(\mu)$; continuous on $\overline{\mathfrak{S}}$ such that $F(\Delta)=0\}$.*

Proof. The linear hull is algebra under multiplication and it separates the point of $\overline{\mathfrak{S}}$. Hence the assertion follows from the Stone-Weierstrass theorem.

Now the infinitely divisible measures are defined in the usual way:

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5) i.e., non-negative Radon measures with total mass ≤ 1.

6) Clearly $L_P(f)$ can be extended as a function on $B^+(S)$=(the set of all strictly positive bounded Borel measurable functions) and has the same continuity property.

Definition 1. 1. $P\in\mathfrak{M}$ is called *infinitely divisible* if for every natural number m, there exists $P_m\in\mathfrak{M}$ such that

$$L_P(f)=[L_{P_m}(f)]^m. \tag{1.4}$$

Our next task is to prove Motoo's result which characterizes completely infinitely divisible measures. For each $n=1, 2, \cdots$, take finite number of non-empty Borel subsets of S, $\{K_i^{(n)}\}$, $i=1, 2, \cdots, \nu_n$ such that

(i) $K_i^{(n)}\cap K_j^{(n)}=\phi$, if $i\neq j$,

(ii) $\bigcup_{i=1}^{\nu_n}K_i^{(n)}=S$,

(iii) diameter $(K_i^{(n)})\leq 1/n$ for all $i=1, 2, \cdots \nu_n$.

Since S is a compact metric space, we can always define such $\{K_i^{(n)}\}$. Choose $x_i^n\in K_i^n$, then clearly $\bigcup_n\{x_i^n\}_{i=1}^{\nu_n}$ is dense in S. Define a mapping η_n; $\mathfrak{S}\to\mathfrak{S}$ by

$$\eta_n(\lambda)=\sum_{i=1}^{\nu_n}\lambda(K_i^n)\delta_{x_i^n}{}^{7)}. \tag{1.5}$$

Following properties of η_n are clear:

$$\eta_n(\lambda+\mu)=\eta_n(\lambda)+\eta_n(\mu), \tag{1.6}$$

$$\eta_n(\lambda)\to\lambda \text{ weakly when } n\to\infty, \tag{1.7}$$

$$\eta_n[\eta_n(\lambda)]=\eta_n(\lambda). \tag{1.8}$$

Let $B^+(S)$ be the set of all strictly positive bounded Borel measurable functions. The dual operator of η_n is a mapping η_n^*: $B^+(S)\to B^+(S)$ define by

$$\eta_n^*f(x)=\sum_{i=1}^{\nu_n}f(x_i^n)I_{K_i^n}(x).^{8)} \tag{1.9}$$

Clearly we have

$$\lambda(\eta_n^*f)=(\eta_n\lambda)(f)=(\eta_n\lambda)(\eta_n^*f), \tag{1.10}$$

where

$$\lambda(f)=\int_S f(x)\lambda(dx).$$

7) δ_x is the unit measure at $x\in S$.

8) $I_K(x)$ is the indicator function of $K\subset S$.

Define a function $\xi(\tilde{\lambda}; f)$ defined on $\tilde{\mathfrak{S}} \times C^+(S)$ by

$$(1.11) \qquad \xi(\tilde{\lambda}; f) = \begin{cases} (1 - e^{-\bar{\lambda}\cdot\lambda_0(f)})\dfrac{1+\bar{\lambda}}{\bar{\lambda}}, & 0<\bar{\lambda}<\infty \\ \lambda_0(f), & \bar{\lambda}=0 \\ 1, & \bar{\lambda}=\infty \end{cases}$$

where $\tilde{\lambda} = (\bar{\lambda}, \lambda_0) \in \tilde{\mathfrak{S}} \equiv [0, \infty] \times \tilde{\mathfrak{S}}_0$ and $\lambda_0(f) = \int_S f(x)\lambda_0(dx)$. It is easy to see that, for each fixed $f \in C^+(S)$, there exist constants $0<C_1<C_2$ such that

$$(1.12) \qquad C_1 \leq \xi(\tilde{\lambda}; f) \leq C_2 \qquad \text{for all } \tilde{\lambda} \in \tilde{\mathfrak{S}},$$

and $\xi(\tilde{\lambda}; f)$ is continuous in $\tilde{\lambda} \in \tilde{\mathfrak{S}}$.

Theorem 1.2.[9] (M. Motoo [13]). *$P \in \mathfrak{M}$ is infinitely divisible if and only if*

$$(1.13) \qquad -\log L_P(f) = \int_{\tilde{\mathfrak{S}}} \xi(\tilde{\lambda}; f) n(d\tilde{\lambda})$$

by some bounded non-negative measure $n(d\tilde{\lambda})$ on $\tilde{\mathfrak{S}}$.

Proof. Let P be infinitely divisible and define, for each $n=1, 2, \cdots$, $P^{(n)} = \eta_n \circ P$.[10] For each $m=1, 2, \cdots$, there exists $P_m \in \mathfrak{M}$ such that

$$L_P(f) = [L_{P_m}(f)]^m.$$

Set $P_m^{(n)} = \eta_n \circ P_m$, then,

$$(1.14) \qquad L_{P^{(n)}}(f) = \int_{\mathfrak{S}} e^{-\eta_n(\lambda)(f)} P(d\lambda)$$
$$= \int_{\mathfrak{S}} e^{-\lambda(\eta_n^*(f))} P(d\lambda)$$
$$= L_P(\eta_n^*(f)) = [L_{P_m}(\eta_n^*(f))]^m = [L_{P_m^{(n)}}(f)]^m.$$

$P^{(n)}$ and $P_m^{(n)}$ can be identified with substochastic measures on $\eta^n(\mathfrak{S}) \simeq R_+^{\nu_n}$ where $R_+^{\nu_n}$ is the positive part of ν_n-dimensional Euclidean

9) Cf., also Jiřina [8].
10) i.e., $P^{(n)}(B) = P(\eta_n^{-1}(B))$ for every $B \in \mathcal{B}(\mathfrak{S})$.

space R^{ν_n}.[11] (1.14) implies that $P^{(n)}$, considered as a measure on $R_+^{\nu_n}$, is infinitely divisible. Hence by the classical result,

$$L_{P^{(n)}}(f)=\exp\left(-\int_{\tilde{\mathfrak{S}}}\xi(\tilde{\lambda};\,f)n^{(n)}(d\tilde{\lambda})\right)$$

where $n^{(n)}(d\tilde{\lambda})$ is a non-negative bounded measure concentrated on $[0,\infty]\times\eta_n(\mathfrak{S}_0)$. For each $f\in C^+(S)$, $L_{P^{(n)}}(f)\to L_P(f)$ and by (1.12) it is easy to see that $\sup_n n^{(n)}(\tilde{\mathfrak{S}})<\infty$. Then, clearly,

$$L_P(f)=\exp\left(-\int_{\tilde{\mathfrak{S}}}\xi(\tilde{\lambda};\,f)n(d\tilde{\lambda})\right),$$

where $n(d\tilde{\lambda})$ is a weak limiting point of $n^{(n)}(d\tilde{\lambda})$.

Conversely, given a bounded non-negative measure $n(d\tilde{\lambda})$ on $\tilde{\mathfrak{S}}$, we shall show that

$$\exp\left(-\int_{\tilde{\mathfrak{S}}}\xi(\tilde{\lambda};\,f)n(d\tilde{\lambda})\right)$$

is the Laplace transform of an infinitely divisible measure $P\in\mathfrak{M}$. For this, it is sufficient to show that the above function is the Laplace transform of some $P\in\mathfrak{M}$, since then, $L_P(f)=[L_{P_m}(f)]^m$, where $P_m\in\mathfrak{M}$ corresponds to $n_m(d\tilde{\lambda})=\frac{1}{m}n(d\tilde{\lambda})$. Again, by the well-known result for finite dimensional case,

$$\begin{aligned}
&\exp\left(-\int_{\tilde{\mathfrak{S}}}\xi(\tilde{\lambda};\,\eta_n^*(f))n(d\tilde{\lambda})\right)\\
&=\exp\left(-\int_{\tilde{\mathfrak{S}}}\xi(\eta_n\tilde{\lambda};\,f)n(d\tilde{\lambda})\right)^{12)}\\
&=\exp\left(-\int_{[0,\infty]\times\eta_n(\mathfrak{S}_0)}\xi(\tilde{\lambda};\,f)n^{(n)}(d\tilde{\lambda})\right)\\
&=\int_{\eta_n(\mathfrak{S})}e^{-(\lambda,f)}P_n(d\lambda),
\end{aligned}$$

where $P_n(d\lambda)$ is a substochastic measure concentrated on $\eta_n(\mathfrak{S})$. P_n, considered as a probability measure on $\bar{\mathfrak{S}}$, has a weak limiting point

11) i.e., $R_+^{\nu_n}=\{(x_1,\cdots,x_{\nu_n});\ x_i\geq 0,\ i=1,2,\cdots,\nu_n\}$.

12) $\eta_n\tilde{\lambda}=(\bar{\lambda},\eta_n\lambda_0)$ for $\tilde{\lambda}=(\bar{\lambda},\lambda_0)$.

P and then, it is clear that

$$\exp\left(-\int_{\tilde{\mathfrak{S}}}\xi(\tilde{\lambda};\ f)n(d\tilde{\lambda})\right)=\lim_{n\to\infty}\exp\left(-\int_{\tilde{\mathfrak{S}}}\xi(\tilde{\lambda};\ \eta_n^*(f))n(d\tilde{\lambda})\right)$$
$$=\int_{\mathfrak{S}}e^{-(\lambda,f)}P(d\lambda),$$

which completes the proof.

Definition 1.2.

(1.15) $$\Psi=\left\{\psi(f)=\int_{\tilde{\mathfrak{S}}}\xi(\tilde{\lambda};\ f)n(d\tilde{\lambda})\ ;\ n(d\tilde{\lambda}),\ \text{non-negative bounded measure on } \tilde{\mathfrak{S}}\right\}.$$

Thus the above theorem states that $P\in\mathfrak{M}$ is infinitely divisible iff $-\log L_P(f)\in\Psi$. By (1.12), it is easy to see that we have the following

Proposition 1.3. *If* $\psi_n\in\Psi$, $n=1, 2, \cdots$, *and* $\psi_n(f)\to\psi(f)$ *for every* $f\in D$ *where* D *is a non-empty open subset*[13] *of* $C^+(S)$, *then there exists a unique extension of* ψ *such that* $\psi\in\Psi$.

Definition 1.3. A function $\psi\equiv\psi(x;\ f)$ defined on $S\times C^+(S)$ is called a *Ψ-function* if

(i) for fixed $x\in S$, it is an element of Ψ,

(ii) for fixed $f\in C^+(S)$, it is an element of $C^+(S)$.

The set of all Ψ-functions is denoted by $\boldsymbol{\Psi}$. Given two Ψ-functions ψ_1 and ψ_2, the composition $\psi_3=\psi_1(\psi_2)$ is defined by

(1·16) $$\psi_3(x;\ f)=\psi_1(x;\ \psi_2(\cdot\ ;\ f)).$$

Lemma 1.2. *If* $\psi_i\in\boldsymbol{\Psi}$, $i=1, 2$, *then* $\psi_1(\psi_2)\in\boldsymbol{\Psi}$.

Proof. For any $\mu\in\mathfrak{S}$ and $\psi\in\boldsymbol{\Psi}$, $\int_S\psi(x;\ f)\mu(dx)\in\Psi$. Therefore given ψ_i, $i=1, 2$, and μ, there exists a unique $P_\mu^i\in\mathfrak{M}$ such that

$$\exp\left(-\int_S\psi_i(x;\ f)\mu(dx)\right)=\int_{\mathfrak{S}}e^{-(\lambda,f)}P_\mu^i(d\lambda).$$

13) With respect to the uniform topology.

Define, for each $x\in S$, $P_x(d\lambda)\in\mathfrak{M}(\mathfrak{S})$ by

$$P_x(d\lambda)=\int_{\mathfrak{S}}P^1_{\delta_x}(d\mu)P^2_{\mu}(d\lambda),$$

then

$$\int_{\mathfrak{S}}e^{-(\lambda,f)}P_x(d\lambda)=\int\exp\Big(-\int_S\psi_2(x;\ f)\mu(dx)\Big)P^1_{\delta_x}(d\mu)$$
$$=\exp[-\psi_1(x;\ \psi_2(\cdot\ ;\ f))],$$

which proves $\psi_1(\psi_2)\in\boldsymbol{\Psi}$.

Definition 1.4. A one-parameter family $\{\psi_t\}_{t\in[0,\infty)}$ of Ψ-functions is called a *Ψ-semi-group* if

$$\psi_{t+s}=\psi_t(\psi_s), \qquad (1.17)$$
$$\psi_0(\cdot\ ;\ f)=f.$$

§2. Continuous state branching processes

Definition 2.1. Let $X=(\mu_t(dx,\omega),\ \Omega,\ \mathcal{F}_t,\ P_\mu)$[14] be a Markov process on $\bar{\mathfrak{S}}=\mathfrak{S}\cup\{\Delta\}$ with Δ as a trap. X is called a *continuous state branching process* (C. B-process) if it satisfies, for every $t\geq 0$, $f\in C^+(S)$ and $\mu_1,\ \mu_2\in\mathfrak{S}$,

$$E_{\mu_1+\mu_2}(e^{-(\mu_t,f)})=E_{\mu_1}(e^{-(\mu_t,f)})E_{\mu_2}(e^{-(\mu_t,f)}).\text{[15]} \qquad (2.1)$$

The property (2.1) is called the *branching property.*

Definition 2.2. A C. B-process is called *regular* if $E_\mu(e^{-(\mu_t,f)})$ is continuous in $\mu\in\bar{\mathfrak{S}}$ for each $t\geq 0$ and $f\in C^+(S)$.

Theorem 2.1. (Jiřina) *There is a one-to-one correspondence between a regular C. B-process $X=\{\mu_t,\ P_\mu\}$ and Ψ-semi-group $\{\psi_t\}_{t\in[0,\infty)}$: the correspondence is given by*

$$E_\mu(e^{-(\mu_t,f)})=\exp\Big(-\int\psi_t(x;\ f)\mu(dx)\Big). \qquad (2.2)$$

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14) Ω is an abstract space, $\mathcal{F}_t$ is an increasing family of Borel fields on Ω, $\mu_t(dx,\omega)$ is a mapping $[0,\infty)\times\Omega\ni(t,\omega)\to\mu_t(dx,\omega)\in\bar{\mathfrak{S}}$ adapted to $\mathcal{F}_t$ and $\{P_\mu,\ \mu\in\mathfrak{S}\}$ is a family of probability measures on $\{\Omega,\bigvee_{t\geq 0}\mathcal{F}_t\}$ such that $P_\mu\{\omega:\ \mu_0(dx,\omega)=\mu(dx)\}=1$.

15) $E_\mu(\cdot)=\int\cdot P_\mu(d\omega)$. We set always $e^{-(\Delta,f)}=0$ for every $f\in C^+(S)$.

Proof. Let $X=\{\mu_t, P_\mu\}$ be a regular C. B-process and set

$$E_{\delta_x}(e^{-(\mu_t, f)})=\exp(-\psi_t(x; f)).$$

Then, for each $x\in S$ and $t\geq 0$, $\psi_t(x; f)\in\Psi$ since

$$\exp(-\psi_t(x; f))=E_{\delta_x}(e^{-(\mu_t, f)})=[E_{\frac{1}{m}\delta_x}(e^{-(\mu_t, f)})]^m.$$

By the regularity of X, it is easy to see that, for each $t\geq 0$ and $f\in C^+(S)$, $\psi_t(\cdot; f)\in C^+(S)$. Now we claim that

$$E_\mu(e^{-(\mu_t, f)})=\exp(-\int\psi_t(x; f)\mu(dx)). \tag{2. 3}$$

When μ is of the form $\mu=\sum_{i=1}^{\nu}\frac{m_i}{n_i}\delta_{x_i}$, ($m_i$, n_i natural numbers, $x_i\in S$) this follows from the branching property. Then, by regularity, (2. 3) holds for every $\mu\in\overline{\mathfrak{S}}$. Now,

$$\begin{aligned} E_{\delta_x}(e^{-(\mu_{t+s}, f)}) &= E_{\delta_x}(E_{\mu_t}(e^{-(\mu_s, f)})) \\ &= E_{\delta_x}\left[\exp(-\int\psi_s(x; f)\mu_t(dx))\right] \\ &= \exp[-\psi_t(x; \psi_s(\cdot; f))]. \end{aligned}$$

Hence, $\psi_{t+s}=\psi_t(\psi_s)$. Thus, $\psi_t=\psi_t(x; f)$ is a Ψ-semi-group.

Conversely, suppose we are given a Ψ-semi-group $\{\psi_t\}$. Then, just as in the proof of Lemma 1. 2, there exists a unique $P_\mu^t\in\mathfrak{M}$ such that

$$\exp(-\int\psi_t(x; f)\mu(dx))=\int_{\mathfrak{S}}e^{-(\lambda, f)}P_\mu^t(d\lambda), \tag{2. 4}$$

for each $t\geq 0$ and $\mu\in\mathfrak{S}$. Now

$$\begin{aligned} &\int_{\mathfrak{S}}e^{-(\nu, f)}\left[\int_{\mathfrak{S}}P_\mu^t(d\lambda)P_\lambda^s(d\nu)\right] \\ &= \int_{\mathfrak{S}}P_\mu^t(d\lambda)\int_{\mathfrak{S}}e^{-(\nu, f)}P_\lambda^s(d\nu) \\ &= \int P_\nu^t(d\lambda)\left[\exp(-\int\psi_s(x; f)\lambda(dx))\right] \\ &= \exp\left[-\int\psi_t(x; \psi_s(\cdot; f))\mu(dx)\right] \end{aligned}$$

$$=\exp\left(-\int \psi_{t+s}(x;\, f)\mu(dx)\right)$$

$$=\int_{\mathfrak{S}} e^{-(\nu, f)} P_\mu^{t+s}(d\nu),$$

and hence

$$\int_{\mathfrak{S}} P_\mu^t(d\lambda) P_\lambda^s(d\nu) = P_\mu^{t+s}(d\nu),$$

i.e., $\{P_\mu^t(d\lambda)\}$ is a transition function on $\mathfrak{S}$. Thus $\{P_\mu^t(d\lambda)\}$ defines a unique Markov process on $\overline{\mathfrak{S}} = \mathfrak{S} \cup \{\Delta\}$ with Δ as a trap, (cf. Dynkin [2]). The branching property is clear from (2. 4). q.e.d.

When S is a single point or finite number of points, $\mathfrak{S}$ can be identified as the positive part of the finite dimensional Euclidean space and in these cases the structure of C. B-processes are completely determined under a slight regularity condition in time t, cf. Lamperti [11], Silverstein [16] and Watanabe [17]. Following the method of [17], we shall now describe a large class of C. B-processes. Some examples were obtained already by Motoo [13].

Let T_t be a non-negative strongly continuous semi-group of bounded operators on $C(S)$ and let A be the infinitesimal generator in Hille-Yosida sense of T_t. Let $D(A)$ be the domain of A. Let $\varphi(x;\, f)$ be a Ψ-function and $\sigma(x)$ be a non-negative function in $C(S)$. Now, consider the following non-linear evolution equation for $\psi_t(x) \in C(S)$:

(2. 5)
$$\frac{\partial \psi_t}{\partial t} = A\psi_t + \sigma[\varphi(\cdot\,;\, \psi_t) - \psi_t]$$
$$\psi_0 = f.$$

In practice, we consider the equivalent integral equation:

(2. 5)′ $$\psi_t(x) = T_t^\sigma f(x) + \int_0^t ds \int_S T_s^\sigma(x, dy)\sigma(y)\varphi(y;\, \psi_{t-s}),$$

where T_t^σ is the semi-group with infinitesimal generator $A-\sigma$.[16)]

16) It is well known that there exists a unique semi-group T^σ with infinitesimal generator $A^\sigma = A - \sigma$ with $D(A^\sigma) = D(A)$. T^σ is non-negative and strongly continuous. $T_s^\sigma(x, dy)$ is the kernel of T^σ.

Proposition 2.2. *For $f \in C^{+}(S)$, the solution $\psi_t = \psi_t(x; f)$ of (2.5)′ exists and unique. Further ψ_t defines a Ψ-semi-group.*

For the proof we need the following

Lemma 2.1. *Let $C_{\epsilon}^{+}(S) = \{f \in C^{+}(S);\ \min_{x \in S} f(x) > \epsilon\}$.*[17] *For every $\epsilon > 0$, there exists $K = K(\epsilon) > 0$ such that*

$$\|\varphi(\cdot\,; f) - \varphi(\cdot\,; g)\| \leq K \|f - g\|$$

for every f, $g \in C_{\epsilon}^{+}(S)$.

Proof. By the mean value theorem,

$$|\xi(\tilde{\lambda}; f) - \xi(\tilde{\lambda}; g)| \leq |e^{-\bar{\lambda}\cdot\lambda_0(f)} - e^{-\bar{\lambda}\cdot\lambda_0(g)}| \cdot \frac{1+\bar{\lambda}}{\bar{\lambda}}$$

$$\leq \bar{\lambda} |\lambda_0(f) - \lambda_0(g)| e^{-\bar{\lambda}\epsilon} \cdot \frac{1+\bar{\lambda}}{\bar{\lambda}}.$$ [18]

Hence, by taking $K(\epsilon) = \sup_{\bar{\lambda} \in (0, \infty)} (1+\bar{\lambda}) e^{-\bar{\lambda}\epsilon}$, the lemma is proved.

Proof of the proposition. Let $f \in C^{+}(S)$ then for some $\epsilon > 0$, $f \in C_{2\epsilon}^{+}(S)$. Then, there exists $t_0 > 0$ such that $T_t^{\sigma} f \in C_{\epsilon}^{+}(S)$ for all $t \in [0, t_0]$. Define $\psi_t^{(n)}(x)$, $t \in [0, t_0]$, $x \in S$, successively by

$$\text{(2.6)} \quad \begin{aligned} \psi_t^{(1)}(x) &= T_t^{\sigma} f(x) \\ \psi_t^{(n)}(x) &= T_t^{\sigma} f(x) + \int_0^t ds \int_S T_s^{\sigma}(x, dy) \sigma(y) \varphi(y; \psi_{t-s}^{(n-1)}). \end{aligned}$$

Then $\psi_t^{(n)} \in C_{\epsilon}^{+}(S)$ for all $t \in [0, t_0]$ and $n = 1, 2, \cdots$, and also, by Lemma 1.2, $\psi_t^{(n)} \in \Psi$. Using Lemma 2.1, we can show by the standard argument that

$$\sup_{0 \leq t \leq t_0} \|\psi_t^{(n)} - \psi_t\| \to 0$$

for some $\psi_t \in C^{+}(S)$. Then $\psi_t \in \Psi$ by Proposition 1.3. Also, by Lemma 2.1, we can show that ψ_t is the unique solution in $C^{+}(S)$

17) Thus $C^{+}(S) = \bigcup_{\epsilon > 0} C_{\epsilon}^{+}(S)$.

18) $\tilde{\lambda} = (\bar{\lambda}, \lambda_0) \in [0, \infty] \times \mathfrak{S}_0$.

of (2.5) in $[0, t_0]$. We denote this solution as $\psi_t = \psi_t(x; f)$ then, by the uniqueness of the solution, $\psi_{t+s} = \psi_t(\psi_s)$ for $t+s \leq t_0$. If we define $\psi_t = \psi_t(x; f)$ in the interval $[t_0, 2t_0]$ by

$$\psi_t(x; f) = \psi_{t-t_0}(x; \psi_{t_0}(\cdot; f)),$$

then, $\psi_t \in \boldsymbol{\Psi}$ by Lemma 1.2 and $\{\psi_t\}$, $t \in [0, 2t_0]$ defines a solution of (2.5) in the interval $[0, 2t_0]$. This is the unique solution in $C^+(S)$ by virtue of Lemma 2.1. Continuing this process, we get the unique solution ψ_t, $t \in [0, \infty)$ of (2.5)′ in $C^+(S)$ and clearly ψ_t is a Ψ-semi-group. q.e.d.

More interesting class of Ψ-semi-groups can be obtained by the following limiting procedure. Let $h(x; f)$ be a function defined on $S \times C^+(S)$ such that $h(\cdot; f) \in C(S)$ for each fixed $f \in C^+(S)$. We assume that $h(x; f)$ is locally Lipschitz continuous in f, i.e., for every $f \in C^+(S)$, there exist a neighborhood[19] $D = D(f)$ and a constant $K > 0$ such that

$$\|h(\cdot; f) - h(\cdot; g)\| \leq K \|f - g\| \tag{2.7}$$

for every $f, g \in D$. We assume further that there exist a non-empty open set $D_0 \subset C^+(S)$, a sequence $\{\varphi_n(x; f)\}$ of Ψ-functions, and a sequence $\{\sigma_n(x)\}$ of non-negative functions in $C(S)$ such that

$$\sup_{f \in D_0} \|\sigma_n \{\varphi_n(\cdot; f) - f\} - h(\cdot; f)\| \to 0 \tag{2.8}$$

when $n \to \infty$. Let T_t be, as before, a non-negative strongly continuous semi-group on $C(S)$ and A be the infinitesimal generator with the domain $D(A)$. Now, consider the following evolution equation for $\psi_t(x) \in C(S)$:

$$\begin{aligned} \frac{\partial \psi_t}{\partial t} &= A\psi_t + h(\cdot; \psi_t), \\ \psi_0 &= f. \end{aligned} \tag{2.9}$$

In practice, we consider the equivalent integral equation:

19) With respect to the uniform topology.

(2. 9)′ $$\psi_t(x)=T_t f(x)+\int_0^t ds\int_S T_s(x, dy)h(y;\ \psi_{t-s}).$$

Theorem 2. 3. *There exists a unique solution $\psi_t(x)$ in $C^+(S)$ of the equation* (2. 9)′. *If we write this solution as $\psi_t=\psi_t(x; f)$, then ψ_t defines a Ψ-semi-group.*

Proof. We first remark that, if there exists a solution $\psi_t(x)$ of (2. 9)′ in $C^+(S)$, then it is a unique solution. This can be proved by the usual argument using the local Lipschitz continuity of h. We shall show, therefore, the existence of the solution $\psi_t=\psi_t(x; f)\in\Psi$. By the local Lipschitz continuity, the solution $\psi_t(x; f)$ of (2. 10) exists in $C^+(S)$ for each $f\in C^+(S)$ in sufficiently small time interval $[0, t_0]$. For each $n=1, 2, \cdots$, let $\psi_t^{(n)}=\psi_t^{(n)}(x; f)$ be the solution of

$$\psi_t^{(n)}(x)=T_t f(x)+\int_0^t ds\int_S T_s(x, dy)\sigma_n(y)\,[\varphi_n(y;\ \psi_{t-s}^{(n)})-\psi_{t-s}^{(n)}(y)].$$

Then $\psi_t^{(n)}$ is the solution of

$$\psi_t^{(n)}(x)=T_t^{\sigma_n}f(x)+\int_0^t ds\int_S T_s^{\sigma_n}(x, dy)\sigma_n(y)\varphi_n(y;\ \psi_{t-s}^{(n)}),$$

and hence, by Proposition 2. 2 $\psi_t^{(n)}$ is a Ψ-semi-group. Now, using (2. 8), we can show, by the same proof as in Lemma 2, §2 of [17], that there exists a non-empty open set $D_1\subset C^+(S)$ and $t_0>0$ such that

$$\sup_{0\le t\le t_0}\ \sup_{f\in D_1}\|\psi_t^{(n)}(\cdot\,; f)-\psi_t(\cdot\,; f)\|\to 0$$

when $n\to\infty$. By Proposition 1. 3, $\psi_t=\psi_t(x; f)\in\Psi$ for $t\in[0, t_0]$. Then ψ_t can be extended as a solution in $t\in[0, \infty)$ just as in the proof of Proposition 2. 2 and it clearly defines a Ψ-semi-group.

Corollary. *Let $F(\xi)$ be a function defined on $\xi\in(0, \infty)$ given by*

(2. 10) $$F(\xi)=C_0+C_1\xi-C_2\xi^2-\int_0^\infty\left(e^{-\xi u}-1+\frac{\xi u}{1+u}\right)n(du).$$

where C_i, $i=0, 1, 2$, *are constants such that* $C_0 \geq 0$, $C_2 \geq 0$ *and* $n(du)$ *is a non-negative measure on* $(0, \infty)$ *such that*

$$\int_0^\infty \frac{u^2}{1+u^2} n(du) < \infty .$$

Let $\sigma(x)$ *be a non-negative continuous function on* S *and define* $h(x; f)$, $x \in S$, $f \in C^+(S)$ *by*

$$h(x; f) = \sigma(x) F(f(x)). \tag{2.11}$$

Then the solution ψ_t *of* (2.9) *or* (2.9)' *defines a* Ψ*-semi-group.*

Proof. It is clear that $h(\cdot\,; f) \in C(S)$ for each $f \in C^+(S)$ and it is locally Lipschitz continuous. Also it is not difficult to show that there exists a sequence of functions $F_n(\xi)$ of the form

$$F_n(\xi) = C_n(\varphi_n(\xi) - \xi),$$

where $C_n > 0$ and $\varphi_n(\xi) = \int_{[0,\infty]} (1 - e^{-u\xi}) \frac{1+u}{u} N_n(du)$ with a non-negative bounded measure $N_n(du)$ on $[0, \infty]$, such that $F_n(\xi) \to F(\xi)$ uniformly on each compact interval in $(0, \infty)$ when $n \to \infty$. Now, $\psi(x; f) \equiv \varphi_n(f(x)) \in \boldsymbol{\Psi}$ for $n = 1, 2, \cdots$, since

$$\varphi_n(f(x)) = \int_{\tilde{\mathfrak{S}}} \xi(\tilde{\lambda}; f) \overline{N}_n^x(d\tilde{\lambda}),$$

where $\overline{N}_n^x(d\tilde{\lambda})$ is the image measure of N_n under the mapping $u \in [0, \infty] \to (u, \delta_x) \in \tilde{\mathfrak{S}}$. Thus, $h(x; f)$ satisfies the condition (2.8).

Example. For a non-negative continuous function $\sigma(x)$ on S, the solutions of the following equations define Ψ-semi-groups:

$$\frac{\partial \psi_t}{\partial t} = A\psi_t - \sigma \cdot \{\psi_t\}^\alpha, \qquad 1 < \alpha \leq 2,$$

or,

$$\frac{\partial \psi_t}{\partial t} = A\psi_t + \sigma \cdot \{\psi_t\}^\alpha, \qquad 0 < \alpha \leq 1.$$

Given T_t and h, we have constructed a Ψ-semi-group in

Theorem 2.3 and by Theorem 2.1, there is the unique C. B-process corresponding to it. Let $\boldsymbol{T}_t$ be the semi-group of this C. B-process. Since

$$\boldsymbol{T}_t\varphi_f(\mu)=\varphi_{\psi_t(\cdot;f)}(\mu)=\exp\left[-\int\psi_t(x;\,f)\mu(dx)\right]^{20)},$$

it is easy to see that $\boldsymbol{T}_t(C_0(\mathfrak{S}))\subset C_0(\mathfrak{S})$, where $C_0(\mathfrak{S})=\{F(\mu)$; continuous on $\bar{\mathfrak{S}}$ and $F(\Delta)=0\}$. Since $\|\psi_t(\cdot\,;f)-f\|\to 0$, we see easily that $\boldsymbol{T}_t$ is strongly continuous on $C_0(\mathfrak{S})$. Hence the C. B-process is a Hunt process; in particular, we may assume that it is a strong Markov process with right continuous and d_1-discontinuous sample functions (cf. Dynkin [2]). The case of diffusion processes will be discussed in the next section. We shall now study the infinitesimal generator of the semi-group $\boldsymbol{T}_t$ on $C_0(\mathfrak{S})$. Let $\boldsymbol{A}$ be the infinitesimal generator in Hille-Yosida sense of $\boldsymbol{T}_t$ with the domain $\boldsymbol{D}(\boldsymbol{A})$. A linear manifold $D\subset\boldsymbol{D}(\boldsymbol{A})$ is called a *core*[21] of $\boldsymbol{A}$ if $\boldsymbol{A}$ is the smallest closed extension of $\boldsymbol{A}|_D$.[22]

Theorem 2.4. *Let T_t and h be as in Theorem 2.3 and let $\boldsymbol{D}$ be the linear hull of $\{\varphi_f(\mu)\,;\,f\in C^+(S)\cap D(A)\}$. Then $\boldsymbol{D}\subset\boldsymbol{D}(\boldsymbol{A})$ and $\boldsymbol{A}\varphi_f$, $f\in C^+(S)\cap D(A)$, is given by*

$$\boldsymbol{A}\varphi_f(\mu)=e^{-(\mu,f)}\int_S\{h(x;\,f)-Af(x)\}\,\mu(dx). \tag{2.12}$$

Furthermore, $\boldsymbol{D}$ is a core of $\boldsymbol{A}$.

Conversely, if $f\in C^+(S)$ is such that $\varphi_f\in D(\boldsymbol{A})$ then $f\in D(A)$ and hence, $\boldsymbol{A}\varphi_f$ is given by (2.12).

Remark. If $D\subset D(A)$ is a core of A, then the linear hull $\boldsymbol{D}'$ of $\{\varphi_f(\mu)\,;\,f\in C^+(S)\cap D\}$ is a core of $\boldsymbol{A}$. In fact, as is easily seen, the smallest closed extension $\overline{\boldsymbol{A}|_{\boldsymbol{D}'}}$ of $\boldsymbol{A}|_{\boldsymbol{D}'}$ satisfies $\overline{\boldsymbol{A}|_{\boldsymbol{D}'}}\supset\boldsymbol{A}|_{\boldsymbol{D}}$.

Proof. We first show that $\boldsymbol{D}\subset\boldsymbol{D}(\boldsymbol{A})$ and $\boldsymbol{A}|_{\boldsymbol{D}}$ is given by

20) $\varphi_f(\mu)=e^{-(\mu,f)}\equiv\exp(-\int f(x)\mu(dx))$, $f\in C^+(S)$.

21) Cf. Kato [9], p. 166.

22) $\boldsymbol{A}|_D$ is the restriction of $\boldsymbol{A}$ on D.

(2.12) and also that, if $f\in C^+(S)$ is such that $\varphi_f\in \boldsymbol{D}(\boldsymbol{A})$, then $f\in D(A)$ and hence, $\varphi_f\in \boldsymbol{D}$. First, if $\psi_t(x; f)$ is the solution of (2.9) for $f\in C^+(S)\cap D(A)$, then

$$\left\| \frac{\psi_t(\cdot\,;f)-f}{t}-(Af-h(\cdot\,;f))\right\| \to 0 \qquad \text{when } t\to 0.$$

This can be proved by the same way as in the proof of Theorem 4.10 of Ikeda-Nagasawa-Watanabe [6]. Then, as is easily seen,

$$\left\| \frac{1}{t}\{e^{-(\mu,\psi_t)}-e^{-(\mu,f)}\}-e^{-(\mu,f)}\int \mu(dx)\{h(x;f)-Af(x)\}\right\|_{\mathfrak{S}} \to 0$$

when $t\to 0$. Thus, $\varphi_f\in \boldsymbol{D}(\boldsymbol{A})$ and $A\varphi_f$ is given by (2.12). The second assertion can be proved by exactly the same way as in the proof of Theorem 4.10 of [6].

It remains only to show that $\boldsymbol{D}$ is a core of $\boldsymbol{D}(\boldsymbol{A})$. First of all, we remark that if $f\in C^+(S)\cap D(A)$ then $\psi_t=\psi_t(x;f)\in C^+(S)\cap D(A)$ for each $t\geq 0$; in fact, $f\in C^+(S)\cap D(A)$ implies $\varphi_f\in \boldsymbol{D}(\boldsymbol{A})$, then $\boldsymbol{T}_t\varphi_f(\mu)=\varphi_{\psi_t}(\mu)\in \boldsymbol{D}(\boldsymbol{A})$. This implies, again by the above result, that $\psi_t\in D(A)$. From this, it is clear that $\boldsymbol{T}_t(\boldsymbol{D})\subset \boldsymbol{D}$. Also, by Lemma 1.1 $\boldsymbol{D}$ is dense in $C_0(\mathfrak{S})$. Now the assertion is a consequence of the following general

Lemma 2.2. *Let U_t be a strongly continuous semi-group of bounded operators on a Banach space $\boldsymbol{B}$ such that $\|U_t\|\leq M\cdot e^{\beta t}$ for some $M>0$ and $\beta>0$. Let G be the infinitesimal generator of U_t with the domain $D(G)$. Let D be a linear manifold of $\boldsymbol{B}$ such that*

(i) $D\subset D(G)$,

(ii) *D is dense, i.e., $\overline{D}=\boldsymbol{B}$,*

(iii) *D is U_t-invariant, i.e., $U_t(D)\subset D$.*

Then D is a core of G.

Proof. It is sufficient to show that for some $\alpha>\beta$, $(\alpha I-G)(D)$[23]

23) I is the identity.

is dense in $\boldsymbol{B}$. In fact, if this is true, then for every $u\in D(G)$ there exists $h_n\in D$ such that $\alpha h_n-Gh_n\rightarrow\alpha u-Gu$. Let $R_\alpha=\int_0^\infty e^{-\alpha t}U_t dt$, then R_α is a bounded operator and hence,

$$h_n=R_\alpha(\alpha h_n-Gh_n)\rightarrow R_\alpha(\alpha u-Gu)=u$$

and also

$$Gh_n=\alpha\cdot h_n-(\alpha h_n-Gh_n)\rightarrow\alpha u-(\alpha u-Gu)=Gu,$$

proving that G is the smallest closed extension of $G|_D$.

In order to prove $(\alpha I-G)(D)$ is dense, it is sufficient to show that, for every continuous linear functional L on $\boldsymbol{B}$ such that $L(\alpha u-Gu)=0$ for every $u\in D$, L is identically 0. Assume, therefore

$L(\alpha u-Gu)=\alpha\cdot L(u)-L(Gu)=0$, for every $u\in D$. Since $U_t(D)\subset D$, we have

$$\alpha L(U_t u)-L(GU_t u)=\alpha L(U_t u)-L\left(s-\frac{d}{dt}U_t u\right)^{24)}$$

$$=\alpha L(U_t u)-\frac{d}{dt}L(U_t u)=0.$$

Hence $L(U_t u)=C\cdot e^{\alpha t}$ for some constant C. But $|L(U_t u)|\leq\|L\|\|U_t u\|\leq K'e^{\beta t}$ and, since $\beta<\alpha$, we must have $C=0$. Therefore $L(U_t u)=0$ for every t and, in particular, $L(u)=0$ for every $u\in D$. Since D is dense in $\boldsymbol{B}$, this implies $L=0$. q.e.d.

§3. The case of diffusion processes

In §2, we have shown that, for a given non-negative strongly continuous semi-group T_t on $C(S)$ and a given non-negative continuous function $\sigma(x)$ on S, the solution of the equation:

$$\frac{\partial\psi_t}{\partial t}=A\psi_t-\sigma\{\psi_t\}^2 \qquad (3.1)$$

$$\psi_0=f,\qquad f\in C^+(S),$$

defines a Ψ-semi-group. We shall show that the corresponding

24) $s-d/dt$ stands for strong derivative.

C. B-process is a diffusion process, i.e., almost all sample functions are continuous in the topology of $\overline{\mathfrak{S}}$. Unfortunately, we can not prove this fact without certain restrictions on T_t and σ; the conclusion seems to be true in general, however.

Theorem 3.1. *We assume that there exists a dense subset D of $C(S)$ such that $D \subset D(A)$ and for every $f \in D$, there exist constants $K>0$ and $\alpha>0$ such that*

$$\left\| \frac{T_t f - f}{t} - Af \right\| + \| T_t(\sigma f^2) - \sigma f^2 \| \leq K \cdot t^{\alpha} \tag{3.2}$$

for all sufficiently small t. Then, the C. B-process corresponding to the equation (3.1) *is a diffusion process.*

Proof. For $f \in C(S)$, define $\psi_t^{(n)}$ successively by

$$\begin{aligned} &\psi_t^{(1)}(x) = T_t f(x) \\ &\psi_t^{(n)}(x) = T_t f(x) - \int_0^t T_s \{\sigma(\psi_{t-s}^{(n-1)})^2\}(x) ds. \end{aligned} \tag{3.3}$$

If we choose t_0 such that

$$4t_0 C^2(t_0) \|\sigma\| \|f\| \leq 1,$$

where we set $(\sup_{0 \leq t \leq t_0} \|T_t\|) \vee 1 = C(t_0)$, then, since

$$\|\psi_t^{(n)}\| \leq C(t) [\|f\| + t\|\sigma\| \sup_{0 \leq s \leq t} \|\psi_{t-s}^{(n-1)}\|^2],$$

we have, by induction, that

$$\begin{aligned} \sup_{0 \leq t \leq t_0} \|\psi_t^{(n)}\| &\leq C(t_0) [\|f\| + t_0 \|\sigma\| \cdot 4C^2(t_0) \|f\|^2] \\ &\leq C(t_0) [\|f\| + \|f\|] = 2\|f\| C(t_0) \end{aligned}$$

for every $n = 1, 2, \cdots$.

Since (for $t \leq t_0$)

$$\begin{aligned} \|\psi_t^{(n+1)} - \psi_t^{(n)}\| &\leq C(t_0) \|\sigma\| \int_0^t \|\psi_s^{(n)2} - \psi_s^{(n-1)2}\| ds \\ &\leq 4C^2(t_0) \cdot \|\sigma\| \|f\| \int_0^t \|\psi_s^{(n)} - \psi_s^{(n-1)}\| ds, \end{aligned}$$

we have

(3. 4) $$\sup_{0\leq t\leq t_0} \|\psi_t^{(n)}-\psi_t^{(n-1)}\| \leq \frac{K^n t_0^n}{n!},$$

where

$$K=4C^2(t_0)\|\sigma\|\|f\|.$$

From now on, we fix $g\in D\cap C_1^+(S)$.[25] For $f=\lambda\cdot g$, we define $\psi_t^{(n)}(x)=\psi_t^{(n)}(x;\lambda)$ by (3. 3). Then clearly, $\psi_t^{(n)}(x;\lambda)$ is a polynomial in λ and by (3. 4), we have, for some $t_0>0$[26] and $\epsilon>0$, that

$$\sup_{\substack{0\leq t\leq t_0\\ |\lambda|\leq 1+\epsilon}} \|\psi_t^{(n)}(\cdot\,;\lambda)-\psi_t(\cdot\,;\lambda)\|\to 0.$$

Hence $\psi_t(x;\lambda)\equiv\psi_t(x;\lambda g)$ is analytic in $|\lambda|\leq 1+\epsilon$. Set

$$\psi_t(x;\lambda)-\lambda\cdot g(x)=t[\lambda Ag(x)-\lambda^2\sigma(x)g^2(x)]+t\cdot H(t,x;\lambda),$$

then for fixed $t\in[0,t_0]$ and $x\in S$, $H(t,x;\lambda)$ is analytic in λ and

(3. 5) $$\|H(t,\cdot\,;\lambda)\|\leq\left\|\frac{\psi_t(\cdot\,;\lambda)-\lambda\cdot g}{t}-\lambda Ag+\lambda^2\sigma\cdot g^2\right\|$$
$$\leq\left\|\frac{T_t(\lambda\cdot g)-\lambda\cdot g}{t}-\lambda\cdot Ag\right\|+\left\|\frac{1}{t}\int_0^t T_s\{\sigma\cdot\psi_{t-s}^2\}\,ds-\lambda^2\sigma\cdot g^2\right\|$$
$$\leq\lambda\left\|\frac{T_tg-g}{t}-Ag\right\|+\left\|\frac{1}{t}\int_0^t[T_s\{\sigma\psi_{t-s}^2-\lambda^2\sigma\cdot g^2\}]\,ds\right\|$$
$$+\left\|\frac{1}{t}\int_0^t[T_s(\lambda^2\sigma\cdot g^2)-\lambda^2\sigma\cdot g^2]\,ds\right\|$$
$$\leq\lambda K\cdot t^\alpha+C(t_0)\sup_{0\leq s\leq t}\|\sigma\cdot\psi_{t-s}^2-\lambda^2\sigma g^2\|+\lambda^2K\frac{1}{t}\int_0^t s^\alpha ds$$
$$\leq K'\cdot t^\alpha,$$

since $\|\sigma\cdot\psi_{t-s}^2-\lambda^2\cdot\sigma\cdot g^2\|\leq C\cdot\|\psi_{t-s}-\lambda\cdot g\|\leq C'\cdot t$. Clearly, $K'=K'(\lambda)$ is bounded in $|\lambda|\leq 1+\epsilon$. Now,

$$H(t,x;\lambda)=\sum_{n=1}^{\infty}\lambda^n\cdot a_n(t,x),$$

where

$$a_n(t,x)=\frac{1}{2\pi}\int_0^{2\pi}H(t,x;e^{i\theta})e^{-in\theta}d\theta.$$

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25) $C_1^+(S)=\{f$: continuous on S and $0<f\leq 1\}$

26) Clearly, this t_0 can be taken common to all $g\in D\cap C_1^+(S)$.

Hence, by (3.5),

(3.6) $$\sup_{x\in S} |a_n(t, x)| \leq K''\cdot t^{\alpha}, \qquad n=1, 2, \cdots.$$

Now, if $t\leq t_0$,

$$E_\nu\{e^{-\lambda(\mu_t, g)}\} = \exp(-\int\psi_t(x; \lambda)\nu(dx))$$

and this is analytic in $|\lambda|\leq 1+\epsilon$. Hence all moments of (μ_t, g) exist. In particular, this implies that,

(3.7) $$\sup_{0\leq t\leq t_0} E_\nu\{\mu_t(S)^m\}<\infty, \qquad m=1, 2, \cdots.$$

We have, finally,

$$\begin{aligned} &E_\nu(\exp[-\lambda\{(\mu_{t+s}, g)-(\mu_s, g)\}]) \\ &\quad= E_\nu(\exp\{-\int_S \mu_s(dx)\,[\psi_t(x; \lambda)-\lambda g]\}) \\ &\quad= \sum_{n=0}^{\infty} E_\nu\left[\frac{(-1)^n}{n!}\left\{\int_S (\psi_t(x;\lambda)-\lambda g)\mu_s(dx)\right\}^n\right] \\ &\quad= \sum_{n=0}^{\infty} E_\nu\left[\frac{(-1)^n}{n!}\left\{\int_S \{t[\lambda A g(x)-\lambda^2\cdot\sigma g^2(x)]+\sum_{k=1}^{\infty}\lambda^k t a_k(t, x)\}\mu_s(dx)\right\}^n\right] \\ &\quad= \sum_{n=0}^{\infty} b_n\lambda^n, \end{aligned}$$

then, as is easily seen by (3.6) and (3.7), we have

$$b_4 = -t\cdot E_\nu\left[\int a_4(t, x)\mu_s(dx)\right]+O(t^2)=O(t^{1+\alpha}).$$

Thus,

$$E_\nu\{[(\mu_{t+s}, g)-(\mu_s, g)]^4\} = 4!\,b_4 = O(t^{1+\alpha})$$

if $0\leq t+s\leq t_0$, where $O(t^{1+\alpha})$ is independent of s. By Kolmogorov's theorem (cf. Neveu [15]), this implies that

$$P_\nu\{(\mu_t, g) \text{ is continuous in } t\in[0, t_0]\} = 1.$$

Since $C_1^+(S)\cap D$ is dense in $C_1^+(S)$, this implies that

$$P_\nu\{\mu_t \text{ is continuous in } t\in[0, t_0]\} = 1$$

and hence, by the Markov property,

$$P_\nu\{\mu_t \text{ is continuous in } t\in[0,\infty)\}=1,$$

i.e., μ_t is a diffusion process.

Example. Let $S=\widehat{R^n}$, one point compactification of R^n and T_t be the strongly continuous semi-group on $C(\widehat{R^n})$ defined by

$$(3.8)\qquad T_t f(x)=\frac{1}{(2\pi t)^n}\int_{R^n}\exp\Big(-\frac{|x-y|^2}{2t}\Big)f(y)dy.$$

Let σ be a positive constant, then the condition of Theorem 3.1 is satisfied: if we take $D=C^\infty(\widehat{R^n})=\{f\in C(\widehat{R^n})$; all of its derivatives $\in C(\widehat{R^n})\}$, (3.2) is clearly satisfied. Note also that A is the smallest closed extension of $\frac{1}{2}\Delta$ on $C^\infty(\widehat{R^n})$. Hence, there is a unique diffusion C. B-process $X=(\mu_t, P_\mu)$ on $\overline{\mathfrak{S}}$ such that

$$E_\mu(\exp[-(\mu_t,f)])=\exp(-\int\psi_t(x;f)\mu(dx))$$

where $\psi_t(x;f)$ is the solution of

$$(3.9)\qquad \begin{aligned}&\frac{\partial\psi_t}{\partial t}=A\psi_t-\sigma\cdot\psi_t^2,\\ &\psi_0=f.\end{aligned}$$

One interesting property of these diffusion processes is the following: We have shown, in the proof of Theorem 3.1, that, for every $f\in C(S)$, the solution of (3.9) exists uniquely for sufficiently small time-interval $[0,t_0]$. If $f=\lambda g$, the solution $\psi(t;\lambda)=\psi(t;\lambda g)$ is analytic in $|\lambda|\le 1+\epsilon$ for sufficiently small $[0,t_0]$. It is easy to see that

$$E_\mu(\exp[(\mu_t,g)])=\exp(\int_{R^n}\varphi_t(x;g)\mu(dx)),\quad t\in[0,t_0],$$

where

$$\varphi_t(x;g)=-\psi_t(x;-g).$$

φ_t is the solution of

$$(3.10)\qquad \frac{\partial\varphi_t}{\partial t}=A\varphi_t+\sigma\cdot\varphi_t^2,\qquad \varphi_0=g.$$

Hence, by Fujita's result [4] (cf. also Nagasawa-Sirao [14]),
(i) if $n=1$, for every non-negative $g\in C(R^n)$ such that $\{g>0\}$ has an interior point,

$$(3.11)\qquad E_{\delta_x}\{\exp(\mu_t, g)\}$$

blows up in a finite time, i.e., (3.11) cannot be finite for all $t\in[0,\infty)$,
(ii) if $n\geq 3$, for every $\gamma>0$ there exists $\delta>0$ such that, for all $g\in C(R^n)$ satisfying $0\leq g(x)\leq\delta\cdot(2\pi\gamma)^{-N/2}\exp(-|x|^2/2\gamma)$, (3.11) is finite for all $t\in[0,\infty)$. Furthermore,

$$\log[E_{\delta_x}\{\exp[(\mu_t, g)]\}]\leq M[2\pi(\gamma+t)]^{-N/2}\exp\left[-\frac{|x|^2}{2(t+\gamma)}\right],\ {}^\forall t\in[0,\infty)$$

for some positive constant M.

The behavior of (3.11) for the critical case $n=2$ is not known.

§4. A limit theorem

Consider the following branching process (cf. Harris [5], Chap. III, §1.6): an object at $x\in R^n$ has the probability p_k of having k children ($k=0, 1, 2, \cdots$); assume that each child, independently of others, has a probability distribution $\sigma(dy)$ for being in $x+dy$. Let $Z_n(dx)$ be the number of objects in dx in the n-th generation. $Z_n(dx)$ defines a discrete-time Markov process whose state space is the set of all non-negative, integer-valued measures. We shall call this process the (F, σ)-process, where $F(s)=\sum_{k=0}^{\infty}p_k s^k$, since it is uniquely determined by F and σ.

Now, consider a sequence of (F, σ)-processes; $\{Z_n^{(m)}(dx), \widetilde{P}_\mu^{(m)}, \mu\in\mathfrak{N}\}$: (F_m, σ_m)-process, $m=1, 2, \cdots$, where $\mathfrak{N}$ is the set of all non-negative, integer-valued measures:

(4.1) $$\mathfrak{N}=\{\mu=\sum_{i=1}^{l}\delta_{x_i};\ x_i\in R^n\}.$$

For each $m=1, 2, \cdots$, let $\mathfrak{S}^{(m)}$ be

(4.2) $$\mathfrak{S}^{(m)}=\frac{1}{m}\mathfrak{N}=\left\{\mu=\frac{1}{m}\sum_{i=1}^{l}\delta_{x_i};\ x_i\in R^n\right\}$$

and define a continuous-time stochastic process $\{\mu_t^{(m)}(dx), P_\mu^{(m)}\}$ on $\mathfrak{S}^{(m)}$ by

(4.3) $$\mu_t^{(m)}(dx)=\frac{1}{m}Z_{[mt]}^{(m)}(dx),$$

$$P_\mu^{(m)}=\widetilde{P}_{m\mu}^{(m)},\qquad \mu\in\mathfrak{S}^{(m)}.$$

Now, let $\{\mu_t, P_\mu\}$ be the diffusion C. B-process discussed in Example of §3, i.e., $S=\widehat{R^n}$ and μ_t is a C. B-process defined by

$$E_\mu(\exp[-(\mu_t, f)])=\exp(-\int\psi_t(x; f)\mu(dx)),$$

where $\psi_t(x; f)$ is the solution of

(4.4) $$\frac{\partial\psi_t}{\partial t}=A\psi_t-\rho\cdot\psi_t^2 \text{ [27]}$$

$$\psi_0=f.$$

We shall assume that F_m and σ_m satisfy the following conditions:

(4.5) $$m\cdot\int_{R^n}[f(x+y)-f(x)]\sigma_m(dy)\rightarrow\frac{1}{2}\Delta f(x)$$

uniformly when $m\rightarrow\infty$, for every $f\in C^\infty(\widehat{R^n})$,

(4.6) $$-\log(F_m(e^{-u/m}))=\frac{u}{m}-\rho\cdot\frac{u^2}{m^2}+o\left(\frac{1}{m^2}\right),$$

where $\rho>0$ is a constant and $o(1/m^2)$ is uniform in $u\in[u_1, u_2]$ for

27) A is the smallest closed extension of $\frac{1}{2}\Delta$ on $C^\infty(\widehat{R^n})$. ρ is a positive constant.

every $0<u_1<u_2$.[28)]

Theorem 4.1. *Under the assumptions* (4.5) *and* (4.6), *finite dimensional distributions of* $\{\mu_t^{(m)}, P_{\delta_x}^{(m)}\}$ *converge to those of* $\{\mu_t, P_{\delta_x}\}$ *for every* $x\in R^n$ *when* $m\to\infty$.

Proof. Let $\overline{\mathfrak{S}}^{(m)}=\mathfrak{S}^{(m)}\cup\{\varDelta\}\subset\overline{\mathfrak{S}}$ and $\boldsymbol{C}^m\equiv C_0(\mathfrak{S}^{(m)})=\{F(\mu)$; continuous on $\overline{\mathfrak{S}}^{(m)}$ and $F(\varDelta)=0\}$. Let P_m: $\boldsymbol{C}\equiv C_0(\mathfrak{S})\to\boldsymbol{C}^m$, be the restriction operator;

$$(P_mF)(\mu)=F\Big|_{\overline{\mathfrak{S}}^{(m)}}(\mu).$$

Let $T^{(m)}(\mu, d\lambda)$ be the probability kernel on $\overline{\mathfrak{S}}^{(m)}\times\overline{\mathfrak{S}}^{(m)}$ defined by

$$\begin{aligned}&\int_{\mathfrak{S}^{(m)}}T^{(m)}(\mu, d\lambda)(P_m\varphi_f)(\lambda)\\ (4.7)\qquad &=\exp\left(-m\int_{R^n}\psi^{(m)}(x; f)\mu(dx)\right),\quad \mu\neq\varDelta\\ &T^{(m)}(\varDelta, d\lambda)=\delta_{\{\varDelta\}}(d\lambda),\end{aligned}$$

where

$$(4.8)\qquad \psi^{(m)}(x; f)=-\log F_m\left(\int_{R^n}\exp\left[-\frac{1}{m}f(x+y)\right]\sigma_m(dy)\right).$$

It is easy to verify that, for $k=1, 2, \cdots$,

$$P_\mu^{(m)}\left[\mu_{(k+1)/m}^{(m)}\in d\lambda\,|\,\mu_t;\ t\leq\frac{k}{m}\right]=T^{(m)}(\mu_{k/m}^{(m)}, d\lambda),\quad \text{a.s.}.$$

Now, we shall apply Trotter's result (cf. Kato [9], IX, §3, Kurtz [10]); if there exists a core $\boldsymbol{D}$ of $\boldsymbol{A}$[29)] such that

$$(4.9)\qquad \|A^{(m)}P_mF-P_mAF\|_{\boldsymbol{C}^m}\to 0\quad (m\to\infty),\ \text{for all } F\in\boldsymbol{D},$$

where

28) (4.5) and (4.6) are satisfied, e.g., if $\sigma_m(dy)=\sigma(\sqrt{m}\cdot dy)$, $m=1, 2, \cdots$, where $\sigma(dy)$ is a probability measure on R^n such that $\int_{R^n}x^ix^j\sigma(dx)=\delta_{ij}$ and $\int_{R^n}x^i\sigma(dx)=0$, and $F_m(s)\equiv F(s)$, $m=1, 2, \cdots$, where $F'(1)=1$ and $0<F''(1)/2=\rho<\infty$.

29) $\boldsymbol{A}$ is the infinitesimal generator of the semi-group of (μ_t, P_μ) acting on $\boldsymbol{C}\equiv C_0(\mathfrak{S})$.

(4. 10) $$A^{(m)}=m(T^{(m)}-I)$$ [30)]

then,

(4. 11) $$\lim_{m\to\infty}\sup_{0\le s\le t}|T_s^{(m)}P_mF-P_mT_sF|_{C^m}=0$$

for every $F\in C$, where

(4. 12) $$T_s^{(m)}=(T^{(m)})^{[sm]}$$

and T_s is the semi-group of (μ_t, P_μ) acting on $C_0(\mathfrak{S})$. We shall verify (4. 9). Let D be the linear hull of $\{\varphi_f(\mu);\ f\in C^+(\widehat{R^n})\cap C^\infty(\widehat{R^n})\}$, then, by Theorem 2. 4 and Remark, D is a core of A. Also,

$$A\varphi_f(\mu)=e^{-(f,\mu)}\int_{\widehat{R^n}}\Big[[\rho\cdot f^2(x)-\frac{1}{2}\Delta f(x)\Big]\mu(dx).$$

Let $\mu=\frac{1}{m}\sum_{i=1}^{l}\delta_{x_i}\in\mathfrak{S}^{(m)}$, then if $f\in C^+(\widehat{R^n})\cap C^\infty(\widehat{R^n})$,

$$A^{(m)}P_m\varphi_f(\mu)=m\Big\{\exp(-\sum_{i=1}^{l}\psi^{(m)}(x_i;f))-\exp\Big(-\frac{1}{m}\sum_{i=1}^{l}f(x_i)\Big)\Big\},$$

where

$$\psi^{(m)}(x;f)=-\log F_m\Big(\int_{R^n}\exp\Big[-\frac{1}{m}f(x+y)\Big]\sigma_m(dy)\Big).$$

By (4. 5) and (4. 6),

$$\int_{R^n}\exp\Big[-\frac{1}{m}f(x+y)\Big]\sigma_m(dy)$$
$$=1-\frac{1}{m}f(x)-\frac{1}{2m^2}\Delta f(x)+\frac{1}{2m^2}f^2(x)+o\Big(\frac{1}{m^2}\Big)$$

and hence

$$\psi^{(m)}(x;f)=\frac{1}{m}f(x)+\frac{1}{m^2}\Big[\frac{1}{2}\Delta f(x)-\rho\cdot f^2(x)\Big]+o\Big(\frac{1}{m^2}\Big),$$

where $o(1/m^2)$ is uniform in $x\in\widehat{R^{(n)}}$. Therefore,

$$|A^{(m)}P_m\varphi_f(\mu)-A\varphi_f(\mu)|=$$

30) $T^{(m)}$ is the operator given by the kernel $T^{(m)}(\mu, d\lambda)$.

$$=\left|\exp\left(-\frac{1}{m}\sum_{i=1}^{l} f(x_i)\right)\left[m\left\{\exp\left(-\sum_{i=1}^{l}\left[\frac{1}{m^2}\left[\frac{1}{2}\Delta f(x_i)-\rho\cdot f^2(x_i)\right]\right.\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left.+o\left(\frac{1}{m^2}\right)\right]\right)-1\right\}+\sum_{i=1}^{l}\frac{1}{m}\left[\frac{1}{2}\Delta f(x_i)-\rho\cdot f^2(x_i)\right]\right]\right|$$

$$\leq K\cdot e^{-(l/m)\epsilon}\left(o(1)\frac{l}{m}+\frac{1}{m}\left(\frac{l}{m}\right)^2 e^{(l/m^2)\cdot c}\right),$$

where K and c are positive constants and $\epsilon=\inf_{x\in R^n} f(x)>0$. Hence, $\sup_l |A^m P_m\varphi_f(\mu)-A\varphi_f(\mu)|\to 0$ when $m\to\infty$, proving (4. 9). Now the convergence of finite-dimensional distributions follows from (4. 11) by a standard argument. q. e. d.

By changing the conditions on F_m and σ_m, various different limit theorems may be obtained.

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NOTE: Just after I finished my present manuscript, I received from Dr. M. L. Silverstein of Princeton University a preprint of his new paper "Continuous state branching semigroups", where a nice existence theorem for C. B-processes was obtained.