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Local times for transforms of Markov processes

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In this paper transformations of standard processes possessing local times are investigated and the local times, when they exist, of the transformed processes are identified in terms of the original local times. The results of this paper extend some of the results of [4].

1. Preliminaries

We shall follow the notation and terminology of Dynkin [2] unless otherwise specified.

Let $X=(\Omega, x_t, \zeta, \mathcal{M}_t, P_x, \theta_t)$ be a standard process with state space $(E, \mathcal{B})$. That is, E is a locally compact separable metric space with $\bar{E}=E\cup\{\Delta\}$ where Δ is the point "at infinity" adjoined to E in the one-point compactification if E is not compact, otherwise Δ is an isolated point. $\mathcal{B}$ and $\bar{\mathcal{B}}$ are the Borel sets of E and $\bar{E}$, respectively. Ω is the sample space of paths ω which are maps ω: $[0, \infty]\to\bar{E}$. The lifetime, ζ, is a map ζ: $\Omega\to[0, \infty]$ such that $\omega(t)\in E$ for $t<\zeta(\omega)$ and $\omega(t)=\Delta$ for $t\geq\zeta(\omega)$. Also $\omega(t)$ is assumed right continuous and with left-hand limits on $[0, \zeta(\omega))$. We write $x_t(\omega)=x(t, \omega)=\omega(t)$. We let $\mathcal{M}^0$ be the smallest σ-algebra with respect to which the maps x_t: $\Omega\to E$ are measurable for all $t\geq 0$ and let $\mathcal{M}_t^0$ be the smallest σ-algebra with respect to which x_s: $\Omega\to E$ are measurable for $0\leq s\leq t$. For each x in $\bar{E}$, P_x is a

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probability measure on $\mathcal{N}^0$ such that $P_x(x(0)=x)=1$ and $x\to P_x(\Lambda)$ is $\bar{\mathcal{B}}$-measurable for each Λ in $\mathcal{N}^0$. For each finite measure μ on $(\bar{E}, \bar{\mathfrak{B}})$ we define a measure P_μ on $\mathcal{N}^0$ by $P_\mu(\Lambda)=\int P_x(\Lambda)d\mu(x)$. We define the σ-algebra $\mathcal{N}_t$ (respectively, $\mathcal{N}$) as the intersection of the P_μ-completions of the σ-algebra $\mathcal{N}_t^0$ (respectively, $\mathcal{N}^0$) taken over all finite measures μ on $(\bar{E}, \bar{\mathcal{B}})$. The shift operators θ_t: $\Omega\to\Omega$ are defined by $x_s(\theta_t\omega)=x_{s+t}(\omega)$. A Markov time is a function τ: $\Omega\to[0, \infty]$ such that $\{\omega: \tau(\omega)\leq t\}\in\mathcal{N}_t$ for each $t\geq 0$. For each Markov time τ let $\mathcal{N}_\tau$ be the σ-algebra of sets Λ in $\mathcal{N}$ such that $\Lambda\cap\{\tau\leq t\}\in\mathcal{N}_t$ for all $t\geq 0$. We define $\theta_\tau\omega$ by $x_t(\theta_\tau\omega)=x_{t+\tau(\omega)}(\omega)$.

X then is a *standard process* if the following conditions are satisfied:

(i) X is a *strong Markov process*, that is, for each Markov time τ and each bounded real valued random variable F on $(\Omega, \mathcal{N})$ we have

$$M_x\{F(\theta_\tau\omega);\ \Lambda\}=M_x\{M_{x(\tau)}(F);\ \Lambda\}$$

for all x in $\bar{E}$ and $\Lambda\in\mathcal{N}_\tau$. We denote $M_x(F;\ \Lambda)=\int_\Lambda FdP_x$.

(ii) $\mathcal{N}_{t+}=\bar{\mathcal{N}}_t=\mathcal{N}_t$ for each $t\geq 0$, where $\mathcal{N}_{t+}=\bigcap_{s>t}\mathcal{N}_s$ and $\bar{\mathcal{N}}_t$ is the intersection of the P_x-completions of $\mathcal{N}_t$ taken over all $x\in E$.

(iii) X is *quasi-left-continuous*, that is, if $\{\tau_n\}$ is an increasing sequence of Markov times with limit τ, then $x(\tau_n)\to x(\tau)$ almost surely on $\{\tau<\zeta\}$. (An expression is said to be true almost surely if it is true almost everywhere P_x for each x; almost surely is abbreviated a. s.)

Given a set $B\in\bar{\mathfrak{B}}$ we define the *hitting time*, τ_B, of B as $\tau_B(\omega)=\inf\{t>0: x_t(\omega)\in B\}$. τ_B is then a Markov time. Let C_0 be the class of sets $G\subseteq\bar{E}$ such that for every x in G there exists a set Γ in $\mathfrak{B}$ such that $x\in\Gamma\subseteq G$ and such that if $\tau=\tau_{\bar{E}-\Gamma}$ then $P_x(\tau>0)=1$. C_0 forms a topology on $\bar{E}$ called the *fine topology* for the process X.

A family $\varphi=\{\varphi(t), t\geq 0\}$ of functions $\varphi(t, \cdot): \Omega\to[0, \infty]$ for each $t\geq 0$ is said to be a *continuous additive functional, CAF,* of a standard process X if

(a) $\varphi(t)$ is $\mathcal{N}_t$-measurable for each $t \geq 0$.

(b) The following statements hold almost surely: $\varphi(0)=0$, $t \to \varphi(t)$ is continuous and non-decreasing, and $\varphi(s)=\lim_{t \to \zeta} \varphi(t)$ whenever $s \geq \zeta$.

(c) For each t, $s \geq 0$ we have $\varphi(t+s, \omega)=\varphi(t, \omega)+\varphi(s, \theta_t\omega)$, a.s.

A CAF φ is said to be *perfect* if $\varphi(t)$ is $\mathcal{N}_t^0$-measurable for each $t \geq 0$.

Meyer [5] has proved that every CAF φ satisfies the *strong Markov property*:

$$\varphi(\tau+\sigma, \omega)=\varphi(\tau, \omega)+\varphi(\sigma, \theta_\tau\omega), \text{ a.s.,} \tag{1.1}$$

for each Markov time τ and each non-negative random variable σ on $(\Omega, \mathcal{N})$.

CAF's φ and γ are *equivalent* if the functions $t \to \varphi(t)$ and $t \to \gamma(t)$ are identical a.s. We do not distinguish between equivalent CAF's φ and γ and we write $\varphi=\gamma$.

Refer to [2] for any terms used below that are not defined in this paper.

2. Fine Supports and Local Times

Let φ be a CAF of a standard process X. We say φ *vanishes* on a nearly Borel set D provided

$$M_x \int_0^\infty e^{-\lambda t} I_D(x_t)\, d\varphi(t)=0$$

for all x and for some $\lambda \geq 0$.[2] The *fine support* of φ is defined to be the smallest set closed in the fine topology on whose complement φ vanishes. Furthermore, if ρ is the Markov time defined by $\rho=\inf\{t \geq 0: \varphi(t)>0\}$, then the fine support of φ is the set $F=\{x: P_x(\rho=0)=1\}$. For a proof of this result see [1, Chapter 5]

2) $I_D(x)=1$ if $x \in D$ and $I_D(x)=0$ if $x \notin D$.

or [3].

A point x is said to be *regular for* $\{x\}$ with respect to a standard process X if $P_x(\tau_x=0)=1$ where $\tau_x=\inf\{t>0: x_t=x\}$ is the hitting time of the set $\{x\}$.

Let x_0 be a fixed point of E. Then a CAF γ of a standard process X is called a *local time* of X at x_0 if the fine support of γ is the set $\{x_0\}$.

The following theorem is proved in [1, Chapter 5]. See also [3].

Theorem 2.1. *Let X be a standard process. There exists a local time of X at a point x_0 if and only if x_0 is regular for $\{x_0\}$. Also, if γ_1 and γ_2 are local times at x_0 then there exists a constant $c>0$ such that $\gamma_1=c\gamma_2$.*

By Theorem 2.1 we can consider a local time as unique up to a multiplicative constant and it is determined by its fine support. For this reason we will not distinguish between two local times at the same point and we will speak of *the* local time at x_0.

3. Random Time Changes

Let φ be a CAF of a standard process X. The *inverse*, τ, of φ is defined by $\tau_t(\omega)=\inf\{s: \varphi(s, \omega)>t\}$ or $\tau_t(\omega)=\infty$ if $\{s: \varphi(s, \omega)>t\}$ is empty. For each t, τ_t is a Markov time and $\tau_{t+s}=\tau_t+\tau_s(\theta_{\tau_t})$, a.s., for all $s, t\geq 0$. Also, τ_t is right continuous and strictly increasing in t almost surely.

We say a CAF φ is strictly increasing if $t\to\varphi(t)$ is strictly increasing a.s. on $(0, \zeta)$. If φ is strictly increasing then clearly $t\to\tau_t$ is continuous almost surely.

Given a CAF φ, define $\tilde{X}=(\tilde{\Omega}, \tilde{x}_t, \tilde{\zeta}, \tilde{\mathcal{N}}_t, \tilde{P}_x, \tilde{\theta}_t)$ as follows: We let $\tilde{\Omega}=\Omega$, $\tilde{x}_t(\tilde{\omega})=x_{\tau_t(\omega)}(\tilde{\omega})$, $\tilde{\zeta}(\tilde{\omega})=\varphi_{\zeta(\omega)-0}(\omega)$, $\tilde{\mathcal{N}}_t=\mathcal{N}_{\tau_t}$, $\tilde{P}_x=P_x$ and $\tilde{\theta}_t(\tilde{\omega})=\theta_{\tau_t(\omega)}(\omega)$. $\tilde{X}$ is then a strong Markov process.

If, in addition, φ is perfect and strictly increasing then $\widetilde{X}$ is a standard process with state space $(E, \mathfrak{B})$.

See [2, Chapter 10, §5] for proofs of these statements. $\widetilde{X}$ is said to be obtained by a *random time change* via φ from the process X.

Lemma 3.1. *Let φ be a strictly increasing, perfect CAF of a standard process X. Let $\widetilde{X}$ be obtained from X as above.*
(1) *If $\gamma=\{\gamma(t), t\geq 0\}$ is a CAF of X then $\tilde{\gamma}=\{\tilde{\gamma}(t), t\geq 0\}$, where $\tilde{\gamma}(t, \tilde{\omega})=\gamma(\tau_t(\omega), \omega)$, defines a CAF of $\widetilde{X}$.*
(2) *If x_0 is regular for $\{x_0\}$ with respect to X then x_0 is regular for $\{x_0\}$ with respect to $\widetilde{X}$.*

Proof.
(1) Clearly (a) and (b) of the definition of a CAF are satisfied. To prove (c) we must show $\tilde{\gamma}(t+s, \tilde{\omega})=\tilde{\gamma}(t, \tilde{\omega})+\tilde{\gamma}(s, \tilde{\theta}_t\tilde{\omega})$, a.s. We have, almost surely, that

$$\begin{aligned}\tilde{\gamma}(t+s, \tilde{\omega}) &= \gamma(\tau_{t+s}(\omega), \omega)\\ &= \gamma(\tau_t(\omega)+\tau_s(\theta_{\tau_t}\omega), \omega)\\ &= \gamma(\tau_t(\omega), \omega)+\gamma(\tau_s(\theta_{\tau_t}\omega), \theta_{\tau_t}\omega) \quad \text{(by (1.1))}\\ &= \tilde{\gamma}(t, \tilde{\omega})+\tilde{\gamma}(s, \tilde{\theta}_t\tilde{\omega}).\end{aligned}$$

(2) The hitting time of $\{x_0\}$ for X is $\sigma_{x_0}(\omega)=\inf\{t>0: x_t(\omega)=x_0\}$. By assumption, $P_{x_0}(\sigma_{x_0}=0)=1$. The hitting time of $\{x_0\}$ for $\widetilde{X}$ is $\tilde{\sigma}_{x_0}(\tilde{\omega})=\inf\{t>0: \tilde{x}_t(\tilde{\omega})=x_0\}$. It is easy to see $\tilde{\sigma}_{x_0}=\varphi(\sigma_{x_0})$. Since φ is strictly increasing $P_{x_0}(\varphi(\sigma_{x_0})=0)=P_{x_0}(\sigma_{x_0}=0)=1$ and so $\widetilde{P}_{x_0}(\tilde{\sigma}_{x_0}=0)=1$, that is, x_0 is regular for $\{x_0\}$ with respect to $\widetilde{X}$.
Q.E.D.

The following theorem shows how local times are transformed under a random time change.

Theorem 3.2. *Suppose X is a standard process. Let φ be a strictly increasing, perfect continuous additive functional of X*

and let $\widetilde{X}$ *be the (standard) process obtained from* X *by random time change via* φ. *If the local time,* γ_{x_0}, *of* X *at* x_0 *exists then the local time,* $\tilde{\gamma}_{x_0}$, *of* $\widetilde{X}$ *at* x_0 *exists and is given by* $\tilde{\gamma}_{x_0}=\tilde{\gamma}$, *where* $\tilde{\gamma}=\{\tilde{\gamma}(t)=\gamma_{x_0}(\tau_t),\ t\geq 0\}$.

Proof. If γ_{x_0} exists then, by Theorem 2.1, x_0 is regular for $\{x_0\}$ with respect to X and hence x_0 is regular for $\{x_0\}$ with respect to $\widetilde{X}$ by Lemma 3.1, so that again by Theorem 2.1 the local time, $\tilde{\gamma}_{x_0}$, of $\widetilde{X}$ at x_0 exists.

By Lemma 3.1, $\tilde{\gamma}=\{\tilde{\gamma}(t)=\gamma_{x_0}(\tau_t),\ t\geq 0\}$ is a CAF of $\widetilde{X}$. The fine support, $\widetilde{F}$, of $\tilde{\gamma}$ is given by $\widetilde{F}=\{x:\ \widetilde{P}_x(\tilde{\rho}=0)=1\}$ where $\tilde{\rho}=\inf\{t\geq 0:\ \tilde{\gamma}(t)>0\}$. Now, $\tilde{\rho}=\varphi(\rho)$ where $\rho=\inf\{t\geq 0:\ \gamma_{x_0}(t)>0\}$. Thus, $\widetilde{P}_x(\tilde{\rho}=0)=1$ if and only if $P_x(\varphi(\rho)=0)=1$ and so $\widetilde{P}_x(\tilde{\rho}=0)=1$ if and only if $P_x(\rho=0)=1$ since φ is strictly increasing.

However, $\{x_0\}=\{x:\ P_x(\rho=0)=1\}$ is the fine support of γ_{x_0} and hence $\widetilde{F}=\{x_0\}$. By the comments following Theorem 2.1 we can assume $\tilde{\gamma}=\tilde{\gamma}_{x_0}$. Q.E.D.

4. α-subprocesses

Let X be a standard process with state space $(E, \mathfrak{B})$. A family $\alpha=\{\alpha_t,\ t\geq 0\}$ of real valued random variables $\alpha_t(\omega)$ on $(\Omega, \mathcal{N})$ is said to be a *multiplicative functional* (MF) of X if

(i) α_t is $\mathcal{N}_t$-measurable for each $t\geq 0$

(ii) $t\rightarrow\alpha_t$ is right continuous a.s.

(iii) $0\leq\alpha_t(\omega)\leq 1$ for all t and ω

(iv) $\alpha_{t+s}=\alpha_t\alpha_s(\theta_t)$, a.s., for each $t, s\geq 0$.

Given a MF α of X we define a new process $\widetilde{X}=(\tilde{\Omega}, \tilde{x}_t, \tilde{\zeta}, \widetilde{\mathcal{N}}_t, \widetilde{P}_x, \tilde{\theta}_t)$ as follows: we let $\tilde{\Omega}=\Omega\times[0, \infty]$ and denote a point $\tilde{\omega}$ in $\tilde{\Omega}$ by $\tilde{\omega}=(\omega, u)$. Let $\mathfrak{R}$ denote the Borel subsets of $[0, \infty]$ and set $\widetilde{\mathcal{N}}=\mathcal{N}\times\mathfrak{R}$. We define $\tilde{\zeta}(\omega, u)=\zeta(\omega)\wedge u$[3] and let $\tilde{x}_t(\omega, u)=x_t(\omega)$ if $0\leq t<\tilde{\zeta}(\omega, u)$ and $\tilde{x}_t(\omega, u)=\Delta$ if $t\geq\tilde{\zeta}(\omega, u)$. Denote by $\widetilde{\mathcal{N}}_t$ the

3) $a\wedge b$ denotes minimum of a and b.

class of all subsets of the form $\Lambda \times (t, \infty]$ where $\Lambda \in \mathcal{N}_t$. Define Q as the set of $\omega \in \Omega$ such that $\alpha_0(\omega)=1$ and $t \to \alpha_t(\omega)$ is right continuous and non-increasing. For $\omega \in Q$ let ψ_ω be the measure on $\mathfrak{R}$ induced by the function $\psi((t, \zeta(\omega)]) = \alpha_t(\omega)$, so that $\psi_\omega((s, t]) = \alpha_s(\omega) - \alpha_t(\omega)$ for $s, t \leq \zeta(\omega)$. We let $\psi_\omega(\{0\})=0$ and $\psi_\omega((\zeta(\omega), \infty])=0$. For $\omega \notin Q$ we let ψ_ω be the unit measure concentrated at the point 0. For $\tilde{\Lambda} \in \tilde{\mathcal{N}}$ and $\omega \in \Omega$ we define the ω-section of $\tilde{\Lambda}$ as $\tilde{\Lambda}_\omega = \{u \in [0, \infty] : \tilde{\omega} = (\omega, u) \in \tilde{\Lambda}\}$. We then define $\tilde{P}_x$ on $\tilde{\mathcal{N}}$ as $\tilde{P}_x(\tilde{\Lambda}) = M_x[\psi_\omega(\tilde{\Lambda}_\omega)]$. Finally, let $\tilde{\theta}_t(\tilde{\omega}) = (\theta_t(\omega), (u-t) \vee 0)$[4] where $\tilde{\omega} = (\omega, u)$.

We then have that $\tilde{X} = (\tilde{\Omega}, \tilde{x}_t, \tilde{\zeta}, \tilde{\mathcal{N}}_t, \tilde{P}_x, \tilde{\theta}_t)$ defines a Markov process with transition probabilities

$$\tilde{P}_x(\tilde{x}_t \in \Gamma) = M_x[I_\Gamma(x_t)\alpha_t].$$

The process $\tilde{X}$ is said to be the (canonical) *α-subprocess* of X.

The following theorem is useful for our purposes.

Theorem 4.1. *Let X be a standand process. Let α be a multiplicative functional of X snch that*
(1) $P_x(\alpha_0=1)=1$ *for all* x.
(2) α_t *is* $\mathcal{N}_t^0$*-measurable for each* $t \geq 0$.
Then the α-subprocess, $\tilde{X}$, of X is a standand process.

For a proof of this theorem see [1, Chapter 3] or [3, Theorem 10.7].

Lemma 4.2. *Let X, α and $\tilde{X}$ be as in Theorem 4.1. If x_0 is regular for $\{x_0\}$ with respect to X then x_0 is regular for $\{x_0\}$ with respect to $\tilde{X}$.*

Proof. We are given $P_{x_0}(\tau_{x_0}=0)=1$ where $\tau_{x_0}(\omega) = \inf\{t>0: x_t(\omega)=x_0\}$. Let $\tilde{\tau}_{x_0}(\tilde{\omega}) = \inf\{t>0: \tilde{x}_t(\tilde{\omega})=x_0\}$ be the hitting time of $\{x_0\}$ for $\tilde{X}$. Now, $\tilde{\tau}_{x_0}(\tilde{\omega}) \wedge u = \tau_{x_0}(\omega) \wedge u$ where $\tilde{\omega} = (\omega, u)$. Let $\tilde{\Lambda} =$

4) $a \vee b$ denotes the maximum of a and b.

$\{\tilde{\omega}: \tilde{\tau}_{x_0}(\tilde{\omega})=0\}$. Then $\widetilde{P}_{x_0}(\tilde{\Lambda})=M_{x_0}[\psi_{\omega}(\tilde{\Lambda}_{\omega})]$ where $\tilde{\Lambda}_{\omega}$ is the ω-section of $\tilde{\Lambda}$. But, $P_{x_0}(\tau_{x_0}=0)=1$ and $\tilde{\tau}_{x_0}(\omega, u)\wedge u=\tau_{x_0}(\omega)\wedge u$ imply $\tilde{\Lambda}_{\omega}=(0, \infty]$ for almost all $\omega[P_{x_0}]$. Hence,

$$\begin{aligned}\widetilde{P}_{x_0}(\tilde{\tau}_{x_0}=0) &= M_{x_0}[\psi_{\omega}((0, \infty])] \\ &= M_{x_0}[\alpha_0]=1.\end{aligned}$$

Thus, x_0 is regular for $\{x_0\}$ with respect to $\widetilde{X}$. Q.E.D.

Theorem 4.3. *Let X, α and $\widetilde{X}$ be as in Theorem* 4.1. *If the local time, γ_{x_0}, of X at x_0 exists then the local time, $\tilde{\gamma}_{x_0}$, of $\widetilde{X}$ at x_0 exists and is given by $\tilde{\gamma}_{x_0}=\tilde{\gamma}$, where $\tilde{\gamma}=\{\tilde{\gamma}(t, \tilde{\omega})=\gamma_{x_0}(t\wedge\tilde{\zeta}(\tilde{\omega}), \omega), t\geq 0\}$.*

Proof. $\widetilde{X}$ is a standard process; hence, by Theorem 2.1 and Lemma 4.2, $\widetilde{X}$ has local times at all points at which X has local times.

Define $\tilde{\gamma}=\{\tilde{\gamma}(t), t\geq 0\}$ by $\tilde{\gamma}(t, \tilde{\omega})=\gamma_{x_0}(t\wedge\tilde{\zeta}(\tilde{\omega}), \omega)$, where γ_{x_0} is the local time of X at x_0. It can easily be seen that $\tilde{\gamma}$ is a CAF of $\widetilde{X}$; see for example [2, Theorem 10.8].

The fine support of $\tilde{\gamma}$ is $\widetilde{F}=\{x: \widetilde{P}_x(\tilde{\rho}=0)=1\}$ where $\tilde{\rho}(\tilde{\omega})=\inf\{t\geq 0: \tilde{\gamma}(t, \tilde{\omega})>0\}$ and the fine support of γ_{x_0} is $\{x_0\}=\{x: P_x(\rho=0)=1\}$ where $\rho(\omega)=\inf\{t\geq 0: \gamma_{x_0}(t, \omega)>0\}$. By the definition of $\tilde{\gamma}$ we see that $\tilde{\rho}(\omega,u)=\rho(\omega)\wedge u$. As in Lemma 4.2 we see $\widetilde{P}_x(\tilde{\rho}=0)=1$ if and only if $P_x(\rho=0)=1$, that is, $\tilde{\gamma}$ and γ_{x_0} have same fine support. Thus $\widetilde{F}=\{x_0\}$, and so $\tilde{\gamma}=\tilde{\gamma}_{x_0}$ where $\tilde{\gamma}_{x_0}$ is the local time of $\widetilde{X}$ at x_0.
Q.E.D.

The method of this paper is now well established and can easily be applied to other transforms of Markov processes.

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