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On iterated suspensions III.

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Introduction.

The present paper is the third part of the series [8] with the same title.

In section 13, we shall treat the stable homotopy of some complexes using the relations of Yamamoto [9]. The results (Theorem 13.5) of section 13 will be applied to obtain our main result Theorem 14.1 which states, briefly, the existence of unstable elements of the fourth type: $\gamma \notin \operatorname{Im} S^2$, $S^{2p}\gamma \neq 0$ and $S^{2p+2}\gamma = 0$. As a consequence, we shall have a generalization (Theorem 14.2) of Theorem 12.5 (with a minor correction) for meta-stable groups. Theorem 14.1 also implies the existence of new generators $\varepsilon_i'(1 \leq i \leq p-2)$ and $\varepsilon_j(1 \leq j \leq p-1)$ which together with α'_{p^2}, α_{p^2+t} $(1 \leq t \leq p-1)$, $\beta_1^{p-s}\beta_s$, $\alpha_1\beta_1^{p-s}\beta_s(2 \leq s \leq p-1)$ and β_1^{p+1} give a system of generators of the p-primary components of the k-stem groups π_k^S for $2p^2(p-1)-1 \leq k < 2(p^2+p)(p-1)-5$. Our proof is independent of algebraic theory of the stable groups as in [3], [5]. Moreover, for the above range of k, the unstable groups $\pi_{2m+1+k}(S^{2m+1}: p)$ are determined in Theorem 15.2. For the cases $k=2p^2(p-1)-3$, $=2p^2(p-1)-2$, the unstable groups are computed (Theorem 15.1) by dividing into two possibilities: (I) $\alpha_1\beta_1^p = 0$, (II) $\alpha_1\beta_1^p \neq 0$. (The case (II) is negative by author's recent note in Proc. Japan Acad. 13 (1967), 839-842.) It is remarkable that in the case (I) and $p=3$ the relations $\beta_1^4 = \alpha_1\varepsilon_1'$ and $\beta_1^6 = 0$ hold (Proposition 15.6).

13. Stable mod p homotopy of a special complex.

We shall discuss stable homotopy of a complex of a special kind, a model of which is the complex $K(m,k)$ of Proposition 3.6 for $m \equiv 0 \pmod{p}$.

We shall use the notations in section 4.

According to Yamamoto [9], we define generators

$$\beta_{(s)} \in \pi^S_{2(sp+s-1)(p-1)-1}(Y_p;\ Y_p),\ 1 \leq s \leq p-1$$

by the following rule.

(13.1). (i). *The functional reduced power operation $\mathcal{P}^p_f$ for a representative f: $Y_p^{2N+2p(p-1)-1} \to Y_p^{2N}$ of $\beta_{(1)}$ has the coefficient* 1 *for the orientation of the top cell of $Y_p^{2N+2p(p-1)-1}$.*

(ii). $\beta_{(s)} \in \langle \beta_{(1)}, \alpha, \beta_{(s-1)} \rangle$ *for* $2 \leq s \leq p-1$.

(iii). $\alpha\beta_{(s)} = \beta_{(s)}\alpha = 0$ *for* $1 \leq s \leq p-2$ *and for* $s = p-1,\ p > 3$.

The first condition will be used as follows. Let $g: Y_p^{2N} \to X$ be a map inducing trivial homomorphism of $H^*(\ ;Z_p)$ and satisfying $g \circ f \simeq 0$. Let $C_g = X \cup CY_p^{2N}$ be a mapping cone of g *and* $\tilde{f}$: $Y_p^{2n+2p(p-1)} \to C_g$ a coextension of f and let $C_{\tilde{f}} = C_g \cup CY_p^{2N+2p(p-1)}$ be a mapping cone of $\tilde{f}$. Let $a \in \mathrm{H}^{2N}(C_{\tilde{f}};\ Z_p)$ and $b \in H^{2N+2p(p-1)}(C_{\tilde{f}};\ Z_p)$ be given by the natural orientations of the cells. Then

$$\mathcal{P}^p a = b, \text{ and } \mathcal{P}^p \Delta a = \Delta b$$

by the Adem relation $\Delta\mathcal{P}^p - \mathcal{P}^p\Delta = \mathcal{P}^{p-1}\Delta\mathcal{P}^1$ and $\mathcal{P}^1 a = 0$. If we change f by a representative of the element α in (4.2), then we have

$$\mathcal{P}^1 \Delta a = b \in H^{2N+2(p-1)+1}(C_{\tilde{f}};\ Z_p).$$

Yamamoto has also proved the followings.

(13.2). $i^*\pi_*\beta_{(s)} \neq 0$, *so we can choose a generator* β_s *of* $(\pi^S_{2(sp+s-1)(p-1)-2}:\ p) \approx Z_p$ *such that* $\beta_s = i^*\pi_*\beta_{(s)}$ $\quad 1 \leq s \leq p-1$.

(13.3), (i). $\alpha\delta\beta_{(s)} = \beta_{(s)}\delta\alpha$ *for* $1 \leq s \leq p-2$ *and for* $s = p-1,\ p > 3$.

(ii). *If* $p > 3$, *then* $\beta_{(s)}\beta_{(t)} = 0$ *for* $s + t \leq p-1$ *and*

$$-\frac{s-1}{s}\beta_{(s)} \in \langle \alpha, \beta_{(1)}, \beta_{(s-1)} \rangle \text{ for } 2 \leq s \leq p-1$$

In [9], he has no proof of the second relation of (ii) for the case $s=p-1$, but the relation $-\frac{s-1}{s}\beta_{(s)}\in\langle\beta_{(s-1)},\beta_{(1)},\alpha\rangle$ is obtained for $2\leq s\leq p-1$. Consider the formula $0\in\langle\langle\alpha,\beta,\gamma\rangle,\delta,\varepsilon\rangle+(-1)^{\deg\alpha}\langle\alpha,\langle\beta,\gamma,\delta\rangle,\varepsilon\rangle+(-1)^{\deg\alpha\beta}\langle\alpha,\beta,\langle\gamma,\delta,\varepsilon\rangle\rangle$ for $\varepsilon=\alpha$, $\beta=\delta=\beta_{(1)}$, $\gamma=\beta_{(p-3)}$. Then we have

$$\beta_{(p-1)}\in\langle\beta_{(p-2)},\beta_{(1)},\alpha\rangle=\langle\alpha,\beta_{(1)},\beta_{(p-2)}\rangle+\langle\alpha,\theta,\alpha\rangle,$$

where $\theta\in\pi^S_{2(p^2-4)(p-1)-2}(Y_p;\ Y_p)=\{\alpha^{p^2-5}\delta\alpha\delta\}$. Since $\alpha\theta=0$ and $\alpha(\alpha^{p^2-5}\delta\alpha\delta)\neq 0$, we have $\theta=0$. This shows that (13.3) (ii) is true for $s=p-1$.

For $p=3$, his methods prove only

(13. 3)′. $\beta_{(1)}\beta_{(1)}\equiv 0 \bmod \{\delta\alpha(\delta\beta_{(1)})^2\delta\}$, $\alpha\beta_{(2)}\equiv\beta_{(2)}\alpha\equiv 0 \bmod \{\beta_{(1)}(\delta\beta_{(1)})^2\}$ *and* $\alpha\delta\beta_{(2)}\equiv\beta_{(2)}\delta\alpha \bmod \{(\delta\beta_{(1)})^3,(\beta_{(1)}\delta)^3\}$.

The following list of independent Z_p-bases of $\pi^S_k(Y_p;\ Y_p)$ is given in [9].

(13. 4). δ, ι, α^t, $\alpha^t\delta$, $\alpha^{t-1}\delta\alpha$, $\alpha^{t-1}\delta\alpha\delta$ *for* $1\leq t\leq p^2-1$; $(\beta_{(1)}\delta)^r\beta_{(s)}$, $\delta(\beta_{(1)}\delta)^r\beta_{(s)}$, $(\beta_{(1)}\delta)^r\beta_{(s)}\delta$, $\delta(\beta_{(1)}\delta)^r\beta_{(s)}\delta$, $\alpha\delta(\beta_{(1)}\delta)^r\beta_{(s)}$, $\delta\alpha\delta(\beta_{(1)}\delta)^r\beta_{(s)}$, $\alpha\delta(\beta_{(1)}\delta)^r\beta_{(s)}\delta$, $\delta\alpha\delta(\beta_{(1)}\delta)^r\beta_{(s)}\delta$ *for* $0\leq r$, $1\leq s$ *and* $r+s\leq p-1$; *and* $(\beta_{(1)}\delta)^{p-1}\beta_{(1)}$, $(\delta\beta_{(1)})^p$, $(\beta_{(1)}\delta)^p$, $\delta(\beta_{(1)}\delta)^p$.

We denote by

$$K_n(k)=Y_p^n\cup CY_p^{n+2(p-1)-1}\cup\cdots\cup CY_p^{n+2(k-1)(p-1)-1}$$

a complex satisfying the following condition.

(13. 5). *For* $1\leq k'<k$, $K_n(k')$ *is a subcomplex of* $K_n(k)$ *and* $K_n(k'+1)=K_n(k')\cup CY_p^{n+2k'(p-1)-1}$ *is a mapping cone of a map* $h_{k'}$: $Y_p^{n+2k'(p-1)-1}\to K_n(k')$ *such that for the projection* $\pi_{k'-1}$: $K_n(k')\to Y_p^{n+2(k'-1)(p-1)}=K_n(k')/K_n(k'-1)$ *the composition* $\pi_{k'-1}\circ h_{k'}$ *represents* $k'\cdot\delta\alpha-(k'-1)\cdot\alpha\delta\in\pi^S_{2p-3}(Y_p;Y_p)$.

By Propositions 3.6, 4.5, we may consider that

(13. 5).′ $K(m,k)=K_{2mp-2}(k)$ *if* $m\equiv 0 \pmod p$ *and* $k(p-1)<mp-1$.

Now, we assume the existence of a complex $K_n(p+2)$ and compute stable groups $\pi^S(Y_p^{n+i};K_n(k))$, $k<p+2$. We denote by i_k: $K_n(k)$

$\to K_n(k+j)$ the inclusion. We shall use the following homotopy exact sequences:

(13. 6). $\pi^S_{i-2k(p-1)+1}(Y_p;\ Y_p)\xrightarrow{h_{k*}}\pi^S(Y_p^{n+i};K_n(k))\xrightarrow{i_{k*}}\pi^S(Y_p^{n+i};K_n(k+1))$
$\xrightarrow{\pi_{k*}}\pi^S_{i-2k(p-1)}(Y_p;\ Y_p)\xrightarrow{h_{k*}}\pi^S(Y_p^{n+i-1};\ K_n(k))\to\cdots.$

By (13.5) and (4.6), we have $\pi_{k-1*}h_{k*}(\alpha^r)=(k\cdot\delta\alpha-(k-1)\alpha\delta)\alpha^r$ $=k((r+1)\alpha^r\delta\alpha-r\cdot\alpha^{r+1}\delta)-(k-1)\ (r\cdot\alpha^r\delta\alpha-(r-1)\alpha^{r+1}\delta)=(k+r)\alpha^r\delta\alpha$ $-(k+r-1)\alpha^{r+1}\delta$ and $\pi_{k-1*}h_{l*}\ (\alpha^{r-1}\delta\alpha)=(k+r-1)\alpha^{r-1}\delta\alpha\delta\alpha=(k+r-1)$ $\alpha^r\delta\alpha\delta$. Thus

(13. 7). $\pi_{k-1*}h_{k*}(\alpha^r)=(k+r)\alpha^r\delta\alpha-(k+r-1)\alpha^{r+1}\delta,$
$\pi_{k-1*}h_{k*}(\alpha^r\delta)=(k+r)\alpha^r\delta\alpha\delta,$
$\pi_{k-1*}h_{k*}(\alpha^{r-1}\delta\alpha)=(k+r-1)\alpha^r\delta\alpha\delta$ *and* $\pi_{k-1*}h_{k*}(\alpha^{r-1}\delta\alpha\delta)=0.$

Lemma 13.1. (i). $\pi^S(Y_p^{n+2p(p-1)-2};\ K_n(p))\approx Z_p+Z_p$ *generated by* $i_{1*}(\beta_{(1)}\delta)$ *and* $i_{1*}(\delta\beta_{(1)})$.

(ii). $\pi^S(Y_p^{n+2p(p-1)-1};\ K_n(p))\approx Z_p+Z_p+Z_p$ *generated by* $i_{1*}(\alpha^{p-1}\delta\alpha)$, $i_{1*}(\beta_{(1)})$ *and* $h_{l*}(\iota)$.

Proof. By (13.7), $\pi_{k-1*}h_{k*}(\alpha^{p-k})=\alpha^{p-k+1}\delta$ and $\pi_{k-1*}h_{k*}(\alpha^{p-k-1}\delta\alpha)$ $=-\alpha^{p-k}\delta\alpha$ for $1\leq k\leq p-1$. In the exact sequences (13.6) for $i=2p(p-1)-1$, $=2p(p-1)-2$, these elements play a similar role as unstable elements of the first type, and we can cancell them. It remains $\beta_{(1)}\delta$, $\delta\beta_{(1)}$, $\delta\alpha\delta$ and $\alpha^{p-1}\delta\alpha$, $\beta_{(1)}$, $\alpha\delta=\pi_{p-1*}h_{l*}(\iota)$. Then it is sufficient to prove that $\delta\alpha\delta$ is not a π_{p-1*}-image. Assume that $\delta\alpha\delta$ $=\pi_{p-1*}(d)$, and consider a mapping cone C_d of d. Then we have $\mathcal{P}^1\mathcal{P}^{p-1}H^{n-1}(C_d;\ Z_p)\neq 0$, but this contradicts to the Adem relation $\mathcal{P}^1\mathcal{P}^{p-1}=0$. Thus $\delta\alpha\delta$ is not a π_{p-1*}-image (it must be cancelled with $\delta\beta_{(1)}\delta$ as in Lemma 10.2).

Lemma 13.2. *There exists an element* $\widetilde{\beta}_{(1)}$ *of* $\pi^S(Y_p^{n+2(p+1)(p-1)-1};$ $K_n(2))$ *such that*

$$\pi_{1*}(\widetilde{\beta}_{(1)})=\beta_{(1)}.$$

Then we have

$\pi^S(Y_p^{n+2(p+1)(p-1)-2};\ K_n(p)) \approx Z_p + Z_p + Z_p$ *generated by*

$i_{1*}(\alpha\delta\beta_{(1)}),\ i_{2*}(\widetilde{\beta_{(1)}\delta})$ *and* $h_{p*}(\alpha\delta)$,

$\pi^S(Y_p^{n+2(p+1)(p-1)-1};\ K_n(p)) \approx Z_p + Z_p + Z_p$ *generated by*

$i_{1*}(\alpha^{p+1}\delta),\ i_{2*}(\widetilde{\beta}_{(1)})$ *and* $h_{p*}(\alpha)$.

Proof. Since $\pi_{k-1*}h_{k*}(\alpha^{p-k+1}) = \alpha^{p-k+1}\delta\alpha$ and $\pi_{k-1*}h_{k*}(\alpha^{p-k+1}\delta) = \alpha^{p-k+1}\delta\alpha\delta$, by (13.7), these elements are cancelled in the exact sequences (13.6) for $i = 2(p+1)(p-1)-1, = 2(p+1)(p-1)-2$. Then it remains $\alpha\delta\beta_{(1)}$, $\delta\beta_{(1)}$, $\beta_{(1)}\delta$, $\alpha\delta\alpha\delta = \pi_{p-1*}h_{p*}(\alpha\delta)$ and $\alpha^{p+1}\delta$, $\beta_{(1)}$, $\alpha\delta\alpha = \pi_{p-1*}h_{p*}(\alpha)$ (and also $\delta\alpha\delta\beta_{(1)}\delta$ if $p=3$). We have also, by (13.5), (13.1) and (13.3), $h_{1*}(\delta\beta_{(1)}) = \delta\alpha\delta\beta_{(1)}$, $h_{1*}(\beta_{(1)}\delta) = \delta\alpha\beta_{(1)}\delta = 0$ and $h_{1*}(\beta_{(1)}) = \delta\alpha\beta_{(1)} = 0$ $\left(h_{1*}(\delta\alpha\delta\beta_{(1)}\delta) = h_{1*}\pi_{1*}h_{2*}\left(\frac{1}{2}\delta\beta_{(1)}\delta\right) = 0 \text{ if } p=3\right)$. Then the lemma follows from the exactness of (13.6). q. e. d.

Denote by

$$C_\alpha = Y_p^{n+2p(p-1)-2} \cup_\alpha C Y_p^{n+2(p+1)(p-1)-2}$$

the mapping cone of (a representative of) α. Let

$$i_0:\ Y_p^{n+2p(p-1)-2} \to C_\alpha \quad \text{and} \quad \pi_0:\ C_\alpha \to Y_p^{n+2(p+1)(p-1)-1}$$

be the inclusion and the projection respectively.

Lemma 13.3. *There exists an element* $\bar{\beta}_{(1)}$ *of* $\pi^S(C_\alpha;\ Y_p^{n-1})$ *such that*

$$i_0^*(\bar{\beta}_{(1)}) = \beta_{(1)}.$$

Then we have

$\pi^S(C_\alpha;\ K_n(p)) \approx Z_p + Z_p$ *generated by* $i_{1*}(\delta\bar{\beta}_{(1)})$ *and* $\pi_0^* i_{2*}(\widetilde{\beta}_{(1)})$.

Proof. From Lemmas 13.1, 13.2, we obtain the following Puppe's exact sequence:

$$\{i_{1*}(\alpha^{p-1}\delta\alpha),\ i_{1*}(\beta_{(1)}),\ h_{p*}(\iota)\} \xrightarrow{\alpha^*} \{i_{1*}(\alpha^{p+1}\delta),\ i_{2*}(\widetilde{\beta}_{(1)}),\ h_{p*}(\alpha)\} \xrightarrow{\pi_0^*}$$
$$\pi^S(C_\alpha : K_n(p)) \xrightarrow{i_0^*} \{i_{1*}(\beta_{(1)}\delta),\ i_{1*}(\delta\beta_{(1)})\} \xrightarrow{\alpha^*} \{i_{1*}(\alpha\delta\beta_{(1)}),\ i_{2*}(\widetilde{\beta_{(1)}\delta}),\ h_{p*}(\alpha\delta)\}.$$

By use of (4.6), (13.1) and (13.3), we have $\alpha^*(i_{1*}(\alpha^{p-1}\delta\alpha)) = i_{1*}(\alpha^{p-1}\delta\alpha^2) = 2\cdot i_{1*}(\alpha^p\delta\alpha) - i_{1*}(a^{p+1}\delta) = -i_{1*}(\alpha^{p+1}\delta)$, $\alpha^*(i_{1*}(\beta_{(1)})) = i_{1*}(\beta_{(1)}\alpha) = 0$,

$\alpha^*(h_{p*}(\iota)) = h_{p*}(\alpha)$, $\alpha^*(i_{1*}(\beta_{(1)}\delta)) = i_{1*}(\beta_{(1)}\delta\alpha) = i_{1*}(\alpha\delta\beta_{(1)})$ and $\alpha^*(i_{1*}(\delta\beta_{(1)})) = i_{1*}(\delta\beta_{(1)}\alpha) = 0$. Thus we have a short exact sequence

$$0 \to \{i_{2*}(\widetilde{\beta_{(1)}})\} \xrightarrow{\pi_0^*} \pi^S(C_\alpha; K_n(p)) \xrightarrow{i_0^*} \{i_{1*}(\delta\beta_{(1)})\} \to 0.$$

Since $\alpha^*(\beta_{(1)}) = 0$, the existence of $\overline{\beta}_{(1)}$ follows. Then $i_0^*(i_{1*}(\delta\overline{\beta}_{(1)})) = i_{1*}(\delta(i_0^*\overline{\beta}_{(1)})) = i_{1*}(\delta\beta_{(1)})$. Since $p(i_{1*}(\delta\overline{\beta}_{(1)})) = i_{1*}((p\cdot\delta)\overline{\beta}_{(1)})) = 0$, the above short sequence splits. q. e. d.

Note that $\pi^S(C_\alpha;\ X)$ is a Z_p-module since $\pi^S(C_\alpha;\ C_\alpha) \approx Z_p$.

Let $a_i \in H^{n+2i(p-1)-1}(K_n(k);\ Z_p)$ be a generator given by the natural orientations of the cells in $K_n(k)$. Then it follows from (13.5) (see the proof of Proposition 4.5)

(13.8). $\mathcal{P}^1 a_i = (i+1)a_{i+1}$ *and* $\mathcal{P}^1 \Delta a_i = i\cdot\Delta a_{i+1}$ *for* $0 \leq i < k$.

By Corollary 8.4, the following relations hold in $K(lp,\ p+2)$:

$$\mathcal{P}^p a_0 = -\binom{lp(p-1)-1}{p} a_p = (l+1)a_p,$$

$$\mathcal{P}^p \Delta a_0 = -\binom{lp(p-1)}{p} \Delta a_p = l\cdot\Delta a_p,$$

$$\mathcal{P}^p a_1 = -\binom{(lp+1)(p-1)-1}{p} a_{p+1} = l\cdot a_{p+1},$$

$$\mathcal{P}^p \Delta a_1 = -\binom{(lp+1)(p-1)}{p} \Delta a_{p+1} = l\cdot\Delta a_{p+1}$$

So, in the following, we add the following condition to $K_n(p+2)$.

(13.9). $\mathcal{P}^p a_0 = (l+1)a_p$, $\mathcal{P}^p \Delta a_0 = l\cdot\Delta a_p$, $\mathcal{P}^p a_1 = l\cdot a_{p+1}$ *and* $\mathcal{P}^p \Delta a_1 = l\cdot\Delta a_{p+1}$.

Now consider the attaching map

$$h_{p+1}:\ Y_p^{n+2(p+1)(p-1)-1} \to K_n(p+1) = K_n(p) \cup_{h_p} CY_p^{n+2p(p-1)-1}.$$

Since we are considering stable groups, we may assume that n is sufficiently large.

Then the existence of such a map h_{p+1} satisfying (13.5) is equivalent to $h_{p*}(\delta\alpha) = 0$, and h_{p+1} can be chosen, in its homotopy class, as a coextension of $\delta\alpha$, *i.e.*, h_{p+1} maps the lower cone $C_- Y_p^{n+2(p+1)(p-1)-2}$ into $K_n(p)$ and maps the upper cone $C_+ Y_p^{n+2(p+1)(p-1)-2}$ by the composition of the canonical extension (cone) : $C_+ Y_p^{n+2(p+1)(p-1)-2} \to CY_p^{n+2p(p-1)-1}$ of $\delta\alpha$ and the characteristic map : $CY_p^{n+2p(p-1)-1} \to K_n(p+1)$. Consider the mapping cone

$$C_{\delta\alpha} = Y_p^{n+2p(p-1)-1} \cup_{\delta\alpha} CY_p^{n+2(p+1)(p-1)-2}$$

of $\delta\alpha$, and define a map

$$h: C_{\delta\alpha} \to K_n(p)$$

by putting $h|Y_p^{n+2p(p-1)-1} = h_p$ and by extending over $CY_p^{n+2(p+1)(p-1)-2}$ by $h_{p+1}|C_- Y_p^{n+2(p+1)(p-1)-2}$ identifying CY_p with $C_- Y_p$. We also define a map

$$D:\ C_\alpha = Y_p^{n+2p(p-1)-2} \cup_\alpha CY_p^{n+2(p+1)(p-1)-2} \to C_{\delta\alpha}$$

by putting $D|Y_p^{n+2p(p-1)-2} = \delta = i \circ \pi$ and extending over $CY_p^{n+2(p+1)(p-1)-2}$ identically. Clearly, for the projection $\pi': C_{\delta\alpha} \to Y_p^{n+2(p+1)(p-1)-1}$, we have

(13. 10). $\pi' \circ D = \pi_0$.

Proposition 13.4. (i). *$K_n(p+2)$ is the mapping cone of the map $h: C_{\delta\alpha} \to K_n(p)$, up to homotopy equivalence, and $(-h_{p+1}) \circ \pi'$ is homotopic to $i_p \circ h$.*

(ii). *$h \circ D$ represents $(l+1) i_{1*}(\delta\overline{\beta_{(1)}}) - l \cdot i_{2*}\pi_0^*(\widetilde{\beta_{(1)}}) \in \pi^S(C_\alpha;\ K_n(p))$.*

Proof. The proof of (i) is directforwards (see Chapter 1 of [5]). Put

$$b_1 = i_{1*}(\delta\overline{\beta_{(1)}}) \text{ and } b_2 = i_{2*}\pi_0^*(\widetilde{\beta_{(1)}}) = \pi_0^* i_{2*}(\widetilde{\beta_{(1)}})$$

and consider mapping cones

$$C_{b_1} = K_n(p) \cup_{b_1} C(C_\alpha),\ C_{b_2} = K_n(p) \cup_{b_2} C(C_\alpha) \text{ and } C_{h\circ D} = K_n(p) \cup_{h\circ D} C(C_\alpha).$$

Let $d_1 \in H^{n+2p(p-1)-1}(C_{b_1}; Z_p)$, $d_2 \in H^{n+2p(p-1)-1}(C_{b_2}; Z_p)$ and $d \in H^{n+2p(p-1)-1}(C_{h\circ D}; Z_p)$ be given by the bottom cell of $C(C_\alpha)$, then $\{d_1,\ \Delta d_1,\ \mathcal{P}^1\Delta d_1,\ \Delta\mathcal{P}^1\Delta d_1\}$ is *a* Z_p-basis of $H^*(C_{b_1}; Z_p)/H^*(K_n(p); Z_p)$ and so for d_2 and d. By the remark after (13.1), we have

$$\mathcal{P}^p a_0 = \Delta d_1,\ \mathcal{P}^p \Delta a_0 = \mathcal{P}^p a_1 = \mathcal{P}^p \Delta a_1 = 0 \text{ in } C_{b_1}$$

and $\mathcal{P}^p a_0 = \mathcal{P}^p \Delta a_0 = 0$, $\mathcal{P}^p a_1 = \mathcal{P}^1 \Delta d_2$, $\mathcal{P}^p \Delta a_1 = \Delta \mathcal{P}^1 \Delta d_2$ in C_{l_2}. By Lemma 13.3, there exists integers x, y such that $h \circ D$ represents $x \cdot d_1 + y \cdot d_2$. Then we can easily construct a map

$$f:\ C_{h\circ D} \to C_{b_1} \cup C_{b_2} = K_n(p) \cup_{b_1} C(C_\alpha) \cup_{b_2} C(C_\alpha)$$

such that $f | K_n(p) = \text{identity}$ and $f^*(d_1) = x \cdot d$, $f^*(d_2) = y \cdot d$. By the naturality of $\mathcal{P}^p$, we have

$$\mathcal{P}^p a_0 = x \cdot \Delta d \text{ and } \mathcal{P}^p a_1 = y \cdot \mathcal{P}^1 \Delta d.$$

On the other hand, identifying $K_n(p+2)$ with C_h by (i), we obtain a map

$$g:\ C_{h\circ D} \to C_h = K_n(p+2)$$

such that $g | K_n(p) = \text{identity}$ and $g^*(a_p) = \Delta d$, $g^*(\Delta a_p) = 0$, $g^*(a_{p+1}) = -\mathcal{P}^1 \Delta d$, $g^*(\Delta a_{p+1}) = -\Delta \mathcal{P}^1 \Delta d$, where the sign is caused of the same reason as in the proof of Proposition 4.5. It follows then from (13.9) that $\mathcal{P}^p a_0 = g^*(\mathcal{P}^p a_0) = g^*((l+1) a_p) = (l+1) \Delta d$ and $\mathcal{P}^p a_1 = g^*(\mathcal{P}^p a_1) = g^*(l \cdot a_{p+1}) = -l \cdot \mathcal{P}^1 \Delta d$. Thus $x \equiv l+1$, $y \equiv -l \pmod p$, and $h \circ D$ represents $(l+1) b_1 - l \cdot b_2$. q. e. d.

The main purpose of this section is to prove the following

Theorem 13.5. *Let* $2 \leq s \leq p-1$ *and* $K_n(p+2)$ *satisfy* (13. 5) *and* (13. 9). *Then the following relation holds in* $\pi^s(Y_p^{n+2(sp+s-1)(p-1)-2}; K_n(p+1))$:

$$h_{p+1*}(\beta_{(s-1)}) = \frac{1}{s}(l+s) i_{1*}(\delta \beta_{(s)}).$$

We shall given two proofs, the first one covers the case $p > 3$ and the second one does the case $s = 2$. First of all we show

Lemma 13.6. *There exists a coextension* $\widetilde{\beta}_{(s-1)} \in \pi^s(Y_p^{n+2(sp+s-1)(p-1)-1}; C_\alpha)$ *of* $\beta_{(s-1)}$ *such that* $\pi_{0*}(\widetilde{\beta}_{(s-1)}) = \beta_{(s-1)}$ *and*

$$\begin{aligned} h_{p+1*}(\beta_{(s-1)}) &= \{-(l+1) i_{1*}(\delta \overline{\beta}_{(1)}) + l \cdot i_{2*} \pi_0^*(\widetilde{\beta}_{(1)})\} \circ \widetilde{\beta}_{(s-1)} \\ &= -(l+1)\, i_{1*}(\delta \overline{\beta}_{(1)} \circ \widetilde{\beta}_{(s-1)}) + l \cdot i_{2*}(\widetilde{\beta}_{(1)} \circ \beta_{(s-1)}). \end{aligned}$$

Proof. By (13. 1), $\alpha \beta_{(s-1)} = 0$, hence $\widetilde{\beta}_{(s-1)}$ exists. By (13. 10) and Proposition 13.4, we have

$$-h_{p+1*}(\beta_{(s-1)}) = -h_{p+1*}\pi_{0*}(\widetilde{\beta}_{(s-1)}) = -h_{p+1*}\pi'_*D_*(\widetilde{\beta}_{(s-1)}) = i_{p*}h_*D_*(\widetilde{\beta}_{(s-1)})$$
$$= \{(l+1)i_{1*}(\delta\overline{\beta}_{(1)}) - l\cdot i_{2*}\pi_0^*(\widetilde{\beta}_{(1)})\}\circ\widetilde{\beta}_{(s-1)}$$
$$= (l+1)i_{1*}(\delta\overline{\beta}_{(1)}\circ\widetilde{\beta}_{(s-1)}) - l\cdot i_{2*}(\widetilde{\beta}_{(1)}\circ\beta_{(s-1)}).$$

Proof of Theorem 13.5 *for* $p>3$. $\langle\beta_{(1)}, \alpha, \beta_{(s-1)}\rangle = \lim(-1)^{n-1}\{\beta_{(1)},\alpha,\beta_{(s-1)}\}$ by definition. To consider the coextension $\widetilde{\beta}_{(s-1)}$ for fixed n, we must take the sign $(-1)^n$ since $\widetilde{S\beta}_{(s-1)}$ is a coextension of $-S\beta_{(s-1)}$. Then from Proposition 1.7 of [6], we have $\overline{\beta}_{(1)}\circ\widetilde{\beta}_{(s-1)}\in -\langle\beta_{(1)}, \alpha, \beta_{(s-1)}\rangle$. Similarly, from Proposition 1.8 and (3.5) of [6], we have $\widetilde{\beta}_{(1)}\circ\beta_{(s-1)}\in -i_{1*}\langle\delta\alpha, \beta_{(1)}, \beta_{(s-1)}\rangle\supset i_{1*}(\delta\circ\langle\alpha, \beta_{(1)}, \beta_{(s-1)}\rangle)$. Then it follows from (13.1) and (13.3)

$$\overline{\beta}_{(1)}\circ\widetilde{\beta}_{(s-1)} \equiv -\beta_{(s)} \qquad \mathrm{mod}\ \beta_{(1)}\circ G_1 + G_2\circ\beta_{(s-1)}$$

and

$$\widetilde{\beta}_{(1)}\circ\beta_{(s-1)} \equiv -\frac{s-1}{s}\, i_{1*}(\delta\beta_{(s)}) \qquad \mathrm{mod}\ i_{1*}(\delta\alpha)\circ G_3 + i_{1*}G_4\circ\beta_{(s-1)}$$

where $G_1 = \pi^S_{2(s-1)(p+1)(p-1)}(Y_p;\ Y_p)$ generated by $\alpha^{(p+1)(s-1)}$, $G_2 = \pi^S_{2(p+1)(p-1)}(Y_p;\ Y_p)$ generated by α^{p+1}, $G_3 = \pi^S_{2(sp+s-2)(p-1)-1}(Y_p;\ Y_p)$ generated by $\alpha^{sp+s-2}\delta$ and $\alpha^{sp+s-3}\delta\alpha$, and $G_4 = \pi^S_{2(p+1)(p-1)-1}$ generated by $\alpha^{p+1}\delta$ and $\alpha^p\delta\alpha$. $\beta_{(1)}\circ\alpha^{(s-1)(p+1)} = \alpha^{p+1}\circ\beta_{(s-1)} = (\alpha^p\delta\alpha)\circ\beta_{(s-1)} = 0$ and $\alpha^{p+1}\delta\circ\beta_{(s-1)} = \alpha^p\beta_{(s-1)}\delta\alpha = 0$ by (13.1) and (13.3). By (4.6), $i_{1*}(\delta\alpha)\alpha^{sp+s-2}\delta$ and $i_{1*}(\delta\alpha)\alpha^{sp+s-3}\delta\alpha$ are some multiples of $i_{1*}(\alpha^{sp+s-2}\delta\alpha\delta)$ which are in the $\pi_{1*}h_{2*}$-image by (13.7). Thus all the indeterminacy vanishes, and

$$\overline{\beta}_{(1)}\circ\widetilde{\beta}_{(s-1)} = -\beta_{(s)} \text{ and } \widetilde{\beta}_{(1)}\circ\beta_{(s-1)} = -\frac{s-1}{s} i_{1*}(\delta\beta_{(s)}).$$

Then, by Lemma 13.6 we have

$$h_{p+1*}(\beta_{(s-1)}) = (l+1)i_{1*}(\delta\beta_{(s)}) - \frac{l(s-1)}{s}\, i_{1*}(\delta\beta_{(s)})$$
$$= \frac{1}{s}(l+s)i_{1*}(\delta\beta_{(s)}). \qquad \text{q. e. d.}$$

For the case $p=3$, (13.3) is not valid, but $s=2$ in the case. We shall give a proof for the case $s=2$ without use of the relation (13.3).

Lemma 13.7. $i_{2*}(\widetilde{\beta}_{(1)})\circ\beta_{(1)} = i_{2*}\pi_0^*(\widetilde{\beta}_{(1)})\circ\widetilde{\beta}_{(1)}$ *is a multiple of*

$i_{1*}(\delta\beta_{(2)})$ *in* $K_n(p)$.

Proof. First compute the group $\pi^S(Y_p^{2(2p+1)(p-1)-2}; K_n(p))$ as in Lemma 13.1. Then the i_{2*}-image is generated by $i_{1*}(\delta\beta_{(2)})$ and $i_{1*}(\delta\beta_{(2)}\delta)$ and possibly by $i_{2*}(\pi_{1*}^{-1}(\delta\alpha(\delta\beta_{(1)})^2\delta)$ if $p=3$. But the last element can be cancelled since $\pi_{1*}h_{2*}((\delta\beta_{(1)})^2\delta)=2\cdot\delta\alpha(\delta\beta_{(1)})^2\delta$. Thus $i_{2*}(\widetilde{\beta_{(1)}})\beta_{(1)} =x\cdot i_{1*}(\delta\beta_{(2)})+y\cdot i_{1*}(\beta_{(2)}\delta)$ for some integers x, y. Since $\beta_{(1)}\alpha=0$, we have by (13.1) and (13.3)$'$, $y\cdot i_{1*}(\alpha\delta\beta_{(2)})\equiv y\cdot i_{1*}(\beta_{(2)}\delta)\alpha=-x\cdot i_{1*}(\delta\beta_{(2)}\alpha)$ $\equiv 0$ mod 0 ($p>3$) or mod $i_{1*}\{(\delta\beta_{(1)})^3, (\beta_{(1)}\delta)^3\}$ ($p=3$). The possibility to killing $\alpha\delta\beta_{(2)}$ by i_{1*} is $h_{1*}(\beta_{(2)})$ and if $p=3$ $h_{2*}(\delta\alpha(\delta\beta_{(1)})^2)$ and $h_{2*}(\alpha\delta(\beta_{(1)}\delta)^2)$. But, $h_{1*}(\beta_{(2)})=\delta\alpha\beta_{(2)}\equiv 0$ mod $(\delta\beta_{(1)})^3$, $\delta\alpha\delta =\frac{1}{2}\pi_{2*}h_{3*}(\delta)$ and $\alpha\delta\beta_{(1)}=-\pi_{2*}h_{3*}(\beta_{(1)})$. Thus $i_{1*}(\alpha\delta\beta_{(2)})\not\equiv 0$, and we have $y\equiv 0 \pmod p$. Then the lemma is proved.

Proof of Theorem 13.5 *for* $s=2$. Put $b_1=i_{1*}(\delta\overline{\beta_{(1)}})$, $b_2 =i_{2*}\pi_0^*(\widetilde{\beta_{(1)}})$ and let $f\colon C_\alpha\to K_n(p)$ be a representative of $x\cdot b_1+y\cdot b_2$. Consider the mapping cone $C_f=K_n(p)\cup_f C(C_\alpha)$ of f. As is seen in the proof of Proposition 13.4, we have

$$\mathcal{P}^p a_0=x\cdot\Delta d \text{ and } \mathcal{P}^p a_1=y\cdot\mathcal{P}^1\Delta d.$$

First consider the case $x=1$, $y=0$ and assume that $f_*\beta_{(1)} =i_{1*}(\delta\overline{\beta_{(1)}}\circ\widetilde{\beta_{(1)}})=0$. Then there exists a coextension $g\colon Y_p^{n+2(2p+1)(p-1)-1}\to C_f$ of $\beta_{(1)}$. In the mapping cone C_g of g, we have $\mathcal{P}^p\mathcal{P}^1\mathcal{P}^p a_0\neq 0$ and $\mathcal{P}^p\mathcal{P}^p\mathcal{P}^1 a_0=\mathcal{P}^p\mathcal{P}^p a_1=0$. But this contradicts to Adem relation

$$(*)\qquad \mathcal{P}^p(2\mathcal{P}^1\mathcal{P}^p-\mathcal{P}^p\mathcal{P}^1)=\mathcal{P}^1(2\mathcal{P}^{2p}+\mathcal{P}^{2p-2}\mathcal{P}^1\mathcal{P}^1)$$

since $(2\mathcal{P}^{2p}+\mathcal{P}^{2p-2}\mathcal{P}^1\mathcal{P}^1)a_0=0$ (*cf.* the proof of Theorem 10.8). Thus $i_{1*}(\delta\overline{\beta_{(1)}}\circ\widetilde{\beta_{(1)}})=-i_{1*}(\beta_{(2)})\neq 0$. (This proves also that $\beta_{(2)}=-\overline{\beta_{(1)}}\circ\widetilde{\beta_{(1)}}$ is an independent generator.)

By Lemma 13.7, we can put $b_2\circ\widetilde{\beta_{(1)}}=z\cdot b_1\circ\widetilde{\beta_{(1)}}$ for some integer z. By putting $x=-z$ and $y=1$, we have in C_g that $\mathcal{P}^p\mathcal{P}^1\mathcal{P}^p a_0=-z\cdot u$ and $\mathcal{P}^p\mathcal{P}^p\mathcal{P}^1 a_0=-u$ for some generator u of $H^{n+2(2p+1)(p-1)}(C_g; Z_p)$. Then by use of (*) we have $(2z-1)u=0$ and $z\equiv\frac{1}{2} \pmod p$. It follows from Lemma 13.6

$$h_{p+1*}(\beta_{(s-1)})=\{-(l+1)b_1+l\cdot b_2\}\circ\widetilde{\beta}_{(1)}=\{-(l+1)+\frac{l}{2}\}b_1\circ\widetilde{\beta}_{(1)}$$
$$=\frac{1}{2}(l+2)i_{1*}(-\delta\bar{\beta}_{(1)}\circ\widetilde{\beta}_{(1)})=\frac{1}{2}(l+2)i_{1*}(\delta\beta_{(2)}).$$

This completes the proof of Theorem 13.5.

14. Unstable elements of the fourth type.

We shall prove the following theorem which generalizes Theorem 10.8.

Theorem 14.1. *Assume* $l\geq 1$ *and* $2\leq s\leq p-1$. *Then there exists elements*

$$\gamma\in\pi_{2lp^2+2(sp+s-1)(p-1)-2}(S^{2lp+1}:p)$$

and
$$\gamma'\in\pi_{2lp^2+2(sp+s)(p-1)}(Q_2^{2lp+2p+1}:p)$$

such that

$H^{(2)}\gamma=x\cdot(l+s)\cdot I'(\beta_s(2lp^2-1))$ *for some integer* $x\not\equiv 0 \pmod p$

$S^{2p}\gamma=p_*\gamma'$, $I\gamma'=\beta_{s-1}(2(lp+p+1)p+1)$ *and* $S^{2p+2}\gamma=0$.

In the proof of Theorem 12.5, we have used Theorem 10.8. So, Theorem 12.5 is not valid when $s\equiv 0 \pmod p$, and we have

Correction to Theorem 12.5. *The last condition* "$2\leq r<p-1$" *in* (iii), (iv), (v) *of Theorem* 12.5 *should be read*

"$a_s\leq r<p-1$",

where $a_s=2$ *if* $s\not\equiv 0 \pmod p$ *and* $a_s=1$ *if* $s\equiv 0 \pmod p$.

Assume that $l+s\not\equiv 0 \pmod p$ and $l\geq 1$, and denote the element γ of Theorem 14.1 by

$$u_4(l,\beta_s)=\gamma\in\pi_{2lp+1+k}(S^{2lp+1}:p),\ k=2((l+s)p+s-1)(p-1)-3.$$

Since $H^{(2)}u_4(l,\beta_s)=x(l+s)\cdot I'\beta_s(2lp^2-1)=x(l+s)\cdot Q^{lp}(\beta_s)\neq 0$ in the notation of (6.3), $u_4(l,\beta_s)\neq 0$. Consider the exact sequences

$$\pi_{2m+2+k}(S^{2m+1}:p)\xrightarrow{H^{(2)}}\pi_{2m-1+k}(Q_2^{2m-1}:p)\xrightarrow{p_*}\pi_{2m-1+k}(S^{2m-1}:p)\xrightarrow{S^2}\pi_{2m+1+k}(S^{2m+1}:p)$$

for $m=lp+1, lp+2, \cdots, lp+p$. The groups $\pi_{2m-1+k}(Q_2^{2m-1}: p)$ are generated by $Q^m(\alpha'_{(l+s)p+s-m-1})$ and additionally by $Q^{lp+1}(\alpha_1\beta_1^2)$ if $p=3$, $s=2$. These elements are $H^{(2)}$-images by Theorems 5.2, 5.1. It follows that the above S^2 are monomorphisms. Thus we have

(14.1). *Assume* $l+s\not\equiv 0 \pmod p$, $l\geq 1$ *and* $2\leq s\leq p-1$, *then up to non-zero coefficients,*

$$H^{(2)}(u_4(l, \beta_s))=Q^{lp}(\beta_s),\ p\cdot u_4(l, \beta_s)=0,\ S^{2p+2}(u_4(l, \beta_s))=0$$
$$\text{and } p_*\overline{Q}^{lp+p+1}(\beta_{s-1})=S^{2p}(u_4(l, \beta_s))\neq 0,$$

hence $u_4(l, \beta_s)$ *is an unstable element of the fourth type.*

The cokernel of the above S^2 is a subgroup of $\pi_{2m-2+k}(Q_2^{2m-1}: p)$ which vanishes for $m=lp+1, \cdots, lp+p-1$ and generated by $\overline{Q}^{lp+p}(\alpha_1\beta_{s-1})$ for $m=lp+p$. By Theorem 12.5 or more precisely by Theorem 10.6, $p^*\overline{Q}^{lp+p}(\alpha_1\beta_{s-1})=S^{2p-2}(\overline{u}_3(l, \beta_1\beta_{s-1}))\neq 0$. Thus the above S^2 are isomorphisms. It follows

(14.2). *Under the assumption of* (14.1), $S^{2j}(u_4(l, \beta_s))$ *generates a direct summand*

$$U_4(m, k) \text{ of } \pi_{2m+1+k}(S^{2m+1}: p),\ k=2((l+s)p+s-1)(p-1)-3,$$

isomorphic to Z_p *for* $0\leq j\leq p$ *and* $m=lp+j$.

The above discussion for S^2 valids for the case $l+s\equiv 0 \pmod p$. Then it follows from Theorem 14.1

(14.3). *Assume* $l+s\equiv 0 \pmod p$, $l\geq 1$ *and* $2\leq s\leq p-1$, *Put* $k=2((l+s)p+s-1)(p-1)-3$.

(i). $S^{2j}: \pi_{2lp+1+k}(S^{2lp+1}: p)\to \pi_{2lp+2j+1+k}(S^{2lp+2j+1}: p)$ *is an isomorphism for* $j=1, 2, \cdots, p$.

(ii). $p_*(\overline{Q}^{lp+p+1}(\beta_{s-1}))\in S^{2p+2}(\pi_{2lp-1+k}(S^{2lp-1}: p))$.

(iii). *If there exists an element* γ *of* $\pi_{2lp+1+k}(S^{2lp+1}: p)$ *such that* $H^{(2)}\gamma=Q^{lp}(\beta_s)$, *then* $S^{2p+2}(\gamma)\neq 0$.

We define $U_4(m, k)=0$ if it is not the case of (14.2). Then Theorem 12.5 (corrected) is generalized as follows.

Theorem 14.2. *Assume that* $k\geq 2p^2(p-1)$, $k\not\equiv -1, -2, -3$

(mod $2p^2(p-1)$) *and* $k-2r(p-1)\not\equiv -2,-3,-4$ (mod $2p^2(p-1)$) *for* $1\leq r<p-1$. *Then the group* $\pi_{2m+1+k}(S^{2m+1}: p)$ *is isomorphic to*

$$(\pi_k^S: p)+\sum_{i=1}^{4}U_i(m, k) \quad \textit{for } m>\frac{k+4}{2p-2}-p^2+p.$$

The proof of the theorem is similar to that of Theorem 12.5.

Now, consider Theorem 14.1. The proof is quite easy if l is sufficiently large and is done by use of Proposition 3.6 and Theorem 13.5. In order to prove Theorem 14.1 for smaller value of l, we prepare the following lemmas.

Lemma 14.3. *Let Q be a 3-connected space, $a\in H^{2r-1}(Q; Z_p)$, and let K be a finite CW-complex having a structure as in Theorem* 1.1. *Assume that the natural map $\Lambda(a)\otimes Z_p[\Delta a]\to H^*(Q; Z_p)$ is isomorphic for deg $<N$ and monomorphic for deg $\leq N$ and that a map $g: Y_p^{2r}\to Q$ induces an epimorphism of $H^*(\ ; Z_p)$. Then $g_*: \pi(SK; Y_p^{2r})\to\pi(SK; Q)$ is onto if dim $K\leq N-2$, and g_* maps the image of $S: \pi(K; Y_p^{2r-1})\to\pi(SK; Y_p^{2r})$ one-to-one into $\pi(SK; Q)$ if dim $K\leq N-3$. In particular, this assertion holds for $Q=\Omega^{2k}Q_2^{2m-1}$, $2m>k+3$, $r=mp-k-1$ and for $N=2pr-2=2p(mp-k-1)-2$.*

Proof. By mapping cyinlder arguments, we may assume that g is the inclusion. $F=\Omega(Q, S^{2r-1})$ is a fibre of a fibering $\Omega(Q; S^{2r-1}, Q)\to Q$, where S^{2r-1} is a deformation retract of $\Omega(Q; S^{2r-1}, Q)$. Consider the spectral sequence $\{E_r^*\}$ associated with the fibering; $E_2^*=H^*(Q; Z_p)\otimes H^*(F; Z_p)$ and $E_\infty^*=H^*(S^{2r-1}; Z_p)=\Lambda(a)$. Then it is verified that $H^*(F; Z_p)=\Lambda(a')$ for deg$\leq N-2$, $d_{2r}(1\otimes a')=\Delta a\otimes 1$. Let Z be the mapping cylindre of a map $S_0^{2r-1}\to S^{2r-1}$ of degree p, $\pi_0: Z\to Y_p^{2r}=Z/S_0^{2r-1}$ the shrinking map, and put $g_0=g\circ\pi_0$: $(Z; S^{2r-1}, S_0^{2r-1})\to(Y_p^{2r}; S^{2r-1},*)\to(Q; S^{2r-1}, *)$. Consider the following commutative diagram of fiberings:

$$\begin{array}{ccccc}
\Omega S^{2r-1} & \to & \Omega(Z, S_0^{2r-1}) & \to & \Omega(Z; S^{2r-1}, S_0^{2r-1}) \\
\| & & \downarrow{\scriptstyle \Omega g_0} & & \downarrow{\scriptstyle \Omega g_0} \\
\Omega S^{2r-1} & \to & \Omega Q & \longrightarrow & \Omega(Q, S^{2r-1})=F.
\end{array}$$

The natural map $S_0^{2r-1}\times I\to Z$ defines an inclusion $S_0^{2r-1}\to\Omega(Z;S^{2r-1}, S_0^{2r-1})$ which is a homotopy equivalence, since S^{2r-1} is a deformation retract of Z. It is easily seen that $(\Omega g_0|S_0^{2r-1})^*a\neq 0$. Then it follows from (1.8) that $(\Omega g_0)_*:\pi_i(\Omega(Z;S^{2r-1},S_0^{2r-1}))\to\pi_i(\Omega(Q,S^{2r-1}))$ is a $\mathcal{C}_p$-isomorphism for $i<N-2$ and a $\mathcal{C}_p$-epimorphism for $i\leq N-2$. The same is true for $(\Omega g_0)_*:\pi_i(\Omega(Z,S_0^{2r-1}))\to\pi_i(\Omega Q)$ by the five lemma for the homotopy exact sequences associated with the above diagram. Then, by Theorem 1.2, we have that $(\Omega g_0)_*:\pi(K;\Omega(Z, S_0^{2r-1}))\to\pi(K;\Omega Q)$ is one-to-one if dim $K\leq N-3$ and onto if dim $K\leq N-2$. Since $(\Omega g_0)_*=(\Omega g)_*\circ(\Omega\pi_0)_*:\pi(K;\Omega(Z,S_0^{2r-1}))\to\pi(K;\Omega Y_p^{2r})\to\pi(K;\Omega Q)$ and since $(\Omega g)_*$ is equivalent to g_*, we have that g_* is onto if dim $K\leq N-2$. Next it is easily seen that the canonical inclusion $i_{0*}: Y_p^{2r-1}\to Y_p^{2r}$ is homotopic to the composition of maps: $Y_p^{2r-1}\to\Omega(Z,S_0^{2r-1})\to\Omega Y_p^{2r}$. Then, if dim $K\leq N-3$ $(\Omega g)_*$ maps $i_{0*}(\pi(K;Y_p^{2r-1}))$ one-to-one into $\pi(K;\Omega Q)$, so by (1.2), g_* maps $S(\pi(K;Y_p^{2r-1}))$ one-to-one into $\pi(SK;Q)$.

The space $Q_2^{2m-1}=\Omega(\Omega^2S^{2m+1},S^{2m-1})$ is $(4m-4)$-connected, so $\Omega^{2k}Q_2^{2m-1}$ is 3-connected since $2m>k+3$. Then $\Omega^{2k}Q_2^{2m-1}=Q$ satisfies the assumption for $r=mp-k-1$ and $N=2pr-2$ by Corollary 2.4 and Lemma 2.5.

Lemma 14.4. *Let K be a finite CW-complex and $r\geq 2$. Then $S^\infty:\pi(SK;Y_p^{2r})\to\pi^S(K;Y_p^{2r-1})$ is an epimorphism if dim $K\leq 2pr-4$, and it maps $S(\pi(K;Y_p^{2r-1}))$ monomorphically if dim $K\leq 2pr-5$.*

Proof. Let n be sufficiently large. $S^n:\pi(SK;Y_p^{2r})\to\pi(S^{n+1}K;Y_p^{n+2r})$ is equivalent to the homomorphism $g_*:\pi(SK;Y_p^{2r})\to\pi(SK;\Omega^nY_p^{n+2r})$ induced by the canonical inclusion g: $Y_p^{2r}\to\Omega^nY_p^{n+2r}$. Put $Q=\Omega^nY_p^{n+2r}$, and consider the map $\Omega g_0:\Omega(Z,S_0^{2r-1})\to\Omega Q$ of the previous proof. Then it is sufficient to prove that $(\Omega g_0)_*:\pi_i(\Omega(Z, S_0^{2r-1}))\to\pi_i(\Omega Q)$ is an isomorphism for $i\leq 2pr-5$ and an epimorphism for $i\leq 2pr-4$. The homomorphism $(\Omega g_0)_*$ is equivalent to the composition $S^n\circ\pi_{0*}$ in the following commutative diagram:

$$\begin{array}{ccc} \pi_{i+1}(Z, S_0^{2r-1}) & \xrightarrow{\pi_{0*}} & \pi_{i+1}(Y_p^{2r}) \\ \downarrow S^n & & \downarrow S^n \\ \pi_{n+i+1}(S^n Z, S_0^{n+2r-1}) & \xrightarrow{S^n\pi_{0*}} & \pi_{n+i+1}(Y_p^{n+2r}). \end{array}$$

$S^n\pi_{0*}$ is an isomorphism since n is large. The S^n of the left-side is an isomorphism for $i \leq 2pr-5$ and an epimorphism for $i \leq 2pr-4$ by (2.8) and the five lemma as in the proof of Theorem 2.2. q. e. d.

As an application, we generalize Proposition 3.6 to the metastable case.

Proposition 14.5. *Assume $1 \leq k < mp-1$ and put $t = mp-k-1$. Then exists a complex $K(m, k)$ satisfying the following conditions.*

(i). *$K(m, k) = S^{2t}K'$ for some complex K', so we may write*

$$S^{-2j}K(m, k) = S^{2t-2j}K' \text{ for } j \leq t.$$

(ii). *$K(m, 1) = Y_p^{2mp-2}$ and $K(m, k)$ is a mapping cone*

$$K(m, k) = Y_p^{2mp-2} \cup_h C(S^{-3}K(m+1, k-1))$$

of a map $h = S^{2t}h'$, h': $S^{-2t-3}K(m+1, k-1) \to Y_p^{2mp-2t-2}$, where $K(m+1, k-1)$ has been given inductively.

(iii). *There exists a map G_k: $K(m, k) \to Q_2^{2m-1}$ such that G_k^*: $H^*(Q_2^{2m-1}; Z_p) \to H^*(K(m, k); Z_p)$ is an epimorphism and the following diagram is homotopy commutative*:

$$\begin{array}{ccccccc} S^{-3}K(m+1, k-1) & \xrightarrow{h} & Y_p^{2mp-2} & \xrightarrow{i} & K(m, k) & \xrightarrow{\pi} & S^{-2}K(m+1, k-1) \\ \downarrow \Omega^3 G_{k-1} & & \downarrow G_1 & & \downarrow G_k & & \downarrow \Omega^2 G_{k-1} \\ \Omega^3 Q_{2k-2}^{2m+1} & \xrightarrow{d} & Q_2^{2m-1} & \xrightarrow{i} & Q_{2k}^{2m-1} & \xrightarrow{j} & \Omega^2 Q_{2k-2}^{2m+1}. \end{array}$$

Proof. The case $k=1$ is obvious (Lemma 2.5). Assume that $K(m+1, k-1) = S^{2(t+p+1)}K''$ and G_{k-1}: $K(m+1, k-1) \to Q_{2k-2}^{2m+1}$ have been given. Choose a map $G_1 = g: Y_p^{2mp-2} \to Q_2^{2m-1}$ of Lemma 2.5 and consider the induced map $\Omega^{2t}G_1$: $Y_p^{2mp-2t-2} \to \Omega^{2t}Q_2^{2m-1}$. Since dim $S^{-2t-3}K(m+1, k-1) = 2(m+1)p-2+2(k-2)(p-1)-2t-3 = 2kp-2p+1 < 2kp-4 = 2p(mp-t-1)-4$, we have by Lemma 14.3 that there exists a map h': $S^{-2t-3}K(m+1, k-1) \to Y_p^{2mp-2t-2}$ such that

$\Omega^{2t}(d\circ\Omega^3 G_{k-1})$ is homotopic to $\Omega^{2t}G_1\circ h'$. Then the commutativity of the left-side square of (iii) follows. $K(m, k)$ is defined by (ii), and (i) is obvious. The map $\Omega^3 G_{k-1}$ and the above homotopy define a map of $C(S^{-3}K(m+1, k-1))$ into Q_{2k}^{2m-1} which extends $i\circ G_1\circ h$. Then G_k is defined by this map, and (iii) is proved as in the proof of Theorem 3.1.

Lemma 14.6. *Let K and L be finite CW-complexes, f: $L\to Y_p^{2r}$ a map and $C_f=Y_p^{2r}\cup CL$ the mapping cone of f. If S^∞: $\pi(K; L)\to\pi^S(K; L)$ is onto and if* $\dim K\leq 2p(r+1)-5$, then S^∞: *$\pi(S^3K; S^2C_f)\to\pi^S(SK; C_f)$ is onto.*

In particular, if $n\leq 2(mp-j-1)p-5$ and $j<mp-k-1$ then

$$S^\infty:\pi(Y_p^n; S^{-2j}K(m, k))\to\pi^S(Y_p^{n+2j}; K(m,k))$$

is onto for the complex $K(m, k)$ of Proposition 14.5 $(1\leq k<mp-1)$.

Proof. Consider the following exact sequence:

$$\pi^S(SK; Y_p^{2r})\xrightarrow{i_*}\pi^S(SK; C_f)\xrightarrow{\pi_*}\pi^S(SK; SL)\xrightarrow{Sf_*}\pi^S(SK; Y_p^{2r+1}).$$

For arbitrary element α of $\pi^S(SK; C_f)$, there exists by assumption an element β of $\pi(K; L)$ such that $S^\infty\beta=\pi_*\alpha$. Since $S^\infty(f_*\beta)=Sf_*\pi_*\alpha=0$, it follows, by Lemma 14.4, that $S^2f_*(S^2\beta)=S^2(f_*\beta)=0$. As a coextension of $S^2\beta$, we have an element $\widetilde{\beta}$ of $\pi(S^3K; S^2C_f)$ such that $(S^2\pi)_*\widetilde{\beta}=S^3\beta$. Then $\pi_*(\alpha-S^\infty\widetilde{\beta})=\pi_*\alpha-S^\infty\beta=0$. By the exactness of the above sequence there exists an element γ of $\pi^S(SK; Y_p^{2r})$ such that $i_*\gamma=\alpha-S^\infty\widetilde{\beta}$. By Lemma 14.4, there exists an element δ of $\pi(S^3K; Y_p^{2r+2})$ such that $S^\infty\delta=\gamma$. Then we have $S^\infty(\widetilde{\beta}+(S^2i)_*\delta)=S^\infty\widetilde{\beta}+i_*\gamma=\alpha$, and the first half of the lemma is proved.

By (i) of Proposition 14.5, $S^{-2(j+3i+1)}K(m+i, k-i)$ is defined for $0\leq i\leq k-1$, and it is a mapping cone of a map $S(S^{-2(j+3i+3)}K(m+i+1, k-i-1))\to Y_p^{2mp-4+2i(p-3)-2j}$. Then by descending induction on i the first half implies that S^∞: $\pi(Y_p^{n-4i}; S^{-2(j+3i)}K(m+i, k-i))\to\pi^S(Y_p^{n+2i+2j}; K(m+i, k-i))$ is onto for $0\leq i\leq k-1$. This proves the second half of the lemma.

Let $Y \cup_f CX$ be the mapping cone of a map $f: X \to Y$, and let $Z \cup_g C(Y \cup_f CX)$ be the mapping cone of a map $g: Y \cup_f CX \to Z$. The reduced join $I \wedge I$, with the base point $(0) \in I$, can be identified with $S^1 \wedge I = CS^1$ such that $I \wedge (1)$ and $(1) \wedge I$ correspond to the cones of upper and lower hemi-circle respectively. Then $C(CX)$ is identified with $C(SX)$, and we have

(14.4). $Z \cup_g C(Y \cup_f CX) = (Z \cup_{g'} CY) \cup_{\tilde{f}} CSX$ *for* $g' = g | Y$ *and a coextension* $\tilde{f}: SX \to Z \cup_{g'} CY$ *of* f.

For example, we have

$$(14.5).\quad K(m, k+1) = Y_p^{2mp-2} \cup_{h} C(S^{-3}K(m+1, k-1)) \cup_{h_k} CY_p^{2mp-3+2k(p-1)}$$
$$= K(m, k) \cup_{h_k} CY_p^{2mp-2+2k(p-1)}.$$

It is directly verified

(14.6). $(-\tilde{f}) \circ \pi: Y \cup_f CX \to SX \to Z \cup_{g'} CY$ *is homotopic to* $i \circ \pi: Y \cup_f CX \to Z \to Z \cup_{g'} CY$.

Proof of Theorem 14.1. Consider the complex $K(lp, p+2)$ of Proposition 14.5. By (14.5) and (14.6) we have the following exact and commutative diagram.

$$\begin{array}{ccccc}
\pi^S(Y_p^n; S^{-3}K(lp+1, p+1)) & \xrightarrow{\pi_*} & \pi^S(Y_p^n; Y_p^r) & \xrightarrow{Sh_{p*}} & \pi^S(Y_p^n: S^{-2}K(lp+1, p)) \\
\downarrow -h_* & & \downarrow h_{p+1*} & & \downarrow id \\
\pi^S(Y_p^n; Y_p^{2lp^2-2}) & \xrightarrow{i_{1*}} & \pi^S(Y_p^n; K(lp, p+1)) & \xrightarrow{\pi_*} & \pi^S(Y_p^n; S^{-2}K(lp+1, p)),
\end{array}$$

$r = 2lp^2 - 3 + 2(p+1)(p-1)$. Then it follows from the relation in Theorem 13.5 that there exists an element ξ of $\pi^S(Y_p^n; S^{-3}K(lp+1, p+1))$ for $n = r + 2((s-1)p + s - 2)(p-1) - 1 = 2lp^2 - 4 + 2(sp+s-1)(p-1)$ such that $\pi_*\xi = \beta_{(s-1)}$ and $i_{1*}h_*\xi = -\frac{1}{s}(l+s)i_{1*}(\delta\beta_{(s)})$. Thus the relation $h_*\xi = -\frac{1}{s}(l+s)\delta\beta_{(s)} + x \cdot \alpha^{sp+s-2}\delta\alpha\delta$ follows from

(14.7). *The kernel of* i_{1*} *is generated by* $\alpha^{sp+s-2}\delta\alpha\delta$.

For, as in the proof of Lemma 13.1, the kernel is generated by $h_{1*}(\alpha^{sp+s-2}\delta) = (s-1)\alpha^{sp+s-2}\delta\alpha\delta$ and additionally when $p=3$ by

$h_{1*}(\alpha(\delta\beta_{(1)})^2\delta)=h_{1*}\pi_{1*}h_{2*}(-(\beta_{(1)}\delta)^2)=0$, $h_{1*}(\delta\alpha(\delta\beta_{(1)})^2)=0$ and possibly by $h_{2*}(\delta(\beta_{(1)}\delta)^2)$. The last element is independent since its π_{1*}-image is $2\cdot\delta\alpha\delta(\beta_{(1)}\delta)^2\neq 0$. So, (14.7) is obtained.

Next we pull back $\beta_{(s)}$ to unstable range. This is done as in the construction of $\alpha(4)$ in (4.7). Consider an extension $\bar{\beta}\in\pi(Y_p^{2p+4+u}; S^{2p+3})$, $u=2(sp+s-1)(p-1)-2$, of the element $\beta_s(2p+3)$ of Lemma 11.2, and then consider a coextension $\beta'\in\pi(Y_p^{2p+6+u}; Y_p^{2p+5})$ of $S\bar{\beta}$. Then $i^*\pi_*\beta'=\beta_s(2p+5)$ *and* $i^*\pi_*(S^\infty\beta'-\beta_{(s)})=0$. Since $\pi_{u+1}^S(Y_p; Y_p)$ is generated by $\beta_{(s)}$, $\alpha^{sp+s-1}\delta$, $\alpha^{sp+s-2}\delta\alpha$ and since $i^*\pi_*(\alpha^{sp+s-1}\delta)=i^*\pi_*(\alpha^{sp+s-2}\delta\alpha)=0$, we have $\beta_{(s)}=S^\infty(\beta'+x\cdot\alpha^{sp+s-1}\delta(2p+5)+y\cdot\alpha^{sp+s-2}\delta\alpha(2p+5))$. Thus we have

(14.8). *For* $n\geq 2p+5$ $(n\geq 2p+3$ *if* $s=1)$, *there exists a series of elements* $\beta_{(s)}(n)\in\pi(Y_p^{n+2(sp+s-1)(p-1)-1}; Y_p^n)$ *such that* $\beta_{(s)}(n+1)=S(\beta_{(s)}(n))$, $i^*\pi_*\beta_{(s)}(n)=\beta_s(n)$ *and* $S^\infty\beta_{(s)}(n)=\beta_{(s)}$. $S^\infty(\delta\beta_{(s)}(n+1))=\delta\beta_{(s)}$ *for* $\delta\beta_{(s)}(n+1)=\delta(n+1)\circ\beta_{(s)}(n)$.

By Lemma 14.6, there exists an element $\xi'\in\pi(Y_p^{n-1}; S^{-4}K(lp+1, p+1))$ such that $S^\infty\xi'=\xi$. By Lemma 14.4, $S^\infty:\pi(Y_p^n; Y_p^{2lp^2-2})\rightarrow\pi_{n-2lp^2+2}^S(Y_p; Y_p)$ maps $\operatorname{Im} S$ monomorphically, since $n-1=2lp^2-3+2(sp+s-1)(p-1)\leq 2lp^2-3+2(lp^2-2)(p-1)+2p-6=2(lp^2-1)p-5$. This shows that the above relation on $h_*\xi$ implies $h_*(S\xi')=-1/s(l+s)\delta\beta_{(s)}(2lp^2-2)+x\cdot\alpha^{sp+s-2}\delta\alpha\delta(2lp^2-2)$. Similarly we have $\pi_*(S\xi')=\beta_{(s-1)}(r)$. Since $i^*\delta(n)=0$, we have, using (14.6),

$$\begin{aligned}h_{p+1*}(i^*\beta_{(s-1)}(r))&=\pi_*h_{p+1*}i^*(S\xi')=-i_{1*}i^*h_*(S\xi')\\&=\frac{1}{s}(l+s)i_{1*}i^*\delta\beta_{(s)}(2lp^2-2)=\frac{1}{s}(l+s)i_{1*}i_*(i^*\pi_*\beta_{(s)}(2lp^2-3))\\&=\frac{1}{s}(l+s)i_{0*}\beta_s(2lp^2-3)\end{aligned}$$

for the inclusion i_0: $S^{2lp^2-3}\subset K(lp, p+1)$. Now consider the following commutative diagram:

$$\begin{array}{ccccccc}\pi_n(S^{2lp-1}) & \xrightarrow{S^2} & \pi_{n+2}(S^{2lp+1}) & \xrightarrow{H^{(2)}} & \pi_{n-1}(Q_2^{2lp-1}) & \xrightarrow{p_*} & \pi_{n-1}(S^{2lp-1})\\ \| & & \downarrow S^{2p} & & \downarrow i_* & & \|\\ \pi_n(S^{2lp-1}) & \xrightarrow{S^{2p+2}} & \pi_{n+2p+2}(S^{2lp+2p+1}) & \xrightarrow{H^{(2p+2)}} & \pi_{n-1}(Q_{2p+2}^{2lp-1}) & \xrightarrow{p_*} & \pi_{n-1}(S^{2lp-1})\\ & & \uparrow p_* & & \uparrow d_* & & \\ & & \pi_{n+2p+2}(Q_2^{2lp+2p-1}) & \xrightarrow{\Omega^{2p+3}} & \pi_{n-1}(\Omega^{2p+3}Q_2^{2lp+2p+1}). & & \end{array}$$

Put $\Omega^{2p+3}\gamma'=g'_*(i^*\beta_{(s-1)}(r))$, then $I\gamma'=S^{2p+6}(i^*\pi_*\beta_{(s-1)}(r))=\beta_{s-1}(2(lp+p+1)p+1)$ and $H^{(2p+2)}p_*\gamma'=d_*\Omega^{2p+3}\gamma'=d_*g'_*(i^*\beta_{(s-1)}(r))=G_{p+1*}h_{p+1*}(i^*\beta_{(s-1)}(r))=(1/s)(l+s)G_{p+1*}i_{0*}\beta_s(2lp^2-3)=(1/s)(l+s)i_*G_{1*}i'_*\beta_s(2lp^2-3)=x(l+s)i_*I'\beta_s(2lp^2-1)$, $x\not\equiv 0 \pmod p$, by Propositions 3.6, 14.5 and Lemma 2.5. Since $p_*(x(l+s)I'\beta_s(2lp^2-1))=p_*H^{(2p+2)}p_*\gamma'=0$, $H^{(2)}\gamma_0=x(l+s)I'\beta_s(2lp^2-1)$ for some γ_0. Also, since $H^{(2p+2)}(p_*\gamma'-S^{2p}\gamma_0)=0$, $p_*\gamma'-S^{2p}\gamma_0=S^{2p+2}\gamma_1$ for some γ_1. Put $\gamma=\gamma_0+S^2\gamma_1$, then $H^{(2)}\gamma=x(l+s)I'\beta_s(2lp-1)$ and $S^{2p}\gamma=p_*\gamma'$.

15. The groups $\pi_{2m+1+k}(S^{2m+1}: p)$ for $2p^2(p-1)-3\leq k<2(p^2+p)(p-1)-5$.

The groups $\pi_{2m+1+k}(S^{2m+1}: p)$ are determined for $k<2p^2(p-1)-3$ by Theorem 11.1. We shall determine the groups for $2p^2(p-1)-2<k<2(p^2+p)(p-1)-5$, and partially for $k=2p^2(p-1)-3$, $2p^2(p-1)-2$, by dividing into two cases:

Case (I): $\alpha_1\beta_1^p=0$,

Case (II): $\alpha_1\beta_1^p\neq 0$.

We shall use the notation $Q^m(\gamma), \bar{Q}^m(\gamma)\in\pi_i(Q_2^{2m-1}: p)$ of (6.3).

Theorem 15.1. *Let* $h=2p^2(p-1)-1$.

(i).

$$\pi_{2m+1+h-2}(S^{2m+1}: p)\approx\begin{cases} Z_p+Z_p & \text{for } m=(s-1)p+s,\ s=1,2,\cdots,p-1 \\ & \text{and for } p^2-2\geq m\geq p(p-1) \text{ of case (II)} \\ Z_p & \text{for } (p-2)p+p-1>m\geq 2,\ m-1\not\equiv 0 \\ & (\text{mod } p+1), \\ & \text{for } m=p^2-2 \text{ of case (I)} \\ & \text{and for } m\geq p^2-1 \text{ of case (II)} \\ Z_{p^2} & \text{for } p^2-3\geq m\geq (p-1)p \text{ of case (I)} \\ 0 & \text{for } m\geq p^2-1 \text{ of case (I)}. \end{cases}$$

(ii).

$$\pi_{2m+1+h-1}(S^{2m+1}: p)\approx\begin{cases} Z_p+Z_p & \text{for } m=1 \\ Z_{p^2}+Z_p & \text{for } m=2 \\ Z_{p^3}+Z_p & \text{for } p\geq m\geq 3 \\ Z_{p^3} & \text{for } p^2-3\geq m\geq p+1 \end{cases}$$

$$\begin{cases} & \textit{and for } m=p^2-2 \textit{ of case (II)} \\ Z_{p^2} & \textit{for } m=p^2-2 \textit{ of case (I)} \\ & \textit{and for } m=p^2-1 \textit{ of case (II)} \\ Z_p & \textit{for } m=p^2-1 \textit{ of case (I)} \\ & \textit{and for } m\geq p^2 \textit{ of case (II)} \\ 0 & \textit{for } m\geq p^2 \textit{ of case (I).} \end{cases}$$

(iii).

$$\pi_{2m+1+h}(S^{2m+1}:p)\approx\begin{cases} Z_p & \textit{for } m=1 \\ Z_{p^2} & \textit{for } m=2 \\ Z_{p^3} & \textit{for } m\geq 3 \end{cases}$$

and S^2 is injective for these groups.

Proof. We prove the case (I), the case (II) is rather easier and omitted. In the case (I) $\pi_{2m+1+h-2}(S^{2m+1}:p)=\pi_{2m+1+h-1}(S^{2m+1}:p)=0$ for stable $m\geq p^2$, by [4] [6]. By (2.5) and Theorem 9.3 (cf. (6.1)), we have the following list of the generators of $\pi_{2m+1+h-i}(Q_2^{2m-1}:p)$, $i=1,2,3$:

$i=1$; $Q^{p^2}(\iota)$, $\bar{Q}^t(\alpha_{p^2-t})$, $1\leq t\leq p^2-1$, $\bar{Q}^p(\alpha_1\beta_1^{p-1})$, $Q^{p+1}(\beta_1^{p-1})$,

$i=2$; $Q^t(\alpha'_{p^2-t})$, $1\leq t\leq p^2-1$, $\bar{Q}^{(s-1)p+s+1}(\beta_{p-s})$, $1\leq s\leq p-1$, $\bar{Q}^1(\beta_1^p)$, $Q^p(\alpha_1\beta_1^{p-1})$,

$i=3$; $\bar{Q}^{(s-1)p+s}(\alpha_1\beta_{p-s})$, $1\leq s\leq p-1$, $Q^1(\beta_1^p)$, $Q^{p^2-p}(\beta_1)$.

By (i) and (ii) of Theorem 5.1, $H^{(2)}p_*\bar{Q}^{(s-1)p+s+1}(\beta_{p-s})=\bar{Q}^{(s-1)p+s}(\alpha_1\beta_{p-s})$ and $H^{(2)}p_*Q^{p+1}(\beta_1^{p-1})=Q^p(\alpha_1\beta_1^{p-1})$, up to non-zero coefficients. So, we can neglect these elements together with the corresponding summands (of the first type) of $\pi_{2m+1+h-i}(S^{2m+1}:p)$ generated by $p_*\bar{Q}^{(s-1)p+s+1}(\beta_{p-s})$ and $p_*Q^{p+1}(\beta_1^{p-1})$. Then, by the exactness of (1.7), we have that the group $\pi_{2m+1+h-2}(S^{2m+1}:p)$ has at most p^2 elements and that $p_*Q^t(\alpha'_{p^2-t})=0$ except just two values of t. If $p_*Q^t(\alpha'_{p^2-t})=0$ for some $t>1$, $Q^t(\alpha'_{p^2-t})=H^{(2)}\gamma_t$ for some $\gamma_t\in\pi_{2t+1+h-1}(S^{2t+1}:p)$. Then by (ii) of Theorem 5.3, there exists an element γ_{t-1} such that $S^2\gamma_{t-1}=p\cdot\gamma_t$ and $H^{(2)}\gamma_{t-1}=x\cdot Q^{t-1}(\alpha'_{p^2-t+1})$, $x\not\equiv 0 \pmod p$, hence $p_*Q^{t-1}(\alpha'_{p^2-t+1})=0$. Thus we have in the case (I)

(15.1). $Q^{p^2-1}(\alpha'_1)\neq 0$, $Q^{p^2-2}(\alpha'_2)\neq 0$, *and there exists a series of* *elements* $\gamma_m \in \pi_{2m+1+h-1}(S^{2m+1}: p)$ *for* $1 \leq m \leq p^2-3$ *such that, up to non-zero coefficients,*

$S^2\gamma_m = p\cdot\gamma_{m+1}$ *for* $1 \leq m \leq p^2-3$ *and* $H^{(2)}\gamma_m = Q^m(\alpha'_{p^2-m})$.

Then the assertion (i) for the case (I) follows from the exactness of (1.7), where the cyclicity of the groups is provided by (ii) of Theorem 10.4. Remark that

(15. 2) *in the case* (I) $\alpha_1\beta_1^p(2m+1) \neq 0$ *for* $1 \leq m \leq p^2-2$ *and it is divisible by* p *for* $p^2-p \leq m \leq p^2-2$.

We have seen in Theorems 7.5, 7.6 the existence of an element $\bar{u}_3(0, \beta_1^2) \in \pi_{3+k}(S^3: p)$ for $k=2(2p+1)(p-1)-4$ satisfying

(15. 3) $H^{(2)}\bar{u}_3(0, \beta_1^2) = \bar{Q}^1(\beta_1^2)$, $S^{2p-4}\bar{u}_3(0, \beta_1^2) = p_*\bar{Q}^p(\alpha_1\beta_1)$ *and* $S^{2p-2}\bar{u}_3(0, \beta_1^2) = 0$ *up to non-zero coefficients.*

Put $\bar{u}_3(0, \beta_1^p) = \bar{u}_3(0, \beta_1^2)\circ\beta_1^{p-2}(2(2p+1)(p-1)-1)$, then we have

$$H^{(2)}\bar{u}_3(0, \beta_1^p) = \bar{Q}^1(\beta_1^p),\ S^{2p-4}\bar{u}_3(0, \beta_1^p) = p_*\bar{Q}^p(\alpha_1\beta_1^{p-1})$$

and $S^{2p-2}\bar{u}_3(0, \beta_1^p) = 0$.

By (15.1) and the exactness of (1.7), we have that the groups $\pi_{2m+1+h-1}(S^{2m+1}: p)$ are generated by γ_m, $p_*Q^{p+1}(\beta_1^{p-1})$, $\bar{u}_3(0, \beta_1^p)$ and their suspensions, and

(15. 4). *Let* $U_2(m, 2p^2(p-1)-2)$ *be generated by* γ_m *for* $1 \leq m \leq p^2-3$ *and by* $S^{2i}\gamma_{p^2-3}$ *for* $m=p^3-3+i$ *and* $i \geq 1$. *Then* $U_2(m, h-1)$, $h=2p^2(p-1)-1$, *is a direct summand of* $\pi_{2m+1+h-1}(S^{2m+1}: p)$ *and* $\pi_{2m+1+h-1}(S^{2m+1}: p)/U_2(m, h-1) = 0$ *for* $m \geq p$, $=Z_p$ *for* $m=p$, *and* $=Z_p$ *or* 0 *generated by* $S^{2m-2}\bar{u}_3(0, \beta_1^p)$ *for* $1 \leq m \leq p-1$.

Consider $p_*\bar{Q}^t(\alpha_{p^2-t})$ for $1 \leq t \leq p^2$, where $\bar{Q}^{p^2}(\alpha_0)$ stands for $Q^{p^2}(\iota)$. Since $S^\infty\alpha_{p^2}(3) = \alpha_{p^2} \neq 0$ we have $H^{(2)}\alpha_{p^2}(3) = \bar{Q}^1(\alpha_{p^2-1})$ up to non-zero coefficient. If $p_*\bar{Q}^s(\alpha_{p^2-s}) = 0$, then $\bar{Q}^s(\alpha_{p^2-s})$ is an $H^{(2)}$-image, and Theorem 5.4, (ii) implies that $\bar{Q}^r(\alpha_{p^2-r})$ are $H^{(2)}$-images for $r < s$ and α_{p^2} is divisible by p^{s-1}. It follows from (4.3) that $s \leq 3$. Thus

$$p_*Q^{p^2}(\iota) \neq 0 \text{ and } p_*\bar{Q}^t(\alpha_{p^2-t}) \neq 0 \text{ for } 4 \leq t \leq p^2-1.$$

Since $\pi_{2m+1+h-1}(S^{2m+1}: p) = 0$ for $m \geq p^2$, it follows from (16.4) that, up to non-zero coefficient, $S^4\gamma_{p^2-3} = p_*Q^{p^2}(\iota)$, $p\cdot S^2\gamma_{p^2-3} = p_*\bar{Q}^{p^2-1}(\alpha_1)$,

$p^2\cdot\gamma_{p^2-3}=p_*\overline{Q}^{p^2-2}(\alpha_2)$, and moreover $p^2\cdot\gamma_m=p_*\overline{Q}^{m+1}(\alpha_{p^2-m-1})$ for $p\leq m\leq p^2-3$. Next, by Theorem 5.3,(i), $p_*\overline{Q}^{m+1}(\alpha_{p^2-m-1})=p\cdot S^{-2}(p_*\overline{Q}^{m+2}(\alpha_{p^2-m-2}))=p^2\cdot\gamma_m$ for $1\leq m\leq p-1$ by descending induction on m. Since γ_1 and γ_2 are at most of degree p and p^2, it follows $p_*\overline{Q}^m(\alpha_{p^2-m})=0$ for $m=1,2,3$. Therefore, by the exactness of (1.7), we have obtained that the order of γ_m is p^3 for $3\leq m\leq p^2-3$, p^2 for $m=2$, p^2-2 and p for $m=1$, p^2-1, and that $S^{2p-4}\overline{u}_3(0,\beta_1^p)=p_*\overline{Q}^p(\alpha_1\beta_1^{p-1})\neq 0$. This together with (15.4) proves (ii) of the case (I).

We have also seen that the cokernel of $S^2\colon\pi_{2m-1+h}(S^{2m-1}\colon p)\to\pi_{2m+1+h}(S^{2m+1}\colon p)$ is trivial for $m\geq 4$ and isomorphic to Z_p for $m=1$, 2, 3. It is known [1] that the stable group π_h^S contains Z_{p^3} as the p-component of the J-image. Then (iii) of the theorem follows immediately.

(The case (II) can be proved similarly, but we simply remark that in the case (II) γ_{p^2-2} exists and $S^\infty\gamma_{p^2-2}$ generates $(\pi_{h-1}^S\colon p)\approx Z_p$.)

q. e. d.

Now, we describe further results. In the following, we always assume

$$2p^2(p-1)\leq k<2(p^2+p)(p-1)-5.$$

We shall define subgroups $A(m,k)$, $B(m,k)$, $E(m,k)$ $U_1(m,k)$ and $U_3(m,k)$ of $\pi_{2m+1+k}(S^{2m+1}\colon p)$. First we define

$$A(m,k)\approx\begin{cases}Z_p & \text{generated by } \alpha_i(2m+1) \text{ for } k=2i(p-1)-1,\ i=p^2+1,\cdots,p^2+p-1,\\ 0 & \text{otherwise.}\end{cases}$$

For $2\leq s\leq p-1$, we define

$$B(m,k)\approx\begin{cases}Z_p & \text{generated by } \beta_1^{p-s}\beta_s(2m+1) \text{ for } k=2(p^2+s-2)(p-1)+2s-4 \text{ and } m\geq p-1,\\ Z_p & \text{generated by } \alpha_1\beta_1^{p-s}\beta_s(2m+1) \text{ for } k=2(p^2+s-1)(p-1)+2s-5 \text{ and } m\geq 1,\\ Z_p & \text{generated by } \beta_1^{p+1}(2m+1) \text{ for } k=2(p^2+p-1)(p-1)-4 \text{ and } m\geq p-1,\\ 0 & \text{otherwise.}\end{cases}$$

Assuming the existsnce of elements ε_i', $1\leq i\leq p-2$, and ε_j, $1\leq j\leq p-1$,

$$E(m,k)\approx\begin{cases} Z_p & \text{generated by } \varepsilon'_i(2m+1) \text{ for} \\ & k=2(p^2+i)(p-1)-3 \text{ and } m\geq p(p-i-1), \\ Z_p & \text{generated by } \varepsilon_j(2m+1) \text{ for} \\ & k=2(p^2+i)(p-1)-2 \text{ and } m\geq p(p-j)+1, \\ 0 & \text{otherwise.} \end{cases}$$

Next we define $U_1(m,k)$ to be the subgroup generated by unstable elements of the first type which are obtained by Theorem 5.1 and Theorem 5.2 from the known results of $A(m,k')$ and $B(m,k')$, $k'<k$. More precisely, the generators of $U_1(m,k)$ are listed as follows:

$p_*\overline{Q}^{m+1}(\alpha'_{p^2+i-m-1})$ for $k=2(p^2+i)(p-1)-2$ and $1\leq m\leq p^2+i-2$,

$p_*Q^{m+1}(\iota)$ for $k=2(p^2+i)(p-1)-2$ and $m=p^2+i-1$,

$p_*Q^{m+1}(\beta_1^r\beta_s)$ for $k=2((r+s+m)p+s-1)(p-1)-2(r+s)-2$ and $m\not\equiv -1 \pmod p$,

$p_*\overline{Q}^{m+1}(\beta_1^r\beta_s)$ for $k=2((r+s+m)p+s-1)(p-1)-2(r+s)-1$ and $m\not\equiv 0 \pmod p$,

where $r\geq 0$, $1\leq s\leq p-1$. Remark that some of these generators are in the same subgroup which are independent and that $H^{(2)}p_*\overline{Q}^{m+1}(\beta_1^p)=\overline{Q}^m(\alpha_1\beta_1^p)$, $1\leq m\leq p-1$, and $H^{(2)}p_*Q^{m+1}(\beta_1^p)=Q^m(\alpha_1\beta_1^p)$, $1\leq m\leq p-2$ are not trivial by (15.2). Then $U_1(m,k)$ will be a Z_p-module having a bases consists of the above elements of the corresponding degrees m and k.

The subgroup $U_3(m,k)$ is defined by (11.7). More precisely, in our case the generators of $U_3(m,k)$ are given as follows:

$S^{2j}(u_3(l,\beta_1^{p-l-s}\beta_s))$ for $0\leq j\leq p-2$, $l\geq 1$ and $2\leq s<p-l$,

$S^{2j}(u_3(l,\beta_1^{p+1-l}))$ for $0\leq j\leq p-2$ and $1\leq l\leq p-2$;

$S^{2j}(\overline{u}_3(l,\beta_1^{p-l-s}\beta_s))$ for $0\leq j\leq p-2$, $l\geq 1$ and $1\leq s\leq p-l$.

$S^{2j}(\overline{u}_3(0,\beta_1^{p-s}\beta_s))$ for $0\leq j\leq p-2$ and $2\leq s\leq p-1$.

For the case $l\geq 1$, the existence of these elements are provided by Theorems 10.4, 10.7. For the case $2\leq s<p-1$, $\overline{u}_3(0,\beta_1^{p-s}\beta_s)$ is defined as the composition of the element $\overline{u}_3(0,\beta_1^2)$ of (15.3) and

$\beta_1^{p-s-2}\beta_s(2(2p+1)(p-1)-1)$. These elements satisfy the relations in (11.7). The existence of an element $\bar{u}_3(0, \beta_1\beta_{p-1})$ should be proved in the proof of the following theorem.

Theorem 15.2. (i). *The following elements exist:*

$\varepsilon_i'(2p(p-i-1)+1)$ *for* $1 \leq i \leq p-2$ *satisfying*
$$H^{(2)}\varepsilon_i'(2p(p-i-1)+1) = Q^{p(p-i-1)}(\beta_{i+1}),$$

$\varepsilon_j(2p(p-j)+3)$ *for* $1 \leq j \leq p-2$ *satisfying*
$$H^{(2)}\varepsilon_j(2p(p-j)+3) = \bar{Q}^{p(p-j)+1}(\beta_j),$$

$\varepsilon_{p-1}(2p+3)$ *for* $p>3$ *satisfying*
$$H^{(2)}\varepsilon_{p-1}(2p+3) = \bar{Q}^{p+1}(\beta_{p-1}),$$

$\bar{u}_3(0, \beta_1\beta_{p-1})$ *satisfying* $H^{(2)}\bar{u}_3(0, \beta_1\beta_{p-1}) = Q^1(\beta_1\beta_{p-1})$.

For the case $p=3$ *there exists either* $\varepsilon_2(9)$ *with* $H^{(2)}\varepsilon_2(9) = \bar{Q}^4(\beta_2)$ *or* $\varepsilon_2(11)$ *with* $H^{(2)}\varepsilon_2(11) = Q^5(\alpha_1\beta_1^2)$.

(ii). *For* $2p^2(p-1) \leq k < 2(p^2+p)(p-1)-5$, *the group* $\pi_{2m+1+k}(S^{2m+1}:p)$ *is isomorphic to the direct sum*

$$A(m,k)+B(m,k)+E(m,k)+U_1(m,k)+U_3(m,k)$$

except the case that $p=3$, $(m,k)=(9,41)$ *or* $=(9,42)$ *and* $\varepsilon_2(9)$ *does not exist, whence we change* $E(9,42)$ *and* $U_3(9,41)$ *to zero.*

The following table indicates the results of $\pi_{n+k}(S^n: 3)$ (* indicates the case (I)).

	$k=33$	$k=34$	$k=35$	$k=36$	$k=37$
$n=3$	Z_3+Z_3	Z_3+Z_3	Z_3	Z_3	Z_3
$n=5$	Z_3	Z_9+Z_3	Z_9	Z_3	Z_3
$n=7$	Z_3	$Z_{27}+Z_3$	Z_{27}	Z_3	Z_3
$n=9$	Z_3	Z_{27}	Z_{27}	Z_3+Z_3	Z_3
$n=11$	Z_3+Z_3	Z_{27}	Z_{27}	Z_3+Z_3	Z_3
$n=13$	Z_9^* *or* Z_3+Z_3	Z_{27}	Z_{27}	Z_3+Z_3	Z_3
$n=15$	Z_3^* *or* Z_3+Z_3	Z_9^* *or* Z_{27}	Z_{27}	Z_3	Z_3
$n=17$	0^* *or* Z_3	Z_3^* *or* Z_9	Z_{27}	Z_3	Z_3
$n \geq 19$	0^* *or* Z_3	0^* *or* Z_3	Z_{27}	Z_3	Z_3
(n:odd)			(α_9'')	$(\beta_1\beta_2)$	(ε_1')

	$k=38$	$k=39$	$k=40$	$k=41$	$k=42$
$n=3$	Z_3	Z_3+Z_3	Z_2	0	Z_3+Z_3
$n=5$	Z_3	Z_3+Z_3	Z_3+Z_3	Z_3	Z_3
$n=7$	Z_3	Z_3+Z_3	Z_3+Z_3	Z_3	Z_3
$n=9$	Z_3+Z_3	$Z_3+Z_3+Z_3$	Z_3	0^* *or* Z_3	Z_3^* *or* Z_3+Z_3
$n=11$	Z_3	Z_3+Z_3	Z_3	0	Z_3+Z_3
$n=13$	Z_3	Z_3+Z_3	Z_3	0	Z_3+Z_3
$n=15$	Z_3+Z_3	Z_3+Z_3	Z_3+Z_3	Z_3	Z_3+Z_3
$n=17$	Z_3+Z_3	Z_3+Z_3	Z_3	0	Z_3+Z_3
$n+19$	Z_3+Z_3	Z_3+Z_3	Z_3	0	Z_3+Z_3
$n=21$	Z_3	Z_3+Z_3	Z_3	0	Z_3+Z_3
$n\geq 23$	Z_3	Z_3+Z_3	Z_3	0	Z_3
(n:odd)	(ε_1)	$(\alpha_{10},\alpha_1\beta_1\beta_2)$	(β_1^4)		(ε_2)

Before proving Theorem 15.2, we prepare some lemmas.

Lemma 15.3. (i). *Let $\gamma\in\pi_i(S^{2p-1})$ be an element of order p, then there exists an element α of $\pi_{i+2}(S^3: p)$ such that*

$$\alpha\in\{\alpha_1(3), p\cdot\iota_{2p}, S\gamma\} \text{ and } H_p(\alpha)=IH^{(2)}(\alpha)=x\cdot S^2\gamma,\ x\not\equiv 0\ (mod\ p).$$

(ii). *Let $\gamma'\in\pi_i(S^{2p^2-5})$ be an element of order p, then there exists an element ε' of $\pi_{i+5}(S^{2p+1}: p)$ such that*

$$\varepsilon'\in\{\beta_1(2p+1), p\cdot\iota_{2p^2-1}, S^4\gamma'\} \text{ and } H^{(2)}\varepsilon'=x'\cdot I'(S^4\gamma'),\ x'\not\equiv 0\ (\mathrm{mod}\ p).$$

Proof. (i). By (1.10), $p\cdot\iota_{2p-1}\gamma=p\cdot\gamma=0$. By Proposition 1.7 of [7], we choose α as the class of the composition $g\circ Sf$, where f: $S^{i+1}\to Y_p^{2p}$ and g: $Y_p^{2p+1}\to S^3$ satisfy that $\pi\circ f$: $S^{i+1}\to Y_p^{2p}\to S^{2p}$ and $g|S^{2p}$: $S^{2p}\to S^3$ represent $S\gamma$ and $\alpha_1(3)$ respectively. Extend the definition of H_p as

$$H_p=\Omega^{-1}\circ h_{p*}\circ\Omega: \pi(SK: S^{2m+1})\approx\pi(SK;\ S_\infty^{2m})\to\pi(K, S_\infty^{2mp})$$
$$\approx\pi(SK;\ S^{2mp+1}),$$

then the relation (i) of (2.12) holds:

$$H_p(\alpha\circ S\beta)=H_p(\alpha\circ S\beta),\ \alpha\in\pi(SK;\ S^{2m+1}),\ \beta\in\pi(L;\ K).$$

Let α' be the class of f, then it is sufficient to prove that $H_p(\alpha')=x\cdot(S\pi)^*\iota_{2p+1}$ for some $x\not\equiv 0$ (mod p). Let a cellular map $f': Y_p^{2p}\to S_\infty^2$ represent $\Omega\alpha'$ and consider the following diagram.

$$\begin{array}{ccccc} Z\approx\pi_{2p}(Y^{2p}_p, S^{2p-1}) & \xrightarrow{\partial} & \pi_{2p-1}(S^{2p-1}) & \underset{\approx}{\xrightarrow{S}} & \pi_{2p}(S^{2p}) \\ \downarrow f'_* & & \downarrow (f'|S^{2p-1})_* & & \downarrow \alpha_1(3)_* \\ Z\approx\pi_{2p}(S^2_\infty, S^2_{p-1}) & \xrightarrow{\partial} & \pi_{2p-1}(S^2_{p-1}) & \xrightarrow{i_*} & \pi_{2p-1}(S^2_\infty)\approx\pi_{2p}(S^3). \end{array}$$

The commutativity is easily obtained. The lower sequence is exact and it is of the form $Z\rightarrow Z\rightarrow Z_p$ up to torsions prime to p. The upper ∂ is of degree p. Then it follows that f'_* is of degree $x\not\equiv 0$ (mod p). Since $h_{p*}: \pi_{2p}(S^2_\infty, S^2_{p-1})\approx\pi_{2p}(S^{2p}_\infty)$, we have $h_{p*}(\Omega\alpha')$ $=x\cdot\pi^*(\Omega\iota_{2p+1})$ and the required relation follows.

(ii). As in (i), let ε' be represented by the composition $g'\circ S^4f'$: $S^{i+5}\rightarrow Y^{2p^2}_p\rightarrow S^{2p+1}$, where $\pi\circ f'$ and $g'|S^{2p^2-1}$ represent $S\gamma'$ and $\beta_1(2p+1)$ respectively. We can choose g' such that $g'|S^{2p^2-1}=S^2g_0$ for a representative g_0 of $\beta_1(2p-1)$. Consider Ω^2g': $(Y^{2p^2-2}_p, S^{2p^2-3})\rightarrow(\Omega^2S^{2p+1}, S^{2p-1})$, where $\Omega^2g'|S^{2p^2-3}=g_0$. Since $\beta_1(2p-1)$ is of order p^2, $\Omega^2g'_*$: $\pi_{2p^2-2}(Y^{2p^2-2}_p, S^{2p^2-3})\rightarrow\pi_{2p^2-2}(\Omega^2S^{2p+1}, S^{2p-1})$ is not trivial on the p-primary component. Thus Ω^3g' is homotopic to a composition $h\circ\pi$: $Y^{2p^2-3}_p\rightarrow S^{2p^2-3}\rightarrow Q^{2p-1}_2=\Omega(\Omega^2S^{2p+1}, S^{2p-1})$ such that h represents a generator of $\pi_{2p^2-3}(Q^{2p-1}_2: p)\approx Z_p$, namely, h represents $x'\cdot I'(\iota_{2p^2-1})$ for some $x'\not\equiv 0$ (mod p). By use of (2.6), it follows that $\Omega^3(g'\circ S^4f')=\Omega^2(g')\circ Sf'$ represents both of $H^{(2)}(\varepsilon')$ and $x'\cdot I'(S^4\gamma')$. q. e. d.

As examples of (i), we have (up to non-zero coefficients)

(15. 5). (i). *$\alpha_{t+1}(3)\in\{\alpha_1(3), p\iota_{2p}, \alpha_t(2p)\}$ satisfies $H_p(\alpha_{t+1}(3))$ $=\alpha_t(2p+1)$ (i.e., $H^{(2)}(\alpha_{t+1}(3))\equiv Q^1(\alpha_t)$).*

(ii). *There exists an element $\bar{u}_3(0, \beta_1\beta_{p-1})$ such that*

$$H_p(\bar{u}_3(0, \beta_1\beta_{p-1}))=\beta_1\beta_{p-1}(2p+1)\,(i.e., H^{(2)}\bar{u}_3(0, \beta_1\beta_{p-1})=\bar{Q}^1(\beta_1\beta_{p-1}))$$

and $\bar{u}_3(0, \beta_1\beta_{p-1})\in\{\alpha_1(3), p\iota_{2p}, \beta_1\beta_{p-1}(2p)\}$.

As an example of (ii), we have (up to non-zero coefficients)

(15. 6). *There exists an element $\varepsilon'_{p-2}(2p+1)$ such that*

$$H^{(2)}\varepsilon'_{p-2}(2p+1)=I'\beta_{p-1}(2p^2-1)\,(i.e., H^{(2)}\varepsilon'_{p-2}(2p+1)=Q^p(\beta_{p-1}))$$

and $\varepsilon'_{p-2}(2p+1)\in\{\beta_1(2p+1), p\iota_{2p^2-1}, \beta_{p-1}(2p^2-1)\}$.

Lemma.15.4. *If $p>3$, then $\langle\alpha_1, p\iota, \beta_{p-1}\rangle=0$.*[1] *If $p>3$ or if $p=3$ and $\langle\alpha_1, 3\iota, \beta_2\rangle=0$, then*

$$S^\infty(\bar{u}_3(0,\beta_1\beta_{p-1}))\equiv 0 \text{ mod } \alpha_1\circ S^\infty(\pi_{5+k}(S^5: p)),$$
$$k=2(p^2+p-2)(p-1)-3.$$

Proof. $\{\alpha_1(7), p\iota_{2p+4}, \beta_{p-1}(2p+4)\}$ is defined. For an element γ of this bracket, we have $S^\infty\gamma\in\langle\alpha_1, p\iota, \beta_{p-1}\rangle$. Theorem 11.1 shows that $S^\infty\gamma=0$. Thus $\langle\alpha_1, p\iota, \beta_{p-1}\rangle=\alpha_1\circ\pi^S_{2(p^2-2)(p-1)-1}+\beta_{p-1}\circ\pi^S_{2p-2}=\{\alpha_1\cdot\alpha_{2p^2-2}\}$ $=0$ by Lemma 4.1. (This proof does not work for $p=3$). Since $p\cdot\beta_1(2p+1)=0$, $S^2\bar{u}_3(0, \beta_1\beta_{p-1})\in\{\alpha_1(5), p\cdot\iota_{2p+2}, \beta_1\beta_{p-1}(2p+2)\}\supset\{\alpha_1(5), p\cdot\iota_{2p+2}, \beta_1(2p+2)\}\circ\beta_{p-1}(2p^2)$. Then the second assertion follows from $\langle\alpha_1, p\iota, \beta_1\rangle=i^*\pi_*(\alpha\beta_{(1)})=0$.

Lemma 15.5. *If $p=3$ and $\langle\alpha_1, 3\iota, \beta_2\rangle\neq 0$, then we have*

$\langle\alpha_1, 3\iota, \beta_2\rangle=\pm\beta_1^3$, $\alpha_1\beta_1^3=0$ (*i.e. the case* (*I*)), $i^*\pi_*(\alpha\beta_{(2)})=\pm\beta_1^3\neq 0$,

$\beta_1^4=\pm\alpha_1\varepsilon_1'=\pm\alpha_1\langle\alpha_1, \alpha_1, \beta_1^3\rangle$, $\beta_1^6=0$,

and $\qquad H^{(2)}p_*(\bar{Q}^3(\beta_2))=\pm Q^2(\beta_1^3)$.

Proof. Since $\langle\alpha_1, 3\iota, \beta_2\rangle$ consists of a single element and belongs to $(\pi^S_{20}: 3)$ which is generated by β_1^3, the first assertion follows. By (13.3)′ and (4.6)

$$-\alpha^2\delta\beta_{(2)}=(\delta\alpha^2+\alpha\delta\alpha)\beta_{(2)}=x(\delta\alpha+\alpha\delta)\delta\alpha\delta(\beta_{(1)}\delta)^2=0.$$

Then $\alpha_1\beta_1^3=\alpha_1\langle\alpha_1, 3\iota, \beta_2\rangle=\langle\alpha_1, \alpha_1, 3\iota\rangle\beta_2=-\alpha_2\beta_2=-i^*\pi_*(\alpha^2)i^*\pi_*(\beta_{(2)})$ $=i^*\pi_*(-\alpha^2\delta\beta_{(2)})=0$. By Theorem 15.1 of the case (I)

$$\pm\beta_1^6=\beta_1^3\langle\alpha_1, 3\iota, \beta_2\rangle=-\langle\beta_1^3, \alpha_1, 3\iota\rangle\beta_2\in(\pi^S_{34}: 3)\circ\beta_2=0.$$

Next, $\pm\beta_1^4=\beta_1\langle\alpha_1, 3\iota, \beta_2\rangle\subset\langle\alpha_1\beta_1, 3\iota, \beta_2\rangle$ and $\alpha_1\varepsilon_1'\in\alpha_1\langle\beta_1, 3\iota, \beta_2\rangle\subset\langle\alpha_1\beta_1, 3\iota, \beta_2\rangle$, and the indeterminacy is $\alpha_1\beta_1\circ(\pi^S_{27}: 3)+(\pi^S_{14}: 3)\circ\beta_2$. Since $(\pi^S_{14}: 3)=0$, $(\pi^S_{27}: 3)$ is generated by α_7 and $\alpha_1\beta_1\alpha_7=\beta_1\alpha_1\alpha_7=0$ by Lemma 4.1, we have $\pm\beta_1^4=\alpha_1\varepsilon_1'$. It is known [7] that $\pm\beta_1=\langle\alpha_1, \alpha_1, \alpha_1\rangle$. Thus $\pm\beta_1^4=\langle\alpha_1, \alpha_1, \alpha_1\rangle\beta_1^3=\alpha_1\langle\alpha_1, \alpha_1, \beta_1^3\rangle$. Since $i^*\pi_*(\alpha\beta_{(2)})$ $=\langle\alpha_1, 3\iota, \beta_2\rangle$ by Proposition 1.7 of [7], we have $i^*\pi_*(\alpha\beta_{(2)})=\pm\beta_1^3$. Then $(\delta\alpha)(i^*\beta_{(2)})=h_*(i^*\beta_{(2)})=\pm i_*\beta_1^3$ for the attaching map h in

1) The proof of the relation $\{\beta_{p-1}, p\iota, \alpha_1\}$ in (4. 13) of [6] is incomplete, since a relation was dropped in Theorem 3.10 of [6].

$K(p, 2)$ of Proposition 4.5. Then the last assertion follows from Proposition 3.6 and (5.2).

Consider the subgroups $A(m, k)$ and $B(m, k)$ of $\pi_{2m+1+k}(S^{2m+1}: p)$ in Theorems 11.1 and 15.2 for sufficiently large m. They are stable, hence we denote these subgroups by

$$A(k),\ B(k) \subset (\pi_k^S: p).$$

For $k=2p^2(p-1)-1$, we put $B(k)=0$ and $A(k) \approx Z_{p^3}$ generated by α'_{p^2}. For $k=2p^2(p-1)-2$, we put $A(k)=B(k)=0$. For $k=2p^2(p-1)-2$, we put $A(k)=0$ and $B(k) \approx Z_p$ generated (formally) by $\alpha_1\beta_1^p$. Then we can use the notation

$$Q^m(\gamma),\ \bar{Q}^m(\gamma) \in \pi_i(Q_2^{2m-1}: p),\ \gamma \in A(k)+B(k)$$

of (6.3), with the convension that $Q^m(\alpha_1\beta_1^p) = I'\alpha_1\beta_1^p(2mp-1)$ and $I\bar{Q}^m(\alpha_1\beta_1^p) = \alpha_1\beta_1^p(2mp+1)$ for $1 \leq m \leq p-1$.

Lemma 15.6. *Let* $2p^2(p-1)-1 \leq k' < 2(p^2+p)(p-1)-5$. *If Theorem* 15.2 *holds for* $k \leq k'-2p$, *then we have the following exact sequence*:

$$0 \to (A(i)+B(i)) \otimes Z_p \xrightarrow{I'} \pi_{2m-1+k'}(Q_2^{2m-1}: p) \xrightarrow{I} \mathrm{Tor}(A(i-1)+B(i-1), Z_p) \to 0,$$

$i=k-2m(p-1)+2$, *with an exception when* $p=3$, $k=38$, $m=1$ *we have* $IQ^1(\alpha'_9) = \alpha_1(7) \circ \alpha_8(10)$.

Proof. The exact sequence comes from (2.5). By dimensional reasons, $E(m, k)$ and $U_4(m, k)$ are independent. So, it is sufficient to prove that subgroups $U_i(m, k)$, $i=1, 2, 3$, are cancelled by the homomorphism $\Delta: \pi_{i+4}(S^{2mp+1}: p) \to \pi_{i+2}(S^{2mp-1}: p)$. Again the elements $p_*Q^m(\iota)$ are independent of our computation. Then Corollaries 9.4, 9.5, the cyclicity of $U_2(m, k)$ and (2.7) give required cancellation. The details are left to the readers.

Proof of Theorem 15.2. The proof is done by induction on k and based on the exact sequence (1.7):

$$\cdots \xrightarrow{H^{(2)}} \pi_{2m-1+k}(Q_2^{2m-1}: p) \xrightarrow{p^*} \pi_{2m-1+k}(S^{2m-1}: p) \xrightarrow{S^2} \pi_{2m+1+k}(S^{2m+1}: p) \xrightarrow{H^{(2)}} \cdots.$$

As in the proof of Theorem 11.1, we cancell generators of $\pi_{2m+1+k}(S^{2m+1}: p)$ with the corresponding pair of elements in $\pi_*(Q_2^*: p)$. First we cancell the generators of $U_1(m, k)$ with the corresponding elements in $\pi_{2m+1+k}(Q_2^{2m+1}: p)$ and $\pi_{2m-2+k}(Q_2^{2m-1}: p)$ by virtue of Theorems 5.1, 5.2 and Lemma 15.6. In the case $p=3$, there is an exceptional element $Q^1(\alpha_9')=I^{-1}(\alpha_1(7)\circ\alpha_8(10))$, which is cancelled with $\bar{Q}^2(\alpha_8)$ since $IH^{(2)}p_*(\bar{Q}^2(\alpha_8))=\pm\alpha_1(7)\circ\alpha_8(10)$ by Propositions 3.6, 4.5 and (5.2). Next consider the generators of $U_3(m, k)$. The non-triviality of these generators are easily checked, after the cancellation of $U_1(m, k)$, except the following two cases. Firstly, there is a possibility of $S^2u_3(1, \beta_1^p)=0$ by a relation $p_*\bar{Q}^{p+1}(\beta_{p-1})=x\cdot u_3(1, \beta_1^p)$, $x\not\equiv 0 \pmod p$. Secondly, the non-triviality of $S^{2p-4}\bar{u}_3(0, \beta_1\beta_{p-1})$ is obtained but the triviality of $S^{2p-2}\bar{u}_3(0, \beta_1\beta_{p-1})$ is not known. Then except these two cases $U_3(m, k)$ is cancelled. Next, stable subgroups $A(m, k)$ and $B(m, k)$ are cancelled with the elements $\bar{Q}^1(\alpha_t)$, $p^2\leq t\leq p^2+p-2$, by (15.5), (i), and $Q^{p-1}(\alpha_1\beta_1^{p-s-1}\beta_s)$, $Q^1(\beta_1^{p-s}\beta_s)$ and $Q^{p-1}(\beta_1^p)$. After these cancellations it remains the following elements:

$$Q^{p(p-i-1)}(\beta_{i+1}) \text{ for } 1\leq i\leq p-2,$$
$$\bar{Q}^{p(p-j)+1}(\beta_j) \text{ for } 1\leq j\leq p-1,$$
$$\bar{Q}^1(\beta_1\beta_{p-1})=H^{(2)}\bar{u}_3(0, \beta_1\beta_{p-1}),\ \bar{Q}^p(\alpha_1\beta_{p-1}),$$

and $$Q^p(\beta_1^p)=H^{(2)}u_3(1, \beta_1^p),\ Q^{2p-1}(\alpha_1\beta_1^{p-1}) \text{ with } p_*Q^{2p-1}(\alpha_1\beta_1^{p-1}) =S^{2p-4}u_3(1, \beta_1^p).$$

The first elements indicate the existence of $\varepsilon_i'(2p(p-i-1)+1)$. Then by (14.3), (ii), $p_*\bar{Q}^{p(p-j)+1}(\beta_j)\in \operatorname{Im} S^{2p+2}=0$ for $1\leq j\leq p-2$. Thus the existence of $\varepsilon_j(2p(p-j)+3)$ is obtained for $1\leq j\leq p-2$. If $p>3$ or if $p=3$ and $\langle\alpha_1, 3\iota, \beta_2\rangle=0$, then Lemma 15.4 and the above computation show that $S^\infty\bar{u}_3(0, \beta_1\beta_{p-1})=0$. Thus $\bar{u}_3(0, \beta_1\beta_{p-1})$ must be cancelled by $\bar{Q}^p(\alpha_1\beta_{p-1})$. If $p=3$ and $\langle\alpha_1, 3\iota, \beta_2\rangle\neq 0$, Lemmas 15.4, 15.5 and the above computation show that $S^\infty\bar{u}_3(0, \beta_1\beta_2)\equiv 0$ mod $\alpha_1\varepsilon_1'=\beta_1^4$. Thus $p_*\bar{Q}^3(\alpha_1\beta_2)$ must give a relation between $\beta_1^4(7)$ and

$S^4\bar{u}_3(0, \beta_1\beta_2)$, and $\bar{u}_3(0, \beta_1\beta_2)$ is cancelled with $\bar{Q}^3(\alpha_1\beta_2)$ in this sense. In this case we have also from the last assertion of Lemma 15.5 that $\bar{Q}^4(\beta_2)$ and $Q^3(\beta_1^3)$ are cancelled with an unstable element $p_*\bar{Q}^4(\beta_2)$ of the first type, and consequently there exists ε_2 (11) wiih $H^{(2)}\varepsilon_2(11) = Q^5(\alpha_1\beta_1^2)$. Finally, to prove the existence of $\varepsilon_{p-1}(2p+3)$ with $H^{(2)}\varepsilon_{p-1}(2p+3)=\bar{Q}^{p+1}(\beta_{p-1})$, for the case that $p>3$ or that $p=3$ and $\langle\alpha_1, 3\iota, \beta_2\rangle=0$, it is sufficient to prove the relation $H^{(2)}p_*\bar{Q}^{p+1}(\beta_{p-1})=0$, the proof of which is similar to that of the last assertion of Lemma 15.5. This completes the proof of Theorem 15.2.

We have seen in the proof of Lemma 15.5 that, in the case (I) and $p=3$, $\beta_1^4\neq 0$ equals to $\pm\alpha_1\langle\alpha_1, \alpha_1, \beta_1^3\rangle$. It follows that $\pm\beta_1^4=\alpha_1\varepsilon_1' =\langle\alpha_1\beta_1, 3\iota, \beta_2\rangle=\beta_1\langle\alpha_1, 3\iota, \beta_2\rangle$. Thus $\langle\alpha_1, 3\iota, \beta_2\rangle\neq 0$, and we have from Lemma 15.5 and (13.3)'

Proposition 15.6. *Let* $p=3$. *Then the case* (I): $\alpha_1\beta_1^3=0$ *is equivalent to* $\langle\alpha_1, 3\iota, \beta_2\rangle=\pm\beta_1^3$, *whence we have* $H^{(2)}\varepsilon_2(11)\neq 0$, $\alpha\beta_{(2)}=\pm\beta_{(1)}(\delta\beta_{(1)})^2$, $\beta_1^4=\pm\alpha_1\varepsilon_1'$, *and* $\beta_1^6=0$.

At the end of the paper, we remark

$$(\pi^S_{2(p^2+p)(p-1)-5}: p)=0 \text{ in the case (I).}$$

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