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On the cohomologies of commutative affine group schemes

By

Masayoshi MIYANISHI *)

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Let k be an algebraically closed field of positive characteristic p, X a proper integral k-scheme of finite type and G be a commutative affine k-group scheme. For a k-prescheme T, the isomorphism classes of principal fibre spaces Y over X_T with group G form an abelian group with the well-known multiplication. We shall denote this abelian group by $\mathrm{PH}(G, X/k)(T)$. Then the functor $T \rightsquigarrow \mathrm{PH}(G, X/k)(T)$ is a contravariant functor from the category of k-preschemes $(\mathbf{Sch}/k)$ to the category of abelian groups $(\mathbf{Ab})$. The associated sheaf of $\mathrm{PH}(G, X/k)$ with respect to the (fpqc)-topology of $(\mathbf{Sch}/k)$ is denoted by $\mathbf{PH}(G, X/k)$.

If G is the multiplicative group G_m, $\mathbf{PH}(G_m, X/k)$ coincides with the Picard functor $\mathbf{Pic}(X/k)$ of X, and $\mathbf{Pic}(X/k)$ is representable by a commutative k-group scheme, locally of finite type over k.

The purpose of this paper is to study the representability of the functor $\mathbf{PH}(G, X/k)$ for an arbitrary commutative affine k-group scheme of finite type. If G is the additive group G_a, $\mathbf{PH}(G_a, X/k)$ is representable by $\mathrm{Lie}(\mathbf{Pic}(X/k))$ which is isomorphic to a direct product of G_a. If G is a simple finite k-group scheme (i.e. $G=\alpha_p$, μ_p, $(\mathbf{Z}/p\mathbf{Z})_k$ and $(\mathbf{Z}/q\mathbf{Z})_k$; q: prime, $(p, q)=1$), $\mathbf{PH}(G, X/k)$ is

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representable by $\mathrm{Ker}(\mathrm{Lie}(\mathbf{Pic}(X/k))\xrightarrow{F}\mathrm{Lie}(\mathbf{Pic}(X/k)))$ if $G=\alpha_p$, ${}_p(\mathbf{Pic}(X/k))$ if $G=\mu_p$, $\mathrm{Ker}(\mathrm{Lie}(\mathbf{Pic}(X/k))\xrightarrow{F-id}\mathrm{Lie}(\mathbf{Pic}(X/k)))$ if $G=(\mathbf{Z}/p\mathbf{Z})_k$ and ${}_q(\mathbf{Pic}(X/k))$ if $G=(\mathbf{Z}/q\mathbf{Z})_k$, where F is the endomorphism of $\mathrm{Lie}(\mathbf{Pic}(X/k))$ induced from the Frobenius endomorphism of G_a (cf. Chapter I, Theorem 1.6).

In general, $\mathbf{PH}(G, X/k)$ is representable by a commutative k-group scheme, locally of finite type over k, if (1) G is a connected commutative algebraic k-group scheme, smooth over k and if (2) G is a commutative finite k-group scheme (cf. Chapter IV, Theorem 4.7).

These results are applied to make a calculation of the fundamental group $F_c(X)$ of X (cf. Chapter III), and to obtain some results on an abelian scheme (i.e. when X is an abelian scheme) (cf. Chapter II).

In this paper, we shall use freely the terminology and the notations of A. Grothendieck. For the references, see EGA, FGA, SGA, SGAD, SGAA and GB (cf. Bibliography). For an abelian group M (resp. an algebraic group G), we denote by ${}_nM$ (resp. ${}_nG$) the kernel of the multiplication by n on M (resp. G), where n is a positive integer. The set of natural numbers is denoted by $\mathbf{N}$ or $\mathbf{Z}^+$.

Contents

Chapter 1. On the PH-functor

1. Topology. In the following, we shall use freely the definitions and the results on Grothendieck topology, for which we refer to [SGAA], [MA] and [SGAD][*].

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*) See Bibliography [2], [1] and [3].

Roughly speaking, "the open coverings" on a prescheme S in the sense of (fpqc)-topology (resp. (fppf)-topology, étale topology are generated by two kinds of families of morphisms:

(1) surjective families of open immersions from affine open sets into S,

(2) finite surjective families of flat morphisms (resp. flat morphisms of finite presentation, étale morphisms).

Then a set-valued contravariant functor F on the category of S-preschemes $(\mathbf{Sch}/S)$ is called a (fpqc)-sheaf (resp. (fppf)-sheaf, étale sheaf) if it satisfies the conditions:

(a) for a surjective family of open immersions $\{U_\alpha \to U\}$ the sequence

$$F(U) \to \prod_{\alpha} F(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(U_\alpha \times_U U_\beta)$$

is exact.

(b) for any (fpqc)-morphism (resp. (fppf)-morphism, étale surjective morphism) $T' \to T$, the sequence,

$$F(T) \to F(T') \rightrightarrows F(T' \times_T T')$$

is exact.

The topologies on $(\mathbf{Sch}/S)$ are ordered as follows: $(\mathrm{can}) \geqslant (\mathrm{fpqc}) \geqslant (\mathrm{fppf}) \geqslant (\mathrm{\acute{e}t}) \geqslant (\mathrm{Zar})$, where one reads the left one is finer than the right one and where (can) means the coarsest topology with which arbitrary prescheme is a sheaf. Therefore, we have the relation of inclusions, $(\mathrm{Sch}) \subseteq (\text{fpqc-sheaf}) \subseteq (\text{fppf-sheaf}) \subseteq (\text{étale sheaf}) \subseteq (\text{Zariski sheaf})$.

Next we shall quote elementary results on the sheafication of a presheaf on a site C whose topology is defined by a pretopology, (cf. SGAA, Exp. I). Let F be a presheaf on C. Then the separated presheaf associated with F is defined by

$$LF(X) = \varinjlim_{\{X_\alpha \to X\} \in J(X)} \mathrm{Ker}\big(\prod_{\alpha} F(X_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(X_\alpha \times_X X_\beta)\big)$$

where $J(X)$ is the set of all coverings in C with the target X. LF possesses the following property; for a covering $\{T_\beta \to T\}$ of C, we have,

$$LF(T) \underset{\text{inclusion}}{\hookrightarrow} \mathrm{Ker}(\prod_{\beta} LF(T_\beta) \rightrightarrows \prod_{\beta,\gamma} LF(T_\beta \underset{T}{\times} T_\gamma)).$$

The sheaf associated with a presheaf F is defined as $L^2F = L(LF)$. We can also define L^2F in one step as follows:

$$L^2F(X) = \varinjlim_{\{X_{\alpha\beta\gamma}, X_\alpha, X, S_{\alpha\beta\gamma}, S_\alpha\} \in J^2(X)} \mathrm{Ker}(\prod_\alpha F(X_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(X_\alpha \underset{X}{\times} X_\beta) \to \prod_{\alpha,\beta,\gamma} F(X_{\alpha\beta\gamma})),$$

where $J^2(X)$ is composed by sets of coverings $\{X_\alpha \xrightarrow{S_\alpha} X\}$ and $\{X_{\alpha\beta\gamma} \xrightarrow{S_{\alpha\beta\gamma}} X_\alpha \underset{X}{\times} X_\beta\}$ for each (α, β). We denote L^2 by $\boldsymbol{a}$ and call it a sheafication functor. $\boldsymbol{a}$ is an exact functor, more generally, $\boldsymbol{a}$ commutes with finite projective limits and inductive limits.

2. Cohomology. Let C be a site, $C^{\sim}$ be the topos formed by sheaves on C and A be a sheaf of commutative rings with unit on C. For two sheaves F, G of A-modules on C (resp. for a sheaf of sets E on C), a cohomology

$\mathrm{Ext}^q_A(C^{\sim};\ F, G)$ (resp. $\mathrm{H}^q(C^{\sim}/E, F)$) or simply

$\mathrm{Ext}^q_A(F, G)$ (resp. $\mathrm{H}^q(E, F)$)

is defined as the q-th right derived functor of the functor $F \rightsquigarrow \mathrm{Hom}_A(F, G)$ (resp. by $\mathrm{H}^q(E, F) = \mathrm{Ext}^q_A(A_E, F)$). Also, for F, G and E as above, we define a q-th local cohomology

$$\mathbf{Ext}^q_A(F, G) \text{ (resp. } \mathbf{H}^q(E, F))$$

as the q-th right derived functor of the functor $F \rightsquigarrow \mathbf{Hom}_A(F, G)$ (resp. by $\mathbf{H}^q(E, F) = \mathbf{Ext}^q_A(A_E, F)$).

Let X be an object of C, and put

$$\mathrm{Ext}^0_A(C^{\sim}/X;\ F, G) = \mathrm{H}^0(C^{\sim}/X, \mathbf{Hom}_A(F, G)).$$

If we denote by $\mathrm{Ext}^q_A(C^{\sim}/X;\ F, G)$ the q-th right derived functor of the functor $F \rightsquigarrow \mathrm{Ext}^0_A(C^{\sim}/X;\ F, G)$, we have by SGAA, Exp. V, Prop. 4.1, a spectral sequence functorial in F, G and X,

(1) $\quad \mathrm{E}_2^{p,q} = \mathrm{H}_p(C^{\sim}/X;\ \mathbf{Ext}^q_A(F, G)) \Longrightarrow \mathrm{Ext}^{p+q}_A(C^{\sim}/X;\ F, G).$

The sheaf $\mathbf{Ext}^q_A(F, G)$ is identified with the sheaf associated with the presheaf $X \rightsquigarrow \mathrm{Ext}^q_A(C^{\sim}/X;\ F, G)$.

Moreover, if we replace F by A_E, we have a spectral sequence functorial in F, E and X.

(2) $\quad \mathrm{E}_2^{p,q}=\mathrm{H}^p(C^{\sim}/X;\ \mathbf{H}^q(E,F))\Longrightarrow \mathrm{Ext}_A^{p+q}(C^{\sim}/X;\ A_E,F).$

Let u: $(C,A)\to(C',A')$ be a morphism of ringed sites. Suppose that the topologies of C and C' are defined by pretopologies, the finite fibre products are representable in C and C' and that u commutes with the finite fibre products. Let F be a sheaf of A'-modules on C' and X be an object of C. Then by SGAA, Exp. V, Cor. 5.3, we have a spectral sequence functorial in F and X,

(3) $\quad \mathrm{E}_2^{p,q}=\mathrm{H}^p(C^{\sim}/X;\ R^q u_S(F))\Longrightarrow \mathrm{H}^{p+q}(C'^{\sim}/u(X),\ F),$

where $R^q u_S$ is the q-th right derived functor of the functor of direct image u_S: $C'^{\sim}_{A'}\to C^{\sim}_A$.
Moreover, if G is a sheaf of A-modules on C, then by Prop. 5.5, ibid., we have a spectral sequence functorial in F and G,

(4) $\quad \mathrm{E}_2^{p,q}=\mathrm{Ext}_A^p(C^{\sim};\ G, R^q u_S(F))\Longrightarrow \mathrm{Ext}_{A'}^{p+q}(C'^{\sim};\ u^s(G),F)$

where u^s is the functor of inverse image u^s: $C^{\sim}_A\to C'^{\sim}_{A'}$.

In the following sections, we shall apply the above cohomology and spectral theories to the case where C is the (fpqc)-site (resp. (fppf)-site, étale site, Zariski site) $(\mathbf{Sch}/S)$, $A=\mathbf{Z}$: constant ring of integers, F, G are commutative group preschemes over S and $E=X$ is a S-prescheme. Then we denote $\mathrm{Ext}_A^q(F,G)$, $\mathrm{H}^q(E,F)$, $\mathbf{Ext}_A^q(F,G)$ $\mathbf{H}^q(E,F)$ by $\mathrm{Ext}_{S\text{-}gr}^q(F,G)_{pq}$, $\mathrm{H}_{pq}^q(X,F)$, $\mathbf{Ext}_{S\text{-}gr}^q(F,G)_{pq}$, $\mathbf{H}_{pq}^q((X/S),F)$ (resp. $\mathrm{Ext}_{S\text{-}gr}^q(F,G)_{pl}$, $\mathrm{H}_{pl}^q(X,F)$ $\mathbf{Ext}_{S\text{-}gr}^q(F,G)_{pl}$, $\mathbf{H}_{pl}^q((X/S),F)$, $\mathrm{Ext}_{S\text{-}gr}^q(F,G)_{et}$, $\cdots$).
Finally, we remark that $\mathrm{H}_{pq}^q(X,F)=\mathrm{H}^q(X_{pq},F_{pq})$, $\mathrm{H}_{pl}^q(X,F)=\mathrm{H}^q(X_{pl},F_{pl})$ and $\mathrm{H}_{et}^q(X,F)=\mathrm{H}^q(X_{et},F_{et})$ where the right term of each equality is the cohomology group calculated on the site X_{pq}, X_{pl}, X_{et}, (cf. SGAA, Exp. VII and Exp. VI, §7, Cor. 3.9).

3. Definition of PH-functor. Let k be a field, S be a locally noetherian k-prescheme, X be a S-prescheme, of finite type over S and G be a k-group scheme of finite type. We define a contravariant functor $\mathrm{PH}(G, X/S)$ of $(\mathbf{Sch}/S)$ concerning a triple (G, X, S) by

$$T\in(\mathbf{Sch}/S)\rightsquigarrow\mathrm{PH}(G, X/S)(T)=$$

$$=\left\{\begin{array}{l}\text{isomorphism classes of principal fibre space}\\ Y \text{ over } X_T=X\underset{S}{\times}T \text{ with group } G \text{ whose}\\ \text{canonical projection is (fppf)}\end{array}\right\}.$$

Such a fibre space is a representable one of principal fibre sheaves with the base space X_T and with group G in the sense of (fpqc)-topology on $(\mathbf{Sch}/S)$ and is, sometimes, expressed by a sequence

$$G\times Y\underset{pr_2}{\overset{\sigma}{\rightrightarrows}}Y\underset{p}{\longrightarrow}X, \quad \text{(cf. } SGA\text{, Exp. XI).}$$

Then the canonical projection p is evidently a (fppf)-morphism. $\mathrm{PH}(G, X/S)$ is not, in general, a (fpqc)-sheaf of $(\mathbf{Sch}/S)$. In fact, if $X=S$, an element Y of $\mathrm{PH}(G, S/S)(T)$, $T\in(\mathbf{Sch}/S)$ is trivialized by passing to $\mathrm{PH}(G, S/S)(Y)$, where $Y\to T$ is the canonical projection of Y. The associated sheaf of $\mathrm{PH}(G, X/S)$ in the sense of (fpqc)-topology is denoted by $\mathbf{PH}(G, X/S)$ and is said a $\mathbf{PH}$-functor concerning a triple (G, X, S).

Let $0\to G_1\to G_2\to G_3\to 0$ be an exact sequence of k-group schemes (i.e. G_1 is invariant in G_2 and G_3 is the quotient of G_2 by G_1). Then by SGA, Exp. XI, we have an exact sequence,

1) $$\begin{aligned}&0\to G_1(X_T)\to G_2(X_T)\to G_3(X_T)\to\mathrm{PH}(G_1, X/S)(T)\to\\ &\to\mathrm{PH}(G_2, X/S)(T)\to\mathrm{PH}(G_3, X/S)(T), \text{ for } T\in(\mathbf{Sch}/S).\end{aligned}$$

Then by operating the sheafication functor $\boldsymbol{a}$, we have an exact sequence,

2) $$\begin{aligned}&0\to\mathbf{Hom}_S(X, G_{1,S})\to\mathbf{Hom}_S(X, G_{2,S})\to\mathbf{Hom}_S(X, G_{3,S})\to\\ &\to\mathbf{PH}(G_1, X/S)\to\mathbf{PH}(G_2, X/S)\to\mathbf{PH}(G_3, X/S),\end{aligned}$$

where $\mathbf{Hom}_S(X, G_{1,S})$

is the (fpqc)-sheaf associated with the presheaf $T \rightsquigarrow \mathrm{Hom}_T(X_T, G_{1,T})$ etc.. Suppose now that G is commutative and consider $\mathrm{H}^1_{pq}(X_T, G)$, $\mathrm{H}^1_{pl}(X_T, G)$, $\mathrm{H}^1_{\text{ét}}(X_T, G)$ and $\mathrm{H}^1_{\text{Zar}}(X_T, G)$, $T \in (\mathbf{Sch}/S)$.
Those groups are identified with the Čech-cohomologies calculated in the corresponding sites on $(\mathbf{Sch}/S)$. Then, the usual argument shows that those groups are the abelian groups of isomorphism classes of principal fibre sheaves on X_T with group G in the corresponding sites on $(\mathbf{Sch}/S)$, (cf. SGAA, Exp. VII). When G is affine, we have the next result.

Lemma 1.1. (1) *If T is quasi-compact, we have*

$$\mathrm{H}^1_{pq}(X_T, G) \cong \mathrm{H}^1_{pl}(X_T, G) \cong \mathrm{PH}(G, X/S)(T).$$

These equalities hold for a non-commutative affine group G if $\mathrm{H}^1_{pq}(X_T, G)$ (resp. $\mathrm{H}^1_{pl}(X_T, G)$) is the set of isomorphism classes of principal fibre sheaves of the base X_T with group G in the (fpqc)-site (resp. (fppf)-site) $(\mathbf{Sch}/S)$.
(2) (*cf.* $\mathrm{GB}_{\mathrm{III}}$, (11.7)). *Suppose, moreover, that G is smooth over k. Then we have*

$$\mathrm{H}^q_{pl}(X_T, G) \cong \mathrm{H}^q_{\text{ét}}(X_T, G),$$

in particular, $\mathrm{H}^1_{pl}(X_T, G) \cong \mathrm{H}^1_{\text{ét}}(X_T, G)$.
(3) *If G is special in the sense of J.-P. Serre* [17], *for T as in the assertion* (1), *we have*

$$\mathrm{H}^1_{\text{ét}}(X_T, G) \cong \mathrm{H}^1_{\text{Zar}}(X_T, G).$$

Proof. First, note that under the assumption on T, an arbitrary (fpqc)-(resp. (fppf)-) covering $\{U_\alpha \to X_T\}$ of X_T is dominated by a finer covering $f: X' \to X$ where f is a (fpqc) (resp. (fppf))-morphism. Then the proof of (1) is done with the argument of [14], III, (17.4). For the proofs of (2) and (3), the readers are sent to the references.

For any S-prescheme, $\mathrm{PH}(G, X/S)(T)$ is an abelian group if G is affine and commutative. In fact, let Y_1, Y_2 be elements of

$PH(G, X/S)(T)$. Let F be a sheaf theoretic sum of Y_1 and Y_2 in $H^1_{pq}(X, G)$. Then F admits a (fpqc)-local section $(Y_1\times Y_2)\underset{X\times X}{\times}(X, \Delta_X)\rightarrow F$. Hence, the argument of Lemma 1.1, (1) shows that F is representable.

Therefore, $\mathbf{PH}(G, X/S)$ is a (fpqc)-abelian sheaf included in an (fpqc)-abelian sheaf $\mathbf{H}^1_{pq}(G, X/S)$. Now we have the following result.

Lemma 1.2. (1) *If G is a commutative affine group scheme of finite type over k, then* $\mathbf{PH}(G, X/S)\cong \mathbf{H}^1_{pq}(X, G)$.

(2) *For an arbitrary k-group scheme G,* $\mathbf{PH}(G, S/S)=0$.

(3) *If X is affine over S,* $\mathbf{PH}(G_a, X/S)=0$.

(4) *If $S=Spec(k)$, k: the field and X is finite over k,* $\mathbf{PH}(G_m, X/k)=0$.

Proof. For $T\in(\mathbf{Sch}/S)$, cover T by affine open sets $\{U_\alpha\}$. Then a commutative diagram

$$\begin{array}{ccccc} 0\rightarrow\mathbf{PH}(G,X/S)(T) & \rightarrow & \prod_\alpha \mathbf{PH}(G,X/S)(U_\alpha) & \rightrightarrows & \prod_{\alpha,\beta}\mathbf{PH}(G,X/S)(U_\alpha\cap U_\beta) \\ \downarrow & & \downarrow & & \downarrow\cap \\ 0\rightarrow\mathbf{H}^1_{pq}(X/S,G)(T) & \rightarrow & \prod_\alpha \mathbf{H}^1_{pq}(X/S,G)(U_\alpha) & \rightrightarrows & \prod_{\alpha,\beta}\mathbf{H}^1_{pq}(X/S,G)(U_\alpha\cap U_\beta) \end{array}$$

shows that $\mathbf{PH}(G, X/S)(T)=H^1_{pq}(X/S, G, T)$ if this equality holds for separated quasi-compact sets. If T is so, this can be proved as follows;

$$\begin{aligned} L(PH(G, X/S))(T) &= \varinjlim_{\substack{T'\rightarrow T\\ \text{fpqc}}} \mathrm{Ker}(PH(G, X/S)(T')\rightrightarrows PH(G, X/S)(T'\underset{T}{\times} T')) \\ &\underset{(\text{Lemma } 1.1)}{=} \varinjlim_{\substack{T'\rightarrow T\\ \text{fpqc}}} \mathrm{Ker}(H^1_{pq}(X_{T'}, G)\rightrightarrows H^1_{pq}(X_{T'\times_T T'}, G)) \\ &= L(H)(T), \text{ where H is a functor } T\in(\mathbf{Sch}/S)\rightsquigarrow \end{aligned}$$

$H(T)=H^1_{pq}(X_T, G)$. Note that in the right hand term of the first equality, $T'\underset{T}{\times}T'$ is quasi-compact. The same calculation shows that $\mathbf{PH}(G, X/S)(T)=L^2(PH(G, X/S))(T)=L^2H(T)=\mathbf{H}^1_{pq}(X/S, G)(T)$. Hence follows (1). (2) is put here for

memory. For (3), it is enough to see that $\mathbf{PH}(G_a, X/S)(U)=0$ for affine scheme $U=\mathrm{Spec}(R)$. Then, since $\mathbf{PH}(G_a, X/S)(U) = \mathbf{PH}(G_a, X_U/U)(U)$, the result is easily proved by Serre's theorem, (cf. EGA, III, (1.3.1)). (4) was essentially proved in [13], with supplementary use of Serre's theorem. So, we omit the proof.
q.e.d.

We shall give now an example of $\mathbf{PH}(G, X/S)$. If $G=G_m$, $\mathbf{PH}(G_m, X/S)$ is the Picard functor of X over S, (cf. FGA, $n^0$232). If $G=G_a$, note that $\mathrm{PH}(G_a, X/S)(T)=\mathrm{H}^1(X_T, \mathcal{O}_{X_T})$, if T is quasi-compact.

Let $0\to G_1\to G_2\xrightarrow{p} G_3\to 0$ be an exact sequence of commutative affine group schemes of finite type over k. Then, a sequence of abelian sheaves on the (fpqc)-(resp. (fppf)-)site $(\mathbf{Sch}/S)$,

$$0\to G_{1,S}\to G_{2,S}\xrightarrow{p_S} G_{3,S}\to 0$$

is exact, because p is a (fppf)-morphism. Consider an exact sequence of cohomologies of X_T-sections.

$$0\to G_1(X_T)\to G_2(X_T)\to G_3(X_T)\to \mathrm{H}^1_i(X_T, G_1)\to \mathrm{H}^1_i(X_T, G_2)\to$$
$$\to \mathrm{H}^1_i(X_T, G_3)(\to H^2_i(X_T, G_1)\to\cdots),\quad i=pq \text{ or } pl.$$

This sequence coincides with the sequence 1), if T is quasi-compact. Consider, also, an exact sequence of local cohomologies of X-sections.

$$0\to \mathbf{Hom}_S(X, G_1)\to \mathbf{Hom}_S(X, G_2)\to \mathbf{Hom}_S(X, G_3)\to \mathbf{H}^1_i(X, G_1)\to$$
$$\to \mathbf{H}^1_i(X, G_2)\to \mathbf{H}^1_i(X, G_3)(\to \mathbf{H}^2_i(X, G_1)\to\cdots),\quad i=pq \text{ or } pl.$$

This sequence coincides with sequence 2) if $i=pq$.

4. Connection between the global cohomologies and the local cohomologies. From now on, we put the following assumption (C) on X, unless explicitly mentioned;

(C) *X has a section s over S, (i.e. $f\cdot s=id_S$), and satisfies $(f_T)_*(\mathcal{O}_{X_T})\cong\mathcal{O}_T$ for every $T\in(\mathbf{Sch}/S)$.*

The latter condition will be satisfied by the Künneth formula if (1) f is proper, S is the spectrum of the field k and $\Gamma(X, \mathcal{O}_X)\cong k$, or (2) f is flat, proper and whose fibres are separable (cf. EGA, IV_2 (4.6.2)) and $f_*(\mathcal{O}_X)\cong\mathcal{O}_S$.

First, we shall prove:

Lemma 1.3. *Let T be a quasi-compact S-prescheme and G be a commutative affine k-group scheme of finite type. Then,*
(1) $\mathrm{H}^1_i(X_T, G)\cong \mathbf{H}^1_i(X/S, G)(T)\times \mathrm{H}^1_i(T, G)$ (*direct product*), $i=pq, pl$.
(2) $\mathbf{H}^1_{pq}(X/S, G)(T)\cong \mathbf{H}^1_{pl}(X/S, G)(T)\cong\mathbf{PH}(G, X/S)(T)$
$\cong L(\mathrm{PH}(G, X/S))(T)$.

Proof. By virtue of the spectral theory (2) of §2, we have a spectral sequence,

$$E_2^{pq}=\mathrm{H}^p_{pq}(T, \mathbf{H}^q_{pq}(X/S, G))\Longrightarrow \mathrm{H}^*_{pq}(X_T, G).$$

The exact sequence of terms of low degree is

$$0\to \mathrm{H}^1_{pq}(T, \mathbf{Hom}_S(X, G))\to \mathrm{H}^1_{pq}(X_T, G)\to H^0(T, \mathbf{H}^1_{pq}(X/S, G)).$$

Put H a functor $T\in(\mathbf{Sch}/S)^0 \rightsquigarrow \mathrm{H}^1_{pq}(X_T, G)$. Then, taking account of the quasi-compactness of T, $LH(T)$ is calculated as follows:

$$\begin{aligned}LH(T)&=\varinjlim_{\substack{T'\to T\\ \text{fpqc}}} \mathrm{Ker}(H(T')\rightrightarrows \mathrm{H}(T'\times_T T'))\\ &=\varinjlim_{\substack{T'\to T\\ \text{fpqc}}} \mathrm{Ker}(\mathrm{PH}(G, X/S)(T')\rightrightarrows \mathrm{PH}(G, X/S)(T'\times_T T'))\\ &=L(\mathrm{PH}(G, X/S))(T').\end{aligned}$$

However, it is not difficult from the (fpqc)-descent theory for affine schemes that the canonical morphism $\mathrm{H}^1_{pq}(X_T, G)\to LH(T)$ is surjective. Since the canonical morphism $LH(T)\to L^2H(T)$ is injective, we have an exact sequence from the above exact sequence

$$0\to \mathrm{H}^1_{pq}(T, G)\underset{s_i^*}{\overset{f_i^*}{\rightleftarrows}} \mathrm{H}^1_{pq}(X_T, G)\to LH(T)\to 0$$

where one note that $\mathbf{Hom}_S(X, G) \cong G$ by the assumption (C). The sequence splits. Let $T' \overset{\alpha}{\to} T$ be a (fpqc)-morphism. Then, there exists sections $s_{T'}$: $T' \to X_{T'}$, $s_{T''}$: $T'' = T' \underset{T}{\times} T' \to X_{T''}$ and commutes the following diagram,

$$\begin{array}{ccc} X_{T''} & \overset{p_{1,X}}{\underset{p_{2,X}}{\rightrightarrows}} & X_{T'} \\ \uparrow s_{T''} & & \uparrow s_{T'} \\ T'' & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & T' \end{array} \quad \text{i.e.} \quad p_{i,X} \cdot s_{T''} = s_{T'} \cdot p_i, \quad i = 1, 2,$$

where p_1, p_2 are the canonical projections of T'' to T'. Since T': T'' are quasi-compact, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^1_{pq}(T, G) & \overset{f^*_T}{\underset{s^*_T}{\rightleftarrows}} & \mathrm{H}^1_{pq}(X_T, G) & \longrightarrow & LH(T) & \longrightarrow & 0 \\ & & \alpha^* \downarrow & & \alpha^*_X \downarrow & & LH(\alpha) \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}^1_{pq}(T', G) & \overset{f^*_{T'}}{\underset{s^*_{T'}}{\rightleftarrows}} & \mathrm{H}^1_{pq}(X_{T'}, G) & \longrightarrow & LH(T') & \longrightarrow & 0 \\ & & p^*_1 \downarrow\downarrow p^*_2 & & p^*_{1,X} \downarrow\downarrow p^*_{2,X} & & LH(p_1) \downarrow\downarrow LH(p_2) & & \\ 0 & \longrightarrow & \mathrm{H}^1_{pq}(T'', G) & \overset{f^*_{T''}}{\underset{s^*_{T''}}{\rightleftarrows}} & \mathrm{H}^1_{pq}(X_{T''}, G) & \longrightarrow & LH(T'') & \longrightarrow & 0 \end{array}$$

where the lines are exact and the columns are exact in the middle terms, without the right column. Then the diagram chasing shows that the right column is also exact, i.e. $LH(T) \cong L^2H(T)$.
Since $L^2H \cong \mathbf{PH}(G, X/S)$, we have $LH(T) \cong \mathbf{PH}(G, X/S)(T)$. For the case of $\mathbf{H}^1_{pl}(X/S, G)$, the proof is the same. q.e.d.

Next, we shall prove

Lemma 1.4. *Let T be a quasi-compact S-prescheme and G be a commutative affine smooth k-group scheme of finite type. Then we have* (1) $\mathrm{Lie}(\mathbf{PH}(G, X/S))(T) \cong \mathbf{PH}(\mathrm{Lie}(G), X/S)(T)$, (2) $\mathrm{Lie}(\mathbf{H}^1_i(X/S, G))(T) \cong \mathbf{H}^1_i(X/S, \mathrm{Lie}(G))(T)$, $i = pq, pl$.
For the definition of Lie-*functor of a group functor, see SGAD, Exp. II.*

Proof. We have an exact sequence of k-group schemes

$$0 \longrightarrow \mathrm{Lie}(G) \xrightarrow{i} G' \underset{e}{\overset{p}{\rightleftarrows}} G \longrightarrow 0, \quad G' = \mathbf{T}(G/k)$$

which splits by the unit section of G-group G'. Since G is smooth over k, $\mathrm{Lie}(G)$, hence G', are also smooth over k. Then, we have the following exact sequences of abelian sheaves of (fpqc)-(and (fppf)-) site $(\boldsymbol{Sch}/S)$,

$$0 \longrightarrow \mathbf{PH}(\mathrm{Lie}(G), X/S) \longrightarrow \mathbf{PH}(G', X/S) \rightleftarrows \mathbf{PH}(G, X/S) \longrightarrow 0$$

$$0 \longrightarrow \mathbf{H}^1_{pl}(X/S, \mathrm{Lie}(G)) \longrightarrow \mathbf{H}^1_{pl}(X/S, G') \rightleftarrows \mathbf{H}^1_{pl}(X/S, G) \to 0.$$

We shall prove now, $\mathrm{H}^1_{pl}(X_T, G') \cong \mathrm{H}^1_{pl}(X_{I_T}, G)$. Since G and G' are smooth over k, we have only to prove $\mathrm{H}^1_{et}(X_T, G') \cong \mathrm{H}^1_{et}(X_{I_T}, G)$ by virtue of Lemma 1.1. By the spectral theory (3) of §2, we have a spectral sequence, $\mathrm{E}_2^{p,q} = \mathrm{H}^p_{et}(X_T, R^q(\pi_{k,et})_*(G)) \Longrightarrow \mathrm{H}^*_{et}(X_{I_T}, G)$, where π_k is the canonical projection $I_k = \mathrm{Spec}(k[t]/(t^2)) \to \mathrm{Spec}(k)$. Then, since π_k induces an equivalence on the étale sites $(\mathbf{Sch}/S)$ and $(\mathbf{Sch}/I_S)$, (cf. SGAA, Exp. VIII, Th. 1.1), we know $R^q(\pi_{k,et})_*(G) = 0$, if $q > 0$. Hence, $\mathrm{H}^1_{et}(X_T, (\pi_{k,et})_*(G)) = \mathrm{H}^1_{et}(X_T, G') \cong \mathrm{H}^1_{et}(X_{I_T}, G)$. Since T is quasi-compact,

$$\begin{aligned}
&\mathrm{Lie}(\mathbf{PH}(G, X/S))(T) \cong \mathrm{Ker}(\mathbf{PH}(G, X/S)(I_T) \to \mathbf{PH}(G, X/S)(T)) \\
&\quad \cong \mathrm{Ker}\,(\mathrm{H}^1_{pl}(X_{I_T}, G)/\mathrm{H}^1_{pl}(I_T, G) \to \mathrm{H}^1_{pl}(X_T, G)/\mathrm{H}^1_{pl}(T, G)) \\
&\quad \cong \mathrm{Ker}(\mathrm{H}^1_{pl}(X_{I_T}, G) \to \mathrm{H}^1_{pl}(X_T, G))/\mathrm{Ker}(\mathrm{H}^1_{pl}(I_T, G) \to \mathrm{H}^1_{pl}(T, G)) \\
&\quad \cong \mathrm{H}^1_{pl}(X_T, \mathrm{Lie}(G))/\mathrm{H}^1_{pl}(T, \mathrm{Lie}(G)) \cong \mathbf{PH}(\mathrm{Lie}(G), X/S)(T).
\end{aligned}$$

The process of calculation will be clear without explanations.

q.e.d.

Corollary 1.5. *If T is locally noetherian, we have*

(1) $\mathrm{Lie}(\mathbf{PH}(G, X/S))(T) \cong \mathbf{PH}(\mathrm{Lie}(G), X/S)(T)$,

in particular $\mathrm{Lie}(\mathbf{PH}(G_m, X/S))(T) \cong \mathbf{PH}(G_a, X/S)(T)$.

(2) $\mathbf{PH}(\boldsymbol{Z}/n\boldsymbol{Z}, X/S)(T) \cong \mathrm{Ker}(\mathbf{PH}(G_m, X/S)(T) \xrightarrow{n}$

$\xrightarrow{n} \mathbf{PH}(G_m, X/S)(T))$,

if n is prime to the characteristic p of the field k.

If p is positive,

$$\mathbf{PH}(\mathbf{Z}/p\mathbf{Z}, X/S)(T) \cong \mathrm{Ker}(\mathbf{PH}(G_a, X/S)(T) \xrightarrow{F-id} \mathbf{PH}(G_a, X/S)(T)),$$

where F is the p-power operation of p-Lie algebra $\mathbf{PH}(G_a, X/S)(T)$ of which structure is induced from the Frobenius endomorphism on G_a, cf. SGAD, Exp. VII.

$$\mathbf{PH}(\mu_p, X/S)(T) \cong \mathrm{Ker}(\mathbf{PH}(G_m, X/S)(T) \xrightarrow{p} \mathbf{PH}(G_m, X/S)(T)).$$
$$\mathbf{PH}(\alpha_p, X/S)(T) \cong \mathrm{Ker}(\mathbf{PH}(G_a, X/S)(T) \xrightarrow{F} \mathbf{PH}(G_a, X/S)(T)).$$

Proof. If T is locally noetherian, T is covered by quasi-compact open sets $\{U_\alpha\}$ such that $U_\alpha \cap U_\beta$ is also quasi-compact, (cf. EGA, IV_1, (1.2.8)). Then (1) is proved as follows;

$$\mathbf{PH}(\mathrm{Lie}(G), X/S)(T) = \mathrm{Ker}(\prod_\alpha \mathbf{PH}(\mathrm{Lie(G)}, X/S)(U_\alpha) \rightrightarrows$$
$$\rightrightarrows \prod_{\alpha,\beta} \mathbf{PH}(\mathrm{Lie}(G), X/S)(U_\alpha \cap U_\beta)) = \mathrm{Ker}(\prod_\alpha \mathrm{Lie}(\mathbf{PH}(G, X/S))(U_\alpha) \rightrightarrows$$
$$\rightrightarrows \prod_{\alpha,\beta} \mathrm{Lie}(\mathbf{PH}(G, X/S))(U_\alpha \cap U_\beta)) = \mathrm{Lie}(\mathbf{PH}(G, X/S))(T),$$

where one note that $\mathrm{Lie}(\mathbf{PH}(G, X/S))$ is also a (fpqc)-sheaf on $(\mathbf{Sch}/S)$. The assertion (2) is easy to prove. q.e.d.

Under these preparations, we can state

Theorem 1.6. *Let X, S be as above. If the Picard prescheme* $\mathrm{Pic}(X/S)$ *exists and is locally of finite type over S, the contravariant functors $\mathbf{PH}(G_a, X/S)$, $\mathbf{PH}(\mathbf{Z}/n\mathbf{Z}, X/S)$, $\mathbf{PH}(\mathbf{Z}/p\mathbf{Z}, X/S)$, $\mathbf{PH}(\mu_p, X/S)$ and $\mathbf{PH}(\alpha_p, X/S)$ restricted to the category of locally noetherian S-preschemes, are representable and satisfies the relations on the above-mentioned category,*

$$\mathbf{PH}(G_a, X/S) \cong \mathrm{Lie}(\mathbf{Pic}(X/S)),\ \mathbf{PH}(\mathbf{Z}/n\mathbf{Z}, X/S) \cong {}_n(\mathbf{Pic}(X/S)),$$
$$\mathbf{PH}(\mathbf{Z}/p\mathbf{Z}, X/S) \cong \mathrm{Ker}(\mathrm{Lie}(\mathbf{Pic}(X/S)) \xrightarrow{F-id} \mathrm{Lie}(\mathbf{Pic}(X/S))),$$
$$\mathbf{PH}(\mu_p, X/S) \cong {}_p(\mathbf{Pic}(X/S)), \quad \textit{and}$$
$$\mathbf{PH}(\alpha_p, X/S) \cong \mathrm{Ker}(\mathrm{Lie}(\mathbf{Pic}(X/S)) \xrightarrow{F} \mathrm{Lie}(\mathbf{Pic}(X/S))).$$

Proof. Trivial.

Corollary 1.7. *Suppose T is quasi-compact. Then,*

(1) $\mathrm{Lie}(\mathbf{Pic}(X/S))(T) \cong \mathrm{H}^1(X_T, \mathcal{O}_{X_T})/\mathrm{H}^1(T, \mathcal{O}_T)$; *if* T *is an affine scheme* ($\cong \mathrm{Spec}(A)$), $\mathrm{Lie}(\mathbf{Pic}(X/S))(A) \cong \mathrm{H}^1(X_A, \mathcal{O}_{X_A})$. *The socle of the nilpotent part* $P(\mathrm{H}^1(X_A, \mathcal{O}_{X_A}))$ *of the Fitting decomposition of p-Lie algebra* $\mathrm{H}^1(X_A, \mathcal{O}_{X_A})$ *is equal to* $\mathbf{PH}(\alpha_p, X/S)(A)$. *If* k *is a field which contains* (p-1)*-th primitive root of unity, then* $\mathbf{PH}(\boldsymbol{Z}/p\boldsymbol{Z}, X/k)(\boldsymbol{k})$ *is equal to* $(\boldsymbol{Z}/p\boldsymbol{Z})^N$ *where* N *is equal to the k-dimension of the semi-simple part**) *of the Fitting decomposition of* $\mathrm{H}^1(X, \mathcal{O}_X)$.

(2) ${}_n(\mathbf{Pic}(X/S))(T) \cong \mathrm{H}^1_{et}(X_T, \boldsymbol{Z}/n\boldsymbol{Z})/\mathrm{H}^1_{et}(T, \boldsymbol{Z}/n\boldsymbol{Z})$. *Especially if* k *is separably algebraically closed,* ${}_n(\mathbf{Pic}(X/S))(k) \cong \mathrm{H}^1_{et}(X, \boldsymbol{Z}/n\boldsymbol{Z})$. ${}_p(\mathbf{Pic}(X/S))(T) \cong \mathrm{H}^1_{pl}(X_T, \mu_p)/\mathrm{H}^1_{pl}(T, \mu_p)$. *Especially if* k *is perfect,* ${}_p(\mathbf{Pic}(X/S))(k) \cong \mathrm{H}^1_{pl}(X, \mu_p)$. *Here, we are limited to the cases where* $\mathrm{Spec}(A)$ *and* $\mathrm{Spec}(k)$ *are S-preschemes.*

Proof. Easy. Note that the statement does not require $\mathbf{Pic}(X/S)$ to be locally of finite type over S. q.e.d.

The Picard prescheme is representable by a group scheme locally of finite type over S, if (1) $f: X \to S$ is projective, flat and the geometric fibres of f are integral (=reduced and irreducible), or if (2) $S = \mathrm{Spec}(k)$ and f is proper (cf. FGA n°232 and n°236 and [11]).

We shall treat in Chapter 4 the problem of the representability of a (fpqc)-sheaf $\mathbf{PH}(G, X/S)$.

Appendix to Chapter I.

In Lemma 1.1, Lemma 1.3, Lemma 1.4, Corollary 1.5, Theorem 1.6 and Corollary 1.7, we have used the quasi-compactness of a S-prescheme T. But this assumption is not essential and can be removed, if we note the following fact.

Lemma. *Let* G, X *and* S *be as in Lemma* 1.1. *Then*

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*) According to the terminology of J. Dieudonné [19], it corresponds to the core of p-Lie algebra $\mathrm{H}^1(X_A, \mathcal{O}_{X_A})$.

$\mathbf{PH}(G, X/S)(T)=\mathrm{H}^1_{pq}(X_T, G)$, *for arbitrary S-prescheme T.*

Proof. Let F be a (fpqc)-principal fibre sheaf over X_T with group G and $\mathfrak{U}=\{U_\alpha\}$ be a covering of T by affine open sets U_α. Then by Lemma 1.1, the restriction F_α of F on X_{U_α} is representable by a prescheme Y_α over X_{U_α}. Suppose $U_\alpha \cap U_\beta \neq \phi$. Then the restriction $F_\alpha | U_\alpha \cap U_\beta$ of F_α on $U_\alpha \cap U_\beta$ is representable by $Y_\alpha | U_\alpha \cap U_\beta$. Analogously, $F_\beta | U_\alpha \cap U_\beta$ is representable by $Y_\beta | U_\alpha \cap U_\beta$. Hence there exists a $X_{U_\alpha \cap U_\beta}$-isomorphism $\varphi_{\alpha\beta}$: $Y_\alpha | U_\alpha \cap U_\beta \to Y_\beta | U_\alpha \cap U_\beta$ such that $\varphi_{\beta\gamma} \cdot \varphi_{\alpha\beta} = \varphi_{\alpha\gamma}$ for α, β, γ such that $U_\alpha \cap U_\beta \cap U_\gamma \neq \phi$. Therefore $\{Y_\alpha\}$ defines a principal fibre space Y over X_T with group G which represents F. q.e.d.

Therefore for $G=G_a$, α_p, μ_p, $\boldsymbol{Z}/p\boldsymbol{Z}$, $\boldsymbol{Z}/n\boldsymbol{Z}$: $n \in \boldsymbol{N}$, $\mathbf{PH}(G, X/S)$ is representable on the category $(\mathbf{Sch}/S)$ if $\mathbf{Pic}(X/S)$ exists.

Chapter II. On the generalized Weil-Barsotti formula

In this chapter, we shall assume that S is a locally noetherian prescheme, and X is a projective abelian scheme over S, (cf. [14]). Then $f: X \to S$ satisfies the assumption (C) of chapter I. Let G be a commutative affine k-group scheme of finite type. We shall define a contravariant functor with respect to a triple (G, X, S) which corresponds to $\mathbf{PH}$-functor; for a S-prescheme T, let $\mathrm{Ext}_{T\text{-}gr}(X_T, G_T)$ be a set of isomorphism classes of Yoneda extensions of commutative T-groups

$$0 \longrightarrow G_T \xrightarrow{i} Y \xrightarrow{p} X_T \longrightarrow 0$$

where p is a (fpqc)-morphism, (cf. [12], III, §17). Then, by the (fpqc)-descent theory for affine morphisms, $\mathrm{Ext}_{T\text{-}gr}(X_T, G_T)$ is an abelian group, and for an exact sequence of commutative affine k-group schemes of finite type $0 \to G_1 \to G_2 \to G_3 \to 0$, we have an exact sequence,

$$1) \qquad 0 \to G_1(T) \to G_2(T) \to G_3(T) \to \mathrm{Ext}_{T\text{-}gr}(X_T, G_{1,T}) \to$$
$$\to \mathrm{Ext}_{T\text{-}gr}(X_T, G_{2,T}) \to \mathrm{Ext}_{T\text{-}gr}(X_T, G_{3,T}).$$

The (fpqc)-sheaf associated wiht the presheaf $T \rightsquigarrow \mathrm{Ext}_{T\text{-}gr}(X_T, G_T)$

on $(\mathbf{Sch}/S)$ is denoted by $\mathbf{Ext}_{S\text{-}gr}(X, G)$. Then for the sequence $0\to G_1\to G_2\to G_3\to 0$, we have an exact sequence of (fpqc)-abelian sheaves,

2) $0\to\mathbf{Ext}_{T\text{-}gr}(X, G_1)\to\mathbf{Ext}_{T\text{-}gr}(X, G_2)\to\mathbf{Ext}_{T\text{-}gr}(X, G_3)$.

On the other hand, X, G_S are considered (fpqc)-(resp. (fppf)-, étale) abelian sheaves on the (fpqc)-(resp. (fppf)-, étale) site$(\mathbf{Sch}/S)$. The i-th global Ext-group and the i-th local Ext-group are denoted by

$$\mathrm{Ext}^i_{T\text{-}gr}(X_T, G_T)_{pq},\ \mathbf{Ext}^i_{S\text{-}gr}(X, G)_{pq}(\text{resp. } \mathrm{Ext}^i_{T\text{-}gr}(X_T, G_T)_{pf},$$
$$\mathbf{Ext}^i_{S\text{-}gr}(X, G)_{pf},\ \mathrm{Ext}^i_{T\text{-}gr}(X_T, G_T)_{et},\ \mathbf{Ext}^i_{S\text{-}gr}(X, G)_{et}).$$

Then we have the following results which corresponds to the results of Chap. I, Lemma 1.1.

Lemma 2.1. *Let G, X, S be as above and T be a quasi-compact prescheme over S. Then,*

(1) $\mathrm{Ext}^1_{T\text{-}gr}(X_T, G_T)_{pq}\cong \mathrm{Ext}^1_{T\text{-}gr}(X_T, G_T)_{pf}\cong \mathrm{Ext}_{T\text{-}gr}(X_T, G_T)$,

(2) *if G is smooth over k, then*

$$\mathrm{Ext}^1_{T\text{-}gr}(X_T, G_T)_{pf}\cong \mathrm{Ext}^1_{T\text{-}gr}(X_T, G_T)_{et}.$$

(3) $\mathbf{Ext}_{S\text{-}gr}(X, G)\cong \mathbf{Ext}^1_{S\text{-}gr}(X, G)_{pq}$.

Proof. The assertion (2) only needs a proof. Since an (Yoneda) extension Y of $\mathrm{Ext}_{T\text{-}gr}(X_T, G_T)$; (Y): $0\to G_T\to Y\xrightarrow{p} X_T\to 0$ can be naturally considered a principal fibre space over X_T with group G in the sense of (fpqc)-topology, thus we have a homomorphism of abelian groups,

$$\pi:\ \mathrm{Ext}_{T\text{-}gr}(X_T, G_T)\to\mathbf{PH}(G, X/S)(T).$$

The extension (Y) is the one in the sense of (fpqc)-(resp. (fppf)-, étale) topology if and only if p is an epimorphism of (fpqc)-(resp. (fppf)-, étale) sheaves of sets. It depends only on the image of (Y) by π. If G is smooth over k, as $\mathrm{PH}(G, X/S)(T)\cong \mathrm{H}^1_{pf}(X_T, G)$ $\cong \mathrm{H}^1_{et}(X_T, G)$, the assertion (2) follows immediately. q.e.d.

For an exact sequence $0\to G_1\to G_2\to G_3\to 0$, we have an exact sequence,

$$0\to G_1(T)\to G_2(T)\to G_3(T)\to \mathrm{Ext}^1_{T\text{-}gr}(X_T, G_{1,T})_i\to$$
$$\to \mathrm{Ext}^1_{T\text{-}gr}(X_T, G_{2,T})_i\to \mathrm{Ext}^1_{T\text{-}gr}(X_T, G_{3,T})_i(\to \mathrm{Ext}^2_{T\text{-}gr}(X_T, G_{1,T})_i\to\cdots),$$

for $i=pq,\ pl$. This sequence coincides with the sequence 1) if T is quasi-compact. For the local case, we have an exact sequence,

$$0\to \mathbf{Ext}^1_{S\text{-}gr}(X, G_1)_i\to \mathbf{Ext}^1_{S\text{-}gr}(X, G_2)_i\to$$
$$\to \mathbf{Ext}^1_{S\text{-}gr}(X, G_3)_i\to (\mathbf{Ext}^2_{S\text{-}gr}(X, G_1)_i\to\cdots)$$

for $i=pq,\ pl$. This sequence coincides with the sequence 2) if $i=pq$. Next, we shall state results connecting the local extension groups with the global extension groups.

Lemma 2. 2. *Let G, X, S be as above and T be a quasi-compact S-prescheme. Then, we have,*

$$\mathbf{Ext}^1_{S\text{-}gr}(X, G)_i(T)\cong \mathrm{Ext}^1_{T\text{-}gr}(X_T, G_T)_i, \quad \textit{for } i=pq,\ pl$$

and

$$\mathbf{Ext}_{S\text{-}gr}(X, G)(T)\cong \mathrm{Ext}_{T\text{-}gr}(X_T, G_T)$$
$$\cong \varinjlim_{\substack{T'\to T\\ fpqc}} \mathrm{Ker}(\mathrm{Ext}_{T'\text{-}gr}(X_{T'}, G_{T'}) \rightrightarrows \mathrm{Ext}_{(T'\times_T T')\text{-}gr}(X_{T'\times_T T'}, G_{T'\times_T T'})).$$

Proof. We use here the spectral theory (1) of Chap. I, §2. There exists a spectral sequence,

$$E_2^{p,q}=\mathrm{H}^p_{pq}(T, \mathbf{Ext}^q_{S\text{-}gr}(X, G)_{pq})\Longrightarrow \mathrm{Ext}^*_{T\text{-}gr}(X_T, G_T)_{pq}.$$

The exact sequence of terms of low degree is,

$$0\to \mathrm{H}^1_{pq}(T, \mathbf{Hom}_{S\text{-}gr}(X, G))\to \mathrm{Ext}^1_{T\text{-}gr}(X_T, G_T)_{pq}\to$$
$$\to \mathrm{H}^0_{pq}(T, \mathbf{Ext}^1_{S\text{-}gr}(X, G)_{pq}).$$

Since $(f_T)_*(\mathcal{O}_{X_T})\cong \mathcal{O}_T$ from the hypothesis and since

$$\mathrm{Ext}^1_{T\text{-}gr}(X_T, G_T)_{pq}\to$$
$$\to \varinjlim_{\substack{T'\to T\\ fpqc}} \mathrm{Ker}(\mathrm{Ext}^1_{T'\text{-}gr}(X_{T'}, G_{T'})_{pq} \rightrightarrows \mathrm{Ext}_{(T'\times_T T')\text{-}gr}(X_{T'\times_T T'}, G_{T'\times_T T'})_{pq})$$

is surjective by virtue of the (fpqc)-descent theory for affine morphisms, we can easily get the results. The proof is the same for the (fppf)-case. q.e.d.

The following results correspond to Lemma 1.4 of Chap. I.

Lemma 2.3. *Let T be a quasi-compact S-prescheme and suppose G is smooth over k. Then we have,*

$$\mathrm{Lie}(\mathbf{Ext}_{S\text{-}gr}(X, G))(T) \cong \mathbf{Ext}_{S\text{-}gr}(X, \mathrm{Lie}(G))(T)$$
$$\mathrm{Lie}(\mathbf{Ext}^1_{S\text{-}gr}(X, G)_i)(T) \cong \mathbf{Ext}^1_{S\text{-}gr}(X, \mathrm{Lie}(G))_i(T).$$

for $i=pq$, pl.

Proof. The proof is analogous. We use the spectral theory (4), of Chap. I, §1. Then the corresponding spectral sequence is,

$$E_2^{p,q} = \mathrm{Ext}^p_{T\text{-}gr}(X_T, R^q(\pi_{k,\acute{e}t})_*(G))_{\acute{e}t} \Longrightarrow \mathrm{Ext}^*_{I_T\text{-}gr}(X_{I_T}, G_{I_T})_{\acute{e}t},$$

where the notations of Lemma 1.4 is used. Hence,

$$\mathrm{Ext}^p_{T\text{-}gr}(X_T, G'_T)_{\acute{e}t} \cong \mathrm{Ext}^p_{I_T\text{-}gr}(X_{I_T}, G_{I_T})_{\acute{e}t}.$$

We leave to readers the work to complete the proof. q.e.d.

Corollary 2.4. *If T is locally noetherian, we have*

(1) $\mathrm{Lie}(\mathbf{Ext}_{S\text{-}gr}(X, G))(T) \cong \mathbf{Ext}_{S\text{-}gr}(X, \mathrm{Lie}(G))(T)$.

In particular,

$$\mathrm{Lie}(\mathbf{Ext}_{S\text{-}gr}(X, G_m))(T) \cong \mathbf{Ext}_{S\text{-}gr}(X, G_a)(T).$$

(2) $\mathbf{Ext}_{S\text{-}gr}(X, \mathbf{Z}/n\mathbf{Z})(T) \cong {}_n(\mathbf{Ext}_{S\text{-}gr}(X, G_m))(T)$, *if n is prime to the characteristic p of the field k. If p is positive,*

$$\mathbf{Ext}_{S\text{-}gr}(X, \mathbf{Z}/p\mathbf{Z})(T) \cong \mathrm{Ker}(\mathbf{Ext}_{S\text{-}gr}(X, G_a)(T)) \xrightarrow{F-id} \to \mathrm{Ext}_{S\text{-}gr}(X, G_a)(T))$$

where F is the endomorphism induced from the Frobenius endomorphism of G_a

$$\mathbf{Ext}_{S\text{-}gr}(X, \mu_p)(T) \cong {}_p(\mathbf{Ext}_{S\text{-}gr}(X, G_m))(T),$$

and

$$\mathbf{Ext}_{S\text{-}gr}(X, \alpha_p)(T) \cong \mathrm{Ker}(\mathbf{Ext}_{S\text{-}gr}(X, G_a)(T) \xrightarrow{F} \to \mathbf{Ext}_{S\text{-}gr}(X, G_a)(T)).$$

Theorem 2.5. *Let X, S be as above. Then we have,*

(1) *(the generalized Weil-Barsotti formula)* (*cf.* [14], *III.* §18),

$$\mathbf{Ext}_{S-gr}(X, G_m)(T) \cong \mathbf{Pic}^0(X/S)(T) \hookrightarrow \mathbf{Pic}(X/S)(T)$$

where T is locally noetherian and $\mathbf{Pic}^0(X/S)$ ($\equiv X^t$: *the dual abelian scheme of X*) *is the connected component of* $\mathbf{Pic}(X/S)$ *which contains the unit of* $\mathbf{Pic}(X/S)$.

(2) *The contravariant functors of abelian groups,*

$$\mathbf{Ext}_{S-gr}(X, G_m),\ \mathbf{Ext}_{S-gr}(X, G_a),\ \mathbf{Ext}_{S-gr}(X, \mathbf{Z}/n\mathbf{Z}),$$
$$\mathbf{Ext}_{S-gr}(X, \mathbf{Z}/p\mathbf{Z}),\ \mathbf{Ext}_{S-gr}(X, \mu_p)\ \textit{and}\ \mathbf{Ext}_{S-gr}(X, \alpha_p)$$

restricted to the category of locally noetherian S-preschemes are representable and satisfy the relations on the above-mentioned category,

$$\mathbf{Ext}_{S-gr}(X, G_m) \cong X^t,\ \mathbf{Ext}_{S-gr}(X, G_a) \cong \mathrm{Lie}(X^t),\ \mathbf{Ext}_{S-gr}(X, \mathbf{Z}/n\mathbf{Z}) \cong {}_n(X^t),\ \mathbf{Ext}_{S-gr}(X, \mathbf{Z}/p\mathbf{Z}) \cong \mathrm{Ker}(\mathrm{Lie}(X^t) \xrightarrow{F-id} \mathrm{Lie}(X^t)).$$

$$\mathbf{Ext}_{S-gr}(X, \mu_p) \cong {}_p(X^t)\ \ \textit{and}\ \ \mathbf{Ext}_{S-gr}(X, \alpha_p) \cong \mathrm{Ker}(\mathrm{Lie}(X^t) \xrightarrow{F} \mathrm{Lie}(X^t)).$$

(3) *If G is a commutative finite k-group scheme, then*

$$\mathbf{Ext}_{S-gr}(X, G) \cong \mathbf{PH}(G, X/S)$$

on the above-mentioned category.

Proof. *F.* Oort [14] has proved that if T is a locally noetherian S-prescheme, $\mathrm{Ext}_{T-gr}(X_T, G_{m,T}) \cong X^t(T)$. Then it is easy to see

$$\mathbf{Ext}_{S-gr}(X, G_m)(T) \cong X^t(T).$$

The assertion (2) comes from Corollary 2.4. For the proof of (3), see next corollaries. q.e.d.

Corollary 2.6. *Suppose T is a noetherian prescheme. Then,*

(1) $\mathrm{Ext}_{T-gr}(X_T, G_{a,T}) \cong \mathrm{Lie}(\mathbf{Pic}(X/S))(T) \cong \mathrm{H}^1(X_T, \mathcal{O}_{X_T})/\mathrm{H}^1(T, \mathcal{O}_T)$.
If T is affine (i.e. $T \cong \mathrm{Spec}(A)$), $\mathrm{Ext}_{A-gr}(X_A, G_{a,A}) \cong \mathrm{H}^1(X_A, \mathcal{O}_{X_A})$.
$\mathrm{Ext}_{A-gr}(X_A, (\mathbf{Z}/p\mathbf{Z})_A) \cong \mathbf{PH}(\mathbf{Z}/p\mathbf{Z}, X/S)(A) \cong \mathrm{H}^1_{\text{et}}(X_A, \mathbf{Z}/p\mathbf{Z})/\mathrm{H}^1_{\text{et}}(A,$

Z/pZ). $\mathrm{Ext}_{A\text{-}gr}(X_A, (\alpha_p)_A) \cong \mathbf{PH}(\alpha_p, X/S)(A) \cong$ *the socle**) *of nilpotent part of* $\mathrm{H}^1(X_A, \mathcal{O}_{X_A})$.

(2) $\mathrm{Ext}_{T\text{-}gr}(X_T, (\boldsymbol{Z}/n\boldsymbol{Z})_T) \cong {}_n(\mathbf{Pic}(X/S))(T)$

$$\cong \mathrm{H}^1_{et}(X_T, \boldsymbol{Z}/n\boldsymbol{Z})/\mathrm{H}^1_{et}(T, \boldsymbol{Z}/n\boldsymbol{Z}).$$

$$\mathrm{Ext}_{T\text{-}gr}(X_T, (\mu_p)_T) \cong {}_p(\mathbf{Pic}(X/S))(T) \cong \mathrm{H}^1_{pl}(X_T, \mu_p)/\mathrm{H}^1_{pl}(T, \mu_p).$$

Proof. Combine the results of Theorem 2. 5 with Corollary 1. 7 of Chapter I. Only note that $\mathbf{Pic}(X/S)/\mathbf{Pic}^0(X/S)$ has no torsion cf. [12] and that $\mathrm{Lie}(X^t) \cong \mathrm{Lie}(\mathbf{Pic}(X/S))$.

Corollary 2. 7. (*cf.* [10] *and* [12]). *If k is an algebraically closed field, and X is an abelian scheme over k, we have* $\mathrm{Ext}_{k\text{-}gr}(X, G) \cong \mathrm{H}^1_{pl}(X, G)$, *for any commutative finite group scheme G over k.*

Proof. By virtue of Corollary 2. 6, the assertion is correct for simple commutative finite group schemes over k, hence it is correct for all commutative finite group schemes over k, (cf. [10]). q.e.d.

Chapter III. On the fundamental group

1. In this chapter, the field k is supposed to be algebraically closed and of positive characteristic p. Let X be an integral scheme of finite type over k. In [8], we saw that covariant functors $C^c_f(k) \ni G \rightsquigarrow \mathrm{E}_k(G, X) \in (\mathbf{Ab})$, $C^{\mathrm{inf}}_f(k) \ni G \rightsquigarrow E_k(G, X) \in (\mathbf{Sets})$ and $C_f(k) \ni G \rightsquigarrow E_k(G; X, x) \in (\mathbf{Sets})$ are strictly pro-representable where $C^c_f(k)$ (resp. $C^{\mathrm{inf}}_f(k)$, $C_f(k)$) is a category of commutative finite k-group schemes (resp. infinitesimal k-group schemes, finite k-group schemes) and x is a generic point of X. $E_k(G, X)$ is nothing but $\mathrm{PH}(G, X/k)(k)$. Denote by $F_c(X)$, $F_{\mathrm{inf}}(X)$ and $F(X, x)$ the pro-finite k-group schemes which pro-represent the above functors.

If $F(X, x)$ is an projective limit $\varprojlim_i G^i(X, x)$, where $G^i(X, x)$ are finite k-group schemes, $F_{\mathrm{inf}}(X)$ is isomorphic to the projective limit

*) $P(\mathrm{H}^1(X_A, \mathcal{O}_{X_A}))$.

$\varprojlim_i G^i(X, x)_{\rm inf}$ of maximal infinitesimal subgroup schemes $G^i(X, x)_{\rm inf}$ of $G^i(X, x)$. The quotient $F(X, x)/F_{\rm inf}(X)$ is the fundamental group of X at x in the sense of A. Grothendieck [4]. $F_c(X)$ is isomorphic to the quotient $F(X, x)/[F(X, x), F(X, x)]$ of $F(X, x)$ by its commutator subgroup $[F(X, x), F(X, x)]$.

Now we shall calculate $F_c(X)$ for a proper integral k-scheme X. Since k is algebraically closed, $\mathrm{PH}(G, X/k)(k) \cong \mathrm{H}^1_{pq}(X, G) \cong E_k(G, X) \cong \mathrm{Hom}_{k\text{-}groups}(F_c(X), G)$ for any commutative finite k-group scheme G. $F_c(X)$ is decomposed to a direct product of four subgroups $F_c(X)_{rr}$, $F_c(X)_{rl}$, $F_c(X)_{lr}$ and $F_c(X)_{ll}$, corresponding to the decomposition of the category $C^c_f(k)$ into $\mathcal{A}_{rr} \times \mathcal{A}_{rl} \times \mathcal{A}_{lr} \times \mathcal{A}_{ll}$, (cf. [14]).

Then our result is

Theorem. 3.1. *Let X be a proper integral k-scheme. Then,*
(1) $F_c(X)_{rr} \cong \prod_{\substack{l \neq p \\ l:\ \text{prime}}} \boldsymbol{Z}_l^{2\dim(\mathbf{Pic}^0(X)_{red})}$.
(2) $F_c(X)_{rl} \cong \boldsymbol{Z}_p^{\sigma_1(X)}$, *where $\sigma_1(X)$ is the k-dimension of the semi-simple part of p-*Lie *algebra* $\mathrm{Lie}(\mathbf{Pic}(X/k))(k) \cong \mathrm{H}^1(X, \mathcal{O}_X)$.
(3) $F_c(X)_{lr} \cong (K_\infty)^{\sigma_2(X)}$, *where $\sigma_2(X)$ is the k-dimension of the semi-simple part of p-Lie algebra* $\mathrm{Lie}(\mathbf{Pic}(X/k)_{red})(k)$. *For the definition of K_∞, see* [14]. *(We shall see that $\sigma_2(X)$ is equal to $\sigma_1(X)$ in the proof of* (4)).
(4) $F_c(X)_{ll} \cong \varprojlim_n D(\mathrm{Ker}(\mathbf{Pic}(X/k) \xrightarrow{F^n} \mathbf{Pic}(X/k)))/\boldsymbol{Z}_p^{\sigma_1(X)}$.

The term of the right hand side of the equality (4) *is an extension of the fundamental group $F(\widehat{\mathbf{Pic}^0(X)_{red}/(\widehat{G}_m)^{\sigma_2(X)}})$ (cf.* [9]) *by an quotient of the finite group scheme $D(\boldsymbol{NS}'^0(X))$ where $D(\boldsymbol{NS}'^0(X))$ is the linear dual of the connected component of the unit of the Neron-Severi group scheme $\boldsymbol{NS}'(X)$ (see Footnote of p. 23) of X.*

Proof. (1), (2) and (3) follows from Theorem 1.6 and Corollary 1.7. For the proof of (4), we use the results of T. Oda [12], p. 73 and 74.

Put $P = \mathbf{Pic}(X/k)$, $P^0 = \mathbf{Pic}^0(X/k)$ and ${}_{F^n}P = \mathrm{Ker}(P \xrightarrow{F^n} P)$.

Denote by $H_n(P)$ the dual vector space of $\mathcal{O}_{P,e}/F^n(\mathfrak{M}_{P,e})\mathcal{O}_{P,e}$, $(\mathcal{O}_{P,e}, \mathfrak{M}_{P,e})$ being the local ring of P at the unit e. $H_n(P)$ can be considered as the hyperalgebra of P formed by invariant derivations of height $\leq n$, (cf. [19]). Then $D({}_{F^n}P)=\mathrm{Spec}(H_n(P))$ and $\mathrm{H}^1(X, W_{n,m})\cong\mathrm{Hom}_{k\text{-groups}}(\mathrm{Spec}(H(P)), W_{n,m})$, where $W_{n,m}={}_{F^m}W_n$, W_n being the Witt group scheme of length n and where $\mathrm{Spec}(H(P))=\varinjlim_n \mathrm{Spec}(H_n(P))$, with transition maps $D(i_n)$: $D({}_{F^{n+1}}P)\rightarrow D({}_{F^n}P)$, i_n being the canonical injection i_n: ${}_{F^n}P\rightarrow{}_{F^{n+1}}P$. An easy calculation shows that $F_c(X)_{ll}\cong\varprojlim_n D({}_{F^n}P)/(\boldsymbol{Z}_p)^{\sigma_1(X)}$. Note that ${}_{F^n}P$ is identified with ${}_{F^n}\widehat{P}$, $\widehat{P}$ being the completion of P at the unit.

Consider an exact sequence,

$$0\longrightarrow P^0_{red}\longrightarrow P\longrightarrow \boldsymbol{NS}'(X)\longrightarrow 0.$$

Then for a positive integer n large enough, we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & {}_{F^n}P^0_{red} & \longrightarrow & {}_{F^n}P & \longrightarrow & \boldsymbol{NS}'^0(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_{red}} & & \downarrow{\scriptstyle i} & & \Vert{\scriptstyle id.} & & \\
0 & \longrightarrow & {}_{F^{n+1}}P^0_{red} & \longrightarrow & {}_{F^{n+1}}P & \longrightarrow & \boldsymbol{NS}'^0(X) & \longrightarrow & 0,
\end{array}$$

or a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D(\boldsymbol{NS}'^0(X)) & \longrightarrow & D({}_{F^{n+1}}P) & \longrightarrow & D({}_{F^{n+1}}P_{red}) & \longrightarrow & 0 \\
 & & \Vert{\scriptstyle id.} & & \downarrow{\scriptstyle D(i)} & & \downarrow{\scriptstyle D(i_{red})} & & \\
0 & \longrightarrow & D(\boldsymbol{NS}'^0(X)) & \longrightarrow & D({}_{F^n}P) & \longrightarrow & D({}_{F^n}P_{red}) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & .
\end{array}$$

Replace ${}_{F^n}P$, ${}_{F^n}P_{red}$ by ${}_{F^n}\widehat{P}$, ${}_{F^n}\widehat{P}_{red}$ and take projective limits. Then we have an exact sequence,

$$(*)\qquad 0\longrightarrow D(\boldsymbol{NS}'^0(X))\longrightarrow\varprojlim_n D({}_{F^n}\widehat{P})\longrightarrow\varprojlim_n D({}_{F^n}\widehat{P}_{red})\longrightarrow 0.$$

Put $N_1=D(\boldsymbol{NS}'^0(X))\cap(\boldsymbol{Z}_p)^{\sigma_1(X)}$. Then N_1 is a finite abelian p-

group. From the sequence $(*)$, we have

$$0\to D(NS'^0(X))/N_1\to\varprojlim_n D({}_{F^n}P)/(Z_p)^{\sigma_1(X)}\to\varprojlim_n D({}_{F^n}P_{\mathrm{red}})/(Z_p)^{\sigma_2(X)}\to 0.$$

This is an exact sequence of local profinite k-group schemes and proves the last assertion of (4), since $F(\widehat{\mathbf{Pic}^0(X)_{\mathrm{red}}}/(\widehat{G}_m)^{\sigma_2(X)})\cong\varprojlim_n D({}_{F^n}\widehat{P}_{\mathrm{red}})/(Z_p)^{\sigma_2(X)}$. At the same time, we have obtained an equality $\sigma_1(X)=\sigma_2(X)$. q.e.d.

Consequently, we have a formula,

$$F_c(X)\cong\prod_{\substack{l\neq p\\ l:\ \mathrm{prime}}} Z_l^{2\dim(\mathbf{Pic}(X/k))}\times Z_p^{\sigma(X)}\times K_\infty^{\sigma(X)}$$
$$\times((\varprojlim_n D({}_{F^n}\mathbf{Pic}(X/k)))/(Z_p)^{\sigma(X)})\qquad \sigma(X)\equiv(\sigma_1(X)=\sigma_2(X)).$$

Corollary 3. 2. *Let X be a proper integral k-scheme. Then,*

$$\mathrm{H}^1_{pq}(X, G)\cong\mathrm{Hom}_{k\text{-}groups}(D(G), \mathbf{Pic}(X/k))$$

for any commutative finite k-group scheme G.

Proof. $\mathrm{H}^1(X, G)\cong\mathrm{Hom}_{k\text{-}groups}(F_c(X), G)\cong\mathrm{Hom}_{k\text{-}groups}(D(G), D(F_c(X)))$ where $D(F_c(X))\cong\bigoplus_{\substack{l\neq p\\ l:\ \mathrm{prime}}}(Q_l/Z_l)^{2\dim(\mathbf{Pic}(X/k))}\oplus(Q_p/Z_p)^{\sigma(X)}\oplus(\widehat{G}_m)^{\sigma(X)}\oplus\widehat{\mathbf{Pic}(X/k)}/(\widehat{G}_m)^{\sigma(X)}\cong\varinjlim$(finite group schemes of $\mathbf{Pic}(X/k)$). Hence $\mathrm{Hom}_{k\text{-}groups}(D(G), D(F_c(X)))\cong\mathrm{Hom}_{k\text{-}groups}(D(G), \mathbf{Pic}(X/k))$.

q.e.d.

Remark. The formula of Corollary 3. 2 is stated in [4] without explicit proof.

2. The isomorphism of Corollary 3.2 can be given an explicit form under the additional assumptions:

X is a proper integral k-scheme such that (i) the connected component $\mathbf{Pic}^0(X/k)$ *of the unit in* $\mathbf{Pic}(X/k)$ *is an abelian scheme and such that (ii) the Neron-Severi group**) $NS(X)=\mathbf{Pic}(X/k)/$

*) We can call $NS'(X)=\mathbf{Pic}(X/k)/\mathbf{Pic}^0(X/k)_{\mathrm{red}}$ the real Neron-Severi group and distinguish it from $NS(X)$.

Pic$^0(X/k)$ *is torsion-free.*

The dual abelian variety $(\mathbf{Pic}^0(X/k))^t$ is the Albanese variety $\mathbf{Alb}(X/k)$ of X. We choose a k-rational point x_0 of X and a canonical morphism η: $X\rightarrow\mathbf{Alb}(X/k)$ such that $\eta(x_0)=$ the unit of $\mathbf{Alb}(X/k)$. Let $A=\mathbf{Alb}(X/k)$. Consider a homomorphism η^*: $\mathrm{H}^1_{pq}(A, G)\rightarrow\mathrm{H}^1_{pq}(X, G)$, for a commutative finite k-group scheme G, which sends $B\in\mathrm{H}^1_{pq}(A, G)$ to $Y=B\underset{A}{\times}X\in\mathrm{H}^1_{pq}(X, G)$. Since A is an abelian scheme, $\mathrm{H}^1_{pq}(A, G)$ is canonically identified with $\mathrm{Ext}_{k-gr}(A, G)$ (cf. Corollary 2. 7).

Take an extension B in $\mathrm{Ext}_{k-gr}(A, G)$,

$$0\longrightarrow G\longrightarrow B\longrightarrow A\longrightarrow 0.$$

Let $B'=(B^0)_{red}$ and $N=G\cap B'$. Then $B/B'\cong G/N$ by the Snake Lemma; see the following commutative diagram,

$$\begin{array}{ccccccccc} & & G/N & \longrightarrow & B/B' & & & & \\ & & \uparrow & & \uparrow & & & & \\ 0 & \longrightarrow & G & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & B' & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

The duality of Nishi-Cartier gives an extension,

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \uparrow & & & & & & \\ 0 & \longrightarrow & D(N) & \xrightarrow{j} & A^t=\mathbf{Pic}^0(X/k) & \longrightarrow & B'' & \longrightarrow & 0 \\ & & \text{can.}\uparrow\text{proj.} & \nearrow & & & & & \\ & & D(G) & & & & & & \end{array}.$$

The composite morphism $D(G)\xrightarrow{\text{can. proj.}}D(N)\xrightarrow{j}\mathbf{Pic}^0(X/k)$ defines an element $\varphi(B)$ of $\mathrm{Hom}_{k-gr}(D(G), \mathbf{Pic}^0(X/k))$ (we denote this map by φ_G or simply by φ). Then we have

Lemma 3. 3. *The map* φ_G: $B\in\mathrm{Ext}_{k-gr}(A, G)\rightsquigarrow\varphi(B)\in\mathrm{Hom}_{k-gr}(D(G), \mathbf{Pic}^0(X/k))$ *is an isomorphism of abelian groups.*

Proof. Let B, B' be elements of $\mathrm{Ext}_{k-gr}(A, G)$. Let $\overline{B}=(B^0)_{red}$, $\overline{B}'=(B'^0)_{red}$, $N=\overline{B}\cap G$ and $G'=\overline{B}'\cap G$. Then they give two extensions,

$$0 \longrightarrow N \longrightarrow \overline{B} \longrightarrow A \longrightarrow 0$$
$$0 \longrightarrow N' \longrightarrow \overline{B}' \longrightarrow A \longrightarrow 0.$$

The exact sequences of local Ext-groups are, then,

$$(*) \quad \begin{cases} 0 \rightarrow \mathbf{Hom}_{k\text{-}gr}(N, G_m) \xrightarrow{j} \mathbf{Ext}_{k\text{-}gr}(A, G_m) \rightarrow \mathbf{Ext}_{k\text{-}gr}(\overline{B}, G_m) \\ 0 \rightarrow \mathbf{Hom}_{k\text{-}gr}(N', G_m) \xrightarrow{j'} \mathbf{Ext}_{k\text{-}gr}(A, G_m) \rightarrow \mathbf{Ext}_{k\text{-}gr}(\overline{B}', G_m) \end{cases}$$

The Cartier-Shatz formula and the Weil-Barsotti formula (cf. [14]) show that $\varphi(B) = j \cdot \pi$, $\varphi(B') = j' \cdot \pi'$ where π (resp. π') is the canonical projection $D(G) \rightarrow D(N)$ (resp. $D(G) \rightarrow D(N')$).

Suppose $\varphi(B) = \varphi(B')$. Then $N = N'$, $j = j'$ and $\pi = \pi'$. On the other hand, B(resp. B') is obtained by extending the group N to G from $\overline{B}$(resp. $\overline{B}'$).

Consider a diagram,

$$\begin{array}{ccc} \mathrm{Ext}_{k\text{-}gr}(A, N) & \xrightarrow{\varphi_N} & \mathrm{Hom}_{k\text{-}gr}(D(N), \mathbf{Pic}^0(X/k)) \\ \downarrow & & \downarrow \\ \mathrm{Ext}_{k\text{-}gr}(A, G) & \xrightarrow{\varphi_G} & \mathrm{Hom}_{k\text{-}gr}(D(G), \mathbf{Pic}^0(X/k)). \end{array}$$

It is evidently commutative and the vertical arrows are injective. Then B is isomorphic to B' if $\overline{B}$ is isomorphic to $\overline{B}'$. Therefore, we can assume $N = G$, $B = \overline{B}$ and $B' = \overline{B}'$.

Suppose first that G can be embedded in G_m (resp. G_a). From the exact sequences $(*)$, we have

$$\begin{array}{ccccc} 0 \rightarrow \mathrm{Hom}_{k\text{-}gr}(G, G_m) & \xrightarrow{\varphi(B)(k)} & \mathrm{Ext}_{k\text{-}gr}(A, G_m) & \rightarrow & \mathrm{Ext}_{k\text{-}gr}(B, G_m) \\ \| & & \| & & \\ 0 \rightarrow \mathrm{Hom}_{k\text{-}gr}(G, G_m) & \xrightarrow{\varphi(B')(k)} & \mathrm{Ext}_{k\text{-}gr}(A, G_m) & \rightarrow & \mathrm{Ext}_{k\text{-}gr}(B', G_m) \end{array}$$

resp.

$$\begin{array}{ccccc} 0 \rightarrow \mathrm{Hom}_{k\text{-}gr}(G, G_a) & \xrightarrow{\mathrm{Lie}(\varphi(B))(k)} & \mathrm{Ext}_{k\text{-}gr}(A, G_a) & \rightarrow & \mathrm{Ext}_{k\text{-}gr}(B, G_a) \\ \| & & \| & & \\ 0 \rightarrow \mathrm{Hom}_{k\text{-}gr}(G, G_a) & \xrightarrow{\mathrm{Lie}(\varphi(B'))(k)} & \mathrm{Ext}_{k\text{-}gr}(A, G_a) & \rightarrow & \mathrm{Ext}_{k\text{-}gr}(B', G_a) \end{array}$$

where we shall note that we have $\mathbf{Hom}_{k\text{-}gr}(G, G_m)(k) \cong \mathrm{Hom}_{k\text{-}gr}(G, G_m)$, $\mathbf{Hom}_{k\text{-}gr}(G, G_a)(k) \cong \mathrm{Hom}_{k\text{-}gr}(G, G_a)$, $\mathbf{Ext}_{k\text{-}gr}(A, G_m)(k) \cong$

$\mathrm{Ext}_{k\text{-}gr}(A, G_m)$, $\mathbf{Ext}_{k\text{-}gr}(A, G_a)(k) \cong \mathrm{Ext}_{k\text{-}gr}(A, G_a)$ and $\mathrm{Lie}(\mathbf{Ext}_{k\text{-}gr}(A, G_m))(k) \cong \mathbf{Ext}_{k\text{-}gr}(A, G_a)(k)$. Let i be the injection of G into G_m (resp. G_a). Then $\varphi(B)(k)(i)$ (resp. $\mathrm{Lie}(\varphi(B))(k)(i)$) is the class of the extension which is obtained from B by extending G to G_m (resp. G_a). Since $\varphi(B) = \varphi(B')$, we have $\varphi(B)(k)(i) = \varphi(B')(k)(i)$ and $\mathrm{Lie}(\varphi(B))(k)(i) = \mathrm{Lie}(\varphi(B'))(k)(i)$. On the other hand, a morphism $\mathrm{Ext}(A, i)\colon \mathrm{Ext}_{k\text{-}gr}(A, G) \to \mathrm{Ext}_{k\text{-}gr}(A, G_m)$ (resp. a morphism $\mathrm{Ext}(A, i)\colon \mathrm{Ext}_{k\text{-}gr}(A, G) \to \mathrm{Ext}_{k\text{-}gr}(A, G_a)$) is injective and $\varphi(B)(k)(i) = \mathrm{Ext}(A, i)(B)$ (resp. $\mathrm{Lie}(\varphi(B))(k)(i) = \mathrm{Ext}(A, i)(B)$) the same equality being valid for B'. Hence B is isomorphic to B'. Therefore $\varphi_G\colon \mathrm{Ext}_{k\text{-}gr}(A, G) \to \mathrm{Hom}_{k\text{-}gr}(D(G), \mathbf{Pic}^0(X/k))$ is injective if G can be embedded into G_m (resp. G_a).

For an arbitrary commutative group scheme G, the induction argument on the k-rank of G reduces us to the following situation:

If $0 \to G_1 \to G_2 \to G_3 \to 0$ is an exact sequence of commutative finite k-group schemes such that φ_{G_1} and φ_{G_3} are injective, then φ_{G_2} is also injective. This can be observed from a commutative diagram,

$$
\begin{array}{ccccc}
 & 0 & & 0 & \\
 & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathrm{Ext}_{k\text{-}gr}(A, G_1) & \xrightarrow{\varphi_{G_1}} & \mathrm{Hom}_{k\text{-}gr}(D(G_1), \mathbf{Pic}^0(X/k)) & \\
 & \downarrow & & \downarrow & \\
 & \mathrm{Ext}_{k\text{-}gr}(A, G_2) & \xrightarrow{\varphi_{G_3}} & \mathrm{Hom}_{k\text{-}gr}(D(G_2), \mathbf{Pic}^0(X/k)) & \\
 & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathrm{Ext}_{k\text{-}gr}(A, G_3) & \xrightarrow{\varphi_{G_2}} & \mathrm{Hom}_{k\text{-}gr}(D(G_3), \mathbf{Pic}^0(X/k)) &
\end{array}
$$

where the columns are exact.

Next we shall show the surjectivity of φ_G. Let λ be an element of $\mathrm{Hom}_{k\text{-}gr}(D(G), \mathbf{Pic}^0(X/k))$, L be the image of λ and B be the quotient abelian scheme of $\mathbf{Pic}^0(X/k)$ by G:

$$
\begin{array}{ccccccc}
 & 0 & & & & & \\
 & \uparrow & & & & & \\
0 \longrightarrow & L & \longrightarrow & \mathbf{Pic}^0(X/k) & \longrightarrow & B & \longrightarrow 0 \\
 & \uparrow & \nearrow \lambda & & & & \\
 & D(G) & & & & & .
\end{array}
$$

Dualizing the above diagram and extending the group $D(L)$ to G, we have a commutative diagram of extensions:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & D(L) & \longrightarrow & B^t & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & G & \longrightarrow & B' & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

Then $B'\in \mathrm{Ext}_{k\text{-}gr}(A, G)$ and $\varphi_G(B')=\lambda$, since $(B'^0)_{red}\cong B^t$. φ_G is thus surjective. q.e.d.

From the assumption (ii), $\mathrm{Hom}_{k\text{-}gr}(D(G), \mathbf{Pic}^0(X/k)) = \mathrm{Hom}_{k\text{-}gr}(D(G), \mathbf{Pic}(X/k))$. Define a homomorphism ψ_G (or simply ψ): $\mathrm{Hom}_{k\text{-}gr}(D(G), \mathbf{Pic}(X/k)) \to \mathrm{H}^1_{pq}(X, G)$ by $\psi=\eta^*\varphi^{-1}$:

$$\begin{array}{ccc} \mathrm{H}^1_{pq}(A, G) & \xrightarrow{\eta^*} & \mathrm{H}^1_{pq}(X, G) \\ \wr\downarrow & & \uparrow \psi_G \\ \mathrm{Ext}_{k\text{-}gr}(A, G) & \xrightarrow[\sim]{\varphi_G} & \mathrm{Hom}_{k\text{-}gr}(D(G), \mathbf{Pic}^0(X/k)). \end{array}$$

Let Y be an element of $\mathrm{H}^1_{pq}(X, G)$. Then, by virtue of Theorem 3 [10], the Albanese variety $\mathbf{Alb}(Y/k)$ is an extension of $\mathbf{Alb}(X/k)=A$ by a quotient H of G:

$$\begin{array}{ccccc} G\times Z & \rightrightarrows & Y & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \eta \\ \mathrm{H}\times \mathbf{Alb}(Y/k) & \rightrightarrows & \mathbf{Alb}(Y/k) & \rightarrow & A. \end{array}$$

Let λ be an element of $\mathrm{Hom}_{k\text{-}gr}(D(G), \mathbf{Pic}^0(X/k))$. Then, it is easy to see that $\mathbf{Alb}(\psi_G(\lambda)/k)\cong \varphi_G^{-1}(\lambda)$. Hence, ψ_G is injective. The comparison of the structures of $\mathrm{H}^1_{pq}(X, G)$ and $\mathrm{Hom}_{k\text{-}gr}(D(G), \mathbf{Pic}^0(X/k))$ shows that ψ_G is an isomorphism. We have now proved

Theorem 3.4. *If X is a proper integral k-scheme such that (i) the connected component of the unit $\mathbf{Pic}^0(X/k)$ of $\mathbf{Pic}(X/k)$ is an abelian scheme and such that (ii) the Neron-Severi group $NS(X)=\mathbf{NS}(X)(k)$ is torsion-free, then the homomorphism attached to a canonical morphism η: $X\to \mathbf{Alb}(X/k)$*

$$\eta^*: \mathrm{H}^1_{pq}(\mathbf{Alb}(X/k), G) \longrightarrow \mathrm{H}^1_{pq}(X, G)$$

is an isomorphism for an arbitrary commutative finite k-group scheme G. In other words, any Galois covering Y of X with group G is obtained by pulling back by η *an extension in* $\mathrm{Ext}_{k\text{-}gr}(\mathbf{Alb}(X/k), G)$ *and the extension is obtained from an isogeny B of* $\mathbf{Alb}(X/k)$, *(cf.* [10]).

Remark. The condition (ii) can be removed or weakened by restricting a group G to the category $\mathcal{A}_{ll}$, $\mathcal{A}_{lr}$, $\mathcal{A}_{rl}$ or $\mathcal{A}_{rr}$.

Chapter IV. On the representability of PH-functor

In this chapter, the field k is supposed to be algebraically closed and of positive characteristic p. Let X be a proper integral scheme of finite type over k and G be a commutative, affine algebraic k-group scheme of some type (cf. Lemma 4.2). The purpose of this chapter is to show that $\mathbf{PH}(G, X/k)$ is representable by a commutative group scheme, locally of finite type over k. For this purpose, we shall apply the representability criterion by J. P. Murre [11]. We must verify the conditions $(P_1)\sim(P_7)$.

We shall begin with the condition (P_1).

Lemma 4.1. *If G is a commutative, affine algebraic k-group scheme,* $\mathbf{PH}(G, X/k)$ *satisfies the condition* (P_1).

Proof. Let C be the category consisting of k-algebras of finite length and morphisms of k-algebras, and P be the restriction of $\mathbf{PH}(G, X/k)$ on the dual of C. For the pro-representability of P, we shall apply the criterion by A. Grothendieck, (FGA, 195-09, Théorème 1). The condition (i) and the case (a) of the condition (ii), are easily verified. For the case (b) of the condition (ii). Théorème 2 (ibid.) is available. The case (b) is as follows: Let A be an object of C which is a local k-algebra and $A\rightarrow A'$ be an injective morphism of C such that the quotient module A'/A is a A-module of length 1. Note that in this case, the diagram $A\rightarrow A'\rightrightarrows A'\underset{A}{\otimes}A'$ is exact. Then the diagram

$$P(A)\xrightarrow{i}P(A')\overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}}P(A'\underset{A}{\otimes}A')$$

is exact. Let Y' be an element of $P(A')$. By virtue of Lemma 1. 3, Chapter 1, we can consider Y' as an element of $H^1_{pq}(X_{A'}, G)$, i.e. we have a diagram,

$$G\times Y'\overset{\sigma}{\underset{pr_2}{\rightrightarrows}}Y'\xrightarrow{p}X_{A'}.$$

Put $\mathcal{F}'=p_*(\mathcal{O}_{Y'})$. Then $\mathcal{F}'$ is a quasi-coherent and flat $\mathcal{O}_{X_{A'}}$-Algebra such that $\mathbf{Spec}(\mathcal{F}')\cong Y'$. Since G is affine, the operation σ of G on Y' given by a $\mathcal{O}_{X_{A'}}$-morphism of Algebras, Δ': $\mathcal{F}'\to\mathcal{F}'\underset{k}{\otimes}\mathcal{O}_G$ such that $(\Delta_G\otimes id_{\mathcal{F}'})\Delta'=(id_{\mathcal{O}_G}\otimes\Delta')\Delta'$ and $(id_{\mathcal{F}'}\otimes\varepsilon)\Delta'=id_{\mathcal{F}'}$, where Δ_G (resp. ε) is the diagonal Δ_G: $\mathcal{O}_G\to\mathcal{O}_G\underset{k}{\otimes}\mathcal{O}_G$ (resp. the augmentation ε: $\mathcal{O}_G\to k$) attached to the multiplication (resp. the unit) of G. The elements of $P(A)$ and $P(A'\underset{A}{\otimes}A')$ are interpreted analogously. If $\pi_1(Y')=\pi_2(Y')$, the descent data with respect to the morphism $A\to A'$ are induced on $\mathcal{F}'$ and Δ'. Then, by the result of A. Grothendieck (Théorème 2, ibid.), there exist a quasi-coherent $\mathcal{O}_{X_A}$-Algebra $\mathcal{F}$ and a $\mathcal{O}_{X_A}$-morphism Δ: $\mathcal{F}\to\mathcal{F}\underset{k}{\otimes}\mathcal{O}_G$ such that $\mathcal{F}'\cong\mathcal{F}\underset{\mathcal{O}_{X_A}}{\otimes}\mathcal{O}_{X_{A'}}$ and $\Delta'\cong\Delta\underset{\mathcal{O}_{X_A}}{\otimes}\mathcal{O}_{X_{A'}}$. It is easy to see that $Y=\mathbf{Spec}(\mathcal{F})$ is an element of $P(A)$ such that $i(Y)\cong Y'$. The injectivity of i can be proved by an analogous argument. Thus P is strictly pro-representable on C.

Next, let R_ξ be the local component which pro-represents P at a rational point ξ of P. R_ξ is noetherian if $P(I_k, \xi)=P(\varepsilon)^{-1}(\xi)$ $\cong\mathrm{Lie}(\mathbf{PH}(G, X/k))(k)$ is a k-vector space of finite length, (cf. FGA, 195-07). On the other hand, $\dim_k\mathrm{Lie}(\mathbf{PH}(G, X/k))(k)$ $\leq\dim_k\mathbf{PH}(\mathrm{Lie}(G), X/k)(k)=\dim_k H^1(X, \mathrm{Lie}(G))$ (cf. Chapter I). Since X is proper over k, $H^1(X, G_a)\cong H^1(X, \mathcal{O}_X)$ is a k-vector space of finite length. Therefore, since $H^1(X, \mathrm{Lie}(G))\cong H^1(X, \mathcal{O}_X)^N$ for some integer N, $H^1(X, \mathrm{Lie}(G))$ is also a k-vector space of finite length. q.e.d.

Lemma 4.2. $\mathbf{PH}(G, X/k)$ *satisfies the condition* (P_2), *if* G *is of the following type*:

(1) G *is a connected commutative algebraic k-group scheme, smooth over k.*

(2) G *is a commutative finite k-group scheme.*

Proof. The case (1). G is decomposed to a direct product of a torus $(G_m)^r$ and a unipotent subgroup U. Then, $\mathbf{PH}(G, X/k)$ is isomorphic to $\mathbf{Pic}(X/k)^r \times \mathbf{PH}(U, X/k)$. Since $\mathbf{Pic}(X/k)$ exists and satisfies the condition (P_2), the problem is reduced to the case where G is unipotent. When G is unipotent, we shall proceed by the induction on the length n of a composition series of G. If $n=1$, i.e. $G \cong G_a$, $\mathbf{PH}(G_a, X/k)$ is representable, hence satisfies the condition (P_2). If $n>1$, we have an exact sequence,

$$0 \longrightarrow G_a \longrightarrow G \longrightarrow H \longrightarrow 0$$

where H is unipotent. Assume H satisfies (P_2). Let A be a noetherian, local k-algebra which is complete and separated with respect to the $\mathfrak{M}$-adic topology ($\mathfrak{M}$ is the maximal ideal of A) and let $A_n = A/\mathfrak{M}^{n+1}$ for $n=0, 1, 2, \cdots$. We shall denote by θ_G the canonical morphism $\mathbf{PH}(G, X/k)(A)$*) $\rightarrow \varprojlim_n \mathbf{PH}(G, X/k)(A_n)$ for a commutative k-group scheme G.

From the assumption on H, we have θ_H: $\mathbf{PH}(H, X/k)(A) \xrightarrow{\sim} \varprojlim_n \mathbf{PH}(H, X/k)(A_n)$. On the other hand, the sequence

$$0 \rightarrow \mathbf{PH}(G_a, X/k) \rightarrow \mathbf{PH}(G, X/k) \rightarrow \mathbf{PH}(H, X/k)$$

is exact. Therefore, we have a commutative diagram with exact lines,

$$\begin{array}{ccc} 0 \longrightarrow \mathbf{PH}(G_a, X/k)(A) & \longrightarrow & \mathbf{PH}(G, X/k)(A) \longrightarrow \\ \wr \downarrow \theta_{G_a} & & \downarrow \theta_G \\ 0 \rightarrow \varprojlim_n \mathbf{PH}(G_a, X/k)(A_n) & \rightarrow & \varprojlim_n \mathbf{PH}(G, X/k)(A_n) \rightarrow \end{array}$$

*) $\mathbf{PH}(G, X/k)(A) = \mathbf{PH}(G, X/k)(\text{Spec } A)$. These types of abbreviations will be easily understood unless explicitly mentioned.

$$\begin{array}{l}\rightarrow \mathbf{PH}(H, X/k)(A) \\ \quad\quad \wr \big\downarrow \theta_H \\ \rightarrow \varprojlim_n \mathbf{PH}(H, X/k)(A_n).\end{array}$$

The diagram chasing shows that the canonical homomorphism θ_G is injective.

It remains to prove the surjectivity of θ_G. First, note that we have isomorphisms, $\mathbf{PH}(G, X/k)(A_n) \cong \mathrm{H}^1(X_{A_n}, G)$ and $\mathbf{PH}(G, X/k)(A) \cong \mathrm{H}^1(X_A, G)$. Therefore, an element of $\mathbf{PH}(G, X/k)(A_n)$ corresponds to a principal fibre space Y_n over X_{A_n} with group G. An element of $\varprojlim_n \mathbf{PH}(G, X/k)(A_n)$ corresponds to a projective system $\{Y_n, \varphi_{m,n}: Y_m \rightarrow Y_n \text{ for } m \leqslant n\}$ where Y_n belongs to $\mathrm{H}^1(X_{A_n}, G)$ and satisfies $Y_m \cong Y_n \otimes_{A_n} A_m$ for $m \leqslant n$. Let $X_n = X_{A_n}$, $\mathfrak{X} = \varinjlim_n X_n$ and $\mathfrak{Y} = \varinjlim_n Y_n$. Then $\mathfrak{Y}$ is a principal fibre space over $\mathfrak{X}$ with group G. Next, embed G into a general linear group $G' = GL_N$ $(N \in \boldsymbol{N})$ as a closed subgroup. Then, for every n, we can construct a principal fibre space Y'_n*) over X_n with group G', extending the group G to the group G'. $\varphi_{m,n}: Y_m \rightarrow Y_n$ is also extended to $\varphi'_{m,n}: Y'_m \rightarrow Y'_n$ for $m \leqslant n$. Then Y'_n and $\varphi'_{m,n}$ form a projective system $\{Y'_n, \varphi'_{m,n}\}$ which is considered as an element of $\varprojlim_n \mathrm{H}^1(X_n, G')$. Let $\mathfrak{Y}' = \varinjlim_n Y'_n$. Then $\mathfrak{Y}'$ is a principal fibre space over $\mathfrak{X}$ with group G' and is isomorphic to $\mathfrak{Y} \times_G G'$.

We shall show that $\mathfrak{Y}'$ is algebraizable. In other words, there exists a principal fibre space Y' over X_A with group G' such that $\mathfrak{Y}' \cong Y' \times_{X_A} \mathfrak{X}$ and $Y'_n \cong Y' \times_{X_A} X_n$. We shall recall the fact that a principal fibre space over a prescheme Z with group $G' = GL_N$ corresponds to a locally free $\mathcal{O}_Z$-Module of rank N. Therefore, the projective system $\{Y'_n, \varphi'_{m,n}\}$ corresponds to a projective system $\{\mathcal{M}_n, \theta_{m,n}\}$ consisting of locally free $\mathcal{O}_{X_n}$-Modules $\mathcal{M}_n$ of rank N and

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*) Y'_n is denoted by $Y_n \times_G G'$ according to Serre's terminology [17]. The existence can be proved using the (fpqc)-descent for affine morphisms, (cf. [4]).

isomorphisms $\theta_{m,n}$: $(\varphi'_{m,n})^*\mathcal{M}_n \cong \mathcal{M}_m$ and $\mathfrak{Y}'$ corresponds to a locally free $\mathcal{O}_{\mathfrak{X}}$-Module $\varprojlim_n M_n$ of rank N. Then, there exists a coherent $\mathcal{O}_{X_A}$-Module $\mathcal{M}$ such that $i^*(\mathcal{M}) \cong \varprojlim_n \mathcal{M}_n$, where i: $\mathfrak{X} \to X$ is the canonical morphism (EGA, III_1, 5.1.6). Let $x \in X_0$, $R' = \mathcal{O}_{\mathfrak{X},x}$ and $R = \mathcal{O}_{X,x}$. Then $i^*(\mathcal{M})_x \cong \mathcal{M}_x \otimes_R R'$ is a free R'-module of rank N. We can take a R'-basis $(e_1, \cdots, e_N)$ of $i^*(\mathcal{M})_x$ from $\mathcal{M}_x$ such that $(e_1, \cdots, e_N)$ defines a surjective R-homomorphism $R^N \xrightarrow{g} \mathcal{M}_x$. The kernel L of g is a R-module of finite type. Since R' is R-flat (EGA, I, 10.8.9), $L \otimes_R R' = 0$. Hence, $L = 0$ (EGA, I, 10.8.11), i.e. $\mathcal{M}_x$ is a free R-module of rank N, (cf. Lemma (II. 4) of [11], Prop. 18 and Prop. 30 of [15]). Then, $\mathcal{M}$ is locally free of rank N. If we take a principal fibre space Y' over X_A with group G' which corresponds to $\mathcal{M}$, Y' is then what we wanted to algebraize $\mathfrak{Y}'$ with.

Next we shall show that $\mathfrak{Y}$ is algebraizable. The proof is analogous to that of Prop. 19 of [15]. Let $E_0 = Y' \times_{G'} G'/G$ and $E = \mathfrak{Y}' \times_{G'} G'/G$ where the operation of G' on G'/G comes from the multiplication of G' from the left. Then, E is isomorphic to $\mathfrak{Y} \times_G G'/G$ and E has a section s from $\mathfrak{X}$ which is induced from $\mathfrak{Y} \to \mathfrak{Y} \times \{G\} \subset \mathfrak{Y} \times_{G'} G'$. The completion $\widehat{E_0} = \varinjlim_n (Y'_n \times_{G'} G'/G)$ is isomorphic to E (an isomorphism f: $\widehat{E_0} \to E$). Then $\widehat{E_0}$ has a section $s'_0 = f^{-1}s$ from $\mathfrak{X}$. Since X is proper over k and E_0 is separated over k, s'_0 comes from a A-morphism s_0: $X_A \to E$, (cf. EGA, III, 5.4.1). Let G operate on Y' through the operation of G'. The quotient prescheme is then isomorphic to E_0. Define Y by $Y' \times_{E_0} (X, s_0)$. Now, it is easy to see that $\varinjlim_n Y_n$ is isomorphic to $\mathfrak{Y}$.

The case (2). The canonical homomorphism θ_G: $\mathbf{PH}(G, X/k)(A) \to \varprojlim_n \mathbf{PH}(G, X/k)(A_n)$ is injective. The proof is done by the induction on the k-rank of G, as we have observed that $\mathbf{PH}(G, X/k)$ is representable for $G = \alpha_p$, μ_p, $(\boldsymbol{Z}/p\boldsymbol{Z})_k$ and $(\boldsymbol{Z}/n\boldsymbol{Z})_k$; $n \in \boldsymbol{N}$, $(n, p) = 1$. Therefore, it remains to see the surjectivity of

θ_G. Let $\mathfrak{A}$ be the affine algebra of G and $\{Y_n, \varphi_{m,n}: Y_m \to Y_n$ for $m \leqslant n\}$ be a projective system of $\varprojlim_n \mathbf{PH}(G, X/k)(A_n)$. Since Y_n is affine over X_n, Y_n is of the form $\mathbf{Spec}(\mathcal{F}_n)$ for a coherent $\mathcal{O}_{X_n}$-Algebra $\mathcal{F}_n$. $\mathcal{F}_n$ is given a diagonal Δ_n: $\mathcal{F}_n \to \mathcal{F}_n \otimes_k \mathfrak{A}$ which defines the operation of G on Y_n, and satisfies $\mathcal{F}_n \otimes_{A_m} A_n \cong \mathcal{F}_m$ and $\Delta_n \otimes_{A_n} A_m \cong \Delta_m$ for $m \leqslant n$. Then, $\varprojlim_n \mathcal{F}_n$ and $\varprojlim_n \Delta_n$ define a coherent $\mathcal{O}_{\mathfrak{X}}$-Algebra $\hat{\mathcal{F}}$ with a diagonal $\hat{\Delta}$: $\hat{\mathcal{F}} \to \hat{\mathcal{F}} \otimes_k \mathfrak{A}$ (cf. EGA, I, 10.11.4). Then there exists a coherent $\mathcal{O}_{X_A}$-Algebra $\mathcal{F}$ with a diagonal Δ: $\mathcal{F} \to \mathcal{F} \otimes_k \mathfrak{A}$ (cf. EGA, III_1, 5.1.6), and $\{\mathcal{F}, \Delta\}$ defines a principal fibre space Y over X_A with group G such that $\theta_G(Y) \cong \{Y_n, \varphi_{m,n}\}$. q.e.d.

Remark. The condition (P_2) seems to be true for all commutative, affine k-group schemes G, if we can embed such G into GL_N for some integer N.

Lemma. 4.3. *If G is a commutative, affine algebraic k-group scheme,* $\mathbf{PH}(G, X/k)$ *satisfies the condition* (P_3).

Proof. For the proof, we refer to SGAD, Exp. VI_B, (10.16).

Lemma 4.4. *Let G be as in Lemma* 4.3. *Then* $\mathbf{PH}(G, X/k)$ *satisfies the conditions* (P_4) *and* (P_5).

Proof. Trivial from the definition of $\mathbf{PH}(G, X/k)$.

Lemma. 4.5. *Let G be as in Lemma* 4.3. *Then* $\mathbf{PH}(G, X/k)$ *satisfies the condition* (P_6).

Proof. First, note that G has a composition series whose quotients are elementary k-group schemes (i.e. G_a, G_m, α_p, μ_p, $(\mathbf{Z}/p\mathbf{Z})_k$ and $(\mathbf{Z}/q\mathbf{Z})_k$; q: a prime such that $(p, q)=1$). For these elementary group schemes, the condition (P_6) holds because $\mathbf{PH}(G, X/k)$ is representable. Therefore, for the proof of our assertion, we have only to show the following:

Let $(*)$: $0 \to G_1 \to G_2 \to G_3 \to 0$ be an exact sequence of commutative, affine algebraic k-group schemes. If the condition (P_6) holds for G_1 and G_3, it holds for G_2.

The exact sequence $(*)$ gives an exact sequence,

$$0 \to \mathbf{PH}(G_1, X/k) \xrightarrow{i} \mathbf{PH}(G_2, X/k) \xrightarrow{\pi} \mathbf{PH}(G_3, X/k).$$

Let $T \in (\mathbf{Sch}/k)$ and ξ: $T \to \mathbf{PH}(X_2, G/k)$. Then, applying the condition (P_6) to $\pi \cdot \xi$, there exists a closed subscheme $N(\pi \cdot \xi)$ of T such that for every $T' \in (\mathbf{Sch}/k)$ and every morphism α: $T' \to T$, we have $\pi \cdot \xi \cdot \alpha = 0$ if and only if α factors through $N(\pi \cdot \xi)$:

$$\begin{array}{ccc} T' & \xrightarrow{\alpha} & T \\ & \searrow \quad \nearrow_{j} & \\ & N(\pi \cdot \xi) & \end{array} .$$

Let j be the canonical injection of $N(\pi \cdot \xi)$ into T. Then $\pi \cdot \xi \cdot j = 0$, hence $\xi \cdot j$ factors through i:

$$\begin{array}{ccc} 0 \longrightarrow \mathbf{PH}(G_1, X/k) & \xrightarrow{i} & \mathbf{PH}(G_2, X/k) \\ {}^{\xi'}\nwarrow & & \nearrow_{\xi \cdot j} \\ & N(\pi \cdot \xi) & \end{array} .$$

Let ξ' be the morphism $N(\pi \cdot \xi) \to \mathbf{PH}(G_1, X/k)$ defined from the above diagram. Then, applying the condition (P_6) to ξ', we have a closed subscheme $N(\xi')$ of $N(\pi \cdot \xi)$ such that for every $T'' \in (\mathbf{Sch}/k)$ and every morphism β: $T'' \to N(\pi \cdot \xi)$, we have $\xi' \cdot \beta = 0$ if and only if β factors through $N(\xi')$.

Let $T_1 \in (\mathbf{Sch}/k)$ and γ be a morphism $T_1 \to T$ such that $\xi \cdot \gamma = 0$. Since $\pi \cdot \xi \cdot \gamma = 0$, γ factors through $N(\pi \cdot \xi)$, i.e. there exists a morphism γ': $T_1 \to N(\pi \cdot \xi)$ such that $\gamma = j \cdot \gamma'$. Since $\xi \cdot \gamma = \xi \cdot j \cdot \gamma' = i \cdot \xi' \cdot \gamma' = 0$ and i is injective, $\xi' \cdot \gamma' = 0$. Hence, γ' factors through $N(\xi')$. This shows that $N(\xi)$ exists and is isomorphic to $N(\xi')$. q.e.d.

Lemma 4.6. *Let G be as in Lemma* 4.2. *Then,* $\mathbf{PH}(G, X/k)$ *satisfies the condition* (P_7).

Proof. The same argument as in Lemma 4. 5 reduces the problem to the following:

Let $(*): 0 \to G_1 \to G_2 \to G_3 \to 0$ be an exact sequence of commutative, affine algebraic k-group schemes. If the condition (P_7) holds for G_3 and if $\mathbf{PH}(G_1, X/k)$ is representable, the condition (P_7) holds for G_2.

Let C be a complete, non-singular, irreducible curve in $(\mathbf{Sch}/k)$, T be a finite set of closed points on C and ξ be a morphism from $C' = C - T$ into $\mathbf{PH}(G_2, X/k)$.

Consider the exact sequence,

$$0 \to \mathbf{PH}(G_1, X/k) \xrightarrow{i} \mathbf{PH}(G_2, X/k) \xrightarrow{\pi} \mathbf{PH}(G_3, X/k).$$

Applying (P_7) to $\pi \cdot \xi$, $\pi \cdot \xi$ has a module $\mathfrak{M} = \sum_{P \in T} n_P P$ with support on T. Let $J_{\mathfrak{M}}$ be the generalized Jacobian of C with respect to the module $\mathfrak{M}$ and $\mathfrak{S}'$ be a set of systems of positive integers $(l_P)_{P \in T}$ such that $l_P \geqslant n_P$ for every $P \in T$. Introduce an order on $\mathfrak{S}'$, putting $(l_P)_{P \in T} \geqslant (l'_P)_{P \in T}$ if and only if $l_P \geqslant l'_P$ for every $P \in T$. Take a totally ordered subset $\mathfrak{S}$ of $\mathfrak{S}'$ which is cofinal in $\mathfrak{S}'$. The elements of $\mathfrak{S}$ correspond to modules with support on T and we denote them by $\mathfrak{M}^{(\alpha)}$, where α is an index defined by the total order on $\mathfrak{S}$. Let $J_{\mathfrak{M}^{(\alpha)}}$ be the generalized Jacobian with respect to a module $\mathfrak{M}^{(\alpha)}$. Then for $\mathfrak{M}^{(\alpha)}$ of $\mathfrak{S}$, we have an exact sequence of commutative algebraic groups,

$$0 \longrightarrow K_\alpha \longrightarrow J_{\mathfrak{M}^{(\alpha)}} \xrightarrow{p_\alpha} J_{\mathfrak{M}} \longrightarrow 0,$$

where K_α is the kernel of the canonical surjection $p_\alpha : J_{\mathfrak{M}^{(\alpha)}} \to J_{\mathfrak{M}}$. We must clarify the algebraic structure of K_α. For this purpose, we use the terminology of J.-P. Serre [16]. Then K_α is given in the form, $K_\alpha \cong \mathrm{Ker}(R_{\mathfrak{M}^{(\alpha)}} \to R_{\mathfrak{M}}) = \{$the set of rational functions f on C such that $n_P \leqslant v_P(f-1) < n_P^{(\alpha)}$ for every $P \in T\}$, (cf n^0 13 of Chap. V, ibid.). For $\beta \geqslant \alpha$, we have a commutative diagram with exact lines,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{\beta} & \longrightarrow & J_{\mathfrak{M}^{(\beta)}} & \longrightarrow & J_{\mathfrak{M}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & K_{\alpha} & \longrightarrow & J_{\mathfrak{M}^{(\alpha)}} & \longrightarrow & J_{\mathfrak{M}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} .$$

Then passing to the projective limits, we have an exact sequence of proalgebraic groups,

$$0 \longrightarrow \varprojlim_{\mathfrak{S}} K_{\alpha} \longrightarrow \varprojlim_{\mathfrak{S}} J_{\mathfrak{M}^{(\alpha)}} \longrightarrow J_{\mathfrak{M}} \longrightarrow 0.$$

Let $\mathcal{D}_T^0$ be the abelian group of all divisors on C of degree 0 which have no component on T. Since $J_{\mathfrak{M}^{(\alpha)}}$ is the quotient of $\mathcal{D}_T^0$ by the relation $D \sim D' \Rightarrow D - D' = (f)$, $f \equiv 1 (\mathrm{mod}\, \mathfrak{M}^{(\alpha)})$, we have the canonical surjection p: $\mathcal{D}_T^0 \to \varprojlim_{\mathfrak{S}} J_{\mathfrak{M}^{(\alpha)}} \to 0$. The kernel of p is formed by rational functions f such that $v_P(f-1) \geqslant N$ for any positive integer N and for all $P \in T$. Hence f is constant 1. This means that $\mathcal{D}_T^0$ has a structure of a proalgebraic group $\varprojlim_{\mathfrak{S}} J_{\mathfrak{M}^{(\alpha)}}$.

Let Ω be a universal domain which contains k and let $\mathcal{D}_T^0(\Omega)$ be the abelian group of all divisors on C of degree 0 whose components are Ω-valued points of $C' = C - T$. Then the morphism ξ: $C' \to \mathbf{PH}(G_2, X/k)$ define homomorphisms $\bar{\xi}(\Omega)$: $\mathcal{D}_T^0(\Omega) \to \mathbf{PH}(G_2, X/k)(\Omega)$ and $\xi_0(\Omega)$: $(\varprojlim_{\mathfrak{S}} K_{\alpha})(\Omega) \to \mathbf{PH}(G_1, X/k)(\Omega)$ which commute a diagram,

$$(*) \quad \begin{array}{ccccccc} 0 \to & \mathbf{PH}(G_1, X/k)(\Omega) & \xrightarrow{i} & \mathbf{PH}(G_2, X/k)(\Omega) & \xrightarrow{\pi} & \mathbf{PH}(G_3, X/k)(\Omega) & \\ & \uparrow \xi_0(\Omega) & & \uparrow \bar{\xi}(\Omega) & & \uparrow & \\ 0 \to & (\varprojlim_{\mathfrak{S}} K_{\alpha})(\Omega) & \longrightarrow & \mathcal{D}_T^0(\Omega) & \longrightarrow & J_{\mathfrak{M}}(\Omega) & \end{array} .$$

On the other hand, K_{α} is isomorphic to a direct product $\prod_{P \in T} K_{\alpha, P}$ of affine algebraic groups $K_{\alpha, P}$ whose elements are of the form $(a_{n_P}, \cdots, a_{n_P^{(\alpha)}-1}) \in \Omega^{n_P^{(\alpha)}-n_P}$ and whose multiplication is defined by

$$(a_{n_P}, \cdots, a_{n_P^{(\alpha)}-1})(b_{n_P}, \cdots, b_{n_P^{(\alpha)}-1}) = (a_{n_P}+b_{n_P}, \cdots, a_i+b_i+\sum_{j+k=i} a_j b_k, \cdots).$$

If $t=t_P$ is a generator of the local ring $\mathcal{O}_{C,P}$ of C at P, a rational function f of $\Omega(C)$ is of the form

$$f=a_{-n}t^{-n}+\cdots+a_0+a_1t+\cdots\cdots;\quad a_{-n}, \cdots, a_0, a_1, \cdots \in \Omega.$$

The elements of $K_{\alpha,P}$ is identified to functions f of the form

$$f=1+a_{n_P}t^{n_P}+\cdots+a_{n_P^{(\alpha)}-1}t^{n_P^{(\alpha)}-1}$$

and the elements $\varprojlim_{\mathfrak{S}} K_{\alpha,P}=K_P$ is identified with the functions of the form $f=1+a_{n_P}t^{n_P}+\cdots\cdots$.

Let $i_{\alpha,P}$ be the canonical regular section of $K_{\alpha,P}$ into K_P defined by $(a_{n_P}, \cdots, a_{n_P^{(\alpha)}-1}) \rightsquigarrow (a_{n_P}, \cdots, a_{n_P^{(\alpha)}-1}), 0, 0, \cdots)$. Put $\xi_{\alpha,P}=\xi_0(\Omega)\cdot i_{\alpha,P}$. We shall show that $\xi_{\alpha,P}$ is a regular map for every α and $P\in T$. Let $(a_{n_P}, \cdots, a_{n_P^{(\alpha)}-1})$ be a generic point of $K_{\alpha,P}$, $g_\alpha=1+a_{n_P}t^{n_P}+\cdots+a_{n_P^{(\alpha)}-1}t^{n_P^{(\alpha)}-1}$ and $(g_\alpha)=P_1+\cdots+P_n-P_{n+1}-\cdots-P_{2n}$ $(\in \mathfrak{D}_T^0(\Omega))$. We shall clarify the map $\bar{\xi}(\Omega)$ for (g_α). First, note that a morphism ξ corresponds to a principal fibre space Y over $X_{C'}$ with group G_2;

$$G_2\times Y \rightrightarrows Y \longrightarrow X_{C'}.$$

Let X_i, Y_i be the fibres of $X_{C'}$ and Y over $P_i(i=1, \cdots, 2n)$. Then Y_i is a principal fibre space over X_i with group G_2 defined over the field $k(P_i)$ for $i=1, \cdots, 2n$. Let $K_0=k(a_{n_P}, \cdots, a_{n_P^{(\alpha)}-1})$, $K=k(P_1, \cdots, P_{2n})$, $L=a$ normal closure of K over K_0 and $\mathfrak{G}=\mathrm{Gal}(L/K_0)$. We denote $X_i\otimes_{k(P_i)} L$, $Y_i\otimes_{k(P_i)} L$ by the same letters X_i, Y_i. Then $Y'=\xi(\Omega)((g_\alpha))$ is obtained by changing the groups by a morphism $\overbrace{G_2\times\cdots\times G_2}^{2n}\rightarrow G_2$ $(x_1\times\cdots\times x_{2n})\rightsquigarrow x_1+\cdots+x_n-x_{n+1}-\cdots-x_{2n})$ from $(Y_1\times_L\cdots\times_L Y_{2n}) \times_{(X_1\times_L\cdots\times_L X_{2n})} (X\otimes_k L, \Delta^{2n})$, where $X_1\times_L\cdots\times_L X_{2n}\cong X^{2n}\otimes_k L$ and Δ^{2n} is the diagonal $x\in X\otimes_k L \rightsquigarrow (x, \cdots, x)\in X^{2n}\otimes_k L$.

On the other hand, an element σ of $\mathfrak{G}$ operates on the set $\{Y_1, \cdots Y_{2n}\}$ as a permutation. Then it is easy to see that $\mathfrak{G}$ operates on Y' and that Y' is indeed invariant with respect to this operation. Therefore Y' is defined over K_0, i.e. there exists a principal fibre space Y_0 over X_{K_0} with group G_2 such that $Y' \cong Y_0 \underset{K_0}{\otimes} L$. From the diagram $(*)$, we see that Y_0 comes from an element Z_α of $\mathbf{PH}(G_1, X/k)(K_0)$ which is equal to $\xi_0(\Omega)(g_\alpha)$. Since $\mathbf{PH}(G_1, X/k)$ is representable, the map $\xi_{\alpha,P}$: $g_\alpha \in K_{\alpha,P} \to Z_\alpha \in \mathbf{PH}(G_1, X/k)$ is a rational map which is defined everywhere.

If $\beta \geqslant \alpha$, the locus $\overline{\xi_{\beta,P}(g_\beta)}$ of $\xi_{\beta,P}(g_\beta)$ in $\mathbf{PH}(G_1, X/k)$ contains $\xi_{\alpha,P}(g_\alpha)$. Therefore $\overline{\xi_{\beta,P}(g_\beta)} \supseteq \overline{\xi_{\alpha,P}(g_\alpha)}$, for $\beta \geqslant \alpha$. Therefore, there exists an index α_0 such that for $\gamma \geqslant \alpha_0$, we have $(**)$ $\overline{\xi_{\gamma,P}(g_\gamma)} = \overline{\xi_{\alpha_0,P}(g_{\alpha_0})}$, because $\overline{\xi_{\alpha,P}(g_\alpha)}$ is connected. This α_0 depends on P. However, since T is a finite set, we can suppose the equality $(**)$ holds for all $P \in T$. Then $\xi_{\alpha_0} = \prod_{P \in T} \xi_{\alpha_0,P}$: $K_{\alpha_0} = \prod_{P \in T} K_{\alpha_0,P} \to \mathbf{PH}(G_1, X/k)$ is a morphism of group schemes such that ξ_γ: $K_\gamma \to \mathbf{PH}(G_1, X/k)$ is a composite morphism $K_\gamma \xrightarrow[\text{can. proj.}]{} K_{\alpha_0} \xrightarrow[\xi_{\alpha_0}]{} \mathbf{PH}(G_1, X/k)$ for every $\gamma \geqslant \alpha_0$. Finally we shall note that ξ_β is not necessarily morphism of group schemes if $\beta < \alpha_0$. Then it is easy to see that $\mathfrak{M}^{(\alpha_0)}$ is a module for ξ with support on T. q.e.d.

Consequently, applying the representability criterion of J. P. Murre, we have

Theorem 4.7. *Let X be a proper, integral k-scheme of finite type and G be as in Lemma 4.2. Then* $\mathbf{PH}(G, X/k)$ *is representable by a commutative k-group scheme, locally of finite type over k.*

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