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Behavior of holomorphic functions in the unit disk on arcs of positive hyperbolic diameter

By

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1. Introduction

The classical lemma of Koebe [7] for bounded holomorphic functions in the unit disk D states that if f tends to zero on a sequence of arcs in D which approach a subarc of the boundary C of D then f must be identically zero. Generalizations of this lemma have succeeded in lifting the hypothesis that f be bounded. For example, Bagemihl and Seidel [2, Theorem 1] showed that Koebe's lemma still holds if f is a normal meromorphic function in D, while G. R. Mac Lane [10, Theorem 13] showed that the result is still true for holomorphic functions of his class $\mathcal{A}$. At one or two instances in our paper there are points of contact with arguments used by Bagemihl and Seidel and by Mac Lane in the above cited papers and we acknowledge this.

A variant of Koebe's lemma was proved recently by I. V. Gavrilov [5, Theorem 1] who showed that if f is a normal holomorphic function in D for which

$$(1.0) \qquad \log|f(re^{i\theta_0})| \leq -\frac{1}{(1-r)^{1+\epsilon}}, \quad 0 \leq r < 1, \quad \epsilon > 0,$$

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on some radius $re^{i\theta_0}$, then f must be identically zero. We can view this as Koebe-type result, in that, if γ_n: $re^{i\theta_0}$, $1-\frac{1}{n}\leq r\leq 1-\frac{1}{n+1}$, then we have that a normal function f which tends to zero on the sequence of arcs $\{\gamma_n\}$, with order prescribed in (1.0), must be identically zero. Notice that the requirement of Koebe's lemma which demands that the arcs γ_n approach a subarc of C has been weakened while the condition that f merely tend to zero on γ_n has been strengthened to $|f(z)|\leq\exp(-(n)^{1+\epsilon})$, $z\in\gamma_n$, $n=1,2,\cdots$.

Gavrilov points out that one can replace the condition that f be normal by the assumption that f have angular limit at $e^{i\theta_0}$. Several other theorems are given by Gavrilov in this paper which are similar in spirit to the one mentioned above. This paper presents generalizations (and in some cases improvements as well) of the theorems in Gavrilov's article which in the sequel we refer to as $\mathcal{G}$.

Our aim is to allow a greater variety of sequences of arcs than Koebe arcs and to determine under what conditions f tending to zero on these sequences imply that f is identically zero. So we are confronted with two problems: (1) what restrictions should we impose on the arcs, and (2), what is the proper order for f on these sequences. It is fairly clear that some restrictions on the arcs are necessary. Let $B(z, \{a_n\})$ be a Blaschke product with $0<a_n<a_{n+1}\to 1$; and let $\mu(r)$, $0<r<1$, be any positive monotonically decreasing function. We can certainly determine a sequence of disk

$$D(a_n, r_n)=\{z\in D||z-a_n|<r_n\},\quad n=1,2,\cdots,$$

such that for $z\in D(a_n, r_n)$,

$$|B(z, \{a_n\})|\leq\mu(|a_n|),\quad n=1,2,\cdots.$$

So, if $\{\gamma_n\}$ is any sequence of arcs with $\gamma_n\subseteq D(a_n, r_n)$, then $B(z, \{a_n\})$ tends to zero on γ_n faster than $\mu(|a_n|)$.

A clue to the proper order for f on γ_n is given by the bounded holomorphic function $f(z)=\exp\left(\frac{z+1}{z-1}\right)$. If we take

(1.1) $$\gamma_n: 1-\frac{2}{n}\leq x\leq 1-\frac{1}{n}, \quad n=1, 2, \cdots;$$

then

(1.2) $$|f(z)|\leq \exp\left(-\frac{1}{1-|z|}\right), \quad z\in\gamma_n, \ n=1, 2, \cdots.$$

So we must impose the condition that f tend to 0 on γ_n with order more severe than (1.2). We gather these notions in precise form.

2. Preparations and terminology

The non-Euclidean hyperbolic metric in D is of use to us and so let

$$\rho(a, b)=\frac{1}{2}\log\frac{|1-a\bar{b}|+|a-b|}{|1-a\bar{b}|-|a-b|}, \quad a, b\in D.$$

For $S\subseteq D$ let

$$HD(S)=\sup\{\rho(a, b)\}, \quad a, b\in S.$$

So $HD(S)$ is the hyperbolic diameter of S.

Definition 1. *Let $\{\gamma_n\}$ be a sequence of Jordan arcs in D satisfying*

(2.0)
$$B_1)\quad \frac{1}{2}\leq \min_{z\in\gamma_n}|z|=r_n\to 1, \quad n\to\infty;$$
$$B_2)\quad 0<\varliminf_{n\to\infty} HD(\gamma_n)\leq \varlimsup_{n\to\infty} HD(\gamma_n)<\infty.$$

We call such a sequence of arcs $\{\gamma_n\}$ a positive hyperbolic diameter sequence, hereafter, a PHD sequence.

A sequence of Jordan arcs $\{\gamma_n\}$ in D-not necessarily a *PHD* sequence-has certain parameters associated with it. Let

$$R_n=\max_{z\in\gamma_n}|z|, \quad n=1, 2, \cdots.$$

and let r_n be defined as in B_1 of (2.0). Let E_n be the closed circular sector of $|z|\leq R_n$ of *minimum* angle opening α_n containing

γ_n. To avoid unnecessary complications we always assume there is such an angle α_n and that $\alpha_n < \pi$ for any sequence $\{\gamma_n\}$. (For a *PHD* sequence necessarily $\alpha_n \to 0$, $n \to \infty$.) Thus E_n is of the form

$$(2.1) \quad E_n:\ 0 \leq |z| \leq R_n, \quad \theta_n \leq \arg z \leq \theta_n + \alpha_n,\ 0 \leq \theta_n < 2\pi, \quad n = 1, 2, \cdots.$$

The quadruple $(R_n, r_n, \theta_n, \alpha_n)$ are the *parameters associated with* γ_n. For any given sequence $\{\gamma_n\}$ with associated parameters $\{(R_n, r_n, \theta_n, \alpha_n)\}$, and any fixed $0 < \alpha < 2\pi$, $\alpha_n \leq \alpha$, all n, define for $n = 1, 2, \cdots$,

$$(2.2) \qquad F_n^{(\alpha)}:\ 0 < |z| < R_n;\ \theta_n - \left(\frac{\alpha - \alpha_n}{2}\right) < \arg z < \theta_n + \frac{\alpha + \alpha_n}{2}.$$

So $F_n^{(\alpha)}$ is the circular sector of $|z| < R_n$ of opening α which contains the interior of E_n in a symmetric fashion. Lastly set, for $n = 1, 2, \cdots$,

$$(2.3) \quad L_n^{(\alpha)}:\ \frac{1}{4} r_n < |z| < \frac{1}{2} r_n;\ \theta_n - \left(\frac{\alpha}{4} - \frac{\alpha_n}{2}\right) < \arg z < \theta_n + \left(\frac{\alpha}{4} + \frac{\alpha_n}{2}\right).$$

Then $L_n^{(\alpha)}$ is a wedge-shaped domain of opening $\frac{\alpha}{2}$ symmetric about the line bisecting the angle α_n. See Fig. 1 for the various domains.

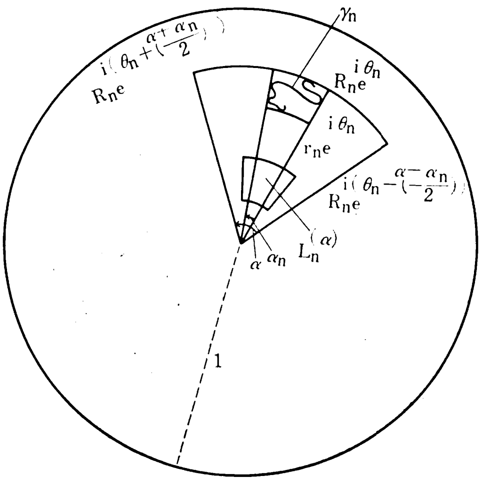

Fig. 1.

We will use the notation E_n, $F_n^{(\alpha)}$ and $L_n^{(\alpha)}$ exclusively in section 3 as defined here and always relative to a given sequence $\{\gamma_n\}$. It is trivial but important to note that because $r_n<1$, $L_n^{(\alpha)}$ is always contained within the disk $|z|<\frac{1}{2}$, all n, regardless of the sequence to which it is associated and regardless of our choice of α.

Now a *PHD* sequence can be characterized by the behavior of its associated parameters. To this end let us call a sequence of Jordan arcs $\{\gamma_n\}$ in D with associated parameters $\{(R_n, r_n, \theta_n, \alpha_n)\}$ a *radial-like sequence* if

$$\text{(2.4)}\quad \begin{aligned} &\text{i)}\quad 0<\varliminf_{n\to\infty}\rho(R_n, r_n)\leq\varlimsup_{n\to\infty}\rho(R_n, r_n)<\infty;\\ &\text{ii)}\quad \varlimsup_{n\to\infty}\rho(R_ne^{i\theta_n}, R_ne^{i(\theta_n+\alpha_n)})<\infty;\end{aligned}$$

or an *arc-like sequence* if

$$\text{(2.5)}\quad \begin{aligned} &\text{i)}\quad \varliminf_{n\to\infty}\rho(R_n, r_n)=0\\ &\text{ii)}\quad 0<\varliminf_{n\to\infty}\rho(R_ne^{i\theta_n}, R_ne^{i(\theta_n+\alpha_n)})\leq\varlimsup_{n\to\infty}\rho(R_ne^{i\theta_n}, R_ne^{i(\theta_n+\alpha_n)})<\infty.\end{aligned}$$

(If γ_n is the segment of the radius $re^{i\theta_0}$ defined by $1-\frac{2}{n}\leq r\leq 1-\frac{1}{n}$, -as in (1.1)- then $\{\gamma_n\}$ is a radial-like sequence; while if $\{\gamma_n\}$ is the arc of $|z|=1-\frac{1}{n}$ defined by $\theta_0\leq\arg z\leq\theta_0+\frac{1}{n}$ an easy calculation shows that this $\{\gamma_n\}$ is an arc-like sequence-hence the nomenclature for each family.)

Proposition 1. *A sequence of Jordan arcs* $\{\gamma_n\}$ *in* D *is a PHD sequence if and only if it satisfies* B_1 *of* (2.0) *and each subsequence contains either a radial-like subsequence or an arc-like subsequence* (*or both*).

Proof. The demonstration of this proposition is elementary but tedious. We sketch the proof. Let $\{\gamma_n\}$ have parameters $\{(R_n, r_n, \theta_n, \alpha_n)\}$. The geometry of the situation gives

(2.6) $$HD(\gamma_n) \leq \rho(r_n, R_n) + \rho(R_n e^{i\theta_n}, R_n e^{i(\theta_n+\alpha_n)}).$$

On the other hand because of our definition of E_n in (2.1), and because of the property of the hyperbolic distance

(2.7) $$HD(\gamma_n) \geq \rho(r_n, R_n).$$

Since

(2.8) $$\rho(R_n e^{i\theta_n}, R_n e^{i(\theta_n+\alpha_n)}) \leq \rho(r_n e^{i(\theta_n+\alpha_n)}, r_n e^{i\theta_n}) + 2\rho(r_n, R_n),$$

then

(2.9) $$HD(\gamma_n) \geq \rho(r_n e^{i(\theta_n+\alpha_n)}, r_n e^{i\theta_n})$$
$$\geq \rho(R_n e^{i\theta_n}, R_n e^{i(\theta_n+\alpha_n)}) - 2\rho(r_n, R_n).$$

The inequalities obtained in (2.6), (2.7) and (2.9) in various combinations prove Proposition 1. Let $\{\gamma_n\}$ be a *PHD* sequence and $\{\gamma_{n_k}\}$ be any subsequence. If $\underline{\lim}_{k\to\infty} \rho(r_{n_k}, R_{n_k}) > 0$ then (2.7) and (2.9) show that it is a radial-like subsequence. If $\underline{\lim}_{k\to\infty} \rho(r_{n_k}, R_{n_k}) = 0$ then the subsequence for which this limit holds is an arc-like subsequence by virtue of (2.6) and (2.9). Conversely assume $\{\gamma_n\}$ is not a *PHD* sequence then there is a subsequence such that either $\lim_{k\to\infty} HD(\gamma_{n_k}) = 0$; or else $\lim_{k\to\infty} HD(\gamma_{n_k}) = \infty$. In the first case (2.7) says $\{\gamma_{n_k}\}$ cannot contain a radial-like subsequence, and (2.9) shows it cannot contain an arc-like sequence either. In the second case (2.6) shows that it cannot contain either an arc-like or a radial-like subsequence.

We now define the order of f on a sequence of Jordan arcs $\{\gamma_n\}$.

Definition 2. *Let f be defined in D taking values in the extended plane W. Let $\{\gamma_n\}$ be a sequence of Jordan arcs in D, $\{A_n\}$ a sequence of positive numbers and $s \geq 0$. We say f has s-exponential order $\{A_n\}$ on $\{\gamma_n\}$ if*

$$|f(z)| \leq \exp \frac{-A_n}{(1-|z|)^s}, \quad z \in \gamma_n, \ n = 1, 2, \cdots.$$

For example, in this terminology Koebe's lemma now reads: Let $\{\gamma_n\}$ be a sequence of Jordan arcs with associate parameters $\{(R_n, r_n, \theta_n, \alpha_n)\}$ such that a bounded holomorphic f has o-exponential order $\{A_n\}$ on $\{\gamma_n\}$. If $\overline{\lim_{n\to\infty}}\alpha_n>0$ and $A_n\to\infty$, $n\to\infty$, then f is identically zero.

Definition 3. *Let $S\subseteq D$ and $0<r\leq 1$. For a complex-valued function defined in D set*

$$\mathcal{M}(r, f, S)=\max(\sup_{\substack{z\in S\\ |z|<r}}\log|f(z)|, 1)$$

If $r=1$ we omit mention of this variable and merely write $M(f, S)$; and if $S=D$ we also abbreviate as $\mathcal{M}(r, f)$. There is no great significance is this somewhat unusual definition. It merely insures that $\mathcal{M}(r, f, S)\geq 1$, so it does not happen that $\mathcal{M}(r, f, S)\to 0$, as $r\to 1$, which is a convenience for us.

Definition 4. *A simple continuous curve $\gamma=\gamma(t)$, $0\leq t<1$, lying in D is said to be a boundary path if $\lim_{t\to 1}|\gamma(t)|=1$; and a boundary path at $\tau\in C$ if $\lim_{t\to 1}\gamma(t)=\tau$.*

One further convention we adopt. Most of the arguments used in this work involve a limiting process, and we are not interested in the first N_0 terms. Rather than keeping a score of the various indices we sometimes use the phrase relative to some sequence "such and such a property holds eventually for the sequence" to replace "there is an integer N_0 such that the property is true for all members of the sequence with index greater than N_0." As long as we use this phrase only finitely often and are otherwise reasonably careful no problem arises.

3. Two known results needed for main theorem

We now state two known results, one rather trivial, the other not so trivial, which are the cornerstones of our theory. The first is the observation by the author [14, Lemma 1], and also by Lappan [8, Lemma 2], concerning the hyperbolic geometry. We give it here in a slightly revised form.

Lemma A. *Let* $a, b \in D$. *Set* $\rho(a, b) = \rho$; $K(\rho) = \frac{e^{2\rho}-1}{e^{2\rho}+1}$; $t(a, b) = \frac{|a-b|}{1-|a|}$; *and* $t(b, a) = \frac{|a-b|}{1-|b|}$. *We then have* $\frac{K(\rho)}{1+K(\rho)} < t(a, b) < \frac{2K(\rho)}{1-K(\rho)}$; *and the same inequality holds also for* $t(b, a)$.

This factorization of the Euclidean distance between a and b gives a simple connection, useful for both computation and intuition, between the Euclidean and non-Euclidean hyperbolic distance. We use this lemma to give an estimate which will be useful in the next two sections. With the notation as given in the lemma we have

$$\frac{1-K(\rho)}{2(1+K(\rho))} < \frac{t(a, b)}{t(b, a)} < \frac{2(1+K(\rho))}{1-K(\rho)}. \tag{3.0}$$

So that, if $\{z_n\}$ and $\{z'_n\}$ are two sequences in D with

$$\overline{\lim_{n\to\infty}}\, \rho(z_n, z'_n) < A < \infty,$$

taking note of the properties of $\frac{1-x}{1+x}$ and $\frac{1+x}{1-x}$, (3.0) becomes for n sufficiently large

$$0 < \frac{1-K(A)}{2(1+K(A))} < \frac{t(z_n, z'_n)}{t(z'_n, z_n)} < \frac{2(1+K(A))}{1-K(A)} < \infty. \tag{3.1}$$

To put it somewhat differently if $\{z_n\}$ and $\{z'_n\}$ are two sequences with $\lim_{n\to\infty}|z_n| = \lim_{n\to\infty}|z'_n| = 1$, and $\overline{\lim_{n\to\infty}}\, \rho(z_n, z'_n) < \infty$, then $(1-|z_n|)$ and $(1-|z'_n|)$ have the same order as $n\to\infty$. If, in addition, $0 < \underline{\lim}_{n\to\infty}\, \rho(z_n, z'_n)$, then $|z_n - z'_n|$ and $(1-|z_n|)$ also have the same order. We use these facts frequently. When there is no need to indicate the dependence on z_n and z'_n we write $t(z_n, z'_n)$ (and $t(z'_n, z_n)$) simply as t_n.

We have listed the elementary result. We now give the more profound result-a form of the Schmidt-Milloux inequality. We state the result in somewhat limited form sufficient for our needs. For a more general statement see, for example, Tsuji [17, p. 306]. If γ is a boundary path at a point $\tau\in C$ we define $\omega(z, \gamma, D-\gamma)$ to be the harmonic measure at z of γ relative to $D-\gamma$.

Theorem A. *Let γ be a boundary path at a point $\tau\in C$. If $\min_{z\in\gamma}|z|=a$ then for $z\in D-\gamma$,*

$$\omega(z, \gamma, D-\gamma)\geq\frac{2}{\pi}\arcsin\frac{(1-a^2)(1-|z|^2)}{16}.$$

This formulation is obtained from the usual form in which $a=0$ by the routine device of mapping D onto D by a linear transformation which takes γ onto a boundary path with $a=0$ and using the conformal invariance of the harmonic measure. Some obvious estimates then produce the above inequality.

4. Main Theorem

Suppose the following situation exists. We have a function f holomorphic in D; a given *PHD* sequence $\{\gamma_n\}$; and a value α, $0<\alpha<2\pi$, such that $\overline{F}_n^{(\alpha)}\supseteq E_n$, $n=1, 2, \cdots$, where these domains are relative to $\{\gamma_n\}$ and are as defined in (2.1) and (2.2).

Theorem 1. *Let f be holomorphic in D such that for some finite value w_0, f-w_0 has 1-exponential order $\{A_n\}$ on a PHD sequence $\{\gamma_n\}$. If*

$$\varliminf_{n\to\infty}\frac{\mathcal{M}(f, F_n^{(\alpha)})}{A_n}=0, \tag{4.0}$$

then f is identically w_0.

Remark. We now see in what direction we have moved from the classical Koebe's lemma by lifting the restriction that the angles α_n associated with γ_n be uniformly bounded away from 0 and by increasing the requirement that f merely tend to 0 on γ_n.

Proof. Since $\mathcal{M}(f, F_n^{(\alpha)})$ and $\mathcal{M}(f-w_0, F_n^{(\alpha)})$ have the same order of growth we can assume, without loss of generality, that $w_0=0$. We suppose (4. 0) holds and let $\{n_k\}$ be the sequence such that

$$(4\cdot1)\qquad \lim_{k\to\infty}\frac{\mathcal{M}(f, F_{nk}^{(\alpha)})}{A_{n_k}}=0\,.$$

We divide the proof into two cases according as to whether $\{\gamma_{n_k}\}$ contains a radial-like subsequence or an arc-like subsequence.

Case i) $\{\gamma_{n_k}\}$ *contains a radial-like subsequence.*

Let this subsequence by $\{\gamma_{n_{k_i}}\}$ and let $j=n_{k_i}$. Then (4. 1) is

$$(4.2)\qquad \lim_{j\to\infty}\frac{\mathcal{M}(f, F_j^{(\alpha)})}{A_j}=0\,.$$

Let $\{(R_j, r_j, \theta_j, \alpha_j)\}$ be the parameters associated with $\{\gamma_j\}$. Furthermore, we can assume that γ_j meets the circle $|z|=R_j$ only at the point $R_je^{i\theta'_j}$ and meets $|z|=r_j$ only at $r_je^{i\theta''_j}$, and that these are the end points of the curve γ_j. (It is clear we can find a subarc of γ_j that satisfies the above condition. Certainly f has the same exponential order on this subarc as on γ_j. These subarcs so chosen for each j have the same parameters $\{(R_j, r_j)\}$ and so are also a radial-like sequence. If θ_j and α_j are altered then of course so is E_j. However we do not need nor use the fact that this new subarc may not meet the left and right boundaries of E_j but use only that this new subarc is contained in the original E_j as described above. So we retain the sets E_j and $F_j^{(\alpha)}$ as defined for the sequence $\{\gamma_j\}$).

Our technique here-and in case (ii)-is to use the two constant theorem of the brothers Nevanlinna [11, p. 42] to estimate the value of f on the domain $F_j^{(\alpha)}$ and in particular on the subdomain $L_j^{(\alpha)}$.

To this end let $\omega(z, \gamma_j, F_j^{(\alpha)}-\gamma_j)$ be the harmonic measure at z of γ_j relative to $F_j^{(\alpha)}-\gamma_j$. The two constant theorem gives for $z\in F^{(\alpha)}-\gamma_j$

$$(4.3)\qquad \log|f(z)|\leq\omega(z, \gamma_j, F_j^{(\alpha)}-\gamma_j)\left(\frac{-A_j}{1-r_j}\right) + (1-\omega(z, \gamma_j, F_j^{(\alpha)}-\gamma_j))\mathfrak{M}(f, F_j^{(\alpha)}),$$

where the estimate of f on γ_j follows from the fact that for $z\in\gamma_j$, $1-|z|\leq 1-r_j$.

Our efforts are now bent toward estimating the harmonic measure for $z\in L_j^{(\alpha)}$. Consequently map $F_j^{(\alpha)}$ onto the unit disk D_w: $|w|<1$, by the conformal map

$$(4.4)\qquad w_j(z)=w(h_j(z))=\frac{i-\left(\dfrac{1+h_j(z)}{1-h_j(z)}\right)^2}{i+\left(\dfrac{1+h_j(z)}{1-h_j(z)}\right)^2},$$

with

$$(4.5)\qquad h_j(z)=\left(\frac{e^{-i\eta_j}z}{R_j}\right)^{\pi/\alpha}, \quad \eta_j=\theta_j-\frac{(\alpha-\alpha_j)}{2}, \quad j=1, 2, \cdots.$$

If $|w_j(r_je^{i\theta_j''})|=a_j$, let γ_j^* be the subarc of $w_j(\gamma_j)$ which connects $|w|=a_j$ to $|w|=1$, and lies, except for endpoints, in $a_j<|w|<1$. By Carleman's principle of Gebeitserweiterung [11, p. 64] we have for $z\in D_w-w_j(\gamma_j)$

$$(4.6)\qquad \omega(w, w_j(\gamma_j), D_w-w_j(\gamma_j))\geq\omega(w, \gamma_j^*, D_w-\gamma_j^*).$$

Theorem A reveals

$$(4.7)\qquad \omega(w, \gamma_j^*, D_w-\gamma_j^*)\geq\frac{2}{\pi}\arcsin\frac{(1-|w|^2)(1-a_j^2)}{16}.$$

By the conformal invariance of the harmonic measure (4.6) and (4.7) allow us to write, for $z\in F_j^{(\alpha)}-\gamma_j$,

$$(4.8)\qquad \omega(z, \gamma_j, F_j^{(\alpha)}-\gamma_j)\geq\frac{2}{\pi}\arcsin\frac{(1-|w_j(z)|^2)(1-|w_j(r_je^{i\theta_j''})|^2)}{16}.$$

To further estimate the right side of (4.8) we now restrict $z\in L_j^{(\alpha)}$. First note

(4.9) $$|w_j(L_j^{(\alpha)})|\leq B_0<1,$$

where B_0 is independent of j. (Simply observe that after applying $z_1=h_j(z)$ the image of $L_j^{(\alpha)}$ is, for all j, contained in a set of the form z_1: $\frac{\pi}{4}<\arg z_1<\frac{3}{4}\pi$; $\left(\frac{1}{8}\right)^{\pi/\alpha}<|z_1|<\left(\frac{1}{2}\right)^{\pi/\alpha}$; and the remaining action of $w_j(z)$ is holomorphic and independent of j and α.)

A tedious but simple calculation gives

(4.10) $$1-|w_j(r_je^{i\theta_j''})|^2\geq\left[\operatorname{Im}\left(\frac{r_je^{i(\theta_j''-\eta_j)}}{R_j}\right)^{\pi/\alpha}\right]\left[1-\left(\frac{r_j}{R_j}\right)^{2\pi/\alpha}\right].$$

Remembering that $\frac{r_j}{R_j}\geq\frac{1}{2}$ we can estimate that

(4.11) $$\operatorname{Im}\left(\frac{r_je^{i(\theta_j''-\eta_j)}}{R_j}\right)^{\pi/\alpha}=\left(\frac{r_j}{R_j}\right)^{\pi/\alpha}\sin\frac{(\theta_j''-\eta_j)\pi}{\alpha}\geq\left(\frac{1}{2}\right)^{\pi/\alpha}\left(1-\frac{\alpha_j}{\alpha}\right).$$

As we have noted before, for a *PHD* sequence $\alpha_j\to 0$, $j\to\infty$, so that eventually $\frac{\alpha_j}{\alpha}\leq\frac{1}{2}$ and (4.11) becomes eventually

(4.12) $$\operatorname{Im}\left(\frac{r_je^{i(\theta_j''-\eta_j)}}{R_j}\right)^{\pi/\alpha}\geq\left(\frac{1}{2}\right)^{\pi/\alpha+1}.$$

The second factor on the right side of (4.10) is estimated by applying the mean-value theorem to $f(x)=x^{2\pi/\alpha}$. Since $r_j\geq\frac{1}{2}$,

(4.13) $$\frac{R_j^{2\pi/\alpha}-r_j^{2\pi/\alpha}}{R_j^{2\pi/\alpha}}\geq\frac{2\pi}{\alpha}\left(\frac{1}{2}\right)^{2\pi/\alpha-1}(R_j-r_j).$$

Setting $K_\alpha=\frac{\pi}{\alpha}\left(\frac{1}{2}\right)^{3\pi/\alpha-1}$, (4.10), (4.12), and (4.13) together give that

(4.14) $$1-|w_j(r_je^{i\theta_j''})|^2\geq K_\alpha(R_j-r_j).$$

Since $\arcsin t>t$, $t>0$, (4.8), (4.9) and (4.14) transform (4.3) to

(4.15) $$\log|f(z)|\leq C_\alpha\frac{(R_j-r_j)}{1-r_j}(-A_j)+\mathcal{M}(f,E_j^{(\alpha)}),$$

for $z\in L_j^{(\alpha)}$, $C_\alpha=\frac{K_\alpha}{8\pi}(1-B_0^2)$, and j sufficiently large.

From here it is but a short trip to the desired conclusion. If we refer to Lemma A (and the remarks thereafter) the fact that $\{\gamma_j\}$ is a radial-like sequence then implies eventually

$$(4.16)\qquad (R_j-r_j)\geq(1-r_j)t_0,\quad t_0>0\,.$$

So (4.15) and (4.16) give for $z\in L_j^{(\alpha)}$, j sufficiently large,

$$(4.17)\qquad \log|f(z)|\leq -A_j\left(C_\alpha t_0-\frac{\mathcal{M}(f, F_j^{(\alpha)})}{A_j}\right).$$

Now (4.2) implies $A_j\to+\infty$, $j\to+\infty$, so that we have for any sequence $\{z_j\}$, $z\in L_j^{(\alpha)}$, that $f(z_j)\to 0$, $j\to\infty$. Since there exists a subsequence $\{j_m\}$ such that $L_{j_m}^{(\alpha)}$ tends to a domain of the form $\frac{1}{4}<|z|<\frac{1}{2}$; $\theta_0-\frac{\alpha}{4}<\arg z<\theta_0+\frac{\alpha}{4}$; and since f cannot be identically zero on this limit domain unless f is zero on all of D this completes the proof of case (i)

Case ii). $\{\gamma_{n_k}\}$ *contains an arc-like subsequence.*

Let this subsequence be $\{\gamma_{n_{k_i}}\}$ and again set $n_{k_i}=j$ so that we have

$$(4.18)\qquad \lim_{j\to\infty}\frac{\mathcal{M}(f, F_j^{(\alpha)})}{A_j}=0\,.$$

We again modify the arcs γ_j slightly in that we select a subarc of γ_j so that this subarc meets $\arg z=\theta_j$ only at $r_j'e^{i\theta_j}$, and meets $\arg z=\theta_j+\alpha_j$ only at $r_j''e^{i(\theta_j+\alpha_j)}$, where these points are the endpoints of the subarc. As in case (i) we retain the notation γ_j for these subarcs which are readily seen to be an arc-like sequence if we but refer to (2.9) which is valid for these subarcs. We also retain the sets E_j and $F_j^{(\alpha)}$ defined for the original sequence $\{\gamma_j\}$. Keep in mind that γ_j now may not meet $|z|=r_j$ or $|z|=R_j$, but this is of no consequence.

Let $q_j^{(1)}$ be the rectilinear segment from $r_j'e^{i\theta_j}$ to $R_je^{i\theta_j}$, and $q_j^{(2)}$ the rectilinear segment from $r_j''e^{i(\theta_j+\alpha_j)}$ to $R_je^{i(\theta_j+\alpha_j)}$ (One or both of these segments may reduce to a point.)

Let $G_j^{(\alpha)}$ be the domain bounded by the radial segments bounding $F_j^{(\alpha)}$; the two arcs of $|z|=R_j$ from these rays to the rays bounding E_j; the segments $q_j^{(1)}$ and $q_j^{(2)}$; and γ_j. Let $L_j^{(\alpha)}$ be defined as in (2.3). For these various domains see Fig. 2.

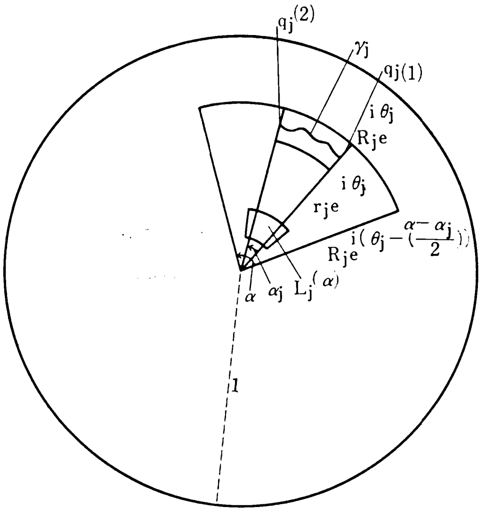

Fig. 2.

Our procedure is basically the same as in case (i). We use harmonic measure to estimate f on $L_j^{(\alpha)}$ by the two constant theorem. Then we estimate the harmonic measure. Let $\omega(z, \gamma_j, G_j^{(\alpha)})$ be the harmonic measure at z of γ_j relative to $G_j^{(\alpha)}$. The two constant theorem gives for $z \in G_j^{(\alpha)} \subseteq F_j^{(\alpha)}$, and $r_j^* = \min|z|$, $z \in \gamma_j$,

$$(4\cdot19) \qquad \log|f(z)| \leq \omega(z, \gamma_j, G_j^{(\alpha)})\left(-\frac{A_j}{(1-r_j^*)}\right) + (1-\omega(z, \gamma_j, G_j^{(\alpha)}))\mathcal{M}(f, F_j^{(\alpha)}).$$

We now estimate from below $\omega(z, \gamma_j, G_j^{(\alpha)})$, in particular for $z \in L_j^{(\alpha)}$.

Carleman's Gebeitserweiterung gives us the estimate

$$(4.20) \qquad \omega(z, \gamma_j \cup q_j^{(1)} \cup q_j^{(2)}, G_j^{(\alpha)}) \geq \omega(z, s_j, F_j^{(\alpha)}),$$

where s_j is the arc of $|z|=R_j$ bounding E_j. Because of the additivity of the harmonic measure, (4.21) $\omega(z, \gamma_j, G_j^{(\alpha)}) \geq \omega(z, s_j, F_j^{(\alpha)}) - \omega(z, q_j^{(1)}, G_j^{(\alpha)}) - \omega(z, q_j^{(2)}, G_j^{(\alpha)})$.

Suppose there was a positive constant C_α such that for $z \in L_j^{(\alpha)}$,

$$\omega(z, s_j, F_j^{(\alpha)}) \geq C_\alpha(1-R_j); \tag{4.22}$$

and further we could show for $z \in L_j^{(\alpha)}$ that eventually both

$$\begin{aligned} &\omega(z, q_j^{(1)}, G_j^{(\alpha)}) \leq \frac{C_\alpha}{4}(1-R_j); \\ &\omega(z, q_j^{(2)}, G_j^{(\alpha)}) \leq \frac{C_\alpha}{4}(1-R_j). \end{aligned} \tag{4.23}$$

If (4.22) and (4.23) were combined with (4.21) the result would be that eventually, $z \in L_j^{(\alpha)}$

$$\omega(z, \gamma_j, G_j^{(\alpha)}) \geq \frac{C_\alpha}{2}(1-R_j). \tag{4.24}$$

If we again harken to Lemma A, because $\{\gamma_j\}$ is arc-like, then eventually $(1-r_j^*) \leq (1-R_j)t_0$, $t_0 < \infty$, and so (4.19) would become eventually, on account of (4.24).

$$\log|f(z)| \leq \frac{C_\alpha}{2}\frac{1}{t_0}(-A_j) + \mathcal{M}(f, F_j^{(\alpha)}), \quad z \in L_j^{(\alpha)}, \tag{4.25}$$

which is (essentially) the same as in case (i), (4.17) and we could again conclude that $f=0$. So our proof will be complete if we demonstrate (4.22) and (4.23), and this we do in the following two lemmas.

Lemma 1. *Let* $\{\gamma_j\}$ *be an arc-like sequence with the situation of case* (ii) *of the proof of Theorem* 1 *prevailing. Then there exists a positive constant* C_α*, depending only on* α*, such that if* $z \in L_j^{(\alpha)}$,

$$\omega(z, s_j, F_j^{(\alpha)}) \geq C_\alpha(1-R_j).$$

Proof. First map $F_j^{(\alpha)}$ onto the upper half disk D'_w: $(|w|<1)$ $(\operatorname{Im} w > 0)$ by $w = h_j(z)$, where this function is defined as in (4.5). As we noted in case (i), $h_j(L_j^{(\alpha)})$ is contained in a set of the form

$\frac{\pi}{4}<\arg w<\frac{3}{4}\pi$, $\left(\frac{1}{8}\right)^{\pi/\alpha}<|w|<\left(\frac{1}{2}\right)^{\pi/\alpha}$; and $h_j(s_j)$ is the arc of $|w|=1$ symmetric about the imaginary axis subtending an angle at the origin of $\frac{\pi\alpha_j}{\alpha}$ radians. By the conformal invariance of the harmonic measure, with $h_j(s_j)=s'_j$

$$\omega(z, s_j, F_j^{(\alpha)})=\omega(h_j(z), s'_j, D'_w). \tag{4.26}$$

The harmonic function $\omega(w, s'_j, D'_w)$ can be continued by reflection across the real axis to all of D_w, and this extended function, which we denote by $w_j^*(w)$, is given by the usual representation

$$w_j^*(re^{i\theta}) =\frac{1}{2\pi}\int_{\frac{\pi}{2}\left(1-\frac{\alpha_j}{\alpha}\right)}^{\frac{\pi}{2}\left(1+\frac{\alpha_j}{\alpha}\right)}\frac{1-r^2 d\varphi}{1+r^2-2r\cos(\theta-\varphi)}\frac{-1}{2\pi}\int_{\frac{3\pi}{2}\left(1-\frac{\alpha_j}{3\alpha}\right)}^{\frac{3\pi}{2}\left(1+\frac{\alpha_j}{3\alpha}\right)}\frac{1-r^2 d\varphi}{1+r^2-2r\cos(\theta-\varphi)}. \tag{4.27}$$

A simple change of variables reduces (4.27) to

$$w_j^*(re^{i\theta})=\frac{1-r^2}{2\pi}\int_{\frac{\pi}{2}\left(1-\frac{\alpha_j}{\alpha}\right)}^{\frac{\pi}{2}\left(1+\frac{\alpha_j}{\alpha}\right)}\frac{4r\cos(\theta-\varphi)d\varphi}{(1+r^2-2r\cos(\theta-\varphi))(1+r^2+2r\cos(\theta-\varphi))}. \tag{4.28}$$

For $w\in h_j(L_j^{(\alpha)})$, $\left(\frac{1}{8}\right)^{\pi/\alpha}<|w|<\left(\frac{1}{2}\right)^{\pi/\alpha}$, and so, if we put

$$K_\alpha=\frac{2\left(1-\left(\frac{1}{2}\right)^{2\pi/\alpha}\right)\left(\frac{1}{8}\right)^{\pi/\alpha}}{\pi\left(1+\left(\frac{1}{2}\right)^{\pi/\alpha}\right)^4}>0,$$

then (4.28) gives the estimate for $w\in h_j(L_j^{(\alpha)})$,

$$w_j^*(w)\geq K_\alpha\int_{\frac{\pi}{2}\left(1-\frac{\alpha_j}{\alpha}\right)}^{\frac{\pi}{2}\left(1+\frac{\alpha_j}{\alpha}\right)}\cos(\theta-\varphi)d\varphi=2K_\alpha\sin\theta\sin\left(\frac{\pi}{2}\frac{\alpha_j}{\alpha}\right) \tag{4.29}$$

$$\geq\frac{2}{\sqrt{2}}K_\alpha\left(\frac{\alpha_j}{\alpha}\right).$$

Of course, $\alpha_j\geq|R_je^{i\theta_j}-R_je^{i(\theta_j+\alpha_j)}|\geq(1-R_j)t_0$, $0<t_0<\infty$, where this last inequality follows from Lemma A and the arc-like property of $\{\gamma_j\}$. Gathering all the constants as $C_\alpha = \frac{2}{\sqrt{2}} \frac{K_\alpha}{\alpha} t_0$, (4.29) together with (4.26) give for $z \in L_j^{(\alpha)}$

$$\omega(z, \gamma_j, F_j^{(\alpha)}) \geq C_\alpha (1 - R_j), \tag{4.30}$$

which is the desired inequality.

Lemma 2. *Let* $\omega(z, q_j^{(1)}, G_j^{(\alpha)})$ *and* $\omega(z, q_j^{(2)}, G_j^{(\alpha)})$ *be defined as in case* (ii) *of the proof of Theorem 1. Then given any number* $\epsilon > 0$ *there is an integer* $J = J(\epsilon)$ *such that for all* $z \in L_j^{(\alpha)}$, $j \geq J$,

$$\omega(z, q_j^{(1)}, G_j^{(\alpha)}) \leq \epsilon (1 - R_j);$$

$$\omega(z, q_j^{(2)}, G_j^{(\alpha)}) \leq \epsilon (1 - R_j).$$

Proof. We consider first $q_j^{(1)}$. By the Gebeitserweiterung, for $z \in G_j^{(\alpha)}$ and $D_R: |z| < R$, we have

$$\omega(z, q_j^{(1)}, G_j^{(\alpha)}) < \omega(z, q_j^{(1)}, D_{R_j}). \tag{4.31}$$

The harmonic measure on the right can be found in explicit form. Let j be fixed. Remembering that $q_j^{(1)}: re^{i\theta_j}$, $r'_j \leq r \leq R_j$, and putting $w_1 = f_1(z) = \frac{(e^{-i\theta_j})z}{R_j}$; $w_2 = f_2(w_1) = \frac{w_1 - r'_j/R_j}{1 - w_1 r'_j/R_j}$; and $f_3(z) = f_2(f_1(z))$, we find, after some elementary calculations,

$$\omega(z, q_j^{(1)}, D_{R_j}) = \frac{2}{\pi} \arcsin \frac{(1 - |f_3(z)|)}{|1 - f_3(z)|} \leq \frac{1 - |f_3(z)|}{|1 - f_3(z)|}, \tag{4.32}$$

where the determination of the square root is $\sqrt{-\frac{1}{4}} = \frac{i}{2}$. It is easy to find the image of $L_j^{(\alpha)}$ by f_3. Observe first that we have

$$f_1(L_j^{(\alpha)}) \subseteq \left\{ |w| < \frac{1}{2} \right\}; \tag{4.33}$$

and if we construe this disk as a non-Euclidean disk of radius $\frac{\log 3}{2}$ with center 0 then $f_2(f_1(L_j^{(\alpha)}))$ is contained in the non-Euclidean disk about $-r'_j/R_j$ with the same non-Euclidean radius

$\frac{\log 3}{2}$ (because f_2 is a linear map). Since $-r'_j/R_j \to -1$, as $j \to \infty$, so also does $f_3(L_j^{(\alpha)})$. This means that we can determine a J_1 so that for $z \in L_j^{(\alpha)}$ and $j \geq J_1$,

$$(4.34) \qquad |1-f_3(z)| \geq \frac{3}{2}.$$

Then (4.32) becomes for $z \in L_j^{(\alpha)}$ and $j \geq J_1$,

$$(4.35) \qquad \omega(z, q_j^{(1)}, D_{R_j}) \leq \frac{2}{3}(1-|f_3(z)|).$$

It is straightforward to estimate for $z \in L_j^{(\alpha)}$

$$(4.36) \quad 1-|f_3(z)| < 1-|f_3(z)|^2 = \frac{(R_j^2-|z|^2)(R_j^2-(r'_j)^2)}{|R_j^2-e^{-i\theta_j}zr'_j|^2} \leq 8(R_j-r'_j).$$

Since $\{\gamma_j\}$ is an arc-like sequence and $r_j \leq r'_j \leq R_j$, Lemma A gives

$$(4.37) \qquad (R_j-r'_j) = (1-R_j)t_j, \quad t_j \to 0, \quad j \to \infty.$$

Thus given $\epsilon > 0$ choose $J_2(\epsilon) \geq J_1$ so that for $j \geq J_2$,

$$(4.38) \qquad t_j \leq \frac{3\epsilon}{16}.$$

Then (4.31), (4.32), (4.35), (4.36), (4.37) and (4.38) all join together to reveal that for $z \in L_j^{(\alpha)}$, $j \geq J_2(\epsilon)$,

$$\omega(z, q_j^{(1)}, G_j^{(\alpha)}) \leq \epsilon(1-R_j).$$

Since the proof with $q_j^{(1)}$ replaced by $q_j^{(2)}$ is identical (with r''_j replacing r'_j) the lemma is proved; and the demonstration of Theorem 1 has been completed.

Condition (4.1) of Theorem 1 cannot be relaxed to allow the limit to be positive. The function $f(z) = \exp\left(\frac{z+1}{z-1}\right)$, given in the introduction, is bounded in D, and has 1-exponential order $\{1\}$ on the *PHD* sequence $\{\gamma_n\}$ defined by (1.1). As we also noted in the introduction, given any positive sequence $\{A_n\}$ there is a non-*PHD* sequence of arcs $\{\gamma_n\}$, and a bounded function which has 1-exponential order $\{A_n\}$, on $\{\gamma_n\}$. So the *PHD* property cannot be omitted.

There are various corollaries inherent in Theorem 1. We mention a few.

Corollary 1. *Suppose γ is a boundary path in D and f is holomorphic in D such that for some finite value w_0, and some positive function $A(r)$, $0\leq r<1$*

(4.39) $$\log|f(z)-w_0|\leq\frac{-A(r)}{(1-|z|)},\quad z\in\gamma,\ |z|\geqq r.$$

If

$$\lim_{\overline{r\to 1}}\frac{\mathcal{M}(r,f)}{A(r)}=0,$$

then $f=w_0$.

Proof. Let $\{R_k\}$ be a sequence, $0<R_k\leq R_{k+1}<1$, $R_k\to 1$, $k\to\infty$, for which

(4.40) $$\lim_{k\to\infty}\frac{\mathcal{M}(R_k,f)}{A(R_k)}=0.$$

It is fairly obvious that for R_k sufficiently close to 1 we may construct a sequence of Jordan arcs $\{\gamma_k\}$ in D, satisfying, for some sequence $\{r_k\}$,

i) $\gamma_k\subseteq\gamma$, all k;

(4.41) ii) $\{\gamma_k\}$ has associated parameters $\{(R_k, r_k, \theta_k, \alpha_k)\}$;

iii) $\{\gamma_k\}$ is a *PHD* sequence.

(For example, choose a sequence $\{r_k\}$, $0<r_k<R_k$, so that

$$0<\varliminf_{k\to\infty}\rho(r_k, R_k)\leq\varlimsup_{k\to\infty}\rho(r_k, R_k)<\infty.$$

Let γ_k be a subarc of γ contained in the annulus $r_k<|z|<R_k$ except for one endpoint on $|z|=R_k$ and the other on $|z|=r_k$. This is possible for large k. If $\{\gamma_k\}$ is a *PHD* sequence all is well. If not, this sequence is too wide, i.e., does not satisfy (ii) of (2.4) because the limit superior is $+\infty$. But then we select subarcs γ_k^*,

$\gamma_k^* \subseteq \gamma_k$, with the same endpoint on $|z|=R_k$ such that $\{\gamma_k^*\}$ is now a *PHD* sequence.)

By (4.39) for $z \in \gamma_k$

$$\log|f(z)-w_0| \leq -\frac{A(R_k)}{1-|z|},$$

or $f-w_0$ has 1-exponential order $\{A(R_k)\}$ on $\{\gamma_k\}$. For any suitable $\alpha>0$ define $F_k^{(\alpha)}$ relative to $\{\gamma_k\}$. We then have

$$\mathcal{M}(f, F_k^{(\alpha)}) \leq \mathcal{M}(R_k, f).$$

From (4.40) and the above

$$\lim_{k\to\infty} \frac{\mathcal{M}(f, F_k^{(\alpha)})}{A(R_k)} = 0.$$

Theorem 1 is now operative so $f=w_0$.

Corollary 2. *Let γ be a boundary path in D and suppose for some $\epsilon>0$ there is a positive constant A_ϵ so that for $z \in \gamma$*

$$\log|f(z)-w_0| \leq \frac{-A_\epsilon}{(1-|z|)^{1+\epsilon}}. \tag{4.42}$$

If

$$\overline{\lim_{r\to 1}} \frac{\mathcal{M}(r, f)}{\log \frac{1}{1-r}} < \infty, \tag{4.43}$$

then $f=w_0$.

Proof. As usual we suppose (4.43) holds and select a sequence $\{R_k\}$ so that

$$\overline{\lim_{R_k\to 1}} \frac{\mathcal{M}(R_k, f)}{\log\left(\frac{1}{1-R_k}\right)} = K < \infty.$$

As in the proof of Corollary 1 define a sequence of Jordan arcs $\{\gamma_k\}$ satisfying (4.41). Because of (4.42) and the *PHD* property, for some $0<t_0<\infty$, and $z \in \gamma_k$,

$$\log|f(z)-w_0| \leq \frac{-1}{(1-|z|)} \frac{A_\epsilon}{(1-|z|)^\epsilon} \leq \frac{1}{1-|z|} \frac{-A_\epsilon}{(1-r_k)^\epsilon} \tag{4.44}$$

$$\leq \frac{1}{1-|z|} \frac{-A_\epsilon}{[(1-R_k)t_0]^\epsilon}.$$

So we have $f-w_0$ has 1-exponential order $\left\{\frac{A_\epsilon}{[(1-R_k)t_0]^\epsilon}\right\}$ on $\{\gamma_k\}$. In order to satisfy the hypothesis of Theorem 1 we need to calculate for some suitable $F_k^{(\alpha}$ that

$$\begin{aligned}&\lim_{k\to\infty} \mathcal{M}(f, F_k^{(\alpha)})(1-R_k)^\epsilon \\ &\leq \lim_{k\to\infty}\left(\frac{\mathcal{M}(R_k, f)}{\log\frac{1}{1-R_k}}\right)(1-R_k)^\epsilon \log\frac{1}{1-R_k} \\ &=0.\end{aligned}$$

Therefore $f=w_0$.

A slight generalization of Koebe's lemma is possible if we use the techniques of the proof of Theorem 1, case (ii), together with Lemma 1.

Definition 5. *The sequence $\{\gamma_n\}$ of Jordan arcs in D with associated parameters $\{(R_n, r_n, \theta_n, \alpha_n)\}$ is said to be Koebe sequence if*

i) $\lim_{n\to\infty} r_n = 1$;

ii) $\varliminf_{n\to\infty} \alpha_n > 0$.

Given such a Koebe sequence $\{\gamma_n\}$ certainly we can select a sequence of subarcs $\{\gamma_n'\}$ such that $\gamma_n' \subseteq \gamma_n$; and such that γ_n' meets the line arg $z=\theta_n$ and the line arg $z=\theta_n+\alpha_n$ only at its endpoints, $r_n' e^{i\theta_n}$ and $r_n'' e^{i(\theta_j+\alpha_j)}$ respectively. Consequently $\{\gamma_n'\}$ is again a Koebe sequence and we shall assume, in fact, that any Koebe sequence has this form. Therefore we can let T_n be the triangular-like domain in D bounded by the line segments arg $z=\theta_n$ and arg $z=\theta_n+\alpha_n$ and the arc γ_n.

Theorem 2. *Let f be holomorphic in D and let $\{\gamma_n\}$ be a Koebe* *sequence in D such that for some finite value* w_0, $f-w_0$ *has* 0-*exponential order* $\{A_n\}$ *on* $\{\gamma_n\}$. *If* T_n *is the domain defined above and*

$$(4.46)\qquad \varliminf_{n\to\infty} \frac{\mathcal{M}(f, T_n)}{A_n}=0,$$

then $f=w_0$.

Proof. We may assume $w_0=0$ and if (4.46) holds there is a sequence $\{n_i\}$ such that

$$(4.47)\qquad \lim_{i\to\infty} \frac{\mathcal{M}(f, T_{n_i})}{A_{n_i}}=0.$$

Let $n_i=j$ and suppose $\{(R_j, r_j, \theta_j, \alpha_j)\}$ are the associated parameters of $\{\gamma_j\}$. In close connection with (2.3) and assuming that $r_j\geq\frac{1}{2}$, define

$$(4.48)\quad L_j: \frac{r_j}{4}<|z|<\frac{r_j}{2};\ \theta_j+\frac{\alpha_j}{4}<\arg z<\theta_j+\frac{3\alpha_j}{4}.$$

The two constant theorem says that for $z\in T_j$

$$(4.49)\quad \log|f(z)|\leq\omega(z, \gamma_j, T_j)(-A_j)+(1-\omega(z, \gamma_j, T_j))\mathcal{M}(f, T_j).$$

By the Gebeitserweiterung for $z\in T_j$

$$(4.50)\qquad \omega(z, \gamma_j, T_j)\geq\omega(z, s_j, V_j),$$

where V_j is

$$0<|z|<R_j;\ \theta_j<\arg z<\theta_j+\alpha_j,$$

and s_j is

$$|z|=R_j;\ \theta_j<\arg z<\theta_j+\alpha_j.$$

We now estimate $\omega(z, s_j, V_j)$, $z\in L_j$, for a fixed but arbitrary j by using Lemma 1. In this lemma put $\alpha=\alpha_j$ (which is allowed by (2.2)). With this choice of α, $F_j^{(\alpha)}$ is V_j and $L_j^{(\alpha)}$ is L_j. The inequality (4.29) of Lemma 1 reproduces here with $K_\alpha=K_{\alpha_j}$, and using (4.26), as

(4.51) $$\omega(z, s_j, V_j) \geq \frac{2}{\sqrt{2}} K_{\alpha_j}, \quad z \in L_j. \quad j=1, 2, \cdots.$$

Since $\underline{\lim}_{j\to\infty} \alpha_j > 0$, then

(4.52) $$\underline{\lim}_{j\to\infty} K_{\alpha_j} = K_0 > 0,$$

where

$$K_{\alpha_j} = \frac{2\left(1-\left(\frac{1}{2}\right)^{\frac{2\pi}{\alpha_j}}\right)\left(\frac{1}{8}\right)^{\frac{\pi}{\alpha_j}}}{\pi\left(1+\left(\frac{1}{2}\right)^{\frac{\pi}{\alpha_j}}\right)^4}, \quad j=1, 2, \cdots.$$

Hence (4.50), (4.51) and (4.52) reduce (4.49) to

(4.53) $$\log|f(z)| \leq 2K_0(-A_j) + \mathcal{M}(f, T_j) = -A_j\left[2K_0 - \frac{\mathcal{M}(f, T_j)}{A_j}\right],$$

for j sufficiently large and $z \in L_j$. But we are now in the situation of (4.25) (or 4.17). We have essentially the same estimate in (4.53) as in (4.25); (4.47) is the same as (4.18) and the definition of L_j in (4.48) together with the fact $\{\gamma_j\}$ is a Koebe sequence allows us to select a "limit domain" of the L_j's which is an open subset of D.

Of course if f is bounded in D then Theorem 2 is precisely Koebe's lemma. We could replace T_j by $F_j^{(\alpha_1)}$ for any $\alpha_1 > \alpha = \underline{\lim}\, \alpha_j$, in the statement of the theorem to obtain a formulation for the original unaltered Koebe sequence which is in the spirit of Theorem 1.

5. Similar theorems in the small

The above group of results demand that we have an estimate on the growth of the maximum modulus of f on fairly large subdomains of D, that is, on the sets $\{F_j^{(\alpha)}\}$ which impinge on $|z|=1$ on a subarc-at least a subsequence of the $\{F_j^{(\alpha)}\}$ does. To obtain generalizations of the theorems in $\mathcal{G}$ we need do little more.

The proof of case (i) of Theorem 1 contains all the necessary estimates and all that is required is to change our point of view. We now consider a *PHD* sequence $\{\gamma_j\}$ where each γ_j is contained in a domain of the form $F_j^{(\alpha)}$ only this time *we suppose that* γ_j *is contained in* $F_j^{(\alpha)}$ *with the exception of one endpoint which is at the origin.* Then tilt the $F_j^{(\alpha)}$ so that the vertex (and consequently the γ_j's) approaches C and we have the situation of the theorems in $\mathcal{G}$. We are not being entirely accurate. Actually we find it more convenient to use domains bounded by arcs of circles rather than triangular domains. Let us proceed to the details.

For a complex number a and real values $0<R<\infty$, $0\leq\theta<2\pi$, $0<\alpha<\pi$, first put $a'=a+\text{Re}^{i\theta}$, then let C_1 and C_2 be the distinct circles of the same radius, each of which meets a and a' and which meet at a with angle α. If L is the perpendicular bisector of the line segment from a to a' then $F(a, R, \theta, \alpha)$ will denote the domain bounded by C_1, C_2 and L, and which contains the point $a+\frac{R}{4}e^{i\theta}$. We shall be concerned with sequences of such domains $\{F(a_n, R_n, \theta_n, \alpha_n)\}$. In the sequel we restrict $\{R_n\}$ and $\{\alpha_n\}$ to be constant sequences which allows somewhat less complicated statements for the results.

Definition 6. *Let* $\{\gamma_n\}$ *be a sequence of Jordan arcs in* D *and* $\{F(a_n, R, \theta_n, \alpha)\}$ *a sequence of domains as defined above. We say that* $\{\gamma_n\}$ *travels in* $\{F(a_n, R, \theta_n, \alpha)\}$ *if*

i) $F(a_n, R, \theta_n, \alpha)\subseteq D$, *all* n;

ii) *For some value* $\epsilon>0$

$$\gamma_n\subseteq F(a_n, R, \theta_n, \alpha-\epsilon),\ \textit{all } n, \tag{5.0}$$

except for one endpoint which coincides with a_n.

Note that $\{\gamma_n\}$ may travel in many different sequences $\{F(a_n, R, \theta_n, \alpha)\}$.

Theorem 3. *Let $\{\gamma_n\}$ be a PHD sequence in D which travels in $\{F(a_n, R, \theta_n, \alpha) \equiv F_n^{(\alpha)}\}$. Suppose f is holomorphic in D and $f-w_0$ has $\frac{\pi}{\alpha}$-exponential order $\{A_n\}$ on $\{\gamma_n\}$ for some finite w_0. If*

$$\varliminf_{n\to\infty} \frac{\mathcal{M}(f, F_n^{(\alpha)})}{A_n} = 0, \tag{5.1}$$

then $f=w_0$.

Proof. We suppose (5.1) holds (and again assume $w_0=0$). Extract a subsequence $n_i=j$ such that

$$\lim_{j\to\infty} \frac{\mathcal{M}(f, F_j^{(\alpha)})}{A_j} = 0. \tag{5.2}$$

Let

$$r_j = \min|z|, \quad z\in\gamma_j, \quad j=1, 2, \cdots. \tag{5.3}$$

With this value r_j we can write

$$\log|f(z)| \leq \frac{-A_j}{(1-r_j)^{\frac{\pi}{\alpha}}}, \quad z\in\gamma_j, \quad j=1, 2, \cdots. \tag{5.4}$$

We now choose a point $b_j \in \gamma_j$ so that

$$\frac{1}{4} HD(\gamma_j) \leq \rho(a_j, b_j) \leq HD(\gamma_j), \quad j=1, 2, \cdots. \tag{5.5}$$

This is clearly possible else γ_j is contained in the non-Euclidean disk about a_j of radius $\frac{1}{4}HD(\gamma_j)$ which implies $HD(\gamma_j) \leq \frac{1}{2}HD(\gamma_j)$, a complete absurdity.

We now invoke the two constant theorem to give for $z\in F_j^{(\alpha)}-\gamma_j$, on account of (5.4),

$$\log|f(z)| \leq \omega(z, \gamma_j, F_j^{(\alpha)}-\gamma_j)\left(\frac{-A_j}{(1-r_j)^{\frac{\pi}{\alpha}}}\right) + (1-\omega(z, \gamma_j, F_j^{(\alpha)}-\gamma_j))\mathcal{M}(f, F_j^{(\alpha)}). \tag{5.6}$$

We proceed just as we did in case (i) of the proof of Theorem 1 after we obtained (4.3). The notation at this point is identical with case (i). Carry $F_j^{(\alpha)}$ onto D_w by

(5.7) $$w_j^*(z)=w(h_j(z)),$$

where w is defined in (4.4) but now

(5.8) $$h_j(z)=e^{-i(\pi-\frac{\alpha}{2})}\left(\frac{z-a_j}{z-a_j'}\right),\quad a_j'=a_j+Re^{i\theta_j},\quad j=1,2,\cdots.$$

If $L^{(\alpha)}:\frac{1}{4}<|w|<\frac{1}{2};\ \frac{\alpha}{4}<\arg w<\frac{3\alpha}{4}$, then set

(5.9) $$L_j^{(\alpha)}=h_j^{-1}(L^{(\alpha)}),\quad j=1,2,\cdots.$$

We must again estimate $\omega(z,\gamma_j,F_j^{(\alpha)}-\gamma_j)$ for $z\in L_j^{(\alpha)}$. Because $r_j\to1$ eventually $L_j^{(\alpha)}\subseteq F_j^{(\alpha)}-\gamma_j$, and we assume this to be the case for all j.

Let γ_j^* be the subarc of $w_j^*(\gamma_j)$ which connects $|w|=|w_j^*(b_j)|$ to $|w|=1$ and lies, except for endpoints, in this annulus. The Gebietserweiterung produces for $w\in D_w-w_j^*(\gamma_j)$, and $j=1,2,\cdots$.

(5.10) $$w(w,w_j^*(\gamma_j),D_w-w_j^*(\gamma_j^*))\geq\omega(w,\gamma_j^*,D_w-\gamma_j^*).$$

Observe that by our construction of $L_j^{(\alpha)}$, there is a B_1 such that

(5.11) $$|w_j^*(L_j^{(\alpha)})|\leq B_1<1,\quad \text{all } j.$$

Precisely as in obtaining (4.7) and (4.8) we again obtain by use of Theorem A, the conformal invariance of the harmonic measure, and (5.10) and (5.11), that

(5.12) $$\omega(z,\gamma_j,F_j^{(\alpha)}-\gamma_j)\geq\frac{2}{\pi}\arcsin\frac{(1-B_1^2)(1-|w_j^*(b_j)|^2)}{16}$$

for $z\in L_j^{(\alpha)}$.

The key estimate (4.10) converts in the present situation to

(5.13) $$1-|w_j^*(b_j)|^2\geq\operatorname{Im}\left[\left(\frac{b_j-a_j}{b_j-a_j'}\right)e^{-i(\pi-\frac{\alpha}{2})}\right]^{\frac{\pi}{\alpha}}\left[1-\left|\frac{b_j-a_j}{b_j-a_j'}\right|^{\frac{2\pi}{\alpha}}\right].$$

Unlike case (i) the last term in brackets in (5.13) contributes not at all to the order estimate because $\rho(a_j,b_j)\leq K_0$ implies $|b_j-a_j|\to0$, $j\to\infty$, which in turn implies, because $|b_j-a_j'|\geq\frac{R}{2}$,

(5. 14) $$1-\left|\frac{b_j-a_j}{b_j-a_j'}\right|^{2\pi/\alpha}\geq\frac{1}{2},\ j \text{ sufficiently large.}$$

The first term is crucial. By considering the geometry of $F_j^{(\alpha)}$ $\left(\text{first transform it by }\left(\frac{z-a_j}{z-a_j'}\right)e^{-i(\pi-\frac{\alpha}{2})}\right)$, it is easy to see that, setting $\rho_j e^{i\varphi_j}=\left(\frac{b_j-a_j}{b_j-a_j'}\right)e^{-i(\pi-\frac{\alpha}{2})}$,

(5. 15) $$\operatorname{Im}\left(\left(\frac{b_j-a_j}{b_j-a_j'}\right)e^{-i(\pi-\frac{\alpha}{2})}\right)^{\pi/\alpha}\geq\left(\frac{|b_j-a_j|}{R}\right)^{\pi/\alpha}\sin\left(\frac{\varphi_j\pi}{\alpha}\right)$$
$$\geq\left(\frac{|b_j-a_j|}{R}\right)^{\pi/\alpha}\sin\frac{\pi\epsilon}{2\alpha}.$$

The last estimate (and value ϵ) is obtained from the definition of $\{\gamma_j\}$ travelling in $\{F_j^{(\alpha)}\}$. According to our definition of b_j in (5. 5), Lemma A obtains and so eventually

(5. 16) $$|a_j-b_j|\geq(1-|a_j|)t_0,\quad t_0>0.$$

With $K_{\alpha,\epsilon}=\frac{1}{2}\left(\frac{t_0}{R}\right)^{\pi/\alpha}\sin\frac{\pi\epsilon}{2\alpha}$, (5. 12) is now because of (5. 13), (5. 14), (5. 15), (5. 16), and the property of arcsin

(5. 17) $$\omega(z,\gamma_j,F_j^{(\alpha)}-\gamma_j)\geq\frac{2}{\pi}\left(\frac{1-B_1^2}{16}\right)K_{\alpha,\epsilon}(1-|a_j|)^{\pi/\alpha},\ z\in L_j^{(\alpha)},$$

and j sufficiently large.

Our last step is to notice because $\{\gamma_j\}$ is a *PHD* sequence

(5. 18) $$1-|a_j|\geq(1-r_j)t_1,\ t_1>0,\ \text{all } j.$$

And so (5. 6) is affected by (5. 17) and (5. 18) and becomes eventually for $z\in L_j^{(\alpha)}$, after setting $C_{\alpha,\epsilon}=\left(\frac{1-B_1^2}{8\pi}\right)t_1^{\pi/\alpha}K_{\alpha,\epsilon}>0$,

$$\log|f(z)|\leq-A_jC_{\alpha,\epsilon}+\mathcal{M}(f,F_j^{(\alpha)})=-A_j\left[C_{\alpha,\epsilon}-\frac{\mathcal{M}(f,F_j^{(\alpha)})}{A_j}\right].$$

Because of (5. 2) $\lim_{j\to\infty}f(z_j)=0$, $z_j\in L_j^{(\alpha)}$. Since the $L_j^{(\alpha)}$ have a "limit domain" of similar form in D then f is identically zero on this limit domain and so must be zero throughout D. This completes the proof.

The theorem is reasonably sharp. Consider the function $f(z) = e^{-\left(\frac{1+z}{1-z}\right)^2}$; and let γ_n: $1-\frac{2}{n}\leq x\leq 1-\frac{1}{n}$ (defined in (1. 1)) which is a *PHD* sequence. If we choose $F_n \equiv F\left(1-\frac{1}{n}, \frac{1}{2}, \pi, \frac{\pi}{2}\right)$, $n=1, 2, \cdots$, $\left(\text{i.e. domains symmetric about the real axis of opening } \frac{\pi}{2}\right)$ then clearly $\{\gamma_n\}$ travels in $\{F_n\}$. It is easily calculated that

i) $\mathcal{M}(f, F_n)=1$, all n;

ii) $\log|f(z)|\leq\frac{-1}{(1-r)^2}$, $z\in\gamma_n$, all n.

Hence f has 2-exponential order $\{1\}$ on the *PHD* sequence $\{\gamma_n\}$ but

$$\varlimsup_{n\to\infty}\frac{\mathcal{M}(f, F_n)}{A_n}>0 .$$

Note that for any $\epsilon>0$ f has $(2-\varepsilon)$ exponential order $\{n^\epsilon\}$ on $\{\gamma_n\}$, and since

$$\varlimsup_{n\to\infty}\frac{\mathcal{M}(f, F_n)}{n^\epsilon}=0 ,$$

we cannot reduce the $\frac{\pi}{\alpha}$-exponential order required in the theorem (or for that matter decrease the angle of opening α).

The *PHD* property cannot be omitted. If we let $a_n=1-\frac{1}{n^2}$, $n=1, 2, \cdots$, and set $B(z, \{a_n\})$ equal to the Blaschke product whose zeroes are at $z=a_n$, $n=1, 2, \cdots$, we can certainly find a sequence of Jordan arcs $\{\gamma_n\}$ of the form $1-\frac{1}{k_n}\leq x\leq 1-\frac{1}{n^2}$, such that f has 2-exponential order $\{n\}$ on $\{\gamma_n\}$. Certainly $\{\gamma_n\}$ travels in the domains $\left\{F\left(1-\frac{1}{n^2}, \frac{1}{2}, \pi, \frac{\pi}{2}\right)\right\}$ and

$$\varlimsup_{n\to\infty}\frac{\mathcal{M}(f, F_n)}{n}=0 .$$

We have chosen $\alpha=\frac{\pi}{2}$ in the above examples but this is only a convenience. For arbitrary $0<\alpha<\pi$ the function $f(z)=\exp\left[-\left(\frac{1+z}{1-z}\right)^{\frac{\pi}{\alpha}}\right]$, with a suitable choice of the root, performs the same role in the general case as does our specific function. Also the Blaschke product given above has $\frac{\pi}{\alpha}$-exponential order $\{n\}$ on the sequence of arcs $\{\gamma_n'\}$, γ_n': $1-\frac{1}{k_n'}\leq z\leq 1-\frac{1}{n^2}$, $n=1, 2, \cdots$, for suitable $\{k_n'\}$.

In Theorem 3 the angle α is restricted to lie between 0 and π. The extreme case $\alpha=\pi$ is really the contents of Theorem 1 where $F(a_n, R, \theta_n, \pi)$ becomes $F_n^{(\alpha)}$. The other extreme case $\alpha=0$ will be treated (albeit in a somewhat restricted form) in Theorem 5.

We list several corollaries which arise from Theorem 3 when we add the requirement that the $\{\gamma_n\}$ approach a single point of C. For $\tau\in C$, and $0<\beta<\pi$, let

$$H(\tau, \beta)=F(\tau, 2, \theta_\tau, \beta),$$

where we choose θ_τ so that $\tau+2e^{i\theta_\tau}=-\tau$ that is, $H(\tau, \beta)$ is the hypercyclic domain at the point τ of opening β.

Corollary 3. *Let $\{\gamma_n\}$ be a PHD sequence travelling in $\{F(a_n, R, \theta_n, \alpha)\}$ such that for some $0<\beta<\pi$, eventually*

$$F(a_n, R, \theta_n, \alpha)\subseteq H(\tau, \beta). \tag{5.19}$$

If f is holomorphic in D and, for some finite w_0, $f-w_0$ has $\frac{\pi}{\alpha}$-exponential order $\{A_n\}$ on *$\{\gamma_n\}$, then*

$$\varliminf_{n\to\infty}\frac{\mathcal{M}(|a_n|, f, H(\tau, \beta))}{A_n}=0\,, \tag{5.20}$$

implies $f=w_0$.

Proof. The hypothesis (and some elementary geometry) demands that eventually

$$F_n^{(\alpha)}=F(a_n, R, \theta_n, \alpha)\subseteq\{|z|<|a_n|\}. \tag{5.21}$$

So (5.19), (5.20) and (5.21) combine to give

$$\lim_{n\to\infty} \frac{\mathcal{M}(f, F_n^{(\alpha)})}{A_n} = 0\,,$$

which validates the hypothesis of Theorem 3.

There are several different types of sequences which satisfy (5.19). Suppose γ is a boundary component of $H(\tau, \psi)$. Let $\{\gamma_n\}$ be a *PHD* sequence with $\gamma_n \subseteq \gamma$, all n. Consider any hypercyclic domain $H(\tau, \beta)$ containing $\{\gamma_n\}$, that is, $\psi < \beta$. Then for suitable choice of R and $\{\theta_n\}$

(5.22) i) $\{\gamma_n\}$ travels in $\{F(a_n, R, \theta_n, \beta)\}$, $n=1, 2, \cdots$;

ii) $F(a_n, R, \theta_n, \beta) \subseteq H(\tau, \beta)$, eventually.

This is a rather obvious geometric fact. In fact γ need not be a circular segment but merely a boundary path at τ that is sufficiently smooth in that it should make the angle ψ at τ with the radius. We organize these observations in

Corollary 4. *Let $\{\gamma_n\}$ be a PHD sequence lying on a boundary path γ at $\tau \in C$, and making angle ψ, $-\frac{\pi}{2} < \psi < \frac{\pi}{2}$, at τ with the radius to τ. Let f be holomorphic in D and suppose for some β, $2|\psi| < \beta < \pi$, and some finite w_0, $f - w_0$ has $\frac{\pi}{\beta}$-exponential order $\{A_n\}$ on $\{\gamma_n\}$. If*

$$\lim_{n\to\infty} \frac{\mathcal{M}(|a_n|, f, H(\tau, \beta))}{A_n} = 0\,,$$

where $|a_n| = \max|z|$, $z \in \gamma_n$, $n=1, 2, \cdots$, then $f = w_0$

Proof. According to the above remarks as summarized in (5.22) we may choose $\alpha = \beta$ in Corollary 3 and the full hypothesis of this corollary is satisfied.

To obtain a somewhat more pleasing formulation of Corollary 4 we abandon the sequence $\{\gamma_n\}$ and replace it by γ.

Corollary 5. *Let γ be a boundary path at τ which makes an angle*

ψ, $-\pi/2<\psi<\pi/2$, with the radius to τ at τ. Suppose for some holomorphic f, some finite value w_0, some $2|\psi|<\beta<\pi$, and some positive $A(r)$, $0\leq r<1$,

$$(5.23)\quad \log|f(z)-w_0|\leq\frac{-A(r)}{(1-|z|)^{\frac{\pi}{\beta}}},\quad z\in\gamma,\ |z|\geqq r.$$

then if

$$(5.24)\qquad \varliminf_{r\to1}\frac{\mathcal{M}(r,f,H(\tau,\beta))}{A(r)}=0,$$

we have $f=w_0$.

Remark. In particular if $\psi=0$ this represents a direct generalization of Theorem 2 of $\mathcal{G}$, as well as an improvement.

Proof. Suppose (5.24) holds and that $\{R_j\}$ is sequence such that

$$(5.25)\qquad \lim_{R_j\to1}\frac{\mathcal{M}(R_j,f,H(\tau,\beta))}{A(R_j)}=0.$$

Choose sequence $\{r_j\}$, $0<r_j<R_j$, such that $1\leq\rho(r_j,R_j)\leq2$ and let γ_j be a subarc of γ which is contained in $r_j\leq|z|\leq R_j$ and satisfies $1\leq HD(\gamma_j)\leq2$, all j. This is certainly possible for large j. According to (5.23), for $z\in\gamma_j$

$$\log|f(z)-w_0|\leq\frac{-A(R_j)}{(1-|z|)^{\frac{\pi}{\beta}}}$$

so that $f-w_0$ has $\frac{\pi}{\beta}$-exponential order $\{A(R_j)\}$ on the *PHD* sequence $\{\gamma_j\}$ and the hypothesis of Corollary 4 is satisfied because of (5.25).

Corollary 6. *Let f be holomorphic in D and suppose for some $\tau\in C$ and all $0<\beta<\pi$*

$$(5.26)\qquad \varlimsup_{r\to1}\frac{\mathcal{M}(r,f,H(\tau,\beta))}{\log\frac{1}{1-r}}<\infty.$$

Let $\{\gamma_n\}$ be a PHD sequence travelling in $\{F(a_n,R,\theta_n,\alpha)\}$, with

$a_n \to \tau$, $n \to \infty$. *If for some* $\eta > 0$, *and finite* w_0, $f - w_0$ *has* $(1+\eta)$-*exponential order* $\{A\}$ *on* $\{\gamma_n\}$ *then* $f = w_0$

Remark. Certainly f satisfies (5.26) if it has finite angular limit at τ. Furthermore if $\log|f(z)| \leq -\frac{1}{(1-|z|)^{1+\eta}}$ on the radius to τ there exists a *PHD* sequence satisfying the above hypothesis (by the usual arguments) and so this represents a generalization of Theorem 1 of $\mathcal{G}$.

Remark. Contrast this corollary with Corollary 2 and note that we can now estimate $\mathcal{M}(r, f)$ over a much smaller subset of D then in Corollary 2 because of our additional restriction on the *PHD* sequence $\{\gamma_n\}$.

Proof. With ϵ given by Definition 6 geometry demands that $F\left(a_n, R, \theta_n, \alpha - \frac{\epsilon}{2}\right)$ be eventually contained in some $H(\tau, \beta_0)$ (by suitably choosing R) and further we have $\alpha - \frac{\epsilon}{2} \leq \beta_0$ because $a_n \to \tau$, $n \to \infty$. Because of this situation we may certainly find a sequence of domains $\{F(a_n, R', \theta'_n, \beta_1)\}$, $\beta_0 \leq \beta_1 < \pi$, so that

(5.27)
i) $\{\gamma_n\}$ travels in $\{F(a_n, R', \theta'_n, \beta_1)\}$;
ii) $\frac{\pi}{\beta_1} < 1 + \frac{\eta}{2}$;
iii) eventually $F(a_n, R', \theta'_n, \beta_1) \subseteq H(\tau, \beta_1)$.

If $r_n = \min|z|$, $z \in \gamma_n$ then

$$\mathcal{M}(f, \gamma_n) \leq \frac{-1}{(1-|z|)^{1+\eta/2}} \frac{A}{(1-|z|)^{\eta/2}} \leq \frac{-1}{(1-|z|)^{1+\eta/2}} \frac{A}{(1-r_n)^{\eta/2}}$$
$$\leq \frac{-1}{(1-|z|)^{\pi/\beta_1}} \frac{A}{(1-r_n)^{\eta/2}},$$

where the last inequality follows (from (5.27 (ii)). This says that f has $\frac{\pi}{\beta_1}$-exponential order $\left\{\frac{A}{(1-r_n)^{\eta/2}}\right\}$ on the *PHD* sequence $\{\gamma_n\}$.

Because of (5.27 (i) and (iii)) the hypothesis of Corollary 3 would be satisfied if we could show

$$\varliminf_{n\to\infty} \mathcal{M}(|a_n|, f, H(\tau, \beta_2))(1-r_n)^{\eta/2}=0\,.$$

Since $\{\gamma_n\}$ is a *PHD* sequence $(1-\gamma_n)\leq(1-|a_n|)t_0$, $0<t_0<\infty$, and on account of this

$$\begin{aligned}&\varliminf_{n\to\infty} \mathcal{M}(|a_n|, f, H(\tau, \beta_1))(1-r_n)^{\eta/2}\\ &\quad=\varliminf_{n\to\infty} \frac{\mathcal{M}(|a_n|, f, H(\tau, \beta_1))}{\log\dfrac{1}{1-|a_n|}}(1-|a_n|)^{\eta/2}\log\frac{1}{1-|a_n|}\\ &\quad=0\,.\end{aligned}$$

Corollary 6 can be changed in the same spirit as Corollary 4 was transformed to produce Corollary 5. Rather than state this as a further corollary it is profitable to indicate that in Corollary 2 if we replace the boundary path γ by a boundary path at $\tau\in C$ making angle ψ at τ, $|\psi|<\frac{\pi}{2}$; and transform (4.43) to $\varlimsup_{r\to1}\dfrac{\mathcal{M}(r, f, H(\tau, \beta))}{\log\dfrac{1}{1-r}}$ $<\infty$ all $0<\beta<\pi$, then we have the result.

We now take a slightly different tack. We wish to eliminate the condition in Corollaries 3 and 4 that the boundary path γ approach τ making some angle with the radius. We can accomplish this if we are willing to abandon both *PHD* notion and the estimate on the growth of the maximum modulus in hypercyclic domains. The proof of this result is a duplicate of the proof of Theorem 3 except for a few revisions which we discuss. This theorem and its corollary also represent generalizations of theorems in $\mathcal{G}$.

Theorem 4. *Let γ be a boundary path at $\tau\in C$ which for some $\epsilon>0$ lies in the hypercyclic domain $H(\tau, \alpha-\epsilon)$, $0<\alpha-\epsilon<\alpha<\pi$. Let f be holomorphic in D and satisfy for some finite value w_0*

$$(5.28)\qquad \lim_{\substack{z\to\tau\\ z\in\gamma}} (1-|z|)^{\pi/\alpha}\log|f(z)-w_0|=-\infty.$$

If f is bounded in $H(\tau, \alpha)$ then $f=w_0$.

Proof. We may as usual assume $w_0=0$. Choose a sequence $\{r_j\}$, $0<r_j<r_{j+1}<1$, $r_j\to 1$, $j\to\infty$. Let γ_j be the subboundary path of γ which is contained in $r_j\leq|z|<1$ with one endpoint b_j on $|z|=r_j$. Then $\{\gamma_j\}$ travels in $\{H(\tau, \alpha)\}$, eventually, and if we set $F_j^{(\alpha)} = H(\tau, \alpha)$ we may take up our proof at (5.2) in the proof of Theorem 3. Let $A_j=\sup(1-|z|)^{\pi/\alpha}\log|f(z)|$, $z\in\gamma_j$. Necessarily $A_j\to-\infty$, and for $z\in\gamma_j$

$$\log|f(z)|\leq\frac{A_j}{(1-|z|)^{\pi/\alpha}}.$$

So f has $\frac{\pi}{\alpha}$-exponential order $\{-A_j\}$ on $\{\gamma_j\}$ and (5.2) is satisfied. Clearly (5.3) is unchanged and (5.4) has been obtained above. We now choose $b_j\in\gamma_j$ to be the endpoint b_j already distinguished (of course (5.5) is not satisfied for this choice of b_j). Let $L_j^{(\alpha)}$ be defined as in (5.9). Everything proceeds exactly the same including (5.14) (since $|b_j-a_j|=|\tau-b_j|\to 0$) until we reach (5.16) which depends on (5.5) which we do not have. In this circumstance (5.16) changes to

$$|a_j-b_j|=|\tau-b_j|\geq(1-|b_j|)K_\alpha=(1-r_j)K_\alpha, \tag{5.29}$$

for some $K_\alpha>0$, all j, because γ lies in $H(\tau, \alpha)$. Note that we do not need (5.18) because of the change in (5.16). All the estimates are now available to complete the proof and we conclude $f=w_0$.

Corollary 7. *Let f be holomorphic in D and suppose for some boundary path γ approaching $\tau\in C$ within some hypercyclic domain at τ,*

$$\lim_{\substack{z\to\tau\\ z\in\gamma}}(1-|z|)^{1+\eta}\log|f(z)-w_0|=-\infty, \tag{5.30}$$

for some finite w_0, and some $\eta>0$. Then if

(5.31) $\mathcal{M}(r, f, H(\tau, \beta)) \leq M_\beta < \infty$,

all $0 < \beta < \pi$, *we have* $f = w_0$.

Proof. Suppose $\gamma \subseteq H(\tau, \beta)$ and choose α, $0 < \beta < \alpha < \pi$, so that also

(5.32) $$\frac{\pi}{\alpha} < 1 + \eta .$$

Because of (5.32) we can reduce (5.30) to

(5.33) $$\lim_{\substack{z \to \tau \\ z \in \gamma}} (1 - |z|)^{\pi/\alpha} \log |f(z) - w_0| = -\infty .$$

We are now in the situation of Theorem 4 so $f = w_0$.

6. Applications to normal functions

In both Corollary 6 and 7 the condition on the growth of the maximum modulus is satisfied if f has angular limit at τ which in turn is assured if f is a normal function because f tends to w_0 on γ. For the sake of completeness and to gather together several ideas on normal functions which have been presented by various authors and which are of use to us we present a digest of these related results. It will also serve to introduce our final applications of Theorem 3 which occur in Theorems 5 and 6 and which correspond, as we have remarked earlier, to the extremal case $\alpha = 0$.

Definition 7. [10, p. 53]. *A function f, meromorphic in D, is said to be a normal function if the family* $\{f(S(z))\}$, $S(z)$ *any* 1-1 *conformal map of D onto D, is a normal family in the sense of Montel.*

An equivalent formulation [10, p. 55] is that f is normal in D if and only if

$$\frac{|f'(z)|}{1 + |f(z)|^2} \leq \frac{C_f}{1 - |z|}, \quad z \in D.$$

Another equivalence was noted by Lappan [8, Theorem 3] who observed that f is normal in D if and only if for any two sequence $\{z_n\}$, $\{z'_n\}$ in D, $\rho(z_n, z'_n)\to 0$ implies $\chi(f(z_n), f(z'_n))\to 0$, $n\to\infty$. (Here $\chi(a, b)$ is the chordal distance on the extended plane W.)

If γ is a boundary path approaching $\tau\in C$ within some hypercyclic domain we say γ approaches τ non-tangentially. Noshiro [13], (and in a more general form Lehto and Virtanen [10, Theorem 2]) proved that if a normal f tends to a value $w_0\in W$ on a nontangential boundary path γ at τ then $f(z_n)\to w_0$ for any sequence $\{z_n\}$ approaching τ provided it approaches τ within a hypercyclic domain, i.e. f has angular limit w_0 at τ. To obtain this result one need not assume that the full family $\{f(S(z))\}$ be normal, but rather that a subfamily be normal. As is shown in [10, p. 57] it is enough to assume that

$$(6.0)\qquad \overline{\lim_{\substack{z\in H(\tau,\beta)\\ z\to\tau}}}\ \frac{|f'(z)|(1-|z|)}{1+|f(z)|^2}\leq C_\beta<\infty,\ 0<\beta<\pi.$$

We come now to Theorem 5. First set $\rho(A, B)=\inf\{\rho(z, w)\}$, $z\in A$, $w\in B$ and $A, B\subseteq D$.

Theorem 5. *Let f be meromorphic in D and suppose for some $\tau\in C$ satisfies* (6.0). *Let $\{\gamma_n\}$ be a PHD sequence travelling in $\{F(a_n, R, \theta_n, \alpha)\}$, some $0<\alpha<\pi$, and $a_n\to\tau$, $n\to\infty$. For some $w_0\in W$ and some $\eta>0$ let $f-w_0$ $\left(or\ \frac{1}{f}\ if\ w_0=\infty\right)$ have $(1+\eta)$-exponential order $\{A\}$ on $\{\gamma_n\}$. If*

$$(6.1)\qquad \overline{\lim_{n\to\infty}}\ \rho(\gamma_n, \gamma_{n+1})<\infty,$$

then $f\equiv w_0$.

Remark. This is a relative of Corollary 7. It is clear that in the hypothesis of Corollary 7 we could replace (5.31) by (6.0) since (6.0) together with (5.30) implies (5.31). Hence Corollary 7 gives us information on the possible exponential order of a nonconstant normal holomorphic f on a non-tangential boundary path at $\tau\in C$. For an arbitray boundary path γ we can infer that for a normal holomorphic f if

$$\log|f(z)-w_0|\leq\frac{-A^*(r)}{(1-|z|)^2}, \quad |z|\geqq r,\ z\in\gamma,\ |w_0|<\infty.$$

and $A^*(r)\to+\infty$, $r\to1$, then $f=w_0$. To demonstrate this first recall that Hayman showed [6, p. 204] that $\mathcal{M}(r,f)\leq\frac{C_f}{1-r}$. Then select any *PHD* sequence $\{\gamma_n\}$, $\gamma_n\subseteq\gamma$, $n=1,2,\cdots$, with associated parameters $\{(R_n, r_n, \theta_n, \alpha_n)\}$. It is trivial that $f-w_0$ has 1-exponential order $\left\{\frac{t_0A^*(r_n)}{1-R_n}\right\}$, $0<t_0<\infty$, on $\{\gamma_n\}$ and so (4.0) of Theorem 1 is satisfied (for suitable α) which gives the result.

Proof. By remarks made previously we have eventually for some $0<\beta_0<\pi$, (with ϵ given in definition 6),

$$F\left(a_n, R, \theta_n, \alpha-\frac{\epsilon}{2}\right)\subseteq H(\gamma, \beta_0) \tag{6.2}$$

If we show that f has angular limit w_0 at τ we can invoke Corollary 6 and so this is our objective. Suppose there is a sequence $\{z_n\}$ tending to τ with

$$z_n\in H(\tau, \beta_1), \text{ some } 0<\beta_1<\pi; \tag{6.3}$$

and

$$\lim_{n\to\infty} f(z_n)=w_1\neq w_0. \tag{6.4}$$

For any z_n let γ_{k_n} be the arc whose corresponding endpoint a_{k_n} is closest to z_n, that is

$$\rho(z_n, a_{k_n})\leq\rho(z_n, a_j), \quad j=1,2,\cdots. \tag{6.5}$$

We assert that

$$\overline{\lim_{n\to\infty}}\ \rho(a_{k_n}, z_n)<\infty. \tag{6.6}$$

Let β_2 be chosen $0<\beta_2<\pi$, so that eventually

$$(6.7) \qquad z_n \in H(\tau, \beta_2)$$
$$\gamma_n \subseteq H(\tau, \beta_2).$$

Denote by z' the unique point on the radius to τ closest to z in the hyperbolic distance. Now choose $w_n \in \gamma_n$, $\hat{w}_n \in \gamma_{n+1}$, $n=1, 2, \cdots$ so that

$$(6.8) \qquad \rho(w_n, \hat{w}_n) = \rho(\gamma_n, \gamma_{n+1}), \quad n=1, 2, \cdots.$$

Then (6.1), (6.7), (6.8) and the *PHD* property collaborate to produce for some suitable $M_1>0$, that eventually

$$(6.9) \quad \rho(a_n', a_{n+1}') \leq \rho(a_n', a_n) + \rho(a_n, w_n) + \rho(w_n, \hat{w}_n) + \rho(w_{n+1}, a_{n+1}')$$
$$\leq M_1.$$

Now for any z_n the corresponding z_n' lies between some a_{l_n}' and a_{l_n+1}', and on account of (6.9) eventually

$$(6.10) \qquad \rho(z_n', a_{l_n}') < M_1$$

But (6.5), (6.7) and (6.10) imply that for some $0<M_2<\infty$, eventually

$$\rho(z_n, a_{k_n}) \leq \rho(z_n, z_n') + \rho(z_n', a_{l_n}') + \rho(a_{l_n}', a_{l_n}) \leq M_2,$$

which verifies (6.6).

As is customary, setting $\zeta = \zeta_n(z) = \dfrac{z-a_{k_n}}{1-\bar{a}_{k_n} z}$, consider the family $\{g_n(\zeta) = f(\zeta_n^{-1}(\zeta))$ $\left(\text{or } \left\{\dfrac{1}{f}(\zeta_n^{-1}(\zeta))\right\} \text{ if } w_0=\infty\right)$. By hypothesis this is a normal family. Now $\{\zeta_n(\gamma_{k_n})\}$ is a *PHD* sequence and each arc has one endpoint at $\zeta=0$. Thus there is a subsequence of functions $\{g_{n_i}(\zeta)\}$ which converges to the constant function w_0, uniformly on compact subsets of $|\zeta|<1$. But (6.6) and the continuous convergence of $\{g_{n_i}\}$ give $\lim_{i\to\infty} f(z_{n_i}) = w_0$ which is incompatible with (6.4), and so f has angular limit w_0 at τ. For any fixed β we may assume that $f\left(\text{or } \dfrac{1}{f}\right)$ is holomorphic in $H(\tau, \beta)$. Since we also have that eventually $F\left(a_n, R, \theta_n, \alpha-\dfrac{\epsilon}{2}\right) \subseteq H(\tau, \beta)$ (for a suitable choice of R) we have that f $\left(\text{or } \frac{1}{f}\right)$ is holomorphic on these domains for large n. A careful reading of Theorem 3 (and Theorem 1) shows that we need assume only that f is meromorphic in D and holomorphic in the domains $F\left(a_n, R, \theta_n, \alpha-\frac{\epsilon}{2}\right)$. Thus we can indeed invoke Corollary 6 and the proof is complete.

Definition 8. *Let f be a continuous function from D into W. The sequence $\{z_n\}$ is said to be a normal sequence for f if the family*

$$\left\{g_n(\zeta)=f\left(\frac{\zeta+z_n}{1+\zeta\bar{z}_n}\right)\right\}$$

is a normal family within some open disk about $\zeta=0$. If $\{g_n(\zeta)\}$ is a normal family in $|\zeta|<\frac{e^{2r}-1}{e^{2r}+1}$ then to emphasize this fact we sometimes write that $\{z_n\}$ is a normal sequence mod r for f.

There is an equivalent formulation for this notion which is similar to Lappan's criteria for a normal function.

Lemma 3. *Let f be a meromorphic function from D into W. A sequence $\{z_n\}$ is a normal sequence for f if and only if $\chi(f(z_n), f(z_n'))\to 0$ whenever $\rho(z_n, z_n')\to 0$, $n\to\infty$.*

Proof. If we set $z_n'=\frac{\zeta_n'+z_n}{1+\zeta_n'\bar{z}_n}$ the lemma now reads: $\{z_n\}$ is a normal sequence for f if and only if $|\zeta_n'|\to 0$ implies $\chi(g_n(\zeta_n'), g_n(0))\to 0$, $n\to\infty$. But this statement is true if and only if the family $\{g_n(\zeta)\}$ is continuously convergent at $\zeta=0$, and this indeed is a necessary and sufficient condition that $\{g_n(\zeta)\}$ be normal in some disk about $\zeta=0$. (See e.g. [3, p. 173 ff] for details)

There is a kind of uniformity that prevails for sets in D.

Lemma 4. *Let $S\subseteq D$. The following propositions are equivalent* *for a meromorphic function* f:

A) *There exists an* $r>0$ *such that each sequence* $\{z_n\}$, $z_n\in S$, *is a normal sequence mod* r *for* f;

B) *Each sequence* $\{z_n\}$, $z_n\in S$, *is a normal sequence for* f.

Proof. We need only prove B implies A. Notice that given $\epsilon>0$ there exists a $\delta(\epsilon)>0$ such that

(6.11) $\rho(z, w)<\delta(\varepsilon)$ implies $\chi(f(z), f(w))<\epsilon$, $z\in S$, $w\in D$.

Otherwise for some $\epsilon>0$ there is a pair $\{z_n, w_n\}$, $z_n\in S$, $w_n\in D$ with $\rho(z_n, w_n)<\frac{1}{n}$, but $\chi(f(z_n), f(w_n))\geq\epsilon$, $n=1,2,\cdots$. But Lemma 3 pronounces that $\{z_n\}$ is not a normal sequence contrary to assumption.

If we select any sequence $\{z_n\}$, $z_n\in S$, then we claim that

$$\left\{g_n(\zeta)=f\left(\frac{\zeta+z_n}{1+\zeta\bar{z}_n}\right)=f(L_n(\zeta))\right\}$$

is a normal family in $\rho(0,\zeta)<\frac{\delta(1/4)}{2}$, where $\delta(1/4)$ is given by letting $\epsilon=1/4$ in (6.11). If a, b are two points in this disk, setting $a_n=L_n(a)$, $b_n=L_n(b)$, then by the triangular inequality both $\rho(a_n, z_n)$ and $\rho(b_n, z_n)$ are less than $\delta(1/4)$ which in turn implies that

$$\chi(f(a_n), f(b_n))=\chi(g_n(a), g_n(b))<\frac{1}{2},$$

because of (6.11). But this means that $\{g_n(\zeta)\}$ is normal in $\rho(0,\zeta)<\frac{\delta(1/4)}{2}$, which disk is independent of the sequence $\{z_n\}$.

Simply put, if every sequence in S is a normal sequence for f then for some $r>0$ every sequence is a normal sequence mod r for f.

Suppose $\{z_n\}$ is a sequence which is not a normal sequence for some meromorphic f. This means thatthe re is a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ for which no subsequence is a normal sequence for f. This leads us to phrase.

Definition 9. *A sequence* $\{z_n\}$ *is said to be an M-sequence for f if no subsequence of* $\{z_n\}$ *is a normal sequence.*

The classical theorem of Montel (in generalized form) enables us to observe that a M(ontel) sequence for f has the following property when f is meromorphic in D. Let $N(z, r) = \{\zeta \mid \rho(z, \zeta) < r\}$.

(6.12) Every subsequence $\{z_{n_k}\}$ of an M-sequence $\{z_n\}$ for a meromorphic f in D has the property that f assumes every value in W infinitely often with at most two possible exceptions in $\bigcup_{k=1}^{\infty} N(z_{n_k}, r)$, and this is true for each choice of $r > 0$.

Given an M-sequence we may certainly choose a subsequence $\{z_{n_k}\}$ such that

$$|z_{n_k}| \leq |z_{n_{k+1}}|;\ \rho(z_{n_k}, z_{n_{k+1}}) \to \infty,\ k \to \infty. \tag{6.13}$$

Gavrilov in $\mathcal{G}$ defined a P-sequence for f to be a sequence satisfying (6.12) and (6.13), where (6.13) must hold for the sequence itself. We see that if a meromorphic f in D possesses an M-sequence it has a P-sequence.

Continuing our brief résumé Anderson in [1, p. 103] called $\tau \in C$ a normal point for a meromorphic f in D if every sequence approaching τ non-tangentially is a normal sequence for f. The standard arguments show that in this case each such sequence must be a normal sequence mod ∞ for f and so it is easily seen that τ is a normal point for f if and only if (6.0) holds. (See Tanaka [16] for related results on normal sequences.)

The following theorem was proved in limited form by Seidel [15, Theorem 4], more generally in $\mathcal{G}$ (Theorem 4), and in its present form was announced by Gauthier [4]. Since our statement is somewhat different than Gauthier (we replace his ρ-sequences by

M-sequences) and because our proof is brief we present it here.

Theorem B. *Let f be meromorphic in D and tend to $w_0 \in W$ on a boundary path γ. Let $P(\gamma, r) = \bigcup_{z \in \gamma} N(z, r)$, $0 < r < \infty$. Then there exists a constant A, $0 \leq A \leq \infty$ such that*

i) *if $0 < A < \infty$ then for all $0 < r < A$*

$$\lim f(z_n) = w_0, \quad |z_n| \to 1, \quad z_n \in P(\gamma, r),$$

while the boundary of $P(\gamma, A)$ contains an M-sequence for f;

ii) *if $A = 0$ then γ contains an M-sequence for f;*

iii) *if $A = \infty$ then for all $0 < r < \infty$,*

$$\lim_{n \to \infty} f(z_n) = w_0, \quad |z_n| \to 1, \quad z_n \in P(\gamma, r).$$

Proof. Suppose every sequence on γ is a normal sequence and define $A = \text{lub } \{r \mid$ every sequence on γ is a normal sequence mod r for $f\}$. Then $A > 0$ by Lemma 4. If $A = \infty$, the usual arguments for normal functions give (iii). If A is finite likewise we have the first part of (i). If the boundary of $P(\gamma, A)$ contained only normal sequence for f, because of Lemma 4 (and some obvious arguments) the definition of A would be compromised. Thus there is an M-sequence for f on the boundary of $P(\gamma, A)$. If $A = 0$ then γ contains an M-sequence for f-again by Lemma 4.

This material enables us to prove the following improvement of Theorem 3 of $\mathcal{G}$.

Theorem 6. *Let f be meromorphic in D and let γ be a boundary path at $\tau \in C$ making angle ψ, $-\frac{\pi}{2} < \psi < \frac{\pi}{2}$, with the radius to τ at τ. Suppose for some $w_0 \in W$ we have*

i) *if w_0 is finite*

$$\log |f(z) - w_0| \leq \frac{-A_s}{(1 - |z|)^s}, \quad z \in \gamma; \tag{6.14}$$

ii) *or if* $w_0=\infty$

$$\log|f(z)|\geq\frac{A_s}{(1-|z|)^s}, \quad z\in\gamma;$$

where A_s *is a positive constant depending on* s *and* (6.14 (i)) *or* (6.14(ii)) *holds for each* $0\leq s<\infty$. *Then either* γ *contains an M-sequence or else* $f=w_0$.

Remark. Together with Corollary 1 and Corollary 5 this theorem completes the analysis of the various s-exponential order ($s\geq1$) of f on non-tangential boundary paths at τ which are smooth at τ.

Proof. If γ does not contain an M-sequence for f Theorem B gives that f tends to w_0 within $P(\gamma, r)$, $0<r<A$, for some $0<A<\infty$. Thus we can find a neighborhood V of τ such that, in $P'\left(\gamma, \frac{A}{2}\right)=P\left(\gamma, \frac{A}{2}\right)\cap V$, f or $1/f$ is bounded. Since γ makes angle ψ with the radius we can determine a *PHD* sequence $\{\gamma_n\}$, $\gamma_n\subseteq\gamma$, and values $0<\alpha<\pi$, $R>0$, $0<\theta_n<2\pi$, such that

i) $\{\gamma_n\}$ *travels in* $\{F(a_n, R, \theta_n, \alpha)\}$

ii) $F(a_n, R, \theta_n, \alpha)\subseteq P'\left(\gamma, \frac{A}{2}\right)$.

If we choose $s>\frac{\pi}{\alpha}$ we can easily show (by using the same inequalities as were used in the proof of Corollary 6) that since f satisfies (6.14) $f-w_0$ or $\frac{1}{f}$, as the case may be, has $\frac{\pi}{\alpha}$-exponential order $\{A_n\}$ on $\{\gamma_n\}$, where $A_n\to\infty$, $n\to\infty$. Thus the hypotheses of Theorem 3 are satisfied and $f=w_0$ which completes the proof of this theorem.

It should be noted that the requirement that γ make some angle ψ at τ can be relaxed in that it is sufficient that, for each $r>0$, $P(\gamma, r)$ contains a *PHD* sequence travelling in domains contained in $P(\gamma, r)$.

When Corollary 7 is combined with Theorem B we obtain

Theorem 7. *Let f be meromorphic in D and suppose for some nontangential boundary path γ at $\tau \in C$, some $w_0 \in W$, and some $\eta > 0$, we have*

i) *if* $w_0 = \infty$

$$\lim_{\substack{z \to \tau \\ z \in \gamma}} (1 - |z|)^{1+\eta} \log |f(z)| = \infty;$$

ii) *if* w_0 *is finite*

$$\lim_{\substack{z \to \tau \\ z \in \gamma}} (1 - |z|)^{1+\eta} \log |f(z) - w_0| = -\infty.$$

Then either f has an M-sequence which lies on a rectilinear segment to τ or else $f = w_0$.

Proof. As in the proof of Theorem 5, with γ_0 equal to the radius to τ, let

$A = \text{lub } \{r \mid$ every sequence on γ_0 is a normal sequence mod r for f$\}$. If $A = 0$ Lemma 4 reveals that γ_0 contains an M-sequence for f. According to the proof of Theorem B if $0 < A < \infty$ there is an M-sequence for f on the boundary of $P(r_0, A)$. It is trivial that if $\{z_n\}$ is an M-sequence for f then so also is $\{z_n'\}$ if $\rho(z_n, z_n') \to 0$, $n \to \infty$. So there is an M-sequence on a rectilinear segment. If $A = \infty$ then f has angular limit w_0 at τ and now (5. 31) of Corollary 7 is satisfied. Then all the hypotheses of Corollary 7 are operative so $f = w_0$.

7. Remarks

All the result in sections 4 and 5 are really theorems about subharmonic functions and are valid under a scheme we present shortly. We need one definition.

Definition 10. *If u is a subharmonic function in D and $\{\gamma_n\}$ is a sequence of Jordan arcs in D we say u has s-order $\{A_n\}$ on $\{\gamma_n\}$, $A_n \geq 0$, $s \geq 0$, if*

$$u(z) \leq \frac{-A_n}{(1-|z|)^s}, \ z \in \gamma_n, \ n=1, 2, \cdots.$$

Relative to the results in Sections 4 and 5 the following substitution rule is valid.

i) Replace "f" by "u" and "holomorphic" by "subharmonic";

ii) Replace "$f-w_0$ has s-exponential order $\{A_n\}$" by "u has s-order $\{A_n\}$", or "$\log|f(z)-w_0|$" by "$u(z)$" depending on the situation; also make this last change in Definition 3;

iii) Replace "$f=w_0$" by "$u=-\infty$". (We agree to allow $u=-\infty$ as an extremal subharmonic function to accommodate the language.)

That this is valid rule can be seen by an analysis of the proofs of Theorems 1, 2, 3 and 4. The function f intruded twice only into the proofs of these theorems. Initially to validate the two constant theorem. (See (4. 3) case (i) of Theorem 1; (4. 19), case (ii) of Theorem 2; (4. 44) Theorem 2; (5. 6) Theorem 3: The proof of Theorem 4 employs the same technique as does the proof of Theorem 2.) And finally the fact f was identically w_0 on a open subset of D was used to infer $f=w_0$ and this weak identity theorem is still true for subharmonic u vis-a-vis the valve $-\infty$. For details see e.g. [13]. The rest of the proofs, and Lemmata 1 and 2, are only concerned with estimating certain harmonic measure. Since the corollaries are only geometric variations of the main theorem they, too, remain true.

The requirement that a *PHD* sequence $\{\gamma_n\}$ have $\overline{\lim_{n\to\infty}} HD(\gamma_n) <\infty$ can be omitted without affecting the validity of the theorems. If a function f has some exponential order on this more general *PHD* sequence $\{\gamma_n\}$ we may certainly find a sequence of subarcs $\{\gamma_n'\}$, $\gamma_n' \subseteq \gamma_n$, all n, which has $\overline{\lim_{n\to\infty}} HD(\gamma_n') < \infty$ and on which f has the same exponential order.

It is possible to derive more general theorems than Theorems 1 and 3 if we abandon entirely the *PHD* notion and characterize sequence of arcs by the behavior of their associated parameters instead. We give now a generalization of Theorem 1 and sketch its proof which is but a duplicate, with minor changes, of the proof of Theorem 1. To generalize the *PHD* notion we need

Definition 11. *A sequence of Jordan arcs* $\{\gamma_n\}$ *in* D *with associated parameters* $\{(R_n, r_n, \theta_n, \alpha_n)\}$, $0\leq\alpha_n<\pi$, $\frac{1}{2}\leq r_n<1$, *is said to be an s-sequence,* $1\leq s<\infty$, *if* $r_n\to 1$, $n\to\infty$, *and any subsequence of* $\{\gamma_n\}$ *contains itself a subsequence* $\{\gamma_j\}$ *which satisfies either* (R) *or* (A);

$$\text{(R)}\quad \varliminf_{j\to\infty}(1-r_j)^{1-s}\frac{K_j}{1+K_j}>0,\ K_j=\frac{e^{2\rho_j}-1}{e^{2\rho_j}+1},\ \rho_j=\rho(R_j, r_j);$$

$$\text{(A)}\quad \lim_{j\to\infty}(1-r_j)^{1-s}\frac{K_j}{1-K_j}=0,\ \text{and}\ \varliminf_{j\to\infty}(1-r_j)^{1-s}\frac{K'_j}{1+K'_j}>0;$$

$$K'_j=\frac{e^{2\rho'_j}-1}{e^{2\rho'_j}+1},\ \rho'_j=\rho(r_je^{i\theta_j}, r_je^{i(\theta_j+\alpha_j)}).$$

Note that when $s=1$ (R) is satisfied if $\{\gamma_j\}$ is a radial-like sequence and (A) is satisfied if $\{\gamma_j\}$ is an arc-like sequence. Lemma A can be used to verify this. Hence $s=1$ defines a *PHD* sequence (although $\varlimsup_{n\to\infty} HD(\gamma_n)\leq\infty$, but as noted this is not serious.)

Theorem 1′. *Let* u *be a subharmonic function in* D *which has s-order* $\{A_n\}$, $1\leq s<\infty$, *on the s-sequence* $\{\gamma_n\}$. *Let* $F_n^{(\alpha)}$ *be defined as in* (2.2). *If, setting* $M(u, F_n^{(\alpha)})=\sup u(z)$, $z\in F_n^{(\alpha)}$, *we have* $A_n\to+\infty$, *and*

$$\text{(7.0)}\qquad \varliminf_{n\to\infty}\frac{M(u, F_n^{(\alpha)})}{A_n}=0,$$

then $u=-\infty$.

Proof. We select a sequence which satisfies (7.0) and then divide the proof into two cases according as to whether this sequence has a subsequence satisfying (R) or (A). These two cases correspond exactly to cases (i) and (ii) respectively in the proof of Theorem 1. In these cases (and in Lemma 1 and 2) the key changes are made in accordance with the following observations. Lemma A says that if we factor $|R_j-r_j|=(1-r_j)t_j=(1-r_j)^s(1-r_j)^{1-s}t_j=(1-r_j)^s t'_j$ and $|r_je^{i\theta_j}-r_je^{i(\theta_j+\alpha_j)}|=(1-r_j)\widehat{t_j}=(1-r_j)^s(1-r_j)^{1-s}\widehat{t_j}=(1-r_j)^s t_j^*$ then the behavior of t'_j and t_j^* is given by conditions (A) and (R), which behavior mirrors precisely the behavior of the corresponding t_j (and $\widehat{t_j}$) in the proof of Theorem 1. A small change occurs in the proof of Lemma 1 where we now estimate $\alpha_j \gtrsim |r_je^{i\theta_j}-r_je^{i(\theta_j+\alpha_j)}|$.

In Theorem 3 (and the resultant corollaries) the method of proof does not allow us to consider, say, an arbitrary *PHD* sequence approaching $\tau \in C$, even non-tangentially. They must travel in the appropriate domains. To subject the $\{\gamma_n\}$ to this mode of travel is to imply that, as viewed from the point τ, the arcs seem quite thin. Whether this "thinness" is necessary we do not know.

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