J. Math. Kyoto Univ.
9-3 (1969) 425-437

Differential modules and derivations of complete discrete valuation rings

By

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(Received August 7, 1969)

In Heerema's [2] and J. Neggers' [3], they discussed problems on inducing of derivations of complete discrete valuation rings to their residue fields (Theorem 1 and Theorem 2). On the other hand, in the author's [4] and A. Grothendieck's [1], they developed general theory of differential modules, using notions of m-adic free modules or formally projective modules. Actually, it is inevitable to treat modules which are not necessarily finitely generated, when we study differential modules of local rings, especially of those of unequal characteristic and they treated them as topological modules and introduced above types of modules as extended notions of free or projective modules. The author's first intention in this paper is to reestablish Heerema and Neggers' theory in terms of differential modules. We will find that in the course of proofs, we can describe it in a simpler way, using our terminology. Secondly, we will discuss on the number $\Delta_{K \mid K^*}$ defined by Neggers, especially on the invariance of those numbers with respect to the choice of prime elements and coefficient rings. Then we will prove that the differential modules of complete discrete valuation rings, satisfying conditions in Theorem 1, over their coefficient rings are independent of the choice of their coefficient rings. In the appendix, we discuss relationship between lifting of derivations and formally left inversibily of homomorphisms of completions of differential modules.

Throughout this paper, we denote by $\Omega_{R/P}$ the differential module of a commutative ring R over a ring P, by $d_{R/P}$ the canonical derivation map of R into $\Omega_{R/P}$, and by $Der_P(R, M)$ the module of derivations of R into an R-module M over P. If P is a prime ring, these are also denoted by Ω_R, d_R and $Der(R, M)$, respectively.

We take this opportunity to thank Professor Paulo Ribenboim, a conversation with whom has given a suggestion to start this work, and Professor Masayoshi Nagata and Professor Tomoharu Akiba whose advices have contributed to improve this work.

§1. Derivations of complete discrete valuation rings

Let R be a local ring with maximal ideal m. For a given derivation ∂ in $Der(R, R)$, we decompose it as $\partial = f \circ d_R$, where f is an R-homomorphism of Ω_R to R. Then, by the commutative diagram:

$$\begin{array}{ccc} R & \xrightarrow{d_R} & \Omega_R \\ \downarrow & & \downarrow \\ R/m & \xrightarrow{d_{(R/m)}} & \Omega_{(R/m)}, \end{array}$$

where vertical arrows are natural surjections, we see that ∂ induces a derivation ∂' in $Der(R/m, R/m)$, if and only if there exists an R/m-homomorphism f' of $\Omega_{(R/m)}$ to R/m such that $\partial' = f' \circ d_{(R/m)}$ and

$$\begin{array}{ccc} \Omega_R & \xrightarrow{f} & R \\ \downarrow & & \downarrow \\ \Omega_{(R/m)} & \xrightarrow{f'} & R/m \end{array} \tag{1}$$

is a commutative diagram. Conversely, for a given derivation ∂' in $Der(R/m, R/m)$ we decompose it as $\partial' = f' \circ d_{(R/m)}$ with an R/m-homomorphism f'. ∂' is induced by a derivation in $Der(R, R)$, if and only if there exists an R-homomorphim f of Ω_R to R, satifying the commutative diagram (1).

We assume here that R is an unramified regular local ring.

Then R is a formally smooth algebra over its prime subring. The canonical topology in Ω_R is the m-adic topology and Ω_R is a formally projective R-module (Suzuki [4], II, Theorem 3 and Theorem 4 or Grothendieck [1], 0_{IV}, (20.4.9)). In our case it is equivalent to saying that the $\Omega_R/m^n\Omega_R$ are projective (hence, free) R/m^n-module for all $n=1,2,\cdots$. Therefore, if we are given an R-homomorphism h_1 of Ω_R to R/m, we can construct, inductively, homomorphisms h_n of Ω_R to R/m^n for all $n=1,2,\cdots$, such that

$$\begin{array}{ccc} & \Omega_R & \\ {\scriptstyle h_{n+1}}\swarrow & & \searrow{\scriptstyle h_n} \\ R/m^{n+1} & \longrightarrow & R/m^n \end{array}$$

are commutative diagrams. Therefore, if we assume that R is complete, taking the projective limit of $\{h_n\}$ we obtain a homomorphism f^* of Ω_R to R which induces h_1. Therefore, if we are given a homomorphism f' of $\Omega_{(R/m)}$ to R/m, we obtain a homomorphism f satisfying the commutative diagram (1). Hence, we get the following theorem which is an extension of Heerema's one ([2]).

Theorem 1. *If R is a complete unramified regular local ring, every derivation in $Der(R/m, R/m)$ is induced by a derivation in $Der(R, R)$.*

We go back to the case of general local rings. The natural surjection: $R \to R/m$ induces a homomorphism:

$$(2) \qquad F : \mathrm{Hom}_R(\Omega_R, R) \to \mathrm{Hom}_R(\Omega_R, R/m),$$

and the natural surjection: $\Omega_R \to \Omega_{(R/m)}$ induces a homomorphism:

$$(3) \qquad G : \mathrm{Hom}_R(\Omega_{(R/m)}, R/m) \to \mathrm{Hom}_R(\Omega_R, R/m).$$

Hence, taking (1) into account, an element f in $\mathrm{Hom}_R(\Omega_R, R)$ induces an element in $\mathrm{Hom}_R(\Omega_{(R/m)}, R/m)$ if and only if $F(f) \in$ Image of G. Conversely, an element f' in $\mathrm{Hom}_R(\Omega_{(R/m)}, R/m)$ is induced by an element in $\mathrm{Hom}_R(\Omega_R, R)$, if and only if $G(f') \in$ Image of F.

We remark here that if M is a complete R-module, we have

$\mathrm{Hom}_R(\Omega_R, M) = \mathrm{Hom}_R(\Omega_R^*, M)$, where Ω_R^* is the completion of Ω_R, because every homomorphism of m-adic modules are continuous. Assume that R is formally smooth (over its prime ring). Let $\{a_\iota\}_{\iota\in I}$ be elements in R such that either the characteristic of R/m is 0 and the classes of a_ι modulo m form a transfscedental base of R/m over its prime field, or the characteristic of R/m equals $p\neq 0$ and the classes of a_ι form a p-independent base of R/m over $(R/m)^p$ then the $d_R a_\iota$ have no linear relation (finite or infinite in the sense of convergence in m-adic topology) in Ω_R^* with coefficients in the completion R^* of R and we have

(4) $\Omega_R^* = \{\sum_{i=1}^{\infty} \alpha_i d_R a_{\iota(i)} : \alpha_i \in R^* \text{ and } \lim_{i\to\infty} \alpha_i = 0,\ \iota(i)\in I\}$ (Suzuki [4]).

Therefore, we have $\mathrm{Hom}_R(\Omega_R, M) \cong \prod_{\iota\in I} M_\iota$ with $M_\iota = M$.

§2. Neggers' Theorem

We assume that R is a complete discrete valuation ring of characteristic 0 with residue field of characteristic $p\neq 0$. Let P be a coefficient ring of R. P is a complete unramifield discrete valuation ring with prime element p. Let K and K^* be quotient fields of R and P, respectively. We remark that R and P are determined uniquely by K and K^*, respectively. Let u be a prime element of R and $f(U)$ be a minimal monic polynomial of u over P. Then, we have $R = P[U]/(f(U))$ and

(5) $\Omega_R = R\otimes_P \Omega_P \oplus RdU/R((d_P f)(u) + f'(u)dU)$,

where dU is an independent element and $(d_P f)(U)$ is a polynomial in $\Omega_P[U]$ obtained from $f(U)$, substituting its coefficients by their images in Ω_P by d_P. We take a set of elements $\{a_\iota\}_{\iota\in I}$ in P, such as their residue classes form a p-independent base of $P/pP(=R/m)$.

It is easy to see that the m-adic (hence, p-adic) completion $(R\otimes_P\Omega_P)^*$ of $R\otimes_P\Omega_P$ is a complete formally projective R-module, the $1\otimes d_P a_\iota$, regarded as elements in $(R\otimes_P\Omega_P)^*$ are free and

$$(R\otimes_P\Omega_P)^* = \{\sum_{i=1}^{\infty} \alpha_i (1\otimes d_P a_{\iota(i)}) \mid \alpha_i \in R \text{ and } \lim_{i\to\infty} \alpha_i = 0,\ \iota(i) = 0,\ \iota(i)\in I\}.$$

Let $(d_P f)(u)=\sum_{i=0}^{\infty}\beta_i(1\otimes d_P a_{\iota(i)})$, with $\beta_i\in R$ be an expression in $(R\otimes_P\Omega_P)^*$. Let v be the valuation of R.

Definition. (Neggers [3])[1)]

$$\Delta_{K\mid K^*}(u)=\min_{\iota} v(\beta_i)-v(f'(u)). \tag{6}$$

We make aconvention that $\Delta_{K\mid K^*}(u)=\infty$, if $(d_P f)(u)=0$ in $(R\otimes_P\Omega_P)^*$.

By (5), we see that

$$\Omega_R^*\cong\text{the completion of } (R\otimes_P\Omega_P)^*\oplus RdU/R((d_P f)(u) + f'(u)dU).$$

Therefore, $\mathrm{Hom}_R(\Omega_R, R)$ is described in the following way. Since we have $\mathrm{Hom}_R(R\otimes_P\Omega_P\oplus RdU, R)\cong\prod_{\iota\in I}R_\iota\times R$ with $R_\iota=R$, $\mathrm{Hom}_R(\Omega_R, R)$ can be regarded as a subset of $\prod_{\iota\in I}R_\iota\times R$, consisting of elements of the form $(c_\iota;\, t)_{\iota\in I}$ with

$$\sum_{i=1}^{\infty}\beta_i c_{\iota(i)}+f'(u)t=0. \tag{7}$$

Then, every h in $\mathrm{Hom}_R(\Omega_R, R)$ is identified with $(c_\iota;\, t)_{\iota\in I}$ if $h(d_R a_\iota)=c_\iota$ and $h(d_R u)=t$. Or, equivalently, a derivation ∂ in $Der(R, R)$ is determined uniquely, if we assign to $(\partial a_\iota;\, \partial u)_{\iota\in I}$ a set of values satisfying

$$\sum_{i=1}^{\infty}\beta_i\partial a_{\iota(i)}+f'(u)\partial u=0. \tag{8}$$

Theorem 2. (Neggers [3]) *Let R be a complete discrete valuation ring with maximal ideal m. Then, every derivation in*

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1) As a matter of fact, this is an alternative definition of Neggers' $\Delta_{K\mid K^*}$. His original definition is as follows. We choose u so that $f(U)$ is an Eisenstein polynomial. Assume that

$$f(U)=U^e+p(f_{e-1}U^{e-1}+\cdots+f_0),\quad f_i\in P.$$

Let $a\in R$. Put $\Delta(a)=\min_{\partial} v^*(\partial(a))$ with $\partial\in Der(P, P)$, where v^* is the valuation of P. Then $\Delta_{K\mid K^*}$ is defined to be

$$\min_{0\le i\le e-1}((\Delta(f_i)+1)e+1)-v(f'(u)).$$

It is not difficult to show that two definitions thus we obtained coincide to each other, when once u is chosen as above.

$Der(R, R)$ induces a derivation in $Der(R/m, R/m)$ if and only if every derivation in $Der(R/m, R/m)$ is induced from a derivation in $Der(R, R)$.

Proof. Our assertion is easily derived from Theorem 1, if R is of equal characteristic. Hence, we assume that R is of unequal charactristic. We use the same notaions and terminology as above. By virtue of what we mentioned below (2) and (3), we have only to prove that either Image(F) in (2) coincides with Image(G) in (3) or there is no relation of inclusion between Image(F) and Image(G). From the exact sequence:

$$0 \to m/m^2 + pR \to (R/m) \otimes_R \Omega_R \to \Omega_{(R/m)} \to 0,$$ [2]

we can easily see that:

$$(R/m) \otimes_R \Omega_R = \bigoplus_{\iota \in I} (R/m)(1 \otimes d_R a_\iota) \oplus (R/m)(1 \otimes d_R u).$$ [2]

It follows that $\mathrm{Hom}_R(\Omega_R, R/m) \cong \prod_{\iota \in I} (R/m)_\iota \times (R/m)$ with $(R/m)_\iota = R/m$, and in this expression we have:

$\mathrm{Image}(G) = \{(\bar{c}_\iota;\ 0)_{\iota \in I} \mid \bar{c}_\iota \in R/m\}$.

Also, we see that:

$\mathrm{Image}(F) = \{(\bar{c}_\iota;\ \bar{t})_{\iota \in I} \mid \bar{c}_\iota$ and $\bar{t}$ are classes of $c_\iota \in R$ and $t \in R$, satisfying (7)$\}$.

In case $\Delta_{K|K^*}(u) \geqq 1$, if we give arbitrary values of c_ι in R, (7) is solved and $t \in m$, which means that Image(F) = Image(G). Next, we consider the case where $\Delta_{K|K^*}(u) \leqq 0$. If we put $t=1$, (7) can be solved. Hence Image(F) $\not\subseteq$ Image(G). On the other hand, let $\bar{i}$ be an integer such that $v(\beta_{\bar{i}}) = \min_{1 \leq i < \infty} v(\beta_i)$. Put $c_\iota \equiv 0$ modulo m if $\iota \neq \iota(\bar{i})$ and $c_{\iota(\bar{i})} \equiv 1$ modulo m. Then (7) can not be solved unless $t \not\equiv 0$ modulo m. Hence, Image(F) $\not\supseteq$ Image(G). Q.E.D.

From what we discussed in the proof of Theorem 2, we have:

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2) The exact sequence is proved, for instance, in E. Kunz, "Die Primidealteiler der Differenten in algemeinen Ringen", J. f. reine *u.* ang. Math. 204 (1960). However, in our case, the equality can be proved directly from (5). if we choose as u one such that $f(U)$ is an Eisenstein polynomial.

Corollary 1. (Neggers [3]) *Conditions in Theorem 2 are true if and only if $\Delta_{K|K^*}(u) \geqq 1$, in case R is of unequal characteristic.*

Since the description in Theorem 2 is independent of the choice of K^* and u, we have:

Corollary 2. *The property that $\Delta_{K|K^*}(u) \geqq 1$ is independent of the choice of K^* and u.*

§3. On $\Delta_{K|K^*}(u)$

Corollary 1 to Theorem 2 gave a characterization of the fact $\Delta_{K|K^*}(u) \geqq 1$. We will make here further investigation on $\Delta_{K|K^*}(u)$.

Proposition 1. *$\Delta_{K|K^*}(u) \geqq 1$ if and only if there exists a prime element w in R such that $\Delta_{K|K^*}(w) = 1$.*

Proof. Assume that $\Delta_{K|K^*}(u) > 1$. By the relation (8), it is easy to see that $\Delta_{K|K^*}(u) = \min_{\partial} v(\partial u)$, where ∂ runs over $Der(R, R)$. On the other hand, we have $v(\partial b) \geqq 1$ for $\partial \in Der(R, R)$ and $b \in m$ by Theorem 2 and its corollary 1. Assume that $\bar{i}$ is an integer >0 such that $v(\beta_{\bar{i}}) = \min_i v(\beta_i)$. Put $a = a_{\iota(\bar{i})}$. Take $\partial \in Der(R, R)$ such that $\partial a_\iota = 0$ for $\iota \neq \iota(\bar{i})$ and $\partial a_{\iota(\bar{i})} = 1$. Put $w = au$. Then we have $v(\partial w) = v(a\partial u + u\partial a) = 1$, because $v(a\partial u) > 1$ and $v(u\partial a) = 1$. Hence $\Delta_{K|K^*}(w) = 1$. By virtue of this fact and Theorem 2, we prove our statement.

Proposition 2. *Assume that $\Delta_{K|K^*}(u) \leqq 0$. Then $\Delta_{K|K^*}(u) = -r$ if and only if for a sufficiently large integer s and for every $\partial \in Der(R, R)$ such that $v(\partial(b)) \geqq s$ for $b \in P$, we have $v(\partial a) \geqq s - r$ for every $a \in R$ and there exists a derivation ∂ with this property such that $\min_{a \in R} v(\partial a) = s - r$.*

Proof. First, by (8), we see that if $v(\partial a_\iota) \geqq s$, $v(\partial u) \geqq s + \Delta_{K|K^*}(u)$ and we can prove as before that if $s \geqq -\Delta_{K|K^*}(u)$, there is a ∂ satisfying, besides the above condition, that $v(\partial u) = s + \Delta_{K|K^*}(u)$. On the other hand, every element in R is written as a polynomial $g(u)$ of

u with coefficients in P. For $\partial \in Der(R, R)$, we have $\partial(g(u)) = (\partial g)(u) + g'(u)\partial u$, where $(\partial g)(U)$ is meant by a polynomial obtained from $g(U)$ substituting its coefficients by their values by ∂. Then, if $v(\partial b) \geq s$ for $b \in P$, we have $v((\partial g)(u)) \geq s$ and $v(g'(u)\partial u) \geq s + \Delta_{K|K^*}(u)$. Hence, $v(\partial(g(u)) \geqq s + \Delta_{K|K^*}(u)$. Our proposition follows from this easily.

Corollary. *If $\Delta_{K|K^*}(u) \leqq 0$, the value of $\Delta_{K|K^*}(u)$ is independent of the choice of the prime element u.*

However, even in case of $\Delta_{K|K^*}(u) \leqq 0$, the value of $\Delta_{K|K^*}(u)$ depends on the choice of K^*, as we see it in the following example.

Example 1. Let Z be the ring of integers and let p be a prime number. Let x be an independent variable over Z_{pZ}. Let P be the p-adic completion of $Z[x]_{pZ[x]}$. Let u be a root of an Eisenstein equation: $U^p + px = 0$. Put $R = P[u]$. Then the relation (8), in our case, is $p\partial x + pu^{p-1}\partial u = 0$ and $\Delta_{K|K^*}(u) = 1 - p$. On the other hand, put $y = u + x$. We get another coefficient ring P' of R such that P' = the p-adic completion of $Z[y]_{pZ[y]}$. u is a root of an Eisenstein equation: $U^p - pU + py = 0$. The relation (8), in this case, is $p\partial y + (pu^{p-1} - p)\partial u = 0$. Therefore $\Delta_{K|K^*}(u) = 0$.

There is a characterization of the property $\Delta_{K|K^*}(u) \geqq 0$ as follows.

Proposition 3. (Neggers [3]). *$\Delta_{K|K^*}(u) \geqq 0$ if and only if every derivation in $Der(P, P)$ is lifted up to a derivation in $Der(R, R)$.*

This can be proved easily from the relation (8).

§4. Invariance of the differential modules

As is shown in the example below, an isomorphism, between two coefficient rings of a complete discrete valuation ring R, can not generally be extended to an automorphism of R, even in the case $\Delta_{K|K^*}(u) \geqq 1$. We will show, however, that $\Omega_{R|P}$ is independent of the choice of K^* in this case.

Example 2. Let Z, x, P, u, y and P' be as in Example 1, except that u is a root of an Eisenstein equation: $U^2-px=0$. Then it is easy to see $\Delta_{K|K^*}(u)=1$. We have an isomorphism $\sigma: P\rightarrow P'$ such that $\sigma x=y$. σ cannot be extended to an automorphism of R, for if there exists such an extension σ', $(\sigma' U)^2=py=p(x+u)$ and $\sigma' u$ is a root of an irreducible quadratic equation over R, hence not in R.

Proposition 4. $\Delta_{K|K^*}(u)\geqq 0$ *if and only if the sequence*:

$$0\longrightarrow(R\otimes_P\Omega_P)^*\xrightarrow{\varphi^*}\Omega_R^*\longrightarrow\Omega_{R/P}\longrightarrow 0$$

is exact and splits.

Remark. The above sequence is deduced from the natural exact sequence:

$$R\otimes_P\Omega_P\xrightarrow{\varphi}\Omega_R\longrightarrow\Omega_{R/P}\longrightarrow 0.$$

We note that since $\Omega_{R/P}$ is a finite module over a complete discrete valuation ring R, $\Omega_{R/P}$ itself is complete.

Proof. Exactness of the sequence: $(R\otimes_P\Omega_P)^*\rightarrow\Omega_R^*\rightarrow\Omega_{R/P}\rightarrow 0$ is always true. With notations in (8), we have

$$\Omega_R^*\cong \text{the completion of } (R\otimes_P\Omega_P)^*\oplus RdU/R(\sum_{i=1}^{\infty}\beta_i(1\otimes d_P a_{\iota(i)}) + f'(u)dU).$$

Assume that $\Delta_{K|K^*}(u)\geqq 0$. Then we can define a homomorphism $\psi: (R\otimes_P\Omega_P)^*\oplus RdU\rightarrow(R\otimes_P\Omega_P)^*$ in such a way that $\psi(a)=a$ for $a\in R\otimes_P\Omega_P$ and $\psi(dU)=-\sum_{i=1}^{\infty}\frac{\beta_i}{f'(u)}(1\otimes d_P a_{\iota(i)})$. It is obvious that ψ induces a homomorphism $\psi^*: \Omega_R^*\rightarrow(R\otimes_P\Omega_P)^*$ and $\psi^*\circ\varphi^*=$the identical automorphism of $(R\otimes_P\Omega_P)^*$. Hence our sequence is a splitting exact sequence. Assume that φ^* is left inversible and ψ^* is a homomorphism: $\Omega_R^*\rightarrow(R\otimes_P\Omega_P)^*$ such that $\psi^*\circ\varphi^*=$the identical automorphism of $(R\otimes_P\Omega_P)^*$. ψ^* induces a homomorphism $\psi: (R\otimes_P\Omega_P)^*\oplus RdU\rightarrow(R\otimes_P\Omega_P)^*$. Then, $\psi(\sum_{i=1}^{\infty}\beta_i(1\otimes d_P a_{\iota(i)})+f'(u)dU)=0$. Hence, $\sum_{i=1}^{\infty}\beta_i(1\otimes d_P a_{\iota(i)})+f'(u)\psi(dU)=0$. From the property of $(R\otimes_P\Omega_P)^*$, it follows that $v(f'(u))\leqq v(\beta_i)$ for $i=1,2,\cdots$. Hence, we have

$\Delta_{K \mid K^*}(u) \geqq 0$.

Remark: Proposition 4 can be proved as a consequence of Proposition 3. We will learn it in Appendix in a more general form.

Theorem 3. *Let R be a complete discrete valuation ring satisfying the condition in Theorem 2. Then the $\Omega_{R/P}$ are determined independently of the choice of the coefficient rings P (up to isomorphisms). Actually, in this case, the $\Omega_{R/P}$ are isomorphic to the submodule of the completion Ω_R^* of Ω_R, consisting of all the torsion elements.*

Proof. Since $(R \otimes_P \Omega_P)^*$ is a completion of a free module, it has no trosion elements. Therefore our Theorem follows from Proposition 4 and the fact that $\Omega_{R/P}$ is a trosion module.

Example 3 shows that in case $\Delta_{K \mid K^*} \lneqq 0$, $\Omega_{R/P}$ is not necessarily determined uniquely, because in that example $\Omega_{R/P} \cong R/(u^{2p-1})$ but $\Omega_{R/P'} \cong R/(u^p)$.

Appedix. Lifting of derivations

We assume that R is a commutative ring with identity and m is an ideal in R. We consider m-adic topology in R and in R-modules. Let M and N are R-modules.

Theorem4. *Let (R, m) be a complete Zariski ring. Let S be a commutative R-algebra, with structure homomorphism ρ, which is a finite free R-module. Assume that Ω_R is a formally projective (m-adic) R-module and $\Omega_{S/R}$ is a torsion module. Then the following two conditions are equivalent.*

(a) *Every derivation in $Der(R, R)$ can be extended to a derivation in $Der(S, S)$.*

(b) *The canonical exact sequence:*

$$S \otimes_R \Omega_R \xrightarrow{\varphi} \Omega_S \longrightarrow \Omega_{S/R}$$

induces a splitting exact sequence:

$$0 \longrightarrow (S \otimes_R \Omega_R)^* \xrightarrow{\varphi^*} \Omega_S^* \longrightarrow \Omega_{S/R} \longrightarrow 0,$$

where $(S \otimes_R \Omega_R)^*$ *and* Ω_S^* *are* (*m-adic*) *completions of* $S \otimes_R \Omega_R$ *and* Ω_S, *respectively.*

Proof. We prove (b)$\Rightarrow$(a). Let ψ^* be a homomorphism : $\Omega_S^* \rightarrow (S \otimes_R \Omega_R)^*$ such that $\psi^* \circ \varphi^* =$ the identical automorphism of $(S \otimes_R \Omega_R)^*$. Let $\partial = f \circ d_R$ be a derivation in $Der(R, R)$. Since S is complete, the homomrphism $1 \otimes f : S \otimes_R \Omega_R \rightarrow S$ induces a homomorphism $(1 \otimes f)^* : (S \otimes_R \Omega_R)^* \rightarrow S$ and we have a commutative diagram:

$$\begin{tikzcd}
R \arrow[rr, "\rho"] \arrow[d, "d_R"] & & S \arrow[d, "d_S"] \\
\Omega_R \arrow[r] \arrow[dd, "f"] & S \otimes_R \Omega_R \arrow[r, "\varphi"] \arrow[d] & \Omega_S \arrow[d, "\alpha"] \\
& (S \otimes_R \Omega_R)^* \arrow[r, "\varphi^*"] \arrow[dr, "(1 \otimes f)^*"] & \Omega_S^* \arrow[d, "(1 \otimes f)^* \circ \psi^*"] \\
R \arrow[rr, "\rho"] & & S
\end{tikzcd}$$

where α is the natural homomorphism. Therefore $\partial' = (1 \otimes f)^* \circ \psi^* \circ \alpha \circ d_S \in Der(S, S)$ is an extension of ∂. Next, we prove (a)$\Rightarrow$(b). We prove it in two steps in the following way.

[1]. For a given S-homomorphism $g^* : (S \otimes_R \Omega_R)^* \rightarrow S$ there exists an S-homomorphism $h^* : \Omega_S^* \rightarrow S$ such that $g^* = h^* \circ \varphi^*$.

[2]. [1] leads to the left inversibility of φ^*.

Proof of [1]. Let $S = Re_1 \oplus \cdots \oplus Re_h$ be a decomposition of S to a direct sum of h copies of R. Let $p_j : S \rightarrow R$ be an R-homomorphism which is the projection of S to the j-th copy of R ($j = 1, 2, \cdots, h$). Then $x = \sum_{j=1}^{h} p_j(x) e_j$ for $x \in S$. The R-homomorphisms $p_j \circ g^* : (S \otimes_R \Omega_R)^* \rightarrow R$ induce R-homomorphisms $g_j^* : \Omega_R^* \rightarrow R$ ($j = 1, 2, \cdots, h$) and we have $g^*(x) = \sum_{j=1}^{h} (1 \otimes g_j^*)(x) e_j$ for all $x \in (S \otimes_R \Omega_R)^*$, because both hand sides coincide to each other for elements in generators $1 \otimes \Omega_R^*$ of $(S \otimes_R \Omega_R)^*$. By (a), there exist S-homomorphisms $h_j^* : \Omega_S^* \rightarrow S$, satifying commutative diagram:

$$\begin{array}{ccc} \Omega_R^* & \longrightarrow & \Omega_S^* \\ g_j^* \downarrow & & \downarrow h_j^* \\ R & \xrightarrow{\rho} & S \end{array} \qquad \text{or equivalently} \qquad \begin{array}{ccc} (S\otimes_R\Omega_R)^* & \longrightarrow & \Omega_S^* . \\ 1\otimes g_j^* \searrow & & \swarrow h_j^* \\ & S & \end{array}$$

Let h^* be a homomorphisms: $\Omega_S^* \to S$ defined as $h^*(x)=\sum_{j=1}^{h} h_j^*(x)e_j$ for all $x \in \Omega_S^*$. Then we get a commutaive diagram:

$$\begin{array}{ccc} (S\otimes_R\Omega_R)^* & \longrightarrow & \Omega_S^* \\ g^* \searrow & & \swarrow h^* \\ & S & \end{array}$$

and we prove our assertion.

Proof of [2]. Since $(S\otimes_R\Omega_R)^*$ is a complete formally projective module, it is a direct summand of an m-adic($=mS$-adic) completion N^* of a free S-module N (Suzuki [5]). Let $\{a_\iota\}_{\iota\in I}$ be a free base of N. Then:

$$N^*=\{\sum_{i=1}^{\infty}\alpha_i a_{\iota(i)} \mid \alpha_i \in S,\ \iota(i)\in I \text{ and } \lim_{i\to\infty}\alpha_i=0\}.$$

Hence $(S\otimes_R\Omega_R)^* \subset \prod_{\iota\in I} S$ $(S_\iota=S)$. Let $\zeta_\iota : (S\otimes_R\Omega_R)^* \to S$ be the projection of Ω_R^* to the ι-th component of $\prod_{\iota\in I} S$. By [1], there exist S-homomorphisms $\psi_\iota : \Omega_S^* \to S$ such that $\zeta_\iota=\psi_\iota\circ\varphi^*$.

$\prod_{\iota\in I}\psi_\iota$ is a map of Ω_S^* into $\prod_{\iota\in I} S_\iota$ such that $(\prod_{\iota\in I}\psi_\iota)\circ\varphi^*=$the identical automorphism of $(S\otimes_R\Omega_R)^*$. Therefore, we have only to prove that Image $(\prod_{\iota\in I}\psi_\iota)$ is contained in $(S\otimes_R\Omega_R)^*$. Since $\Omega_{S/R}$ is a torsion module, for any element $a\in\Omega_S^*$ there exists a non-zero divisor r of S such that $ra\in\varphi^*((S\otimes_R\Omega_R)^*)$. Hence $r(\prod_{\iota\in I}\psi_\iota)(a)\in(S\otimes_R\Omega_R)^*$. Hence if we prove that $(\prod_{\iota\in I}\psi_\iota)(a)\in N^*$, then $(\prod_{\iota\in I}\psi_\iota)(a)\in(S\otimes_R\Omega_R)^*$, because $(S\otimes_R\Omega_R)^*$ is a direct summand of N^* and r is not a zero-divisor in N^*. Since $r(\prod_{\iota\in I}\psi_\iota)(a)\in N^*$, we have an expansion: $r(\prod_{\iota\in I}\psi_\iota)(a)=\sum_{i=0}^{\infty}\alpha_i a_{\iota(i)}$ where $\alpha_i\in S$ and $\lim_{i\to\infty}\alpha_i=0$. Let β_ι be the ι-th coordinate of $(\prod_{\iota\in I}\psi_\iota)(a)$ in $\prod_{\iota\in I} S_\iota$, then it follows that $\beta_\iota=0$ if $\iota\neq\iota(i)$ $i=1,2,\cdots$, and $\lim_{i\to\infty} r\beta_{\iota(i)}=0$. Since S is a Zariski ring, we conclude $\lim_{i\to\infty}\beta_{\iota(i)}=0$ (see, for instance, Zariski-Samuel [6], Ch. VIII Cor. 1 to

Theorem 13). Hence $(\prod_{\iota \in I} \psi_{r_\iota})(a) \in N^*$. Q.E.D.

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