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A remark on submersions and immersions with codimension one or two

By

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Dedicated to Professor Atuo Komatu on the occasion of his 60th birthday

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§1. Introduction

Let M and N be smooth manifolds and $f: M \to N$ a smooth map. We say that f has *maximal rank* if at each point of M the Jacobian matrix of f has maximal rank. If $\dim M < \dim N$, then f is an immersion; while if $\dim M > \dim N$, f is a submersion. According to E. Thomas [13], for convenience we call the integer $|\dim M - \dim N|$ *codimension* of a map $M \to N$. In [13] E. Thomas considers the following problem. Let $g: M \to N$ be a continuous map of codimension one or two. When is g homotopic to a smooth map of maximal rank? By exploiting the work of M. Hirsch [6] and A. Phillips [11] he obtains answers in terms of cohomology invariants of M and N. However, he supposes that the source manifold M satisfy the following condition (*):

Condition (*):

(i) $\dim M \leqq 9$; if $\dim M = 9$, M is open;

(ii) $H^4(M, Z)$ has no 2-torsion;

(iii) $H^8(M, Z)$ has no 6-torsion.

In the present note we shall remark that the above condition (*)

can be a little more weakened.

All manifolds in this note will be smooth, paracompact, connected and without boundary. For any such manifold V we let τ_V denote the tangent bundle of V.

Througout this note, we let a_k denote 1 for k even and 2 for k odd.

We will say that a manifold M satisfies Condition (♯) if it has the following properties:

(a) torsion coefficients of $H^{4k}(M, Z)$ are 0 or relatively prime to $(2k-1)!a_k$, $k=1, 2, \cdots$.

(b) $H^{8k+1}(M, Z_2)=H^{8k+2}(M, Z_2)=0$, for $k=1, 2, \cdots$.

We combine Theorem 1.1 and Theorem 1.2 in Thomas [13] with Theorem 5 in §4 to give the following results. The proofs will be given in §5. For a manifold M, we shall denote by $P^{4i}(M) \in H^{4i}(M, Z)$ the i-th Pontrjagin class of M, and by $W^i(M) \in H^i(M, Z_2)$ the i-th Stiefel-Whitney class of M, $i \geqq 0$.

Theorem 1. *Let M be a manifold satisfying Condition* (♯) *and let $f: M \to N$ be a map of codimension* 1.

(a) *Suppose that* $\dim M < \dim N$. *Then f is homotopic to an immersion if and only if there is a class $u \in H^1(M, Z_2)$ such that*

$$W^i(M)+W^{i-1}(M) \cup u = f^* W^i(N), \quad i=1, 2,$$

and

$$P^{4i}(M)=f^* P^{4i}(N), \quad i=1, 2, \cdots.$$

(b) *Suppose that* $\dim M > \dim N$ *and that M is open. Then f is homotopic to a submersion if and only if there is a class $u \in H^1(M, Z_2)$ such that*

$$W^i(M)=f^* W^i(N)+f^* W^{i-1}(N) \cup u, \quad i=1, 2,$$

and

$$P^{4i}(M)=f^* P^{4i}(N), \quad i=1, 2, \cdots.$$

There are similar results for codimension 2.

We will say that a map $f: M \to N$ is *orientable* if $f^* W^1(N) = W^1(M)$.

Theorem 2. *Let M be a manifold satisfying Condition (#) and let $f: M \to N$ be an orientable map of codimension 2.*

(a) *Suppose that* $\dim M < \dim N$. *Then f is homotopic to an immersion if and only if there is a class $v \in H^2(M, Z)$ such that*

(i) $W^2(M) + f^* W^2(N) \equiv v$, mod 2,

(ii) $P^{4i}(M) + P^{4i-4}(M) \cup v^2 = f^* P^{4i}(N)$, $i = 1, 2, \cdots$.

(b) *Suppose that* $\dim M > \dim N$ *and that M is open. Then f is homotopic to a submersion if and only if there is a class $v \in H^2(M, Z)$ such that*

(i) $W^2(M) + f^* W^2(N) \equiv v$, mod 2,

(ii) $P^{4i}(M) = f^* P^{4i}(N) + f^* P^{4i-4}(N) \cup v^2$, $i = 1, 2, \cdots$.

§2. Examples

We take N to be one of the two projective spaces, real or complex, which we denote respectively by RP^n (of dimension n), CP^n (of dimension $2n$). For a complex X we can compute the set $[X, N]$ of homotopy classes of maps as follows. If $\dim X < n$, then $[X, RP^n] = H^1(X, Z_2)$; if $\dim X \leqq 2n$, then $[X, CP^n] = H^2(X, Z)$. In each case the correspondence is given by $f \to f^* \iota$, where f denotes a map from X into the projective space, and where ι denotes generically the fundamental class of the projective space. Thus, we have

$$\iota \in H^1(RP^n, Z_2), \quad \iota \in H^2(CP^n, Z)$$

depending on which of the two projective spaces we are referring to. We call the cohomology class $f^* \iota$ the *degree* of the map f. Since the characteristic classes of the projective spaces are known, we now can apply Theorems 1 and 2 to determine which degrees can occur as the degree of an immersion from M into a projective space.

As an example we have the following results giving immersions of codimension 1 or 2. We assume below that M is a manifold satisfying Condition (#) given in §1.

Theorem 3. (α) *Let* $f: M^m \to RP^{m+2}$ *be an orientable map,* $3 \leqq m$, *with degree* $x \in H^1(M, Z_2)$. *Then f is homotopic to an immersion if and only if there is a class* $v \in H^2(M, Z)$ *such that*

(i) $W^2(M) + \binom{m+3}{2} x^2 \equiv v, \mod 2,$

(ii) $P^{4i}(M) + P^{4i-4}(M) \cup v^2 = 0, \quad i = 1, 2, \cdots.$

(β) *Let* $f: M^{2q} \to CP^{q+1}$ *be an orientable map,* $2 \leqq q$, *with degree* $y \in H^2(M, Z)$. *Then f is homotopic to an immersion if and only if there is a class* $v \in H^2(M, Z)$ *such that*

(i) $W^2(M) + qy \equiv v, \mod 2,$

(ii) $P^{4i}(M) + P^{4i-4}(M) \cup v^2 = \binom{q+2}{i} y^{2i}, \quad i = 1, 2, \cdots.$

Theorem 4. (α) *Let* $f: M^m \to RP^{m+1}$ *be a map,* $2 \leqq m$, *with degree* $x \in H^1(M, Z_2)$. *Then f is homotopic to an immersion if and only if there is a class* $u \in H^1(M, Z_2)$ *such that*

$$W^i(M) + W^{i-1}(M) \cup u \equiv \binom{m+2}{i} x^i, \mod 2, \quad i = 1, 2,$$

and

$$P^{4i}(M) = 0, \quad i = 1, 2, \cdots.$$

(β) *Let* $f: M^{2q-1} \to CP^q$ *be a map,* $2 \leqq q$, *with degree* $y \in H^2(M, Z)$. *Then f is homotopic to an immersion if and only if there is a class* $u \in H^1(M, Z_2)$ *such that*

$$W^1(M) + u = 0,$$

$$W^2(M) + W^1(M) \cup u \equiv (q+1)y, \mod 2,$$

and

$$P^{4i}(M) = \binom{q+1}{i} y^{2i}, \quad i = 1, 2, \cdots.$$

Examples. (a) For $m \geqq 2$, quaternion projective space QP^m can not be immersed in RP^{4m+2}.

(b) For $m \geqq 2$, QP^m can not be immersed in CP^{2m+1}.

We know the characteristic classes of QP^m (cf. Hirzebruch [7]), therefore, these are obtained by Theorem 3.

§3. Lemmas on characteristic classes

We precede the proofs of Theorem 1 and 2 by a classification theorem.

In this and next sections we shall study the problem of classifying $O(n)$-bundles over complexes K of a certain kind. It is well known (Steenrod [12], Part II) that the set of equivalence classes of $O(n)$-bundles over K is in one-to-one correspondence with the set $[K, B_{O(n)}]$ of homotopy classes of maps from K into the classifying space $B_{O(n)}$ for othogonal group. Thus we have reduced our geometric problem to the computation of $[K, B_{O(n)}]$.

In order to study $[K, B_{O(n)}]$, we need to recall the following results of Bott [3], [4]:

(a) $\pi_i(B_{U(n)}) \cong \begin{cases} 0, & \text{fer } i \text{ odd}, \quad i \leqq 2n, \\ Z, & \text{for } i \text{ even}, \quad i \leqq 2n, \end{cases}$

(b) $\pi_{2n+1}(B_{U(n)}) \cong Z_{n!}$,

(c) the groups $\pi_i(B_{O(n)})$, $2 < i < n$, are as follows;

i mod 8	0	1	2	3	4	5	6	7
$\pi_i(B_{O(n)})$	Z	Z_2	Z_2	0	Z	0	0	0.

We shall denote by $E_{O(n)} = (E_{O(n)}, p_{O(n)}, B_{O(n)})$, $E_{U(n)} = (E_{U(n)}, p_{U(n)}, B_{U(n)})$ the universal $O(n)$-, $U(n)$-bundle, respectively. We shall denote by s^n the fundamental class of $H^n(S^n, Z) \cong Z$.

Lemma 1. *Let $f: S^1 \to B_{O(n)}$ be a representative map of a generator of $\pi_1(B_{O(n)}) \cong Z_2$ $(1<n)$. Then the Stiefel-Whitney class W^1 of the $O(n)$-bundle $f^* E_{O(n)}$ induced by f is equal to s^1* mod 2.

Lemma 2. *Let $f: S^2 \to B_{O(n)}$ be a representative map of a generator of $\pi_2(B_{O(n)}) \cong Z_2$ $(2<n)$. Then the Stiefel-Whitney class W^2 of the $O(n)$-bundle $f^* E_{O(n)}$ induced by f is equal to s^2* mod 2.

Lemma 3. *(α) Let $g: S^{2k} \rightarrow B_{U(n)}$ be a representative map of a generator of $\pi_{2k}(B_{U(n)}) \cong Z$ $(0<k<n)$. Then the k-th Chern class C^{2k} of the $U(n)$-bundle $g^*E_{U(n)}$ induced by g is equal to $-(k-1)!s^{2k}$.*

(β) Let $f: S^{4k} \rightarrow B_{O(n)}$ be a representative map of a generator of $\pi_{4k}(B_{O(n)}) \cong Z$ $(0<4k<n)$. Then the k-th Pontrjagin class P^{4k} of the $O(n)$-bundle $f^*E_{O(n)}$ induced by f is equal to $(-1)^{k+1}(2k-1)!a_k s^{4k}$.

Proof. (α) Let $g: S^{2k-1} \rightarrow U(n)$ be the characteristic map of the $U(n)$-bundle $g^*E_{U(n)}$ and $i: U(k) \rightarrow U(n)$ be the inclusion map. Then $i_*: \pi_{2k-1}(U(k)) \cong \pi_{2k-1}(U(n))$. Therefore, there exists a map $\bar{g}: S^{2k-1} \rightarrow U(k)$ such that the following diagram

$$\begin{array}{ccc} S^{2k-1} & \xrightarrow{g} & U(n) \\ & {}_{\bar{g}}\searrow \quad \nearrow{}_{i} & \\ & U(k) & \end{array}$$

is homotopy commutative, and the homotopy class $\{\bar{g}\}$ of $\bar{g}$ generates $\pi_{2k-1}(U(k)) \cong Z$.

Let $p: U(k) \rightarrow U(k)/U(k-1) = S^{2k-1}$ be the natural projection. Then, as is easily seen, the Chern class $C^{2k}(g^*E_{U(n)})$ is equal to $-(\text{degree of } p\circ\bar{g})s^{2k}$ (cf. Milnor [9]; Steenrod [12], Part II, Theorem 35.12). Now we consider the homotopy exact sequence of the bundle $p: U(k) \rightarrow U(k)/U(k-1) = S^{2k-1}$:

$$\cdots \longrightarrow \pi_{2k-1}(U(k-1)) \xrightarrow{i_*} \pi_{2k-1}(U(k)) \xrightarrow{p_*} \pi_{2k-1}(S^{2k-1})$$
$$\longrightarrow \pi_{2k-2}(U(k-1)) \xrightarrow{i_*} \pi_{2k-2}U(k)) \xrightarrow{p_*} \cdots.$$

Then $\{\bar{g}\}$ generates $\pi_{2k-1}(U(k))$, therefore, we obtain $(\text{degree of } p\circ\bar{g}) = (k-1)!$ by the table (a), (b). Thus (α) is proved.

(β) Let $f: S^{4k-1} \rightarrow O(n)$ be the characteristic map of the $O(n)$-bundle $f^*E_{O(n)}$ and $\rho: O(n) \rightarrow U(n)$ be the canonical injection. By Kervaire [8], we know that the composite map $\rho\circ f: S^{4k-1} \rightarrow U(n)$ represents the class $a_k\sigma$, where σ is the generator of $\pi_{4k-1}(U(n)) \cong Z$. By (α) the Chern class C^{4k} of the $U(n)$-bundle $(\rho(O(n), U(n))\circ f)^*E_{U(n)}$

induces by $\rho(O(n), U(n))\circ f$ is equal to $(2k-1)!a_k s^{4k}$, where $\rho(O(n), U(n))$ denotes the canonical map $B_{O(n)}\to B_{U(n)}$ induced by $\rho: O(n)\to U(n)$. Therefore, by the definition of Pontrajagin classes, we obtain

$$P^{4k}(f^*E_{O(n)})=(-1)^k C^{4k}((\rho(O(n), U(n))\circ f)^*E_{U(n)})$$
$$=(-1)^{k+1}(2k-1)!a_k s^{4k}.$$

Thus the lemma is proved.

Lemma 1 and 2 are easily proved by the same way as the proof of Lemma $3(\alpha)$.

REMARK. This Lemma gives another proof of Theorem 26. 5 in Borel-Hirzebruch [2] (the case of $Sp(n)$-bundles we can easily prove by this method), and Theorem 5. 1 in Peterson [10].

§4. A classification theorem

In this section we shall use the terminologies and notaions in Wu [15].

We shall consider the classifying space $B_{O(n)}$ as the Grassmann manifold $R_{m,n}=O(m+n)/O(m)\times O(n)$, where m is sufficiently large. We shall consider two celluar subdivisions $K_{(x)}$ and $K_{(x^*)}$ of $R_{m,n}$ which are dual to each other (cf. Wu [15], Chapitre I, §4).

Theorem 5. *Let K be a complex of dimension$\leqq n-1$, and ξ_0, ξ_1 be $O(n)$-bundles over K. Assume that*

i) *the torsion coefficients of $H^{4k}(K, Z)$, $k=1, 2, \cdots$,*

are 0 *or relatively prime to $(2k-1)!a_k$, and*

ii) $H^{8j+1}(K, Z_2)=H^{8j+2}(K, Z_2)=0,\quad j=1, 2, \cdots.$

Then ξ_0 and ξ_1 are equivalent if and only if

$$(1)\qquad \begin{cases} W^1(\xi_0)=W^1(\xi_1), \\ W^2(\xi_0)=W^2(\xi_1), \\ P^{4k}(\xi_0)=P^{4k}(\xi_1), \qquad k=1, 2, \cdots. \end{cases}$$

PROOF. Assume that ξ_0, ξ_1 satisfy the relations (1). We know that $O(n)$-bundles ξ_i over K are induced by mappings f_i of K into $B_{O(n)}$ $(i=1, 2)$. Let K^i be the i-dimensional skeleton of K and I be the unit interval. It is sufficient to construct a mapping F of $K\times I$ into $B_{O(n)}$ such that

$$(2) \qquad F(x, 0)=f_0(x), \quad F(x, 1)=f_1(x).$$

We shall construct such a mapping F skeletonwise. Since $R_{m,n}$ is arcwise connected, we can define a mapping

$$F_0:(K\times\partial I)\cup(K^0\times I)\rightarrow R_{m,n},$$

satisfying (2). By the relation $W^1(\xi_0)=W^1(\xi_1)$, there exists a 0-cochain D^0 of K such that[1)]

$$f_1^*\{\omega_1^n\}_2-f_0^*\{\omega_1^n\}_2=\delta D^0.$$

We shall replace F_0 by a mapping $F_0':(K\times\partial I)\cup(K^0\times I)\rightarrow R_{m,n}$ such that

(α) $F_0'|(K\times\partial I)=F_0$,

(β) for a 0-cell $\sigma^0\in K$, $F_0'(\sigma^0\times I)$ and $F_0(\sigma^0\times I)$

from a sphere homotopic to[2)] $\{D^0(\sigma^0)-I_2([\omega_1^{n*}]_2, F_0(\sigma_0\times I))\}\cdot S_2^1$ where I_2 denotes the intersection number mod 2 in $R_{m,n}$ and S_2^2 the spherical cycle mod 2 representing a generator of $\pi_1(R_{m,n})\cong Z_2$. By Lemma 1 we can deduce from this

$$I_2([\omega_1^{n*}]_2, F_0'(\sigma^0\times I))=D^0(\sigma^0).$$

Therefore, for a 1-cell $\sigma^1\in K$,

$$\begin{aligned} I_2([\omega_1^{n*}]_2, F_0'(\partial\sigma^1\times I)) &= D^0(\partial\sigma^1)=(\delta D^0)(\sigma^1) \\ &= f_1^*\{\omega_1^n\}_2(\sigma^1)-f_0^*\{\omega_1^n\}_2(\sigma^1). \end{aligned}$$

Consequently we have

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1) $\{\omega_1^n\}_2$ denotes a cocycle of $W^1(E_{O(n)})\in H^1(K_{(\infty)}, Z_2)$. For the precise definition, see Wu [15], Chapitre I.

2) $[\omega_1^{n*}]_2$ denotes a cycle mod 2 in $K_{(\infty^*)}$ which is dual to $\{\omega_1^n\}_2$. For the precise definition, see Wu [15], Chapitre I.

$$I_2([\omega_1^{n*}]_2, F_0'(\partial(\sigma^1\times I)))=0.$$

By Lemma 1 it follows that

$$F_0'(\partial(\sigma^1\times I))\simeq 0, \quad \text{for any 1-cell } \sigma^1\in K.$$

Therefore, we can extend F_0' over $K^1\times I$. We shall denote it by

$$F_1: (K\times\partial I)\cup(K^1\times I)\rightarrow R_{m,n}.$$

Using Lemma 2 and the relation $W^2(\xi_0)=W^2(\xi_1)$, we can extend F_1 over $K^2\times I$ by the same way as above:

$$F_2: (K\times\partial I)\cup(K^2\times I)\rightarrow R_{m,n}.$$

Moreover, it can be extended over $K^3\times I$, because $\pi_3(R_{m,n})=0$. Thus we have a mapping $F_3: (K\times\partial I)\cup(K^3\times I)\rightarrow R_{m,n}$, satisfying (2). By $P^4(\xi_0)=P^4(\xi_1)$, there exists an integral 3-cochain A^3 in K such that[3)]

$$f_1^*\{\omega_{2,2}^n\}_0-f_0\{\omega_{2,2}^n\}_0=\delta A^3.$$

Suppose that for a 4-cell $\sigma^4\in K$ the sphere

$$F_3(\partial(\sigma^4\times I))\simeq B^4(\sigma^4)S_0^4,$$

where S_0^4 is the spherical cycle representing a generator of $\pi_4(R_{m,n})\cong Z$. Then we can consider B^4 as an integral 4-cochain of K. Let us define another integral cochain C^3 by

$$C^3(\sigma^3)=I_0([\omega_{2,2}^{n*}]_0, F_3(\sigma^3\times I)).$$

Then for any 4-cell $\sigma^4\in K$, by Lemma 3, (β) we have

$$I_0([\omega_{2,2}^{n*}]_0, F_3(\partial(\sigma^4\times I)))=2B^4(\sigma^4).$$

On the other hand

$$\begin{aligned}&I_0([\omega_{2,2}^{n*}]_0, F_3(\partial(\sigma^4\times I)))\\ &=I_0([\omega_{2,2}^{n*}]_0, F_3(\partial\sigma^4\times I))+I_0([\omega_{2,2}^{n*}]_0, F_3(\sigma^4\times\partial I))\\ &=(\delta C^3)(\sigma^4)+(\delta A^3)(\sigma^4).\end{aligned}$$

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3) $\{\omega_{2,2}^n\}_0$ denotes a cocycle in $P^4(E_{O(n)})\in H^4(K_{(x)}, Z)$ and $[\omega_{2,2}^{n*}]_0$ is the cycle in $K_{(x^*)}$ which is dual to $\{\omega_{2,2}^n\}_0$.

Therefore, we have

$$2B^4=\delta(C^3+A^3).$$

By the assumption we have that B^4 is cohomologous to 0. By the classical obstruction theory (Eilenbeg [5]) we can replace the mapping F_3 by another mapping

$$F_3' : (K\times\partial I)\cup(K^3\times I)\rightarrow R_{m,n}\,,$$

such that

i) $F_3'|(K\times\partial I)\cup(K^2\times I)=F_3$,

ii) $F_3'(\partial(\sigma^4\times I))\simeq 0$, for any 4-cell $\sigma^4\in K$.

Consequently this mapping F_3' can be extended over $K^4\times I$, and we denote an extended mapping by F_4. Since $\pi_5(R_{m,n})=\pi_6(R_{m,n})=\pi_7(R_{m,n})=0$, we can extend F_4 over $(K\times\partial I)\cup(K^7\times I)$. We shall denote an extended mapping by F_7. By the same method as in dimension 3, we can extend F_7 to $F_8 : (K\times\partial I)\cup(K_8\times I)\rightarrow R_{m,n}$, using Lemma 3, (β).

We know that $\pi_9(R_{m,n})\cong\pi_{10}(R_{m,n})\cong Z_2$ (for $n>10$), and we assume that $H^9(K, Z_2)=H^{10}(K, Z_2)=0$. Therefore, we can find a mapping $F_{10} : (K\times\partial I)\cup(K^{10}\times I)\rightarrow R_{m,n}$, satisfying (2). In virture of the assumption, the periodicity of $\pi_i(B_{O(n)})$ and Lemma 3, (β), we can easily obtain a mapping $F : K\times I\rightarrow R_{m,n}$, satisfying (2) by repeating this method.

REMARK. By this way we can also prove Peterson's Theorem ([10]), using Lemma 3, (α) (cf. Adachi [1]).

§5. Proof of Theorem 1 and 2

Now we shall prove Theorem 1 and 2.

Recall that 1-plane bundles over a complex X are in $1-1$ correspondence with $H^1(X, Z_2)$. For each class $u\in H^1(X, Z_2)$ let $\eta(u)$ denote the 1-plane bundle such that $W^1(\eta(u))=u$. Similarly oriented 2-plane bundles over X are in $1-1$ correspondence with $H^2(X, Z)$.

For each $v \in H^2(X, Z)$, let $\xi(v)$ denote the oriented 2-plane bundle with Euler class $X^2(\xi(v)) = v$.

For a bundle ξ we let (ξ) denote the stable equivalence class determined by ξ.

Now let M and N be manifolds and $f: M \to N$ a continuous map of codimension one or two. We consider separately these two cases.

Case 1: *Codimension* $f=1$. By the work of Hirsch [6] and Phillips [11], E. Thomas [13], [14] gives the following:

Theorem 6. (a) *Suppose that* $\dim M = \dim N - 1$. *Then* f *is homotopic to an immersion if and only if there is a class* $u \in H^1(M, Z_2)$ *such that* $(\tau_M \oplus \eta(u)) = f^*(\tau_N)$.

(b) *Suppose that* $\dim M = \dim N + 1$ *and that* M *is open. Then* f *is homotopic to a submersion if and only if there is a class* $u \in H^1(M, Z_2)$ *such that* $(\tau_M) = (f^*\tau_N \oplus \eta(u))$.

Case 2: *Codimension* $f=2$, f orientable.

Theorem 7. (a) *Suppose that* $\dim M = \dim N - 2$ *and that* $f: M \to N$ *is an orientable map. Then* f *is homotopic to an immersion if and only if there is a class* $v \in H^2(M, Z)$ *such that* $(\tau_M \oplus \xi(v)) = f^*(\tau_N)$.

(b) *Suppose that* $\dim M = \dim N + 2$, *that* M *is open and that* $f: M \to N$ *is orientable. Then* f *is homotopic to a submersion if and only if there is a class* $v \in H^2(M, Z)$ *such that* $(\tau_M) = (f^*\tau_N \oplus \xi(v))$.

Again E. Thomas [13], [14] shows that the result follows from Hirsch [6] and Phillips [11].

If a manifold M satisfies Condition (#) in §1, it also satisfies the hypotheses of Theorem 5. Consequently, Theorem 1 and 2 now follow by computing the characteristic classes of the bundles in Theorem 6 and 7 and then applying Theorem 5. Here we need the fact that for $v \in H^2(X, Z)$, $P^4(\xi(v)) = v^2$, and that by the assumption

$H^{4k}(M, Z)$ has no 2-torsion for any $k \geqq 1$. We leave the details to the reader.

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