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Holomorphic functions and open harmonic mappings

By

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Introduction. It is well-known that a non-constant holomorphic function is an open mapping.[1] In this paper we consider the converse under the assumption that the real and imaginary parts of a complex-valued function are harmonic functions. Our main purpose is to show the following

Theorem. *Let R be a Riemann surface and let u and v be real-valued harmonic functions on R.*

[I] *Assume that $R \in 0_{AB}$.*[2] *Suppose u is not constant. Then $u+iv$ is an open mapping on R into the complex plane, if and only if u has a single-valued conjugate function u^* on R and $v=\alpha u+\beta u^*+\gamma$, where α, β and γ are certain real numbers and $\beta \neq 0$.*

[II] *Assume that $R \notin 0_{AB}$. Then there exist u and v such that $u+iv$ is an open mapping on R, the conjugate function u^* of u is single-valued on R and $v \neq \alpha u+\beta u^*+\gamma$ for any real numbers α, β and γ.*

1. Let f be a complex-valued function defined on the disk $D=\{z;\ |z|<1\}$. We say that f is *open at the point* z in D, if, for

1) From the viewpoint of openness of a mapping, for exemple, G. T. Whyburn [4] shows theorems about the theory of functions of one complex variable.

2) $R \in 0_{AB}$ means that R is a Riemann surface on which every bounded analytic function reduces to a constant (see, for example, [2], p. 200).

any open set V containing z, the image $f(V)$ contains an open set (with respect to the plane topology) which contains $f(z)$. f is said to be *open* on a subset S of D, if it is open at each point of S. Under these terminologies, we have

Lemma 1. *Suppose that f is continuous on D and is open on a punctured disk $D-\{0\}$. Then f is open at* 0.

Proof. It is sufficient to show that $f(0)$ is contained in the interior of the image $f(|z|<r)$ for any r such that $0<r<1$. We may suppose $f(0)\notin f(|z|=r/2)$. Put $\rho=\operatorname{Min}_{|z|=r/2}|f(z)-f(0)|$ (>0) and $U=\{w;\ 0<|w-f(0)|<\rho\}$. Since the boundary of $f(|z|\leqq r/2)$ is a subset of $f(0)\cup f(|z|=r/2)$, it is disjoint with U. On the other hand, U contains an interior point of $f(|z|\leqq r/2)$. It follows that U is contained in the interior of $f(|z|\leqq r/2)$. Q.E.D.

The following lemma will be frequently used in what follows:

Lemma 2. *Let u and v be harmonic functions on D. Write*

$$f=u+iv,\quad J_f=\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\[2mm] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}\end{vmatrix}$$

and

$$w(z)=\left(\frac{dv+idv^*}{dz}\bigg/\frac{du+idu^*}{dz}\right)(z)$$

where u^* and v^* are harmonic conjugate functions of u and v on D.

Suppose that u is not constant on D. Then the following conditions are equivalent:

(a) *f is open on D,*

(b) *the set $\{z\in D;\ J_f(z)=0\}$ consists of isolated points,*

(c) *$w(z)$ is holomorphic on D and the set $\{z\in D;\ \operatorname{Im} w(z)=0\}$ is empty.*

Proof. For convenience' sake we put $\varphi=u+iu^*$ and $\psi=v+iv^*$. Since $u\not\equiv$const., $w(z)=\dfrac{\psi'(z)}{\varphi'(z)}$ is a meromorphic function on D.

Observing that

$$f=\frac{1}{2}(\varphi+i\psi+\overline{\varphi-i\psi})$$

and $$J_f=|f_z|^2-|f_{\bar z}|^2=\frac{1}{4}(|\varphi'+i\psi'|^2-|\varphi'-i\psi'|^2),$$

we have

$$\{z\in D;\ J_f(z)=0\}$$
$$=\{z\in D;\ \varphi'(z)=0\}\cup\{z\in D;\ \operatorname{Im}w(z)=0 \text{ or } w(z)=\infty\}.$$

We simply denote by E the second part of the right hand side. Since the set $\{z\in D;\ \varphi'(z)=0\}$ consists of isolated points and each connected component of E is clearly a continuum (cf. the footnote on page 387), we see that (b) is equivalent to (c). Also Lemma 1 implies that (b) induces (a). It thus remains to prove that (a)→(c), namely, if w is holomorphic on D and E is not empty, then f is not open on D. Since E consists of continuums, we can find a point z_0 in E such that there exists a small disk with center at z_0 on which φ is one to one. By change of variables:

$$z\longrightarrow\zeta=\varphi(z)-\varphi(z_0)$$

and by $J_f(z)=J_f(\zeta)\cdot\left|\frac{d\zeta}{dz}\right|^2$, we can reduce our assertion as follows:

Let $f(z)=x+iv(z)$, where $z=x+iy$ and let $v(z)$ be a (real-valued) harmonic function on D. If $J_f(0)=0$, then f is not open on D.

To prove this, we may suppose that $v(0)=0$ and that v does not depend on only x, i.e., $\frac{\partial v}{\partial y}\not\equiv 0$ on D. Note that

$$J_f(z)=\frac{\partial v}{\partial y}(z)$$

and denote by C the connected component containing 0 of the set $\{z;\ J_f(z)=0\}$. Since $\frac{\partial v}{\partial y}$ is a nonconstant harmonic function, C is an analytic curve[3)] which does not reduce to a point.

First suppose C contains $Y=\{iy;\ -1<y<1\}$. Then $v\equiv 0$ on Y. Since $v\not\equiv$const. on D, we find a point iy in Y at which $\dfrac{\partial v}{\partial x}\neq 0$. If $\dfrac{\partial v}{\partial x}(iy)>0$, then there exists a small square with center at iy whose image by f is contained the first and the third quadrants. Hence f is not open at iy.

Next suppose C does not contain Y and $\dfrac{\partial^2 v}{\partial y^2}\equiv 0$ on C. Since

$$\psi'=\frac{\partial v^*}{\partial y}-i\frac{\partial v}{\partial y} \quad \text{and} \quad \psi''=-\frac{\partial^2 v}{\partial y^2}-i\frac{\partial^2 v^*}{\partial y^2}$$

we have

$$\operatorname{Im}\psi'=\operatorname{Re}\psi''\equiv 0 \text{ on } C.$$

We find points z on C at which the slope $(=\tan\theta)$ of the tangent of C is not equal to ∞. Since

$$\psi''(z)=\lim_{\substack{h\to 0\\ z+h\in C}}\frac{\psi'(z+h)-\psi'(z)}{h}=\lim_{|h|\to 0}\frac{\psi'(z+h)-\psi'(z)}{|h|e^{i\theta}}\cdot\frac{|h|e^{i\theta}}{h}$$

we see from $\lim_{h\to 0}\dfrac{|h|e^{i\theta}}{h}=1$ that a pure imaginary number $\psi''(z)$ is equal to $\alpha\cdot(1/e^{i\theta})$, ($\alpha$: a real number). Because of $\tan\theta\not\equiv\infty$, we have thus $\psi''(z)=0$. It follows that $\psi''\equiv 0$ on D. Hence $v=\alpha x+\beta$ where α and β are certain real numbers. This contradicts the fact that $\dfrac{\partial v}{\partial y}\not\equiv 0$ on D.

Finally suppose that C does not contain Y and $\dfrac{\partial^2 v}{\partial y^2}\not\equiv 0$ on C. We find a point $z_0=x_0+iy_0$ on C at which $\dfrac{\partial^2 v}{\partial y^2}\neq 0$ and the slope of the tangent of C is not equal to ∞. It is proved that f is not open at z_0. For, if $\dfrac{\partial^2 v}{\partial y^2}(z_0)>0$, then $\dfrac{\partial v}{\partial y}(z_0)=0$ implies that there exists a $\delta>0$ such that

$$v(x_0,\ y)\geqq v(x_0,\ y_0)$$

for any y which satisfies $y_0-\delta<y<y_0+\delta$. It follows that f maps a neighborhood of z_0 into the upper part with respect to the following curve:

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3) C may have branch points.

$\{f(z);\ z\in C$ and $|z-z_0|<\varepsilon$, where ε is a small positive number$\}$.

Q.E.D.

2. We shall now prove the theorem stated in Introduction:

Proof of [I]. If u^* is single-valued and $v=\alpha u+\beta u^*+\gamma$ $(\beta\neq 0)$ on R, then we see that, on each parametric disk: $\{z;\ |z|<1\}$, we have

$$\left(\frac{d(v+iv^*)}{dz}\Big/\frac{d(u+iu^*)}{dz}\right)(z)=\alpha-i\beta.$$

It follows from Lemma 2 that f is open on R. Let us prove the converse under the assumption that $R\in 0_{AB}$. Suppose that $f=u+iv$ is open on R. Consider the following holomorphic differentials on R:

$$\omega=du+i(du^*) \quad\text{and}\quad \sigma=dv+i(dv)^*.$$

Then the quotient σ/ω is a meromorphic function on R, which we denote by w. This notation is compatible with that in the proof of Lemma 2. For, on each parametric disk: $\{z;\ |z|<1\}$, we have

$$\frac{\sigma}{\omega}(z)=w(z)=\left(\frac{d(v+iv^*)}{dz}\Big/\frac{d(u+iu^*)}{dz}\right)(z).$$

On account of Lemma 2, we see that w is holomorphic on R and $\operatorname{Im} w\neq 0$ at each point in R. It follows that $\operatorname{Im} w>0$ on R or $\operatorname{Im} w<0$ on R. Since $R\in 0_{AB}$, the function w must be a constant c such that $\operatorname{Im} c\neq 0$. Hence

$$v=(\operatorname{Re} c)u-(\operatorname{Im} c)u^*+\gamma$$

where γ is a real number.

Proof of [II]. Since $R\notin 0_{AB}$, there exists a nonconstant holomorphic function w on R such that $\operatorname{Im} w>0$ on R. We can choose a single-valued branch of $\log w$. If we set

$$u=\operatorname{Re}(\log w) \quad\text{and}\quad v=\operatorname{Re} w,$$

then we have, on each parametric disk,

$$\frac{d(v+iv^*)}{dz}\Big/\frac{d(u+iu^*)}{dz}=dw/d(\log w)=w.$$

Since $\operatorname{Im} w>0$ on R and w is nonconstant, we see from Lemma 2 that $f=u+iv$ is open and $v\not\equiv\alpha u+\beta u^{*}+\gamma$ for any real numbers α, β and γ.

3. By making use of the theorem and Lemma 2 we find some results:

Corollary 1. *Assume that $R\in 0_{AB}$. Let P be a point in R. Let u and v be harmonic functions on $R-\{P\}$ and have Laurent developments at P as follows:*

$$u(z)=\operatorname{Re}\sum_{n=-\infty}^{\infty}a_{n}z^{n} \quad \textit{and} \quad v(z)=\operatorname{Im}\sum_{n=-\infty}^{\infty}b_{n}z^{n}.$$

If $f=u+iv$ is open on $R-\{P\}$ and $a_{n}=b_{n}\neq 0$ for some $n\neq 0$, then f is holomorphic on $R-\{P\}$.

Proof. Since $R-\{P\}\in 0_{AB}$, Theorem [I] implies that $v=\alpha u+\beta u^{*}+\gamma$. Hence we have, in a neighborhood of P,

$$-i\sum_{n=-\infty}^{\infty}a_{n}z^{n}=\alpha\sum_{n=-\infty}^{\infty}a_{n}z^{n}-i\beta\sum_{n=-\infty}^{\infty}a_{n}z^{n}+c$$

where c is a complex number. We have thus $-ib_{n}=(\alpha-i\beta)a_{n}$ for all $n\neq 0$. Our assumption implies $\alpha=0$ and $\beta=1$. Consequently, $f=u+iu^{*}+i\gamma$.

Corollary 2. *Assume that u and v are harmonic functions on a punctured disk: $D-\{0\}=\{z;\ 0<|z|<1\}$ which have essential singularities at 0. Let they have Laurent developments as in Corollary 1. If $f=u+iv$ is open on $D-\{0\}$ and $a_{-n}=b_{-n}$ for sufficiently large n, then f is holomorphic on $D-\{0\}$.*

Proof. Let $a_{-n}=b_{-n}$ for all $n\geqq n_{0}$ and set

$$w(z)=\frac{dv+i(dv)^{*}}{du+i(du)^{*}}=\frac{-i\sum_{n=-\infty}^{\infty}nb_{n}z^{n-1}}{\sum_{n=-\infty}^{\infty}na_{n}z^{n-1}}.$$

Since f is open on $D-\{0\}$, Lemma 2 implies that $\operatorname{Im} w(z)>0$ on $D-\{0\}$ or <0 on $D-\{0\}$. Hence 0 is a removable singularity of $w(z)$. On the other hand, we have

$$w(z)=\frac{i\sum_{n=n_0}^{\infty}\frac{na_{-n}}{z^{n+1}}-i\sum_{n=-n_0+1}^{\infty}nb_nz^{n-1}}{-\sum_{n=n_0}^{\infty}\frac{na_{-n}}{z^{n+1}}+\sum_{n=-n_0+1}^{\infty}na_nz^{n-1}}=-i+\frac{w_1(z)}{\sum_{n=-\infty}^{\infty}na_nz^{n-1}}$$

where $w_1(z)$ has at most a pole at 0. If we assume that $w_1(z)\not\equiv 0$ on D, then 0 must be an essential singularity of $w(z)$. This is a contradiction. Hence $w_1(z)\equiv 0$, namely, $w(z)\equiv -i$. We have thus $v=u^*+\gamma$ on $D-\{0\}$, where γ is a real number. Q.E.D.

Corollary 3. [I] *Assume that $R\in 0_{AB}$. Suppose that u is a harmonic function on R whose conjugate is not single-valued. Then there is no harmonic function v such that $f=u+iv$ is open on R.*

[II] *If $R\notin 0_{AB}$, we can find a harmonic fnunction u on R which satisfies the following two conditions*:

(a) *the conjugate of u has arbitrarily given periods,*

(b) *there exists a harmonic function v on R such that $u+iv$ in open on R.*[4)]

Proof of [II]. Consider a non-constant holomorphic function w on R such that $\operatorname{Im} w(z)\neq 0$ at each point z in R. Write simply $W(z)=\frac{1}{w(z)}$, which is also holomorphic on R. It is well-known

4) For arbitrary harmonic function u we cannot always find v such that $f=u+iv$ is open. For instance, suppose R is the punctured disk: $\{z;\ 0<|z|<1\}$ and put $u(z)=\log|z|$. Since any harmonic function v on R is of the form:

$$\operatorname{Re}\Big(\sum_{n=-\infty}^{\infty}a_nz^n\Big)+c\log|z|$$

where c is a real number, we have

$$w(z)=\frac{dv+i(dv)^*}{du+i(du)^*}=d\Big(\sum_{n=-\infty}^{\infty}a_nz^n+c\log z\Big)/d(\log z)$$
$$=-\sum_{n=1}^{\infty}\frac{na_{-n}}{z^n}+c+\sum_{n=1}^{\infty}na_nz^n.$$

Then the set $\{z\in R;\ \operatorname{Im}w(z)=0\}$ is not empty. In fact, if 0 is an essential singularity of w, then by the Picard's theorem we find z in R such that $\operatorname{Im}w(z)=0$. Next, if 0 is a pole of w, the image $w(|z|<1)$ contains a neighborhood of ∞ (with respect to the Riemann sphere). Consequently, $\{z\in R;\ \operatorname{Im}w(z)=0\}$ is not empty. Finally, if 0 is a regular point of w, then, observing that c is a real number, we analogously find z in R which satisfies $\operatorname{Im}w(z)=0$. Hence $u+iv$ is not open on R.

that there exists a harmonic function p on R whose conjugate has arbitrarily given periods. Put $\tau=dp+i(dp)^*$ and denote by $\{P_n\}$ and $m(n)$ the set of 0-points of holomorphic differential dW and its order at P_n respectively. By Mittag-Lefflerscher *Anschmiegungssatz* ([3], p. 257) for open Riemann surfaces, there exists a holomorphic function g on R such that the order of zero of the holomorphic differential $dg-\tau$ at P_n is at least $m(n)$. Therefore the quotient

$$\frac{dg-\tau}{dW}$$

is a holomorphic function on R, which we denote by ψ. Since the equality

$$Wd\psi=d(W\psi)-\psi dW=d(W\psi-g)+\tau$$

holds, the holomorphic differential $Wd\psi$ has the periods of τ. If we put

$$u(P)=\int^P \mathrm{Re}(Wd\psi) \text{ and } v=\mathrm{Re}\,\psi$$

then the conjugate of u has the given periods and $f=u+iv$ is open on R. In fact, on each parametric disk, we have

$$\mathrm{Im}\frac{dv+i(dv)^*}{du+i(du)^*}=\mathrm{Im}\frac{d\psi}{Wd\psi}=\mathrm{Im}\,w\neq 0.$$

Consequently, u is one of the desired functions. Q.E.D.

Let E be a compact set in the complex plane. It is well-known that, *if E is linear measure zero, then E is AB-removable* (see [1], p. 121). Using this fact, we shall prove

Corollary 4. *If E is linear measure zero, then E is OB-removable. Namely, let G be a connected open set which contains E and suppose that $f=u+iv$ is a bounded open harmonic mapping on $G-E$. Then it is possible to find an extension of f which is bounded and open harmonic on all of G.*

Proof. Since $f=u+iv$ is open on $G-E$, Lemma 2 implies that, if we put $w(z)=\frac{dv+i(dv)^*}{du+i(du)^*}$, then $w(z)$ is a holomorphic function on $G-E$ and $\operatorname{Im} w(z)>0$ on $G-E$ or <0 on $G-E$. We may suppose $\operatorname{Im} w(z)>0$ on $G-E$. Using the fact that E is removable for all AB-functions, we can find an analytic function $\hat{w}(z)$ on G which is equal to $w(z)$ on $G-E$. By maximum principle we have $\operatorname{Im}\hat{w}(z)>0$ on G. For simplicity we write $\hat{w}(z)=\operatorname{Re}\hat{w}(z)+i\operatorname{Im}\hat{w}(z)=p(z)+iq(z)$ on G. We have on $G-E$,

$$dv+i(dv)^*=(p+iq)(du+i(du)^*)$$

and hence

$$dv=p(du)-q(du)^*.$$

By virtue of $q\neq 0$ at each point in G, we can write

$$(du)^*=\frac{p}{q}(du)-\frac{1}{q}(dv).$$

Observing that

$$\frac{p}{q}(du)=d\left(\frac{p}{q}u\right)-ud\left(\frac{p}{q}\right) \quad \text{and} \quad \frac{1}{q}(dv)=d\left(\frac{1}{q}v\right)-vd\left(\frac{1}{q}\right),$$

we obtain, on $G-E$,

$$(du)^*=d\left(\frac{pu-v}{q}\right)-ud\left(\frac{p}{q}\right)+vd\left(\frac{1}{q}\right).$$

Now, let S be any subregion of G which is bounded by a Jordan curve in $G-E$. Let β be an arbitrary simple closed analytic curve in $S-E$ and denote by S_β the subregion of S which is bounded by β. For a given $\varepsilon>0$ à priori, let $\{\beta_\nu\}$ be the peripheries, of total length $<\varepsilon$, of circles in S_β that enclose the subset of E contained in S_β. Since β is homologous to a cycle $\sum_\nu \beta'_\nu$, where β'_ν is a certain subarc of β_ν, we have

$$\begin{aligned}\int_\beta (du)^* &= \int_{\sum_\nu \beta'_\nu}(du)^* = \int_{\sum_\nu \beta'_\nu} d\left(\frac{pu-v}{q}\right)-ud\left(\frac{p}{q}\right)+vd\left(\frac{1}{q}\right)\\ &=\sum_\nu \int_{\beta'_\nu}(-u)d\left(\frac{p}{q}\right)+vd\left(\frac{1}{q}\right).\end{aligned}$$

On the other hand, by the assumption u and v are bounded on $G-E$, and the function p/q and $1/q$ are continuously differentiable on G.

We have thus, on any arc γ on $S-E$,

$$|(-u)d(p/q)+vd(1/q)|\leqq|u|\,|d(p/q)|+|v|\,|d(1/q)|\leqq M|dz|$$

where $|dz|$ is the line element of γ and

$$M=\sup_{z\in\bar{S}-E}\left\{\begin{array}{l}|u(z)|\sqrt{\left(\frac{\partial(p/q)}{\partial x}(z)\right)^2+\left(\frac{\partial(p/q)}{\partial y}(z)\right)^2}\\+|v(z)|\sqrt{\left(\frac{\partial(1/q)}{\partial x}(z)\right)^2+\left(\frac{\partial(1/q)}{\partial y}(z)\right)^2}\end{array}\right\}(<\infty).$$

Consequently,

$$\left|\int_\beta(du)^*\right|\leqq M\cdot\left(\sum_\nu\int_{\beta'_\nu}|dz|\right)<M\varepsilon.$$

We let $\varepsilon\to 0$ and hence

$$\int_\beta(du)^*=0.$$

Moreover, since the region S is simply connected, it follows that u has a single-valued conjugate function u^* on $S-E$, that is, $u+iu^*$ is an analytic function on $S-E$. Observing that E is an AB-removable singularity and u is bounded, we can find an analytic function $u_S+iu_S^*$ on S which is equal to $u+iu^*$ on $S-E$.

Analogously, there exists an analytic function $v_S+iv_S^*$ on S which is equal to $v+iv^*$ on $S-E$. Obviously, we see that

$$\operatorname{Im}\frac{dv_S+idv_S^*}{du_S+idu_S^*}=\operatorname{Im}\hat{w}(z)>0 \text{ on } S.$$

Hence the mapping u_S+iv_S is open on S.

Since S is arbitrary Jordan subregion of G, if we set

$$\hat{u}=u_S \text{ and } \hat{v}=v_S \text{ on each } S,$$

then $\hat{u}$ and $\hat{v}$ cleary define harmonic functions on G. If we consider $\hat{f}=\hat{u}+i\hat{v}$ on G, then the mapping $\hat{f}$ is the desired extension of f.

Q.E.D.

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