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A theorem of Gutwirth

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The following fact was proved by Gutwirth[1] in the classical case: Let D be a line on $\boldsymbol{P}^2$ and consider the affine plane $S=\boldsymbol{P}^2-D$. Assume that C is an irreducible curve defined over a ground field K and of degree, say d, on $\boldsymbol{P}^2$ such that $C\cap S$ is biregular to an affine line. Then $C\cap D$ contains a unique ordinary point, say P. If we look at also infinitely near points, then all of singular points, say $P_1, \cdots, P_n$ are arranged so that (i) $P=P_1$ and (ii) each P_{i+1} is an infinitely near point of P_i of order 1. Let m_i be the effective multiplicity of P_i on C (that is, the multiplicity of P_i on the proper transform of C by successive quadratic dilatations with centers $P_1, \cdots, P_{i-1}$). On the other hand, let $f(x, y)$ be the irreducible polynomial which defines $C\cap S$ in the affine coordinate ring $K[x, y]$ of S. Then

Theorem. *Consider the linear system L of curves of degree d on $\boldsymbol{P}^2$ which goes through $\sum m_i P_i$. If* $\dim L\geq 1$, *then d is a multiple of $d-m_1$.*

This fact implies also, under the same assumption, that there is a polynomial $g(x, y)$ such that $K[x, y]=K[f, g]$.

The purpose of the present paper is to give a proof of the above theorem without any restriction on the ground field K. We add also

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1) A. Gutwirth, An inequality for certain pencils of plane curves, Proc. Amer. Math. Soc. Vol. **12** (1961) pp. 631-639

some remarks on positive characteristic case. In particular, we give an example which shows that the conclusion of the theorem become false under a slight modification of the assumption, in the positive characteristic case. Therefore we like to restate a well known open question in the classical case in the following form:

Conjecture. *If d is not a multiple of the characteristic p of K, then the assumption of the theorem holds good always, or equivalently, d-times of D belongs to L.*

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1. (d, r)-sequence

When two natural numbers d and r such that $d \geq r$ are given, sequence $r_1, \cdots, r_q$ defined as follows is called the (d, r)-sequence:

Start with $d_0 = d$ and $d_1 = r$. When $d_0, \cdots, d_j$ are defined and if $d_j > 0$, let q_j and d_{j+1} be such that $d_{j-1} = q_j d_j + d_{j+1}$ $(0 \leq d_{j+1} < d_j)$. Then for every k such that $(\sum_{i<j} q_i) + 1 \leq k \leq \sum_{i \leq j} q_i$, r_k is defined to be d_j.

Lemma 1.1. *Under the notation, we have*

$$q = \sum_{i=1}^{\alpha} q_i, \qquad d_\alpha = (d, r) \text{ and}$$

$$\sum r_i = d + r - d_\alpha, \qquad \sum r_i^2 = dr.$$

Proof. We have

$$d_0 = q_1 d_1 + d_2; \qquad d_0 d_1 = q_1 d_1^2 + d_1 d_2,$$

$$d_1 = q_2 d_2 + d_3; \qquad d_1 d_2 = q_2 d_2^2 + d_2 d_3,$$

$$\cdots\cdots \qquad \cdots\cdots$$

$$\cdots\cdots \qquad \cdots\cdots$$

$$d_{\alpha-2}=q_{\alpha-1}d_{\alpha-1}+d_{\alpha}; \qquad d_{\alpha-2}d_{\alpha-1}=q_{\alpha-1}d_{\alpha-1}^2+d_{\alpha-1}d_{\alpha}$$

$$d_{\alpha-1}=q_{\alpha}d_{\alpha}; \qquad d_{\alpha-1}d_{\alpha}=q_{\alpha}d_{\alpha}^2.$$

Summing up these equalities respectively, we have $d_0+d_1 =\sum q_i d_i+d_{\alpha}$; $d_0 d_1=\sum q_i d_i^2$ and we have the required result.

Proposition 1.2. *Let C be an irreducible curve on a non-singular surface F and let P be a point of C such that P corresponds to only one point of the derived normal model of C.*[2] *Let r be the multiplicity of P on C. Let D be another irreducible curve on F which goes through P as a simple point. Let d be the intersection multiplicity of C and D at P, and let c be the* G.C.M. (d, r). *Let the (d, r)-sequence be $r_1, \cdots, r_q$. Then there is a sequence of points $P_1=P, P_2, \cdots, P_q$ which is determined uniquely by d/c, r/c and D such that* (i) *each P_{i+1} is an infinitely near point of P_i of order* 1 *and* (ii) *effective multiplicity of P_i on C is r_i. (The way of determination of P_i is shown by the proof below.)*

Proof. We use an induction argument on d. If $d=r$, then $q=1$, $r_1=r$ and the assertion is obvious. Assume that $d>r$. Consider the quadratic dilatation $\mathrm{dil}_P F$, the proper transforms C', D' of C, D and also the intersection number $(\mathrm{dil}_P P, C')$. Since P is an r-ple point of C, we have $(\mathrm{dil}_P P, C')=r$. Consider the unique common point P_2 of $\mathrm{dil}_P P$ and D'. By our assumption on P, P_2 is the unique common ordinary point of $\mathrm{dil}_P P$ and C'. On the other hand, since the intersection multiplicity at P of C and D is d and since P is r-ple on C, the intersection multiplicity at P_2 of C' and D' is $d-r$. Therefore the multiplicity of P_2 on C' is the minimum of r and $d-r$. Now, if $d-r \geq r$, then considering C' and D' instead of C and D respectively, we have a case with less d, and the proof is completed by our induction argument. On the other hand, if $r>d-r$, then considering $\mathrm{dil}_P P$ and C' instead of D and C respectively, we complete the proof similarly.

2) This is equivalent to that P is an analytically irreducible point of C.

2. The proof of the theorem

Consider C, d, m_i P_i etc. as in the theorem, without assuming that $\dim L \geq 1$. Let (d, m_1)-sequence be $m_1 = r_1, r_2, \cdots, r_q$.

(1) Assume that $(d, m_1) = 1$.[3] Then we see by virtue of Proposition 1.2 that $m_i = r_i$ for any $i \leq n$ and $r_{n+1} = r_{n+2} = \cdots = r_q = 1$. This means that $2(\text{genus of } C) = d^2 - 3d + 2 - \sum r_i^2 + \sum r_i = d(d - m_1 - 2) + m_1 + 1$ by Lemma 1.1. Therefore, by that C is rational, we have $d - m_1 - 2 < 0$, whence $m_1 \geq d - 1$, and we see that $m_1 = d - 1$, and therefore $1 = d - m_1$ divides d in this case.

(2) Assume now that $\delta = (d, m_1) \neq 1$ and that $d - m_1$ does not divide d. Then $n \geq q$ and $m_i = r_i$ for any $i \leq q$ and $m_j \leq \delta$ for any $j > q$. On the other hand,

$$
\begin{aligned}
0 = 2(\text{genus of } C) &= d^2 - 3d + 2 - \sum m_i^2 + \sum m_i \\
&= d^2 - 3d + 2 - \sum_{i \leq q} m_i^2 + \sum_{i \leq q} m_i - \sum_{j > q} m_j^2 + \sum_{j > q} m_j \\
&= d(d - m_1) - 2d + m_1 + 2 - \delta - \sum_{j > q} m_j^2 + \sum_{j > q} m_j.
\end{aligned}
$$

Let $(d, d - m_1)$-sequence be $s_1, \cdots, s_q$. Then $\sum s_i^2 = d(d - m_1)$, $\sum s_i = d + (d - m_1) - \delta$. Therefore

$$
\sum_{j > q} m_j^2 - \sum_{j > q} m_j = \sum s_i^2 - \sum s_i + 2 - 2\delta. \tag{2.1}
$$

Since $d - m_1$ does not divide d, $d - m_1$ is a proper multiple of δ; $d - m_1 = u\delta$ $(u \geq 2)$. On the other hand, let β and γ be integers such

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3) Our computation shows the following fact: Assume that C' is a curve on a non-singular surface F and let P' be a point of C' such that (i) as a curve, C' has no singularity other than P' and (ii) P' is analytically irreducible (i.e., P' is a one-place singularity of C'). Let $r(>1)$ be the multiplicity of P on C'. Assume that there is a curve D' going through P' as a simple point such that the intersection multiplicity d of $C' \cdot D'$ at P' is prime to r. Then (arithmetic genus of C') $-$(genus of C')$=(dr - d - r - 1)/2$. Therefore d is uniquely determined by C' (if exists).

Geometric reason for this is the following. Under the notation of Proposition 1.2, both P_{q_1+1} and P_{q_1+2} lies on $\mathrm{dil}_{P_{q_1}} P_{q_1}$, and therefore no curve, having P as a simple point, goes through $P_1, \cdots, P_{q_1+2}$.

that $\sum_{j>q} m_j = \beta\delta + \gamma$, $0 \leq \gamma < \delta$. Set $\delta_1 = \cdots = \delta_\beta = \delta$, $\delta_{\beta+1} = \gamma$. Then $\sum \delta_i = \sum_{j>q} m_j$ and obviously

$$\sum \delta_i^2 - \sum \delta_i \geq \sum_{j>q} m_j^2 - \sum_{j>q} m_j.$$

Assume for a moment that $\sum_{j>q} m_j \leq \sum s_i + 2\delta$. Then

$$\begin{aligned}\sum s_i^2 - \sum s_i &\geq s_1^2 - s_1 + \sum_{i>u+2} \delta_i^2 - \sum_{i>u+2} \delta_i \\ &= u^2\delta^2 - u\delta + \sum_{i>u+2} \delta_i^2 - \sum_{i>u+2} \delta_i \\ &= (u^2 - u - 2)\delta^2 + 2\delta + \sum \delta_i^2 - \sum \delta_i \\ &\geq 2\delta + \sum_{j>q} m_j^2 - \sum_{j>q} m_j \\ &= 2\delta + \sum s_i^2 - \sum s_i + 2 - 2\delta \qquad \text{(by (2.1).)}\end{aligned}$$

This implies $2 \leq 0$, which is impossible. Therefore we must have

$$\sum_{j>q} m_j > \sum s_i + 2\delta.$$

Then, since $\sum_{i \leqslant q} m_i + \sum s_i = d + m_1 - \delta + d + (d - m_1) - \delta = 3d - 2\delta$ (by Lemma 1.1), we have

$$\sum m_i > 3d.$$

Since $0 = d^2 - 3d + 2 - \sum m_i^2 + \sum m_i$, we have $\sum m_i^2 = d^2 - 3d + 2 + \sum m_i > d^2 + 2$. This implies that two members of L have intersection number bigger than $d^2 + 2$ unless they have common components. Since L has an irreducible member C, we see that $\dim L = 0$.

By these (1) and (2), we completes the proof of the Theorem.

3. A remark

In the case where the characteristic of the ground field K is zero, the condition

(*) There is a linear system L^* of curves such that (i) C is a member of L^* (ii) a generic member of L^* is an irreducible rational curve and (iii) $\dim L^* \geq 1$ implies that $\dim L \geq 1$ for the linear system L in the theorem, because L^* has no variable singularities by a theorem of Bertini whence L^* is contained in L.

But, in the positive characteristic case, one can have an easy counter-example.

Indeed, letting $p(\neq 0)$ be the characteristic of K, consider curve C_b with a parameter t in the affine plane as follows:

$$\begin{cases} x = t^{p2} \\ y = t^{ap} + t + b \end{cases} \quad (a \text{ is a natural number prime to } p,\ b \in K).$$

Since $K[t^{p2}, t^{ap} + t + b] = K[t]$, this C_b satisfies the requirement on singularities. The equation for C_b is $y^{p2} = x^{ap} + x + b^{p2}$. Therefore C_b is a member of the linear system spanned by $C = C_0$ and d-times of the line at infinity, where $d = \deg C_0 = \max\ (p^2, ap)$. Therefore there is an L^* as in (*) but, if $a > 1$ $\dim L = 0$ by virtue of our theorem.

Note that the above example gives an example of a polynomial $f(x, y)$ in the polynomial ring $K[x, y]$ such that (i) $K[x, y]/fK[x, y] \cong K[t]$ but (ii) there is no g such that $K[x, y] = K[f, g]$.

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