J. Math. Kyoto Univ. (JMKYAZ)
11-1 (1971) 89-112

On automorphism groups of ruled surfaces

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(Received July 31, 1970)

Let k be an algebraically closed field of arbitrary characteristic and let X be a complete non-singular irreducible curve defined over k. A complete surface S defined over k is a ruled surface over X if and only if there is a k-morphism $\pi: S \rightarrow X$ such that $\pi^{-1}(x) = \boldsymbol{P}^1$ for all $x \in X$. We know that every ruled surface is locally trivial, that is, $\boldsymbol{P}^1$-bundle over X ([3] Corollary 0.2). On the other hand, any $\boldsymbol{P}^1$-bundle over X is the associated projective bundle $P(E)$ of a vector bundle E of rank 2 over X ([3] Introduction). Thus automorphism groups of ruled surfaces are closely related to those of vector bundles of rank 2 over X. The purpose of the present article is to study automorphism groups of ruled surfaces in this direction.

Notation and convention. All objects such as varieties, bundles etc. in the present article are restricted to those defined over k and therefore, under a point, we understand a k-rational point. I denotes the trivial line bundle over X. $\mathrm{Aut}(E)$ denotes the automorphism group of a vector bundle E of rank 2 over X. (S, X, π) denotes a ruled surface S over X with a canonical morphism $\pi: S \rightarrow X$. $\mathrm{Aut}(S)$ denotes the automorphism group of S and $\mathrm{Aut}_X(S)$ denotes the automorphism group of S over X, that is, $\mathrm{Aut}_X(S) = \{\sigma \in \mathrm{Aut}(S) \mid \pi \cdot \sigma = \pi\}$. $\boldsymbol{E}$, $\boldsymbol{L}$ denote sheaves of germs of regular sections of vector bundles E, L respectively. $\mathcal{O}_X$ denotes the structure sheaf of X and $\mathcal{O}_X^*$ denotes the sheaf of units of $\mathcal{O}_X$. Finally, G_m denotes the multiplicative group $k-\{0\}$ and G_a denotes the additive group k.

The author wishes to thank Professors M. Nagata, H. Matsumura and T. Oda for valuable conversations with them.

1. Let E be a vector bundle of rank 2 over X. We know that E has infinitely many sublinebundles (see [3]). Degrees of sublinebundles of E are bounded above ([3] Lemma 1.1). A sublinebundle of E which has the maximum degree is called a maximal subbundle of E and $M(E)$ denotes the degree. We shall begin with a key lemma which was proved in [3] (Lemma 1.5 of [3]).

Lemma 1. *If L_1, L_2 are distinct maximal subbundles of E and if $L_1\cong L_2$, then $E\cong L_1\oplus L_2$.*

Let L be a maximal subbundle of E. If $\{U_i\}_{1\leqq i\leqq n}$ is a sufficiently fine open covering of X, then we can choose a system of local coordinates $\begin{pmatrix} u_i \\ v_i \end{pmatrix}$ $(i=1, \cdots, n)$ of E such that transition matrices of E are $\left\{\begin{pmatrix} a_{ij} & c_{ij} \\ 0 & b_{ij} \end{pmatrix} \middle| 1\leqq i,\ j\leqq n\right\}$ and L is defined by $v_i=0$ for all i, where the transition matrices of L are $\{(a_{ij})\,|\,1\leqq i,\ j\leqq n\}$.

Suppose that $E\not\cong L\oplus L$. $\mathrm{Aut}(E)\ni\sigma$ induces an automorphism σ_i of the vector bundle $E|_{U_i}$ over U_i. By virtue of Lemma 1, σ must fix L and $\mathrm{Aut}(E|_{U_i})=GL(2, \Gamma(U_i, \mathcal{O}_X))$. Therefore $\sigma_i\in\mathrm{Aut}(E|_{U_i})$ is defined by $\left\{\begin{pmatrix} \alpha_i & \gamma_i \\ 0 & \beta_i \end{pmatrix} \middle| \begin{matrix} \alpha_i, \beta_i\in\Gamma(U_i, \mathcal{O}_X^*) \\ \gamma_i\in\Gamma(U_i, \mathcal{O}_X) \end{matrix}\right\}$, that is, $\sigma_i\begin{pmatrix} u_i \\ v_i \end{pmatrix}=\begin{pmatrix} \alpha_i(x) & \gamma_i(x) \\ 0 & \beta_i(x) \end{pmatrix}\begin{pmatrix} u_i \\ v_i \end{pmatrix}$ in the fibre over $x\in U_i$. There exists a $\sigma\in\mathrm{Aut}(E)$ such that $\sigma|_{U_i}=\sigma_i$ for every i if and only if the following conditions are satisfied;

$$(1)\quad \begin{pmatrix} a_{ij} & c_{ij} \\ 0 & b_{ij} \end{pmatrix}\begin{pmatrix} \alpha_j & \gamma_j \\ 0 & \beta_j \end{pmatrix}=\begin{pmatrix} \alpha_i & \gamma_i \\ 0 & \beta_i \end{pmatrix}\begin{pmatrix} a_{ij} & c_{ij} \\ 0 & b_{ij} \end{pmatrix} \quad \begin{matrix} \text{in } U_i\cap U_j \\ 1\leqq i,\ j\leqq n. \end{matrix}$$

These conditions are equivalent to the following;

(2) $$a_{ij}\alpha_j=\alpha_i a_{ij} \qquad \text{in } U_i\cap U_j$$

(3) $$b_{ij}\beta_j=\beta_i b_{ij} \qquad 1\leqq i,\ j\leqq n.$$

(4) $$a_{ij}\gamma_j+c_{ij}\beta_j=\alpha_i c_{ij}+\gamma_i b_{ij}$$

By the equations (2), (3) we have that $\alpha_1=\cdots=\alpha_n=\alpha$, $\beta_1=\cdots=\beta_n=\beta\in\Gamma(X, \mathcal{O}_X^*)=G_m$. Thus (4) can be rewritten as the following;

(4′) $$a_{ij}\gamma_j-\gamma_i b_{ij}=(\alpha-\beta)c_{ij}$$

Now, assume that $\alpha\neq\beta$ and put $\alpha-\beta=\delta$. Let us observe the coordinate transformation;

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix}=\begin{pmatrix} 1 & -\gamma_i/\delta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_i' \\ v_i' \end{pmatrix} \qquad 1\leqq i\leqq n.$$

Then, the transition matrices by the new coordinates $\begin{pmatrix} u_i' \\ v_i' \end{pmatrix}$ are $\left\{\begin{pmatrix} a_{ij} & 0 \\ 0 & b_{ij} \end{pmatrix}\middle| 1\leqq i,\ j\leqq n\right\}$. Thus we get that $E\cong L\oplus L'$ for some subbundle L' of E. Therefore, if E is indecomposable, then $\alpha=\beta$, whence (4′) reduces to the following;

(4″) $$\gamma_i=(b_{ij}^{-1}a_{ij})\gamma_j.$$

Since $\{(b_{ij})\}$ define the linebundle $(\det E)\otimes L^{-1}$, $\{\gamma_i\}$ is a regular section of $(\det E)^{-1}\otimes L^2$. Consequently, σ determines $\alpha\in G_m$ and a regular section $\{\gamma_i\}$ of $(\det E)^{-1}\otimes L^2$. Conversely, suppose that $\alpha\in G_m$ and a regular section $\{\gamma_i\}$ of $(\det E)^{-1}\otimes L^2$ are given. Then, $\left\{\begin{pmatrix} \alpha & \gamma_i \\ 0 & \alpha \end{pmatrix}\middle| 1\leqq i\leqq n\right\}$ define an automorphism of E. Hence we have

(a) If E is indecomposable, then

$$\operatorname{Aut}(E)=\left\{\begin{pmatrix} \alpha & s \\ 0 & \alpha \end{pmatrix}\middle| \begin{matrix} \alpha\in G_m \\ s\in\Gamma(X, (\det \boldsymbol{E})^{-1}\otimes \boldsymbol{L}^2) \end{matrix}\right\},$$

where L is a maximal subbundle of E.

Next, assume that $E \cong L \oplus L'$ $(L \not\cong L')$, that is, $c_{ij}=0$ in (1). In this case, (4′) reduces also to (4″), whence σ determines $\alpha, \beta \in G_m$ and $s \in \Gamma(X, (\det \boldsymbol{E})^{-1} \otimes \boldsymbol{L}^2)$. Conversely, any $\alpha, \beta \in G_m$ and a regular section $\{\gamma_i\}$ of $(\det E)^{-1} \otimes L^2$ define an automorphism of E; $\left\{\begin{pmatrix} \alpha & \gamma_i \\ 0 & \beta \end{pmatrix} \middle| 1 \leqq i \leqq n \right\}$. Thus we get

(b) If $E \cong L \oplus L'$ $(\deg L \geqq \deg L'$, that is, L is a maximal subbundle of E, see [3]) and if $L \not\cong L'$, then

$$\operatorname{Aut}(E) = \left\{ \begin{pmatrix} \alpha & s \\ 0 & \beta \end{pmatrix} \middle| \begin{array}{l} \alpha, \beta \in G_m \\ s \in \Gamma(X, (\det \boldsymbol{E})^{-1} \otimes \boldsymbol{L}^2) \end{array} \right\}.$$

Finally, assume that $E \cong L \oplus L$, then the transition matrices are $\left\{\begin{pmatrix} a_{ij} & 0 \\ 0 & a_{ij} \end{pmatrix} \middle| 1 \leqq i, j \leqq n \right\}$ and σ is defined by $\begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$. The conditions as (1) assert that $\alpha_1 = \cdots = \alpha_n = \alpha$, $\beta_1 = \cdots = \beta_n = \beta$, $\gamma_1 = \cdots = \gamma_n = \gamma$, $\delta_1 = \cdots = \delta_n = \delta$ and $\alpha, \beta, \gamma, \delta \in k$, $\alpha\delta - \beta\gamma \neq 0$. Thus we have

(c) If $E \cong L \oplus L$, then $\operatorname{Aut}(E) \cong GL(2, k)$.

We introduced in [3] the invariant $N(E) = \deg E - 2M(E)$ and proved the following lemma ([3] Corollary 1.6).

Lemma 2. (1) *If one of the following conditions* (i), (ii) *is satisfied, then E has only one maximal subbundle.*

(i) $N(E) < 0$.

(ii) $N(E) = 0$ *and E is indecomposable.*

(2) *If* $E \cong L_1 \oplus L_2$, $\deg L_1 = \deg L_2$ (i.e. $N(E) = 0$) *and if* $L_1 \not\cong L_2$, *then E has only two maximal subbundles and they are* L_1 *and* L_2.

In the case (a), (b) above, if $\Gamma(X, (\det \boldsymbol{E})^{-1} \otimes \boldsymbol{L}^2) \neq 0$, then $-N(E) = \deg((\det E)^{-1} \otimes L^2) \geqq 0$. Therefore, by virtue of Lemma 2, L is uniquely determined by E.

Now, let us define connected linear groups H_r, H'_r (r is a nonnegative integer) which are subgroups of $(r+1)$-ple product of $GL(2, k)$.

$$H_r=\left\{\left(\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}, \begin{pmatrix}\alpha & t_1\\ 0 & \alpha\end{pmatrix}, \cdots, \begin{pmatrix}\alpha & t_r\\ 0 & \alpha\end{pmatrix}\right)\in GL(2, k)\times\cdots\times GL(2, k)\ \middle|\ \begin{matrix}\alpha\in G_m\\ t_i\in k\end{matrix}\right\}$$

$$H'_r=\left\{\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}, \begin{pmatrix}\alpha & t_1\\ 0 & \beta\end{pmatrix}, \cdots, \begin{pmatrix}\alpha & t_r\\ 0 & \beta\end{pmatrix}\right)\in GL(2, k)\times\cdots\times GL(2, k)\ \middle|\ \begin{matrix}\alpha, \beta\in G_m\\ t_i\in k\end{matrix}\right\}.$$

Then, we have the following theorem.

Theorem 1. *Let E be a vector bundle of rank 2 over a complete non-singular irreducible curve X. Then,*

(1) *If $N(E)>0$, then* $\mathrm{Aut}(E)\cong G_m$

(2) *If $N(E)\leqq 0$, E is indecomposable, and if L is the unique maximal subbundle of E, then* $\mathrm{Aut}(E)\cong H_r$, *where* $r=\dim\Gamma(X, (\det\boldsymbol{E})^{-1}\otimes\boldsymbol{L}^2)$.

(3) *If $E\cong L_1\oplus L_2$, $\deg L_1\geqq\deg L_2$ and if $L_1\not\cong L_2$, then* $\mathrm{Aut}(E)\cong H'_r$, *where* $r=\dim\Gamma(X, (\det\boldsymbol{E})^{-1}\otimes\boldsymbol{L}_1^2)$.

(4) *If $E\cong L\oplus L$, then* $\mathrm{Aut}(E)\cong GL(2, k)$.

Proof. (1) Since $N(E)>0$ implies that E is indecomposable, this is a case of (a). Moreover, $\Gamma(X, (\det\boldsymbol{E})^{-1}\otimes\boldsymbol{L}^2)=0$ because $\deg((\det E)^{-1}\otimes L^2)=-N(E)<0$. Thus (a) shows our assertion.

(2) This is the other case of (a). Fix a basis $(s_1, \cdots, s_r)$ of $\Gamma(X, (\det\boldsymbol{E})^{-1}\otimes\boldsymbol{L}^2)$. Consider the map $g\colon \mathrm{Aut}(E)\to H_r$;

$$g\begin{pmatrix}\alpha & s\\ 0 & \alpha\end{pmatrix}=\left(\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}, \begin{pmatrix}\alpha & t_1\\ 0 & \alpha\end{pmatrix}, \cdots, \begin{pmatrix}\alpha & t_r\\ 0 & \alpha\end{pmatrix}\right),$$

where $s=t_1s_1+\cdots+t_rs_r$. It is obvious that g is a group isomorphism. If $\{\gamma_j^{(i)}\}$ is a representative of s_i by $\gamma_j^{(i)}\in\Gamma(U_j, (\det\boldsymbol{E})^{-1}\otimes\boldsymbol{L}^2)$ for some sufficiently fine open covering $\{U_j\}$ of X. Then, since the action of $\sigma=\left(\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}, \begin{pmatrix}\alpha & t_1\\ 0 & \alpha\end{pmatrix}, \cdots, \begin{pmatrix}\alpha & t_r\\ 0 & \alpha\end{pmatrix}\right)$ on the fibre over $x\in U_j$ is $\sigma\begin{pmatrix}u_j\\ v_j\end{pmatrix}$

$=\begin{pmatrix}\alpha & t_1\gamma_j^{(1)}(x)+\cdots+t_r\gamma_j^{(r)}(x)\\ 0 & \alpha\end{pmatrix}\begin{pmatrix}u_j\\ v_j\end{pmatrix}$, H_r acts rationally on E.

(3) This is the case (b). The proof is similar to that of (2).

(4) is the case (c). q.e.d.

Remark 1. H_r (or H'_r) has a normal subgroup K_r (or, K'_r, respectively) which is isomorphic to r-ple product of G_a, and $H_r/K_r=G_m$, $H'_r/K'_r=G_m\times G_m$. H_r, H'_r are, therefore, solvable groups.

2. Every ruled surface (S, X, π) is isomorphic to $(P(E), X, \pi')$ for some vector bundle E of rank 2 over X; in other words there is an isomorphism $\varphi: S\rightarrow P(E)$ such that $\pi=\pi'\cdot\varphi$. The relation between $\mathrm{Aut}(E)$ and $\mathrm{Aut}_X(P(E))$ is given by the following lemma which is found in [1].

Lemma 3. *Let E be a vector bundle over a connected locally noetherian prescheme Y. Put $\Delta=\{\bar{N}\mid\bar{N}$ is isomorphism class of a linebundle N such that $E\cong E\otimes N\}$. (Clearly, Δ is a group.) Then, we have an exact sequence;*

$$e\rightarrow \mathrm{Aut}(E)/\Gamma(Y, \mathcal{O}_Y^*)\rightarrow \mathrm{Aut}_Y(P(E))\rightarrow \Delta\rightarrow e.$$

In our case, since $Y=X$ and E is a vector bundle of rank 2, $\Gamma(Y, \mathcal{O}_Y^*)=\Gamma(X, \mathcal{O}_X^*)=G_m$ and Δ is a subgroup of the 2-torsion part of the Jacobian variety of X because $E\cong E\otimes N$ asserts $N^2\cong I$. Therefore, Δ is a finite group.

Lemma 4. (1) *If $N(E)\leqq 0$ and $E\not\cong L\oplus(L\otimes N)$ for any linebundle N such that $N^2\cong I$, then $\Delta=\{e\}$.*

(2) *If $E\cong L\oplus(L\otimes N)$, $N^2\cong I$ and if $N\not\cong I$, then $\Delta\cong \mathbf{Z}/2\mathbf{Z}$.*

(3) *If $E\cong L\oplus L$, then $\Delta=\{e\}$.*

Proof. (1) By virtue of Lemma 2 either E has only one maximal subbundle L, or E has just two maximal subbundles L, L' and

$L \not\cong L' \otimes N$ for any linebundle N such that $N^2 \cong I$. Take $\overline{N'} \in \Delta$, $N' \not\cong I$, then $L \otimes N'$ is sublinebundle of $E \otimes N'$, whence it is that of E, which never occurs in any case.

(2) Maximal subbundles of E are just $L, L \otimes N$. Hence if $\overline{N'} \in \Delta$, $N' \not\cong I$, then $L \otimes N' = L \otimes N$, that is, $N' \cong N$. Conversely, $\varphi: \begin{pmatrix} u_i \\ v_i \end{pmatrix} \to \begin{pmatrix} v_i \\ u_i \end{pmatrix}$ is an isomorphism of E to $E \otimes N = (L \otimes N) \oplus L$. Thus $\Delta = \{I, \bar{N}\} = \boldsymbol{Z}/2\boldsymbol{Z}$.

(3) Note that every maximal subbundle of E is isomorphic to L. Then, the proof is the same as above. q.e.d.

A section s of $P(E)$ is called a minimal section of $P(E)$ if and only if s has the smallest self-intersection number (s, s) among sections of $P(E)$. We proved in [3] (Proposition 1.9, Theorem 1.16) the following lemma.

Lemma 5. (1) *$N(E)$ depends only on $P(E)$. So $N(P(E))$ (or $N(S)$) has meaning. $N(P(E))$ is the self-intersection number of a minimal section of $P(E)$.*

(2) *The set of minimal sections of $P(E)$ is in bijective correspondence with that of maximal subbundles of E.*

(3) *If L is a maximal subbundle of E, then the diviser class on X defined by the linebundle $(\det E) \otimes L^{-2}$ is equal to that of $\pi(s \cdot s)$ for the minimal section s of $P(E)$ which corresponds to L by the correspondence in (2).*

Let us define a linear group $\overline{H'_r}$ which is a subgroup of $(r+1)$-ple product of $GL(2, k)$ and isomorphic to H'_r/G_m.

$$\overline{H'_r} = \left\{ \left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \alpha & t_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} \alpha & t_r \\ 0 & 1 \end{pmatrix} \right) \middle| \begin{matrix} t_1, \dots, t_r \in k \\ \alpha \in G_m \end{matrix} \right\}.$$

Theorem 2. *Let (S, X, π) be a ruled surface.*

(1) *If $N(S) > 0$, then* $\mathrm{Aut}_X(S) \cong \Delta$, *where Δ is a finite group*

defined in Lemma 3 *for a vector bundle* E *such that* $(S, X, \pi) \cong (P(E), X, \pi')$.

(2) *If* $N(S) \leqq 0$, S *is indecomposable*[1] *and if* s *is the unique minimal section of* S, *then* $\mathrm{Aut}_X(S) \cong G_a \times \cdots \times G_a$, *r-ple product of* G_a, *where* $r = \dim|-\pi(s \cdot s)|+1$.

(3) *If* S *is decomposable and if* S *does not carry two minimal sections* s, s' *such that* $\pi(s \cdot s) = \pi(s' \cdot s')$, *then* $\mathrm{Aut}_X(S) \cong \overline{H}'_r$, *where* $r = \dim|-\pi(s \cdot s)|+1$ *for a minimal section* s *of* S.

(4) *If* S *is decomposable,* $S \not\cong \boldsymbol{P}^1 \times X$ *and if* S *has two distinct minimal sections* s, s' *such that* $\pi(s \cdot s) = \pi(s' \cdot s')$ (*accordingly,* $N(S)=0$), *then* $\mathrm{Aut}_X(S) \cong \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \middle| \alpha \in G_m \right\} \cup \left\{ \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix} \beta \in G_m \right\}$[2].

(5) *If* $S \cong \boldsymbol{P}^1 \times X$, *then* $\mathrm{Aut}_X(S) \cong PGL(1, k)$.

Proof. (1), (3), (5) are obvious by virtue of Theorem 1 and Lemma 3, 4, 5. If one notes that $H_r/G_m \cong G_a \times \cdots \times G_a$, (2) is obvious by the same reason. If $(S, X, \pi) \cong (P(E), X, \pi')$, in the case (4), E is a vector bundle of type (3) of Theorem 1 and type (2) of Lemma 4. Hence $\mathrm{Aut}(E)/G_m \cong \overline{H}'_r$. but since $(\det E)^{-1} \otimes L^2 = N^{-1}$ (notation is the same as in Lemma 4, (2)), we have that $r=0$. The generator of the group $\varDelta$ corresponds to the isomorphism φ: $\begin{pmatrix} u_i \\ v_i \end{pmatrix} \to \begin{pmatrix} v_i \\ u_i \end{pmatrix}$. Thus

$$\mathrm{Aut}_X(S) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix} \right\}. \qquad \text{q.e.d.}$$

Remark 2. We showed in [3] that (1)—(5) exhaust all cases which may occur and anyone of (1)—(5) can occur if the genus of

1) S is indecomposable (or, decomposable) if E is indecomposable (or, decomposable, respectively) for some vector bundle E such that $(S, X, \pi) \cong (P(E), X, \pi')$. This property is independent of choice of E. S is decomposable if and only if S has two sections which do not meet to each other.

2) The multiplication of this group is defined by

$$\begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \beta\alpha^{-1} \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha\beta \\ 1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & \beta \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \beta' \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \beta\beta'^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha' & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha\alpha' & 0 \\ 0 & 1 \end{pmatrix}.$$

$X \geqq 1$. If X is fixed, the set of isomorphism classes of $\boldsymbol{P}^1$-bundles over X of type (4), (5) of Theorem 2 is in bijective correspondence with the 2-torsion part of the Jacobian variety of X ([3] Corollary 1.12).

Remark 3. Let s be a minimal section of S and let $L(\pi(s \cdot s))$ be a linebundle defined by a divisor of the divisor class $\pi(s \cdot s)$. Let $\{U_j\}$ be a sufficiently fine open covering of X and let $\{\gamma_j^{(1)}\}, \cdots, \{\gamma_j^{(r)}\}$ be representatives of a basis of $\Gamma(X, L(\pi(s \cdot s))^{-1})$. Take the minimal section s as the infinity section of S and let z_j be the coordinate of fibres over U_j.

(i) In the case (2) of Theorem 2, if $\sigma=(t_1, \cdots, t_r) \in G_a \times \cdots \times G_a$, the action of σ on the fibre over $x \in U_j$ is $\sigma(z_j)=z_j+t_1\gamma_j^{(1)}(x)+\cdots+t_r\gamma_j^{(r)}(x)$.

(ii) In the case (3), the action of $\sigma=\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}\alpha & t_1\\ 0 & 1\end{pmatrix}, \cdots, \begin{pmatrix}\alpha & t_r\\ 0 & 1\end{pmatrix}\right)$ is $\sigma(z_j)=\alpha z_j+t_1\gamma_j^{(1)}(x)+\cdots+t_r\gamma_j^{(r)}(x)$.

(iii) In the case (4), the action of $\sigma=\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}$ is $\sigma(z_j)=\alpha z_j$ and the action of $\sigma'=\begin{pmatrix}0 & \beta\\ 1 & 0\end{pmatrix}$ is $\sigma'(z_j)=\beta/z_j$.

(iv) In the case (5), the action of $\sigma=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ is $\sigma(z_j)=\dfrac{\alpha z_j+\beta}{\gamma z_j+\delta}$.

Remark 4. Let V be a variety of dimention 2 defined over k. If the linear part of the connected component of the unit element of $\mathrm{Aut}(V)$ has a positive dimention, then V is birationally isomorphic to a ruled surface ([4]).

3. A relation between $\mathrm{Aut}_X(S)$ and $\mathrm{Aut}(S)$ is given by the following lemma.

Lemma 6. *Let (S, X, π) be a ruled surface. If X is irrational, or if X is rational and $S \not\cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$, then there is an exact sequence;*

$$e \rightarrow \mathrm{Aut}_X(S) \rightarrow \mathrm{Aut}(S) \xrightarrow{f} \mathrm{Aut}(X).$$

Proof. If X is irrational, a fibring of S over X is unique because a rational curve passing through a point of S is unique. In rational case, the uniqueness of a fibring is due to Proposition 5 of [7]. In either case, therefore, an automorphism σ of S sends a fibre to a fibre. Thus we have an automorphism $\bar{\sigma}$ of X such that $\pi\cdot\sigma=\bar{\sigma}\cdot\pi$ and we know that $\mathrm{Aut}_X(S)$ is a normal subgroup of $\mathrm{Aut}(S)$. It is clear that $\bar{\sigma}=e$ if and only if $\sigma\in\mathrm{Aut}_X(S)$. q.e.d.

Remark 5. Fix a section s of S and an isomorphism $i: X\to s$ such that $\pi\cdot i=id_X$. Let us define a subfunctor $\mathcal{A}ut_{S/X}$ of the functor $\mathcal{A}ut_S$: $(\mathbf{Sch}/k)_{\mathrm{red}}=$(reduced algebraic schemes/k)$\to(Gr)$ ([5]). For any $T\in(\mathbf{Sch}/k)_{\mathrm{red}}$ put $\mathcal{A}ut_{S/X}(T)=\{\sigma\in\mathcal{A}ut_S(T)\mid(\pi\times id_T)\cdot\sigma\cdot(i\times id_T)=id_{X\times T}\}$ and define $F(T)(\sigma)=(\pi\times id_T)\cdot\sigma\cdot(i\times id_T)$. Then, noting that k is an algebraically closed field, we have an exact sequence;

$$e\to\mathcal{A}ut_{S/X}(T)\to\mathcal{A}ut_S(T)\xrightarrow{F(T)}\mathcal{A}ut_X(T).$$

Since this sequence is functorial, $\mathrm{Aut}_X(S)$ in Theorem 2 represents $\mathcal{A}ut_{S/X}$ and since $\mathcal{A}ut_S$, $\mathcal{A}ut_X$ are representable (see [5]), the sequence in Lemma 6 is an exact sequence as algebraic groups.

Corollary. *Let* $\mathrm{Bir}(k(V))$ *be the group of birational transformations of a variety* V *defined over* k *onto itself. If* (S, X, π) *is a ruled surface, then* $\dim\mathrm{Bir}(k(S))=\infty$.

Proof. Let n be an arbitrary positive integer. There is a decomposable ruled surface S' with $N(S')=-n$ which is birationally isomorphic to S. Then, by virtue of Theorem 2 and the above Remark, $\dim\mathrm{Aut}(S')\geqq n-g+2$, where g is the genus of X. Thus we have that $\dim\mathrm{Bir}(k(S))=\infty$. q.e.d.

Lemma 7. *If the genus of* $X\geqq 2$, *or if* X *is an elliptic curve and* S *is decomposable with* $N(S)\neq 0$, *then* $\mathrm{Im}\,f$ *of the sequence of Lemma* 6 *is a finite group*

Proof. It is well known that if the genus of $X \geqq 2$, $\mathrm{Aut}(X)$ is a finite group. Thus there is nothing to prove in the former case. In the latter case, S has only one minimal section s and the divisor class $-\pi(s \cdot s)$ has positive degree ([3] Theorem 1. 11). Since an automorphism of $\mathrm{Im}\, f$ fixes the divisor class, $\mathrm{Im}\, f$ is a finite group. q.e.d.

From now on, we denote the connected component of the unit element of $\mathrm{Aut}(S)$ ($\mathrm{Aut}_X(S)$ or $\mathrm{Aut}(X)$) by $\mathrm{Aut}^0(S)$ ($\mathrm{Aut}_X^0(S)$ or $\mathrm{Aut}^0(X)$, respectively).

Corollary. *Under the conditions of Lemma* 7, *we have that* $\mathrm{Aut}_X^0(S) \simeq \mathrm{Aut}^0(S)$. *In particular if* $N(S)>0$ *and if the genus of* $X \geqq 2$, *then* $\mathrm{Aut}(S)$ *is a finite group.*

If the genus of X is greater than 1, then there is an indecomposable ruled surface S such that $\dim|-\pi(s \cdot s)|+1=0$ for the minimal section s of S. On the other hand, if X is a general curve with the genus greater than 2, then we have that $\mathrm{Aut}(X)=\{e\}$. Thus there is a ruled surface S with $\mathrm{Aut}(S)=\{e\}$.

Now, the remaining parts are the following cases;

(i) $S \cong \boldsymbol{P}^1 \times X$

(ii) X is rational and $S \not\cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$

(iii) X is elliptic and S is indecomposable.

(iv) X is elliptic, S is decomposable with $N(S)=0$ and $S \not\cong \boldsymbol{P}^1 \times X$.

It is obvious that if $S \cong \boldsymbol{P}^1 \times X$ and X is irrational, then $\mathrm{Aut}(S) \simeq \mathrm{Aut}_X(S) \times \mathrm{Aut}(X) = \mathrm{PGL}(1) \times \mathrm{Aut}(X)$. If $S \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$, then $\mathrm{Aut}(S) = \{\mathrm{PGL}(1) \times \mathrm{PGL}(1)\} \cup \{V(\mathrm{PGL}(1) \times \mathrm{PGL}(1))\}$ where V is the interchanging transformation (see [7] p. 354 (4)).

Lemma 8. (1) *In the cases* (ii) *or* (iii) *above*, f *in Lemma* 6 *is surjective.*

(2) *In the case* (iv) *above, let* s *be a minimal section of* S *and let*

$\pi(s\cdot s)=x_0-x$, *where* x_0 *is the unit element of* X *as an abelian variety. Then, we have that* $\mathrm{Im}\, f=\mathrm{Aut}^0(X)\cup(\varphi_0\,\mathrm{Aut}^0(X))\cup(\varphi_1\,\mathrm{Aut}^0(X))\cup\cdots\cup(\varphi_r\,\mathrm{Aut}^0(X))$, *where* φ_0(*or*, φ_i, $1\leqq i\leqq r$) *is an automorphism of* X *as an abelian variety such that* $\varphi_0(y)=-y$ *for all* $y\in X$ (*or*, $\varphi_i(x)=x$, $1\leqq i\leqq r$, *respectively*).

Proof. (1) If $\sigma\in\mathrm{Aut}(X)$, the morphism $\tilde{\sigma}:(z,x)\to(z,\sigma(x))$, $(z,x)\in\boldsymbol{P}^1\times X$ is an automorphism of $\boldsymbol{P}^1\times X$. Suppose that X is rational and $N(S)=-n$ $(n>0)$. Then, there is a bundle isomorphism $g: S\to S_e=\mathrm{elm}_{P_1,\dots,P_n}(\boldsymbol{P}^1\times X)$ for some points $P_1,\cdots,P_n\in P\times X(P\in\boldsymbol{P}^1)$ ([3] Proposition 4.1 and Theorem 4.3). Put $S_\sigma=\mathrm{elm}_{\tilde{\sigma}(P_1),\dots,\tilde{\sigma}(P_n)}(\boldsymbol{P}^1\times X)$ for $\sigma\in\mathrm{Aut}(X)$. Then, $T_\sigma=\mathrm{elm}_{\tilde{\sigma}(P_1),\dots,\tilde{\sigma}(P_n)}\cdot\tilde{\sigma}\cdot(\mathrm{elm}_{P_1,\dots,P_n})^{-1}$ is a biregular map of S_e to S_σ. On the other hand, since S_σ is isomorphic to S_e as $\boldsymbol{P}^1$-bundle ([3] Theorem 4.3), there is an isomorphism $h_\sigma: S_\sigma\to S_e$ such that $\pi_\sigma=\pi_e\cdot h_\sigma$, where π_σ, π_e are canonical projections of S_σ, S_e to X respectively. Thus we have an automorphism $h_\sigma\cdot T_\sigma$ of S_e. Then, $\bar{\sigma}=g^{-1}\cdot h_\sigma\cdot T_\sigma\cdot g$ is an automorphism of S such that the image by f is σ. In the case (iii), the same proof as above is available by virtue of Theorem 4.8 of [3].

(2) $S=S_e=\mathrm{elm}_{P_0,P}(\boldsymbol{P}^1\times X)$ as $\boldsymbol{P}^1$-bundle, where $\pi_0(P_0)=x_0$, $\pi_0(P)=x$ (π_0: canonical projection of $\boldsymbol{P}^1\times X$ to X) $P_0\in(0)\times X$, $P\in(\infty)\times X$([2] Proposition 4.1). Furthermore, $S_e\cong S_\sigma=\mathrm{elm}_{\tilde{\sigma}(P_0),\tilde{\sigma}(P)}(\boldsymbol{P}^1\times X)$ as $\boldsymbol{P}^1$-bundle if and only if σ transforms the divisor class $\pi(s\cdot s)$ to either itself or $-\pi(s\cdot s)$. On the other hand, it is easy to see that $\sigma\in\mathrm{Aut}(X)$ satisfies the above condition if and only if σ is contained in the group defined in our lemma. Thus the same proof as in (1) is also valid in this case. q.e.d.

Remark 6. The proof of Lemma 8 shows that if S is decomposable and $S\not\cong\boldsymbol{P}^1\times X$, then $\sigma\in\mathrm{Aut}(X)$ is in the image of f in Lemma 6 if and only if σ transforms the divisor class $\pi(s\cdot s)$ to either itself or $-\pi(s\cdot s)$, where s is a minimal section of S.

Lemma 9. *Let* (S, X, π) *be a ruled surface over an elliptic curve* X *and let* S *be isomorphic to* $\boldsymbol{P}_1$ ([3] p. 66).

(1) *S carries no linear pencil L which satisfies the following conditions*;

a) *A generic member is a non-singular elliptic curve.*

b) *If* $L \ni D, D'$, *then* $(D, D')=0$

c) *If* $L \ni D$, *then* $(D, l)=2$, *where l is a fibre of S.*

(2) *Suppose that the characteristic of k is not equal to 2. Then, S carries a linear pencil L which satisfies the conditions* (a), (b) *above and the following*;

(c′) *If* $L \ni D$, *then* $(D, l)=4$, *where l is a fibre of S.*

Proof. Let π_0 be the canonical projection of $S_0=\boldsymbol{P}^1 \times X$ to X. We know ([3] Theorem 4.8) that $S \cong \boldsymbol{P}_1=\mathrm{elm}_{Q_1,Q_2,Q_3}(S_0)$, where $\pi_0(Q_i) \neq \pi_0(Q_j)$ $(i \neq j)$ and $Q_i \in R_i \times X$ $(R_i \neq R_j$ $(i \neq j)$, $R_i \in \boldsymbol{P}^1)$. Note that a linear system on $\boldsymbol{P}_1$ such that a generic member of it does not contain a fibre of $\boldsymbol{P}_1$ is the proper transform by $\mathrm{elm}_{Q_1,Q_2,Q_3}$ of a linear system on S_0. On the other hand, a positive divisor D on S_0 is linearly equivalent to $m(P\times X)+\sum_{i=1}^{n}\pi_0^{-1}(x_i)$ for some points $x_1, \cdots, x_n \in X$. If $D \sim m(P\times X)+\sum_{i=1}^{n}\pi_0^{-1}(x_i)$ on S_0, then we obtain that $p_a(D)=m+(m-1)(n-1)$ and $(D, D)=2mn$ ([3] Remark 3.3). Thus we know that a linear pencil L on $\boldsymbol{P}_1$ satisfying conditions (a), (b) must be the proper transform by $\mathrm{elm}_{Q_1,Q_2,Q_3}$ of $(|2m(P\times X)+\sum_{i=1}^{3m}\pi_0^{-1}(x_i)| -mQ_1-mQ_2-mQ_3)$[3)]

(1) (Nagata) Since the points $\pi_0(Q_1)$, $\pi_0(Q_2)$, $\pi_0(Q_3)$ are arbitrary if $\pi_0(Q_i) \neq \pi_0(Q_j)$ $(i \neq j)$, we may assume that $2\pi_0(Q_1) \not\sim 2\pi_0(Q_2)$. The condition (c) asserts that the number m above is equal to 1. It is easy to see that $\dim(|2(P\times X)+\sum_{i=1}^{3}\pi_0^{-1}(x_i)|-2Q_1-2Q_2-Q_3)=8-3-3-1=1$ ([3] Lemma 3.1). Assume that every member of $M=(|2(P\times X)$

3) Let M be a linear system and P_i be points. Then $M-\sum P_i$ denotes the linear system which consists of members of M going through all P_i.

$+\sum_{i=1}^{3}\pi_0^{-1}(x_i)|-2Q_1-2Q_2-Q_3)$ goes doubly through the point Q_3. Put $l_{Q_3}=\pi_0^{-1}(\pi_0(Q_3))$ and consider $Tr_{l_{Q_3}}M$. Since $\dim(Tr_{l_{Q_3}}M)=0$, we have that $\dim(M-l_{Q_3})=0$, whence there is a member $E+l_{Q_3}$ of $M-l_{Q_3}$. Hence we obtain that $2\pi_0(Q_1)\sim 2\pi_0(Q_2)$ because $E\cdot(R_1\times X)=2Q_1$, $E\cdot(R_2\times X)=2Q_2$. This contradicts to the assumption that $2\pi_0(Q_1)\not\sim 2\pi_0(Q_2)$. Thus we have that $\dim(|2(P\times X)+\sum_{i=1}^{3}\pi_0(x_i)|-\sum_{i=1}^{3}2Q_i)\leqq 0$, which proves our assertion.

(2) Since the characteristic of k is not equal to 2, the group of 2-torsion part of the Jacobian variety of X is isomorphic to $\boldsymbol{Z}/2\boldsymbol{Z}\oplus\boldsymbol{Z}/2\boldsymbol{Z}$, whence we may assume that $2\pi_0(Q_1)\sim 2\pi_0(Q_2)\sim 2\pi_0(Q_3)$. Consider the linear system $L_1=|2(P\times X)+2\pi_0^{-1}(\pi_0(Q_1))|$ on S_0. $(L_1-2Q_1-2Q_2)$ (or, $(L_1-2Q_1-2Q_3)$) contains $2(R_1\times X)+2\pi_0^{-1}(\pi_0(Q_2))$ and $2(R_2\times X)+2\pi_0^{-1}(\pi_0(Q_1))$ (or, $2(R_1\times X)+2\pi_0^{-1}(\pi_0(Q_3))$ and $2(R_3\times X)+2\pi_0^{-1}(\pi_0(Q_1))$, respectively). Thus we have that $\dim(L_1-2Q_1-2Q_2)\geqq 1$ (or, $\dim(L_1-2Q_1-2Q_3)\geqq 1$, respectively), whence $(L_1-2Q_1-2Q_2-Q_3)$ (or, $(L_1-2Q_1-2Q_3-Q_2)$) contains a divisor D_1 (or, D_2, respectively). Therefore, the linear system $M=(|4(P\times X)+6\pi_0^{-1}(\pi_0(Q_1))|-4Q_1-4Q_2-4Q_3)$ contains $2D_1+2\pi_0^{-1}(\pi_0(Q_3))$ and $2D_2+2\pi_0^{-1}(\pi_0(Q_2))$. Hence we obtain that $\dim M\geqq 1$. Now, we can see easily that the proper transform of M by $\mathrm{elm}_{Q_1,Q_2,Q_3}$ satisfies the conditions (a), (b), (c′).

q.e.d.

Remark 7. By "*reduction*" we know thet Lemma 9 (2) is true even if the characteristic of k is equal to 2. On the other hand, M. Nagata proved directly that Lemma 9 (2) is true if the characteristic of k is equal to 2 and X has non-trivial 2-torsion part (i.e. Hasse invariant of $X\neq 0$).

Calculation of $H^0(S, \Theta_S)$ $(\Theta_S=\mathcal{H}om(\Omega_S^1, \mathcal{O}_S))$ for a ruled surface (S, X, π) over an elliptic curve defined over $\boldsymbol{C}$ has been done by T. Suwa [10]. Similar argument is applicable to the case where k is an arbitrary algebraic closed field and X is rational or elliptic.

Our results are the following;

Let p be the characteristic of k and θ is an element of $H^0(S, \Theta_S)$.

A) The case where $N(S)=-n<0$; There is an open covering $X=U_0\cup U$ such that $\pi^{-1}(U_0)\cong \boldsymbol{P}^1\times U_0$, $\pi^{-1}(U)\cong \boldsymbol{P}^1\times U$ and that $z_0=u^n z$, where z_0 (or, z) is coordinate of fibres over U_0 (or, U, respectively) and where u is a local parameter at a point $Q\in U$.

i) X: rational.

a) If $p|n$, then

$$\theta=g\frac{\partial}{\partial z_0}+az_0\frac{\partial}{\partial z_0}+D,$$

where $g\in \mathscr{L}(nQ)$, $D\in H^0(X, \Theta_X)$ and where a is an arbitrary constant.

b) If $p\nmid n$, then

$$\theta=g\frac{\partial}{\partial z_0}+(a-n\alpha h)z_0\frac{\partial}{\partial z_0}+D,$$

where g, D, a are the same as in (a), h is a fixed non-constant function in $\mathscr{L}(Q)$ and where α is uniquely determined by D and h.

ii) X: elliptic.

a) If $p|n$, then

$$\theta=g\frac{\partial}{\partial z_0}+az_0\frac{\partial}{\partial z_0}+D,$$

where $g\in \mathscr{L}(nQ)$, $D\in H^0(X, \Theta_X)$ and where a is an arbitrary constant.

b) If $p\nmid n$, then

$$\theta=g\frac{\partial}{\partial z_0}+az_0\frac{\partial}{\partial z_0},$$

where g, a are the same as in (a) above.

B) If $S\cong \boldsymbol{P}^1\times X$, then

$$\theta=a_0\frac{\partial}{\partial z_0}+a_1 z_0\frac{\partial}{\partial z_0}+a_2 z_0^2\frac{\partial}{\partial z_0}+D,$$

where a_0, a_1, a_2 are arbitrary constants and where $D\in H^0(X, \Theta_X)$.

C) The case where X is elliptic, $N(S)=0$ and where S is decomposable;

There is an open covering $X=U_0\cup U_1\cup U_2$ such that $\pi^{-1}(U_i)$

$\cong \boldsymbol{P}^1 \times U_i$ $(i=0, 1, 2)$ and that $z_0=u_1 z_1$, $z_0=u_2^{-1}z_2$, where z_i is a coordinate of fibres over U_i and u_1 (or, u_2) is a local parameter at a point $Q_1 \in U_1$ (or, $Q_2 \in U_2$, respectively) and where $Q_1 \not\sim Q_2$. Then, there are $D_0 \in H^0(X, \Theta_X)$, $g_0 \in \mathscr{L}(Q_1+Q_2)$ and

$$\theta=(ag_0+b)z_0\frac{\partial}{\partial z_0}+aD_0,$$

where a, b are arbitrary constants.

D) The case where X is elliptic, $N(S)=0$ and where S is indecomposable, that is, $S \cong \boldsymbol{P}_0$ (see [3] p. 66).

There is an open covering $X=U_0 \cup U$ and $z_0=z+\frac{1}{u}$ (Notation is the same as in (A)).

a) If $p \neq 2$, then there are $D_0 \in H^0(X, \Theta_X)$, $g_0 \in \mathscr{L}(2P)$ and

$$\theta=(ag_0+b)\frac{\partial}{\partial z_0}+\alpha z_0\frac{\partial}{\partial z_0}+aD_0,$$

where a, b are arbitrary constants and α is uniquely determined by g_0.

b) If $p=2$, then

$$\theta=((b+a_2)g_0+a_1)\frac{\partial}{\partial z_0}+\alpha z_0\frac{\partial}{\partial z_0}+a_2 z_0^2\frac{\partial}{\partial z_0}+bD_0,$$

where g_0, D_0, α are the same as in (a) above and where a_1, a_2, b are arbitrary constants.

E) The case where X is elliptic and $N(S)=1$, that is $S \cong \boldsymbol{P}_1$ ([3] p. 66);

There is an open covering $X=U_0 \cup U$ and $z_0=uz+u^{-1}$ (notation is the same as in (A)).

a) If $p \neq 2$, then there are $g_0 \in \mathscr{L}(2Q)$, $D_0 \in H^0(X, \Theta_X)$ and

$$\theta=3ag_0\frac{\partial}{\partial z_0}+\alpha z_0\frac{\partial}{\partial z_0}+az_0^2\frac{\partial}{\partial z_0}+2aD_0,$$

where a is an arbitrary constant and where α is uniquely determined by g_0 and a.

b) If $p=2$, then

$$\theta = ag_0 \frac{\partial}{\partial z_0} + \alpha z_0 \frac{\partial}{\partial z_0} + az_0^2 \frac{\partial}{\partial z_0},$$

where g_0, a, α are the same as in (a) above.

Sumalizing the above results, we have the following.

Lemma 10. *Let (S, X, π) be a ruled surface, Θ_S be the sheaf of germs of regular sections of the tangent bundle of S and let p be the characteristic of k.*

1) *Let X be rational and let $N(S) = -n$.*

a) *If $n \neq 0$, then*

$\dim H^0(S, \Theta_S) = n + 5$

b) *If $n = 0$ (i.e. $S \cong \boldsymbol{P}^1 \times X$), then*

$\dim H^0(S, \Theta_S) = 6$.

2) *Let X be elliptic and let $N(S) = -n$.*

c) *If $p \nmid n$, and $n \neq 0, -1$, then*

$\dim H^0(S, \Theta_S) = n + 1$.

d) *If $p | n$ and $n \neq 0, -1$, then*

$\dim H^0(S, \Theta_S) = n + 2$

e) *If $n = 0$, S is decomposable and if $S \not\cong \boldsymbol{P}^1 \times X$, then*

$\dim H^0(S, \Theta_S) = 2$.

f) *If $n = 0$, S is indecomposable (i.e. $S \cong \boldsymbol{P}_0$) and if $p \neq 2$, then*

$\dim H^0(S, \Theta_S) = 2$.

g) *If $n = 0$, S is indecomposable and if $p = 2$, then*

$\dim H^0(S, \Theta_S) = 3$.

h) *If $n = -1$ (i.e. $S \cong \boldsymbol{P}_1$), then*

$\dim H^0(S, \Theta_S) = 1$.

i) *If $S \cong \boldsymbol{P}^1 \times X$, then*

$\dim H^0(S, \Theta_S) = 4$

Corollary. 1) Aut(S) *in Lemma* 6 (*Cf. Remark* 5) *represents the functor* $\mathscr{A}ut_S$ *over* $(\mathbf{Sch}/k)$ *in the case satisfying one of the conditions* (a), (b), (c), (e), (f), (h), (i) *in Lemma* 10.

2) *In the case* (d), (g) *in Lemma* 10, Aut(S) *in Lemma* 6 *never represents the functor* $\mathscr{A}ut_S$ *over* $(\mathbf{Sch}/k)$, *that is, the group scheme which represents this functor over* $(\mathbf{Sch}/k)$ *is not redued.*

3) *Let* f_* *be the morphism of* Lie(Aut$^0(S)$) *to* Lie(Aut$^0(X)$) *which associates to* f *in Lemma* 6. *Then,* f_* *is surjective if one of the following conditions is satisfied*;

i) X *is rational.*

ii) X *is elliptic and* S *is decomposable with* $N(S)=0$.

iii) X *is elliptic,* S *is indecomposable and the characteristic of* k *is not equal to* 2.

Proof. If one notes that $\dim \mathrm{Aut}(S)=\dim H^0(S, \Theta_S)$ (or, $\dim \mathrm{Aut}(S)<\dim H^0(S, \Theta_S)$) under the conditions of (1) (or, (2), respectively) (see Theorem 2, Lemma 7 and Lemma 8), then our assertions in (1) and (2) are followed from the results of [5]. Since $H^0(S, \Theta_S)\cong \mathrm{Lie}(\mathrm{Aut}^0(S))$ if Aut(S) represents the functor $\mathscr{A}ut_S$ over $(\mathbf{Sch}/k)$ and since $f_*\left(a_0\frac{\partial}{\partial z_0}+a_1 z_0\frac{\partial}{\partial z_0}+a_2 z_0^2\frac{\partial}{\partial z_0}+D\right)=D$ $(D\in H^0(X, \Theta_X))$, we obtain (3) by virtue of (A) (i), (B), (C), (D), (E) above. q.e.d.

Now, we come to the following theorem.

Theorem 3. *Let* (S, X, π) *be a ruled surface.*

(1) *If* X *is rational and* $S\cong F_n$ $(n>0)$ ([7] *or* [3] *Theorem* 4.3), *then we have an exact sequence*[4] *of algebraic groups*;

$$e\to \bar{H}_{n+1}\to \mathrm{Aut}(S)\to \mathrm{PGL}(1)\to e.$$

4) By an exact sequence of algebraic groups

$$e\to G'\to G\xrightarrow{g} G''\to e$$

we mean the following; G' is an algebraic subgroup of G and $\bar{g}: G/G'\xrightarrow{\sim} G''$ as algebraic groups.

(2) *If X is elliptic and if S is decomposable with $N(S)=0$, $S \not\cong \boldsymbol{P}^1 \times X$, then we have an exact sequence of algebraic groups*;

$$e \to G \to \mathrm{Aut}(S) \to \mathrm{Aut}'(X) \to e,$$

where $\mathrm{Aut}'(X)$ *is the group defined in Lemma* 8 *and where G is G_m or the group defined in Theorem* 2, (4) *according to whether S satisfies the condition* (3) *of Theorem* 2 *or not. Moreover*, $\mathrm{Aut}^0(S)$ *is a commutative group and it is isomorphic to $S-s_1-s_2$ with the natural algebraic group structure* (*see* [9] *Ch.* VII §3 *n*° 15, 16), *where s_1, s_2 are the minimal sections of S.*

(3) *If X is elliptic and if $S \cong \boldsymbol{P}_0$, then we have an exact sequence of algebraic groups*;

$$e \to G_a \to \mathrm{Aut}(S) \to \mathrm{Aut}(X) \to e.$$

Moreover, $\mathrm{Aut}^0(S)$ *is a commutative group and it is a non-trivial extension of* $\mathrm{Aut}^0(X)$ *by G_a, i.e.* $\mathrm{Aut}^0(S)$ *is isomorphic to $S-s$ with the natural algebraic group structure* (*see* [9] *Ch.* VII §3 *n*° 15, 17), *where s is the minimal section of S.*

(4) *If X is elliptic, $S \cong \boldsymbol{P}_1$ and if the characteristic of k is not equal to* 2, *then we have an exact sequence of algebraic groups*;

$$e \to \Delta \to \mathrm{Aut}(S) \to \mathrm{Aut}(X) \to e,$$

where Δ is the group of the 2*-torsion part of X. Furthermore, we have that* $\Delta \cap \mathrm{Aut}^0(S) = \Delta$.

Proof. Exactness of the sequences of (1), (2), (3) above is followed from Theorem 2, Lemma 6, Lemma 8, and Corollary to Lemma 10 except for the case where $S \cong \boldsymbol{P}_0$ and the characteristic p of k is equal to 2. In order to prove exactness of the sequence in (3) in the case where $p=2$, we must show that $\mathrm{Aut}^0(S)/G_a \to \mathrm{Aut}^0(X)$ is a separable morphism. In fact we can prove existence of a local section from $\mathrm{Aut}^0(X)$ to $\mathrm{Aut}^0(S)$, but it is very complicated, hence we omit it. By virtue of Corollary 2 to Theorem 13 of [8], we have that $\mathrm{Aut}^0(S)$ in the case (2), (3) is a commutative group. In the case (3) if $\mathrm{Aut}^0(S)\cong\mathrm{Aut}^0(X)\times G_a$, then an orbit of $\mathrm{Aut}^0(X)\times\{e\}$ is a section of S, whence it meets the minimal section s of S. On the other hand, s is also an orbit of $\mathrm{Aut}^0(X)\times\{e\}$, which is impossible. Thus $\mathrm{Aut}^0(S)$ is a non-trivial extension of $\mathrm{Aut}^0(X)$ by G_a. It is easy to see that $S-s$ with the natural group structure (or, $S-s_1-s_2$ in the case (2)) acts regularly on S (or, S in the case (2), respectively) (see [9] Ch. VII $n^\circ 15$). Thus we have the last statements of (2) and (3). By virtue of Lemma 4.5, Theorem 4.7 of [3], if $\boldsymbol{P}_1\cong P(E)$, we have that $E\cong E\otimes N$ if and only if $N^2\cong I$. Thus if $S\cong\boldsymbol{P}_1$, Δ in Theorem 2, (i) is isomorphic to the 2-torsion part of X. Hence, by virtue of Theorem 2, Lemma 6, Lemma 8 and Corollary to Lemma 10, we have an exact sequence in (4). Finally, the latter assertion of (4) is followed from Lemma 9 because the orbit space of $\mathrm{Aut}^0(S)$ is a linear pencil satisfying the conditions (a), (b) in Lemma 9 and because $\Delta\cong\boldsymbol{Z}/2\boldsymbol{Z}\oplus\boldsymbol{Z}/2\boldsymbol{Z}$. q.e.d.

Remark 8. In the case (4) of the above theorem, we have a morphism $\bar{f}\colon \mathrm{Aut}(S)/\Delta\to\mathrm{Aut}(X)$ even if the characteristic p of k is equal to 2 (Δ is the 2-torsion part of X). Moreover, we know that $\Delta\cap\mathrm{Aut}^0(S)=\Delta$ (see Remark 7). But if $p=2$, then $\bar{f}$ is not an isomorphism (i.e. $\bar{f}$ is purely inseparable).

4. In this section we shall consider an application of the above results. A complete non-singular surface S is called an elliptic surface if and only if there is a morphism g of S to a complete non-singular curve C such that $g^{-1}(c)$ is a non-singular elliptic curve for a generic point c of C. Consider the following problem[5]; "*When does a ruled surface become an elliptic surface?*"

Since if a ruled surface S is an elliptic surface, the curve C in

5) This problem was solved by T. Suwa [10] when the base field k is the complex number field.

the above definition is a rational curve, our problem is equivalent to the following; "*When does a ruled surface carry a linear pencil satisfying the conditions* (a), (b) *in Lemma* 9?"

Lemma 11. *Let* (S, X, π) *be a ruled surface. If* S *is also an elliptic surface, then* (i) X *is an elliptic curve and* (ii) $N(S)=0$ *or* 1.

Proof. By virtue of Lemma 23 of [2] (it remains true even if the characteristic of k is positive), we have that $(K^2)=0$ for a canonical divisor K on S. On the other hand, if the genus of X is g, then $(K^2)=8-8g$. Thus we get that $g=1$, whence X is an elliptic curve. In order to prove (ii), assume that $N(S)=-n<0$, then $S=\mathrm{elm}_{P_1,\dots,P_n}(\boldsymbol{P}^1\times X)$ for some points $P_1, \cdots, P_n$ on $P\times X$ $(P\in\boldsymbol{P}^1)$. If S carries a linear pencil satisfying the conditions (a), (b) of Lemma 9, it must be the proper transform of a linear system $L_{r,s}$ on $\boldsymbol{P}^1\times X$ such as $L_{r,s}=\left(\left|r(P\times X)+\sum_{i=1}^{s}\pi_0^{-1}(x_i)\right|-\sum_{i=1}^{n}rP_i\right)$, where $r>0$, $s\geqq 0$, $P\in\boldsymbol{P}^1$, $x_1, \cdots, x_s\in X$ and where π_0 is the projection of $\boldsymbol{P}^1\times X$ to X. Let D be a general member in $L_{r,s}$, then D must be irreducible and D can not be tangent to fibres going through P_i $(1\leqq i\leqq n)$. Since $p_a(D)=r+(r-1)(s-1)$, $D^2=2rs$, we have that

$$1=p_a(\mathrm{elm}_{P_1,\dots,P_n}[D])=r+(r-1)(s-1)-\frac{nr(r-1)}{2},$$

$$0=(\mathrm{elm}_{P_1,\dots,P_n}[D]^2)=2rs-nr^2.$$

$$nr\leqq(D, P\times X)=s.$$

Therefore, we obtain that $r=0$, which is a contradiction. q.e.d.

Theorem 4. *Let* (S, X, π) *be a ruled surface over an algebraically closed field* k *with characteristic* p.

i) *If* S *is an elliptic surface, then* X *is an elliptic curve.*

ii) *Assume that* $p=0$. *Then,* S *is an elliptic surface if and only if one of the following conditions is satisfied;*

1) $S \cong \boldsymbol{P}_1$

2) *S is decomposable with $N(S)=0$ and $\pi(s\cdot s)$ is a torsion element in the Jacobian variety of X, where s is a minimal section of S.*

iii) *Assume that $p>0$. Then, S is an elliptic surface if and only if one of the conditions* (1), (2) *above or* (3) *below is satisfied*;

3) $S \cong \boldsymbol{P}_0$

Proof. (i) is Lemma 11, (i). By virtue of Theorem 3, (4) and Remark 8, S is an elliptic surface if $S \cong \boldsymbol{P}_1$. It is clear that if $S = \boldsymbol{P}^1 \times X$, then S is an elliptic surface. If S is decomposable with $N(S)=0$ and if $S \not\cong \boldsymbol{P}^1 \times X$, then $S = \mathrm{elm}_{P_1,P_2}(\boldsymbol{P}^1 \times X)$ for some points $P_1 \in (0) \times X$, $P_2 \in (\infty) \times X$ and $\pi(s\cdot s) = \pi_0(P_1) - \pi_0(P_2)$ for a minimal section of S, where π_0 is the projection of $\boldsymbol{P}^1 \times X$ to X ([3] Proposition 4.1). A linear pencil on S satisfying the conditions (a), (b) of Lemma 9 is the proper transform of a linear system L_r for some $r>0$ on $\boldsymbol{P}^1 \times X$;

$$L_r = (|r(P\times X) + \sum_{i=1}^{r} \pi_0^{-1}(x_i)| - rP_1 - rP_2).$$

Take a general member D in L_r, then $\pi_0(D\cdot(0)\times X) = r\pi_0(P_1)$ is linearly equivalent to $\pi_0(D\cdot(\infty)\times X) = r\pi_0(P_2)$. Thus $\pi(s\cdot s) = \pi_0(P_1) - \pi_0(P_2)$ is r-torsion element. Conversely, suppose that $\pi(s\cdot s) = \pi_0(P_1) - \pi_0(P_2)$ is a torsion element. Let r be the smallest positive integer such that $r\pi(s\cdot s)=0$. Then,

$$L'_r = (|r(P\times X) + r\pi_0^{-1}(\pi_0(P_1))| - rP_1 - rP_2)$$

contains $r((0)\times X) + r\pi_0^{-1}(\pi_0(P_2))$ and $r((\infty)\times X) + r\pi_0^{-1}(\pi_0(P_1))$, whence $\dim L'_r \geqq 1$. A general member of L'_r is irreducible because r is smallest. It is easy to see that the proper transform of the above L'_r by elm_{P_1,P_2} satisfies the conditions (a), (b) of Lemma 9. Thus if S is decomposable with $N(S)=0$, then S is an elliptic surface if and only if $\pi(s\cdot s)$ is a torsion element for a minimal section s of S.

Lemma 11, (ii) implies that the remaining part of our proof is the case where $S \cong \boldsymbol{P}_0$. If $\boldsymbol{P}_0$ carries the required linear pencil L, then G_a acts non-trivially on L (Theorem 3, (3)). Every member of L is never a section because two sections of $\boldsymbol{P}_0$ meet to each other. If D is a general member of L, then $D \cdot l = P_1 + \cdots + P_t$ ($P_i \neq P_j$ if $p=0$, $i \neq j$), where l is a general fibre of $\boldsymbol{P}_0$. Any element $\sigma \in G_a$ such that $\sigma(P_1) = P_i$ fixes D. This is impossible if $p=0$. If $p>0$, then $\boldsymbol{P}_0$ is an elliptic surface by virtue of Proposition in p. 336 of [6] (take the minimal section as C in the Proposition). q.e.d.

Finally, we add some more remarks.

Remark 9. 1) If $p \neq 2$ and $S \cong \boldsymbol{P}_1$, then S has three singular fibres. They are multiple fibres of the following type; $2C$, C is an elliptic curve with $(C, l)=2$ for a fibre $\pi^{-1}(x)=l$. This is proved by studying the action of automorphisms on the points $s_i \cdot s_j$ $1 \leqq i, j \leqq 4$, where s_i are minimal sections on S such that $\pi(s_1 \cdot s_1) = \cdots = \pi(s_4 \cdot s_4)$.

2) The moduli space of isomorphism classes of decomposable ruled surfaces S with $N(S)=0$ is a curve (see [3] Theorem 4.10). The parameter is $\pi(s \cdot s)$, where s is a minimal section of S. Thus Theorem 4 shows that the cardinal number of the set of isomorphism classes of ruled surfaces which carry the structure of an elliptic surface is countable.

3) If S satisfies the condition (2) of Theorem 4 and if $S \not\cong \boldsymbol{P}^1 \times X$, then S has two singular fibres rs_1, rs_2, where s_1, s_2 are the minimal sections of S and where r is the smallest positive integer such that $r\pi(s_1 \cdot s_1) = 0$.

4) M. Nagata and the author proved directly that if $p>0$ and $S \cong \boldsymbol{P}_0$, then S has a linear pencil L satisfying the conditions (a), (b) of Lemma 9 and $(D, l) = p$ for $D \in L$, $l = \pi^{-1}(x)$.

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