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Some remarks on high order derivations

By

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In this short paper we shall study some problems related to the theory of "high order derivations" introduced and developed by H. Osborn and Y. Nakai.

First we consider the problem in the situation that A and B are commutative k-algebras and there exists an algebra-homomorphism $A \rightarrow B$. After clarifying the relation between the homomorphism $\varphi: B \otimes_A \Omega_k(A) \rightarrow \Omega_k(B)$ and the extension of k-derivation of A to that of B, we shall slightly generalize the localization theorem $\Omega_k{}^{(q)}(A_S) \cong A_S \otimes_A \Omega_k^{(q)}(A)$ of Nakai ([2], Th. I. 15 and Th. II. 11) and give a simplified proof.

In §2 we shall investigate the conditions that the derivation algebra $\mathscr{D}(A/k)$ coincide with $\mathrm{Hom}_k(A, A)$, where A is a ring containing a subring k. Already Nakai has proved that, when k is a field and A is a finite extension of k, $\mathscr{D}(A/k) = \mathrm{Hom}_k(A, A)$ if and only if A is purely inseparable over k ([3], Th. I. 4 and Th. I. 5). We shall strengthen this theorem as follows: when A is an arbitrary extension field of a field k, we have $\mathscr{D}(A/k) = \mathrm{Hom}_k(A, A)$ if and only if A is purely inseparable and finite over k.

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Terminology. We adopt the terminology and notation in [2]. All rings are commutative rings with 1. When A is a k-algebra,

$\Omega_k^{(q)}(A)$ denotes the module of q-th order Kähler differentials of A/k and $\delta_{A/k}^{(q)}$ or $\delta_A^{(q)}$ denotes the canonical q-th order k-derivation $A \to \Omega_k^{(q)}(A)$ of A. The pair $\{\Omega_k^{(q)}(A), \delta_{A/k}^{(q)}\}$ has the universal mapping property with respect to the q-th order k-derivations of A. $I_{A/k}$ or I_A denotes the kernel of the canonical mapping $A \otimes_k A \to A\,(a \otimes b \mapsto ab)$. $\Omega_k^{(q)}(A)$ is identified with I_A/I_A^{q+1}. When we regard $A \otimes_k A$ as a left A-module, it is understood tacitly that we endow it an A-module structure defined by the formula $a(x \otimes y) = (a \otimes 1)(x \otimes y) = ax \otimes y$.

When $g: A \to B$ is a k-algebra homomorphism, we write $\delta_B^{(q)}(a)$ instead of $\delta_B^{(q)}(g(a))$ for any $a \in A$. When we consider the tensor products over two or more ground rings, one of which is k, we write $x \otimes y$ instead of $x \otimes_k y$ (but we don't omit the letter for another ground ring).

$\mathscr{D}_0^{(q)}(A/k)$ denotes the totality of q-th order k-derivations of A into A itself, and $\mathscr{D}_0(A/k)$ means $\bigcup_{q=1}^{\infty} \mathscr{D}_0^{(q)}(A/k)$. And $\mathscr{D}(A/k)$ denotes the derivation algebra of A over k i.e. the subring $A \oplus \mathscr{D}_0(A/k)$ of $\mathrm{Hom}_k(A, A)$.

§1

In this section consistently we assume that A and B are k-algebras and $g: A \to B$ is a k-algebra homomorphism. Then there exists a natural B-homomorphism $\varphi: B \otimes_A \Omega_k^{(q)}(A) \to \Omega_k^{(q)}(B)$ such that

$$\varphi(b \otimes \delta_A^{(q)}(a)) = b \cdot \delta_B^{(q)}(a)$$

for each $b \in B$ and $a \in A$. And we have the exact sequence

$$B \otimes_A \Omega_k^{(q)}(A) \xrightarrow{\varphi} \Omega_k^{(q)}(B) \xrightarrow{j} \Omega_k^{(q)}(B/A) \to 0 \tag{1}$$

of B-modules, where $\Omega_k^{(q)}(B/A)$ denotes the cokernel of φ.

If we set $\delta' = j \circ \delta_{B/k}^{(q)}$, δ' is a q-th order k-derivation of B into $\Omega_k^{(q)}(B/A)$ vanishing on A. Moreover $\Omega_k^{(q)}(B/A)$ and δ' have the universal mapping property with respect to the k-derivations of B vanishing on A. ([2], II. §3).

Because $\delta_{B/A}^{(q)}: B \to \Omega_A^{(q)}(B)$ is such a k-derivation, it is decomposed uniquely in the form $\delta^{(q)}_{B/A}=\psi\circ\delta'$, where ψ is a surjective B-homomorphism $\Omega^{(q)}_k(B/A)\to\Omega^{(q)}_A(B)$. Moreover we obtain the exact sequence

$$\Omega^{(q-1)}_k(B)\otimes_A\Omega^{(q-1)}_k(A)\xrightarrow{\sigma}\Omega^{(q)}_k(B/A)\xrightarrow{\psi}\Omega^{(q)}_A(B)\to 0 \tag{2}$$

of B-modules, where σ is determined by the formula

$$\sigma(\delta^{(q-1)}_B(b)\otimes_A\delta^{(q-1)}_A(a))=[\delta', a](b)$$

for every $b\in B$ and $a\in A$. ([2], Th. II. 11)

Now it is immediate that $\Omega^{(q)}_k(B/A)=0$ implies $\Omega^{(q)}_A(B)=0$. We shall show the converse of this.

Lemma 1. $\Omega^{(q)}_A(B)=0$ *implies* $\Omega^{(q)}_k(B/A)=0$.

Proof. We consider the k-derivation $\delta'_{(q)}=\delta'\colon B\to\Omega^{(q)}_k(B/A)$. We have only to show that $\delta'_{(q)}=0$, for the image $\mathrm{Im}(\delta')$ of δ' generates $\Omega^{(q)}_k(B/A)$.

If $q=1$, this is obvious on account of the exact sequence (2) and the fact $\Omega^{(0)}_k=0$.

Let $q>1$ and $\Omega^{(q)}_A(B)=0$. Then $I^{q+1}_{B/A}=I_{B/A}$ implies $\Omega^{(q-1)}_A(B)=0$. Therefore, by induction assumption, we have $\delta'_{(q-1)}=0$. This means that the null map is the only $(q-1)$-th order k-derivation of B vanishing on A. Hence we have $[\delta'_{(q)}, a]=0$ for each $a\in A$, because this is a $(q-1)$-th order k-derivation and $[\delta'_{(q)}, a](a')=\delta'_{(q)}(aa')-a\cdot\delta'_{(q)}(a')-a'\cdot\delta'_{(q)}(a)=0$ for all $a'\in A$. Moreover, $[\delta'_{(q)}, a]=0$ and $\delta'_{(q)}(a)=0$ imply $\delta'_{(q)}(ax)=a\cdot\delta'_{(q)}(x)$ for any $a\in A$ and $x\in B$. Therefore $\delta'_{(q)}$ is an A-derivation, hence we have $\delta'_{(q)}=0$ by our assumption $\Omega^{(q)}_A(B)=0$.

q.e.d.

Above lemma shows that we have $\Omega^{(q)}_A(B)=0$ if and only if $\Omega^{(q)}_k(B/A)=0$. This property can be restated as follows.

Proposition 1. *Following conditions are equivalent for any integer* $q\geqq 1$:

(1) $\Omega_A^{(q)}(B)=0$

(1′) *There is no non-trivial q-th order A-derivation of B.*

(2) $\Omega_k^{(q)}(B/A)=0$

(2′) *There is no non-trivial q-th order k-derivation of B which vanishes on A.*

(3) *The homomorphism* $\varphi\colon B\otimes_A \Omega_k^{(q)}(A)\to\Omega_k^{(q)}(B)$ *is surjective.*

(4) *Any q-th order k-derivation of A (into a B-module) has at most one extension to that of B.*

Moreover if one of these is satisfied for some q, then so is all for all q.

Proof. It is trivial to check the implication

$$\begin{array}{ccccc} (1') & \Leftrightarrow & (1) & \Leftarrow & (4) \\ & & \Updownarrow & & \Uparrow \\ (3) & \Leftrightarrow & (2) & \Leftrightarrow & (2') \end{array}$$

For example, the canonical A-derivation $\delta_{B/A}^{(q)}\colon B\to\Omega_A^{(q)}(B)$ is an extension of the null derivation $0\colon A\to\Omega_A^{(q)}(B)$. Therefore the assumption (4) implies that $\delta_{B/A}^{(q)}=0$ i.e. $\Omega_A^{(q)}(B)=0$.

On the other hand, we have known already the following

Proposition 2. ([2], Prop. II. 8)

Following conditions are equivalent for any integer $q\geqq 1$.

(1) The homomorphism $\varphi\colon B\otimes_A\Omega_k^{(q)}(A)\to\Omega_k^{(q)}(B)$ is left invertible.

(2) The homomorphism $\varphi\colon B\otimes_A\Omega_k^{(q)}(A)\to\Omega_k^{(q)}(B)$ is injective and the image $\mathrm{Im}(\varphi)$ is a direct summand of B-module $\Omega_k^{(q)}(B)$.

(3) The sequence

$$0\to B\otimes_A\Omega_k^{(q)}(A)\xrightarrow{\varphi}\Omega_k^{(q)}(B)\xrightarrow[j]{}\Omega_k^{(q)}(B/A)\to 0$$

is a split exact sequence.

(4) Every k-derivation of A (into a B-module) of order q can be extended to that of B.

Combining the above two propositions we obtain the next

Proposition 3. *B-homomorphism $\varphi: B\otimes_A \Omega_k^{(q)}(A)\to \Omega_k^{(q)}(B)$ is an isomorphism if and only if every k-derivation of A (into a B-module) of order q can be extended to that of B, in the unique way.*

Now we shall generalize the localization theorem $A_S\otimes_A \Omega_k^{(q)}(A) \cong \Omega_k^{(q)}(A_S)$. Localization A_S possesses two important properties—the one is the fact that the canonical map $A\to A_S$ is epimorphic (in the sense of category theory) and the other is the flatness over A. Therefore it is natural to consider at first the case $A\xrightarrow{g} B$ is an epimorphism (in the category of commutative rings). Then we have $\Omega_A^{(q)}(B)=I_{B/A}/I_{B/A}{}^{q+1}=(0)$ by the following

Lemma 2. ([4] $n^\circ 3$, *Th.* 1) *Following conditions are equivalent.*

(1) *$A\xrightarrow{g} B$ is an epimorphism.*

(2) $I_{B/A}=\mathrm{Ker}\,(B\underset{A}{\otimes} B\to B)=(0)$

(3) *The canonical homomorphism $B\underset{A}{\otimes} B\to B$ is an isomorphism.*

Therefore we obtain the next result from Prop. 1 above.

Proposition 4. *If $A\xrightarrow{g} B$ is an epimorphism (in the category of commutative rings), then the B-homomorphism $\varphi: B\otimes_A \Omega_k^{(q)}(A)\to \Omega_k^{(q)}(B)$ is surjective and any k-derivation of A of order q (into a B-module) has at most one extension to that of B.*

Next we want to seek after any connection between the flatness of $g: A\to B$ and the injectivity of $\varphi: B\otimes_A \Omega_k^{(q)}(A)\to \Omega_k^{(q)}(B)$.

To this purpose we shall describe the homomorphism $\varphi: B\otimes_A\Omega_k^{(q)}(A) \to \Omega_k^{(q)}(B)$ making use of ideals I_B and I_A. If we identify $\Omega_k^{(q)}(A)$ and $\Omega_k^{(q)}(B)$ with I_A/I_A^{q+1} and I_B/I_B^{q+1}, respectively, then the B-homomorphism

$$\varphi: B \otimes_A (I_A/I_A^{q+1}) \to I_B/I_B^{q+1}$$

is nothing but the one induced by the following procedure: first, from the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & I_A & \to & A \otimes_k A & \to & A & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & I_B & \to & B \otimes_k B & \to & B & \to & 0 \end{array}$$

we derive the A-homomorphism $I_A/I_A^{q+1} \to I_B/I_B^{q+1}$, and extend this naturally to the B-homomorphism

$$\varphi: B \otimes_A (I_A/I_A^{q+1}) \to I_B/I_B^{q+1}.$$

On the other hand, $B \otimes_A (I_A/I_A^{q+1})$ is identified with $B \otimes_A I_A/B \otimes_A I_A^{q+1}$. Therefore we can understand that φ is a map induced by the B-homomorphism $B \otimes_A I_A \to I_B/I_B^{q+1}$ which is the extension of the composite A-homomorphism $I_A \to I_B \to I_B/I_B^{q+1}$.

Thus we obtain the next

Lemma 3. *The homomorphism* $\varphi: B \otimes_A \Omega_k^{(q)}(A) \to \Omega_k^{(q)}(B)$ *is injective if and only if we have* $\mathrm{Ker}(B \otimes_A I_A \to I_B/I_B^{q+1}) = B \otimes_A I_A^{q+1}$. *(Of course,* $B \otimes_A I_A^{q+1}$ *means* $\mathrm{Im}(B \otimes_A I_A^{q+1} \to B \otimes_A I_A)$.*)*

Unfortunately the flatness of $g: A \to B$ does not yield the above condition. For example it does not occur if B is a purely inseparable algebraic extension of a field A.

However the flatness of $g: A \to B$ implies some weaker property.

Lemma 4. *If the homomorphism* $g: A \to B$ *is flat (i.e. if* B *is flat with respect to the* A*-module structure induced by* g*) then we have*

$$\mathrm{Ker}(B \otimes_A I_A \to I_B/I_A^{q+1}(B \otimes_k B)) = B \otimes_A I_A^{q+1}$$

for each integer $q \geqq 0$.

Proof. We shall prove by induction on q. This is trivial if $q = 0$.

Assume $q>0$ and x is an element in $\mathrm{Ker}(B\otimes_A I_A \to I_B/I_A^{q+1}(B\otimes B))$. Then x belongs to $\mathrm{Ker}(B\otimes_A I_A\to I_B/I_A^q(B\otimes B))$, hence to $B\otimes_A I_A^q$ by induction assumption. Therefore we can write x in the form

$$x=\sum_{i=1}^n b_i\otimes_A \xi_i,\quad b_i\in B,\quad \xi_i\in I_A^q.$$

And $\mathrm{Im}(x)=0$ means $\sum_i b_i\xi_i\in I_A^{q+1}(B\otimes B)$. Hence there exist the elements β_i in $B\otimes B$ and η_i in I_A^{q+1} such that

$$\sum_{i=1}^n (b_i\otimes 1)\xi_i+\sum_{i=n+1}^m \beta_i\eta_i=0 \qquad \text{in } B\otimes B.$$

However we can easily show that the induced homomorphism $A\otimes A \xrightarrow{g\otimes g} B\otimes B$ is also flat. Therefore, by the famous lemma of Bourbaki ([1], ch. 1, §2, Prop. 13 Cor. 1), there exist the elements λ_{ji} in $A\otimes A$ and ζ_j in $B\otimes B\,(1\leqslant i\leqslant m,\ 1\leqslant j\leqslant p)$ such that

i) $\displaystyle\sum_{i=1}^n \lambda_{ji}\xi_i+\sum_{i=n+1}^m \lambda_{ji}\eta_i=0\quad (1\leqslant i\leqslant p)\qquad \text{in } A\otimes A$

ii) $\displaystyle b_i\otimes 1=\sum_{j=1}^p \lambda_{ji}\zeta_j\quad (1\leqslant i\leqslant n)$

$\displaystyle\beta_i=\sum_{j=1}^p \lambda_{ji}\zeta_j\quad (n+1\leqslant i\leqslant m)$ in $B\otimes B$.

On the other hand, $A\otimes A$ and $B\otimes B$ are decomposed as follows.

$$A\otimes A=(A\otimes 1)\oplus I_A \quad \text{as left } A\text{-module.}$$

$$B\otimes B=(B\otimes 1)\oplus I_B \quad \text{as left } B\text{-module.}$$

Therefore we can write λ_{ji} and ζ_j $(1\leqslant i\leqslant n,\ 1\leqslant j\leqslant p)$ in the form

$$\lambda_{ji}=(a_{ji}\otimes 1)+\lambda'_{ji},\quad a_{ji}\in A,\quad \lambda'_{ji}\in I_A$$

$$\zeta_j=(b'_j\otimes 1)+\zeta'_j,\quad b'_j\in B,\quad \zeta'_j\in I_B.$$

Then we have

$$b_i\otimes 1=\sum_{j=1}^p \lambda_{ji}\zeta_j=\sum_{j=1}^p (a_{ji}\otimes 1+\lambda'_{ji})(b'_j\otimes 1+\zeta'_j)$$

$$=(\sum_{j=1}^{p}a_{ji}b'_j)\otimes 1+\sum_{j=1}^{p}\{\lambda'_{ji}(b'_j\otimes 1)+(a_{ji}\otimes 1)\zeta'_j+\lambda'_{ji}\zeta'_j\}$$

and, on the right hand side, the first term is in $B\otimes 1$ and the second in I_B. Hence we have

$$b_i\otimes 1=(\sum_{j=1}^{p}a_{ji}b'_j)\otimes 1.$$

And this means that $b_i=\sum_{j=1}^{p}a_{ji}b'_j$ $(1\leqslant i\leqslant n)$. Therefore

$$x=\sum_{i=1}^{n}b_i\otimes_A\xi_i=\sum_{i=1}^{n}(\sum_{j=1}^{p}a_{ji}b'_j)\otimes_A\xi_i$$

$$=\sum_{j=1}^{p}b'_j\otimes_A(\sum_{i=1}^{n}a_{ji}\xi_i)=\sum_{j=1}^{p}b'_j\otimes_A(\sum_{i=1}^{n}(a_{ji}\otimes 1)\xi_i)$$

$$=\sum_{j=1}^{p}b'_j\otimes_A\{\sum_{i=1}^{n}(\lambda_{ji}-\lambda'_{ji})\xi_i\}$$

$$=\sum_{j=1}^{p}b'_j\otimes_A\{-\sum_{i=n+1}^{m}\lambda_{ji}\eta_i-\sum_{i=1}^{n}\lambda'_{ji}\xi_i\}.$$

Hence x belongs to $B\otimes_A I_A^{q+1}$, since $\sum_{i=n+1}^{m}\lambda_{ji}\eta_i$ and $\sum_{i=1}^{n}\lambda'_{ji}\xi_i$ are in I_A^{q+1}. Thus we proved that $\mathrm{Ker}(B\otimes_A I_A\to I_B/I_A^{q+1}(B\otimes B))$ is contained in $B\otimes_A I_A^{q+1}$. The opposite inclusion is obvious. q.e.d.

As observed above we failed to connect the flatness of $g\colon A\to B$ directly with the injectivity of $\varphi\colon B\otimes_A\Omega_k^{(q)}(A)\to\Omega_k^{(q)}(B)$. However, Lemma 4 tells us that, under the assumption that g is an epimorphism, the flatness of g yields the injectivity of φ, because we have the next lemma.

Lemma 5. *The homomorphism $g\colon A\to B$ is an epimorphism if and only if we have $I_A(B\otimes_k B)=I_B$.*

Proof. If we start from the definition of tensor products, we can easily see that $\mathrm{Ker}(B\otimes_k B\xrightarrow{\nu}B\otimes_A B)=I_A(B\otimes_k B)$, where ν is the natural map. Therefore from the commutative diagram

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & I_B \arrow[r] \arrow[dr] & B\otimes_k B \arrow[r] & B \arrow[r] & 0 \quad \text{(exact)} \\
& & B\otimes_A B \arrow[ur] \arrow[dr] & & \\
& & & 0 &
\end{tikzcd}$$

we can deduce the following equivalence:

g: $A \to B$ is an epimorphism.

$\Leftrightarrow$ The natural homomorphism $B \underset{A}{\otimes} B \to B$ is an isomorphism.

$\Leftrightarrow$ $I_B = \mathrm{Ker}(B \underset{k}{\otimes} B \xrightarrow{\nu} B \underset{A}{\otimes} B)$

$\Leftrightarrow$ $I_B = I_A(B \underset{k}{\otimes} B)$ q.e.d.

Thus we obtain the next theorem.

Theorem 1. *Let A and B be commutative k-algebras and let g: $A \to B$ be an algebra homomorphism. If g: $A \to B$ is a flat epimorphism (i.e. if g is an epimorphism in the category of commutative rings and B is flat with respect to the A-module structure induced by g), then the B-homomorphism φ: $B \otimes_A \Omega_k^{(q)}(A) \to \Omega_k^{(q)}(B)$ is an isomorphism for any $q \geqq 0$. And every k-derivation of A (into a B-module) of order q can be extended to that of B, in the unique way.*

Proof. φ is surjective by Prop. 4, and injective by Lemmata 3, 4 and 5. The last half follows from Prop. 3.

Remark. g: $A \to B$ is a flat epimorphism, for example, in the following cases.

(i) B is a quotient ring of A, and g is the canonical homomorphism. (More generally, B is a "generalized quotient ring of A" in the sense of F. Richman or T. Akiba.)

(ii) A is a Prüfer domain with the quotient field K, B is a subring of K containing A, g is the canonical injection and k is a subring of A. (Such B is not always a quotient ring.)

§2

Let A be a ring and k a subring of A. Denote by A/k the cokernel of the k-homomorphism $k \to A$ (inclusion map).

We define a k-homomorphism $\delta: A \to I_A$ by the formula

$$\delta(a) = 1 \otimes a - a \otimes 1 \qquad (a \in A),$$

where I_A denotes the kernel of $A \otimes_k A \to A$. Then δ induces the k-homomorphism $A/k \to I_A$, in an obvious manner, which we denote by the same letter δ.

Lemma 6. *Let A, k and δ be as above. Then we obtain the following results.*

(i) *The k-homomorphism $\delta: A/k \to I_A$ induces naturally the A-homomorphism $\delta_*: A \otimes_k (A/k) \to I_A$, which is actually an A-isomorphism between left A-modules.*

(ii) *If A is a free k-module with a basis $\{1, \alpha_\lambda \mid \lambda \in \Lambda\}$, then I_A is a free A-module with a basis $\{\delta(\alpha_\lambda) \mid \lambda \in \Lambda\}$.*

Proof. Because the exact sequence of left A-modules

$$0 \to I_A \to A \otimes_k A \to A \to 0$$

splits, we have the exact sequence

$$0 \to A \xrightarrow{\alpha} A \otimes_k A \xrightarrow{\beta} I_A \to 0$$

where α and β are mappings such that

$$\alpha: a \mapsto a \otimes 1$$

$$\beta: a \otimes b \mapsto a(1 \otimes b - b \otimes 1).$$

On the other hand, from the exact sequence of k-modules

$$0 \to k \to A \xrightarrow{p} A/k \to 0$$

we obtain the exact sequence of A-modules

$$0 \to A \xrightarrow{\alpha} A \otimes_k A \xrightarrow{\gamma} A \otimes_k (A/k) \to 0.$$

And it is obvious that the following diagram is commutative.

$$\begin{tikzcd} A \otimes_k A \arrow[r, "\gamma"] \arrow[dr, "\beta"'] & A \otimes_k (A/k) \arrow[d, "\delta_*"] \\ & I_A \end{tikzcd}$$

Hence δ_* is an A-isomorphism, which proves (i). The statement (ii) is obvious from (i), because A/k can be identified with a free k-module having a basis $\{\alpha_\lambda\}$. q.e.d.

Now we shall investigate the condition for the coincidence of $\mathscr{D}(A/k)$ with $\mathrm{Hom}_k(A, A)$, where $\mathscr{D}(A/k)$ denotes the derivation algebra of A over k.

Proposition 5. *Let A be a ring and k be a subring of A. Then the following three statements are equivalent.*

(i) $\mathscr{D}(A/k) = \mathrm{Hom}_k(A, A)$.

(ii) *Each element of* $\mathrm{Hom}_k(A, A)$ *which vanishes on k is a k-derivation of some order.*

(iii) *Each element of* $\mathrm{Hom}_A(I_A, A)$ *vanishes on some power I_A^{q+1} of I_A.*

Proof. (i)$\Rightarrow$(ii): Let f be an arbitrary element of $\mathrm{Hom}_k(A, A)$ such that $f|_k = 0$. Then, since $\mathscr{D}(A/k) = A \oplus \mathscr{D}_0(A/k)$, f can be written in the form

$$f = a + \varDelta, \quad a \in A, \quad \varDelta \in \mathscr{D}_0(A/k).$$

But, then, we have $0 = f(1) = (a + \varDelta)1 = a + \varDelta(1) = a$. Therefore it follows that $f = \varDelta \in \mathscr{D}_0(A/k)$. Thus f is a k-derivation of some order.

(ii)$\Rightarrow$(iii): Let φ be an element of $\mathrm{Hom}_A(I_A, A)$. Consider the map f which is determined by the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[dr, "f"'] & I_A \arrow[d, "\varphi"] \\ & A. \end{tikzcd}$$

Then f belongs to $\mathrm{Hom}_k(A, A)$ and vanishes on k. Hence, by assumption, f is a k-derivation of some order, say, q. If we define an A-homomorphism $f_*: A \otimes_k A \to A$ by the formula

$$f_*(a \otimes b) = a \cdot f(b)$$

then we can easily see that $f_*|_{I_A} = \varphi$. On the other hand, since f is a q-th order k-derivation, f_* vanishes on I_A^{q+1}. Therefore φ vanishes also on I_A^{q+1}.

(iii)$\Rightarrow$(i): We have only to show that $\mathrm{Hom}_k(A, A) \subset \mathscr{D}(A/k)$. Choose arbitrarily an element g of $\mathrm{Hom}_k(A, A)$ and set $f = g - g(1)_L$ (where $g(1)_L$ means the left multiplication by $g(1)$). Then it is obvious that $f \in \mathrm{Hom}_k(A, A)$ and $f|_k = 0$. Therefore f is factored through $A \xrightarrow{p} A/k$ as follows:

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[dr, "f"'] & A/k \arrow[d, "\bar{f}"] \\ & A. \end{tikzcd}$$

But the k-homomorphism $\bar{f}: A/k \to A$ can be extended naturally to the A-homomorphism $\psi: A \otimes_k (A/k) \to A$, which is again factored into the composition of the isomorphism $\delta_*: A \otimes_k (A/k) \xrightarrow{\sim} I_A$ and an A-homomorphism $\varphi: I_A \to A$ i.e. $\psi = \varphi \circ \delta_*$. Then, by assumption, φ vanishes on some power, say I_A^{q+1}, of I_A. Hence we have a commutative diagram

$$\begin{tikzcd} I_A \arrow[r, "\nu"] \arrow[d, "\varphi"'] & I_A/I_A^{q+1} = \Omega_k^{(q)}(A) \arrow[dl, "\varphi'"] \\ A & \end{tikzcd}$$

of A-homomorphisms naturally induced by φ.

And we can easily verify that the diagram below is also commutative (i.e. $\nu \circ \delta_* \circ i \circ p = \delta_A^{(q)}$).

$$\begin{tikzcd}
A \arrow[dr, "p"] \arrow[rrrrd, "\delta_A^{(q)}"] \arrow[ddr, bend right, "f"'] & & & & \\
& A/k \arrow[r, "i"] \arrow[d, "\bar{f}"] & A\otimes_k(A/k) \arrow[r, "\sim"] \arrow[dl, "\psi"] & I_A \arrow[r, "\nu"'] \arrow[dll, bend left, "\varphi", "\delta_*"'] & \Omega_k^{(q)}(A) \arrow[dlll, bend left, "\varphi'"] \\
& A & & &
\end{tikzcd}$$

Thus we have $f=\varphi'\circ\delta_A^{(q)}$. Therefore f is a k-derivation of order q, and $g=g(1)_L+f$ belongs to $\mathscr{D}(A/k)=A\oplus\mathscr{D}_0(A/k)$. q.e.d.

Next we shall proceed to the reduction theorem.

Lemma 7. *Let A be a subalgebra of a k-algebra B. If B is both A-flat and k-flat, then the natural B-homomorphism $B\otimes_A I_A\to I_B$ is an injection.*

Proof. From the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & I_A & \to & A\otimes_k A & \to & A & \to & 0 \quad \text{(exact)} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I_B & \to & B\otimes_k B & \to & B & \to & 0 \quad \text{(exact)}
\end{array}$$

we obtain the next commutative diagram by virtue of the A-flatness of B:

$$\begin{array}{ccccccccc}
0 & \to & B\otimes_A I_A & \to & B\otimes_k A & \to & B & \to & 0 \quad \text{(exact)} \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & I_B & \to & B\otimes_k B & \to & B & \to & 0 \quad \text{(exact).}
\end{array}$$

On the other hand, by the k-flatness of B, the homomorphism $B\otimes_k A \to B\otimes_k B$ is injective. Hence $B\otimes_A I_A\to I_B$ is also injective. q.e.d.

Proposition 6. *Let k be a subfield of a field L, and let K be an intermediate ring between k and L. Then, $\mathscr{D}(K/k)\neq \mathrm{Hom}_k(K, K)$ implies $\mathscr{D}(L/k)\neq\mathrm{Hom}_k(L, L)$.*

Proof. On account of Prop. 5, $\mathscr{D}(K/k)\neq\mathrm{Hom}_k(K, K)$ means that there exists an element φ in $\mathrm{Hom}_K(I_K, K)$ not vanishing on any power

I_K^q of I_k. Regarding φ as an element of $\mathrm{Hom}_K(I_K, L)$, we can extend it to an L-homomorphism $\varphi': L\otimes_K I_K \to L$ in an obvious manner. On the other hand, since L is K-flat and k-flat, the natural L-homomorphism $L\otimes_K I_K \to I_L$ is an injection. Hence φ' can be extended to an L-homomorphism $\varphi'': I_L \to L$. And φ'' does not vanish on any power of I_L, because we have $I_K^q \subset I_L^q$. Therefore it follows from Prop. 5 that $\mathscr{D}(L/k) \neq \mathrm{Hom}_k(L, L)$. q.e.d.

This proposition is useful to reduce the problem to the smaller field case. Indeed we see immediately, by the next easy lemma, that the field L must be purely inseparable over a subfield k if $\mathscr{D}(L/k) = \mathrm{Hom}_k(L, L)$.

Lemma 8. *Let k be a field and suppose that one of the following conditions is satisfied:*

(1) *K is a separably algebraic simple extension field of k and $k \neq K$.*

(2) *$K = k[x]$ is a polynomial ring over k in one variable.*

Then we have $\mathscr{D}(K/k) \neq \mathrm{Hom}_k(K, K)$.

Proof. In case (1): we have $I_K = I_K^q \neq (0)$ for any $q \geqq 1$. Hence the condition (iii) in Prop. 5 is not satisfied.

In case (2): if we denote by I_n the K-free submodule of $K\otimes_k K$ generated by $\delta(x)^n$, then we have $I_K = \bigoplus_{n=1}^{\infty} I_n$ and $I_K^q = \bigoplus_{n=q}^{\infty} I_n$. Hence there exists an element φ in $\mathrm{Hom}_K(I_K, K)$ not vanishing on any power of I_K.

Thus, in any case, we have $\mathscr{D}(K/k) \neq \mathrm{Hom}_k(K, K)$ by Prop. 5, (iii). q.e.d.

Lemma 9. *Let K be a purely inseparable extension of a field k of characteristic p, and suppose that the exponent of K over k be infinite. Then we have $\mathscr{D}(K/k) \neq \mathrm{Hom}_k(K, K)$.*

Proof. For any non-negative integer n, there exists an element x of K such that $x^{p^n} \notin k$. Set $A = k(x)$. Then we have $I_A^{p^n} \neq (0)$, hence $I_K^{p^n} \neq (0)$ because $I_A^{p^n} \subset I_K^{p^n}$. In other words, the ideal I_K is not nilpotent. Now, if $\bigcap_{q \geq 1} I_K^q \neq (0)$ then it is obvious that there exists an element of $\mathrm{Hom}_K(I_K, K)$ which does not vanish on $\bigcap_{q \geq 1} I_K^q$. If $\bigcap_{q \geq 1} I_K^q = (0)$, then we can choose the elements x_q such that $x_q \in I_K^q - I_K^{q+1}$. Hence there exists an element φ of $\mathrm{Hom}_K(I_K, K)$ such that $\varphi(x_q) \neq 0$ for each q. Therefore we have $\mathscr{D}(K/k) \neq \mathrm{Hom}_k(K, K)$ by Prop. 5, (iii).

Now we shall consider the case in which K is a purely inseparable extension of a field k having finite exponent e over k.

Let $B_1 = \{\alpha_\lambda \mid \lambda \in \Lambda\}$ be a p-basis for K over k (i.e. a maximal p-independent subset of K over $k(K^p)$). Successively, let B_i be a maximal p-independent subset of $B_{i-1}^p = \{\beta^p \mid \beta \in B_{i-1}\}$ over $k(K^{p^i})$, for $i = 2, 3, \cdots, e$. Then it is obvious that B_i is a p-basis for $k(K^{p^{i-1}})$ over k.

By the length of α_λ we shall mean the integer n such that $\alpha_\lambda^{p^{n-1}} \in B_n$ and $\alpha_\lambda^{p^n} \notin B_{n+1}$. (Of course, the length depends on the choice of B_i's.)

Then we obtain the following.

Lemma 10. *Let k be a field, and K a purely inseparable extension of finite exponent e over k. Let $\{\alpha_\lambda \mid \lambda \in \Lambda\}$ be a p-basis for K over k and denote the length of α_λ in the above sense by $l(\alpha_\lambda)$. If we denote by I_n the K-submodule of I_K generated by*

$$\{\delta(\alpha_{\lambda_1})^{n_1} \cdots \delta(\alpha_{\lambda_s})^{n_s} \mid 0 < n_i < p^{l(\alpha_{\lambda_i})},\ n_1 + \cdots + n_s = n\}$$

then I_n is a vector space over K which has the above set as a basis and we have $I_K = \bigoplus_{n=1}^{\infty} I_n$, $I_K^q = \bigoplus_{n=q}^{\infty} I_n$ and $\bigcap_{q=1}^{\infty} I^q = (0)$.

Proof. It is evident that the set $\{1, \prod_{\lambda \in \Lambda} \alpha_\lambda^{n_\lambda} \mid 0 < n_\lambda < p^{l(\alpha_\lambda)}$, almost all $n_\lambda = 0\}$ is a linear basis for K over k. Hence, by Lemma 6, the set $\{\delta(\prod_{\lambda\in\Lambda}\alpha_\lambda^{n_\lambda})\,|\,0<n_\lambda<p^{l(\alpha_\lambda)}$, almost all $n_\lambda=0\}$ is a basis of a left K-module I_K. Let us denote by J_n the submodule of I_K generated by all $\delta(\prod_\lambda\alpha_\lambda^{n_\lambda})$'s such that $\sum_\lambda n_\lambda=n$. Then, from the equation

$$\delta(x_1x_2\cdots x_n)=\delta(x_1)\delta(x_2)\cdots\delta(x_n)$$
$$+\sum_{s=1}^{n-1}\sum_{i_1<\cdots<i_s}x_1\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_s}\cdots x_n\cdot\delta(x_{i_1})\cdots\delta(x_{i_s})$$

we see easily that we have $\sum_{n=1}^{q}I_n=\sum_{n=1}^{q}J_n=\bigoplus_{n=1}^{q}J_n$. Therefore, since the linear independence is a finiteness condition, $\sum_{n=1}^{q}I_n$ is a vector space over K with a basis $\{\prod_\lambda\delta(\alpha_\lambda)^{n_\lambda}\,|\,\sum_\lambda n_\lambda\leqq n\}$. Consequently I_n has a basis $\{\prod_\lambda\delta(\alpha_\lambda)^{n_\lambda}\,|\,\sum_\lambda n_\lambda=n\}$ and $\sum_{n=1}^{q}I_n=\bigoplus_{n=1}^{q}I_n$. Then $I_K=\bigoplus_{n=1}^{\infty}I_n$ follows immediately from $I_K=\bigoplus_{n=1}^{\infty}J_n$, and the rest is obvious. q.e.d.

Lemma 11. *Let k be a field and let K be a purely inseparable extension of finite exponent over k such that $[K:k]=\infty$. Then we have $\mathscr{D}(K/k)\neq\mathrm{Hom}_k(K,K)$.*

Proof. Let $\{\alpha_\lambda\,|\,\lambda\in\Lambda\}$ and I_n be as in Lemma 10. Then, by assumption, Λ is not a finite set and we have $I_n\neq(0)$ for all n. Moreover each I_n is a free module. Hence there exists an element φ in $\mathrm{Hom}_K(I_K,K)$ not vanishing on any power of I_K. Therefore it follows that $\mathscr{D}(K/k)\neq\mathrm{Hom}_k(K,K)$ by Prop. 5, (iii). q.e.d.

Now we arrive at the following main theorem.

Theorem 2.[1] *Let k be a field and L an arbitrary extension field*

1) A. Hattori gave a simplified proof of this theorem. His proof is based on the isomorphism $\mathrm{Hom}_k^0(L,L)\cong\mathrm{Hom}_L(I,L)$ where $\mathrm{Hom}_k^0(L,L)$ denotes the set of the elements f of $\mathrm{Hom}_k(L,L)$ such that $f(1)=0$, and I means I_L. It runs as follows:

$$D(L/k)=\mathrm{Hom}_k(L,L)\Leftrightarrow\bigcup_{q=1}^{\infty}D_0^{(q)}(L/k)=\mathrm{Hom}_k^0(L,L)$$
$$\Leftrightarrow\ \bigcup_{q=1}^{\infty}\mathrm{Hom}_L(I/I^{q+1},L)=\mathrm{Hom}_L(I,L)$$

of k. Then the following three conditions are equivalent.

(1) $\mathscr{D}(L/k)=\mathrm{Hom}_k(L, L)$.

(2) *L is purely inseparable over k and $[L:k]<\infty$.*

(3) *$I_L=\mathrm{Ker}(L\otimes_k L\to L)$ is a nilpotent ideal of $L\otimes_k L$.*

Proof. The implication (1)$\Rightarrow$(2) follows from Prop. 6, Lemma 8, Lemma 9 and Lemma 11. The implication (2)$\Rightarrow$(3) is obvious and (3)$\Rightarrow$(1) follows immediately from Prop. 5, (iii). q.e.d.

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$\Leftrightarrow$ I is nilpotent.

$\Leftrightarrow$ L is purely inseparable over k and $[L:k]<\infty$.

cf. A. Hattori "On high order derivations from the view-point of two-sided modules" (Scientific Papers of the College of General Education, Univ. of Tokyo, Vol. 20 (1970), pp. 1-11).