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# Connected fully reducible affine group schemes in positive characteristic are Abelian

By

Moss Eisenberg SWEEDLER*

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## Introduction

This paper gives the broad middle ground of a theorem of which the two extreme cases were known.

In [2] Hochschild proved that if $L$ is a finite dimensional $p$-Lie algebra all of whose $p$-Lie modules are completely reducible then $L$ is abelian. Let $U$ be the $p$-universal enveloping algebra of $L$. $U$ is a finite dimensional Hopf algebra and its dual $A$ is a finite dimensional commutative local Hopf algebra. $L$ being "fully reducible" is equivalent to $U$ being semi-simple and this is equivalent to $A$ being co-semi-simple; i.e., the (direct) sum of its simple subcoalgebras. By Hochschild's result if $A$ is co-semi-simple it is cocommutative. It is then easy to show that when the ground field is algebraically closed $A$ is (isomorphic to) the group algebra of a group of the form $\boldsymbol{Z}/p\boldsymbol{Z}\times\cdots\times\boldsymbol{Z}/p\boldsymbol{Z}$ (a finite number of times).

In [4] Nagata has shown that a fully reducible connected affine algebraic group in positive characteristic is a torus. Thus if $A$ is the Hopf algebra of regular functions on the group and if the ground field is algebraically closed $A$ is (isomorphic to) the group algebra of a group

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of the form $\boldsymbol{Z}\times\cdots\times\boldsymbol{Z}$ (a finite number of times).

Hochschild's result and Nagata's result are but opposite extremes of the characterization of fully reducible (absolutely) connected affine group schemes in positive characteristic. The complete result is that if $A$ is a commutative Hopf algebra representing a fully reducible absolutely connected affine group scheme in positive characteristic then $A$ is cocommutative. If the ground field is algebraically closed then $A$ is the group algebra of an abelian group all of whose torsion elements have $p$-power torsion.

In order to prove this result we first engage in some general Hopf algebra theory. This theory when applied to affine group schemes generalizes a classical result and shows that a (closed) normal subgroup scheme of a fully reducible affine group scheme is again fully reducible. We use this to show that the "infinitesimal" normal subgroup schemes of a fully reducible group scheme are again fully reducible. Using Hochschild's result and some finite dimensional Hopf algebra structure theory we show that these infinitesimal subgroup schemes are commutative. It follows that the original group scheme is commutative because we prove that the original group scheme can be approximated by the infinitesimal subgroup schemes. This takes the form of a (Krull type) intersection theorem for Hopf algebras.

The intersection theorem is that if $A$ is a finitely generated commutative Hopf algebra representing an absolutely connected affine group scheme then for any proper ideal $I\subset A$ the intersection $\bigcap_n I^n$ is zero. We give an example to show that it is necessary to assume that $A$ is a Hopf algebra.

(We use "fully reducible" to mean that all rational representations are completely reducible. "Absolutely" connected means that the group scheme remains connected when raised to the algebraic closure.)

## §1. Splittings of Hopf algebra maps.

We shall be working over a ground field, say $k$. By $\otimes$, Hom, End, etc. we mean over $k$. For a vector space $V$ we use $V^*$ to denote the linear dual, $\mathrm{Hom}(V, k)$. Often for $f \in V^*$, $v \in V$ we will write $<f, v>$ in place of $f(v)$. We use the terms "coalgebra" and "Hopf algebra" as they are defined in [7]. For a coalgebra the diagonal map is usually denoted $\Delta$ and the counit $\varepsilon$. For Hopf algebras $S$ denotes the antipode. We freely use the comultiplication notation for coalgebras, [7, p. 10] or [6, p. 323–324].

Primarily we shall be working with commutative Hopf algebras. For such a Hopf algebra $A$ the functor

$$G_A(\ )=\{\text{algebra maps from } A \text{ to } (\ )\}$$

is an affine group scheme. We usually shall work in Hopf algebraic terms and (sometimes) indicate what the implications are for the group scheme.

Given a comodule $M \xrightarrow{\psi} M \otimes A$, [7, p. 30] we have the functor $M \otimes (\ )$. Given a commutative algebra $B$, the $B$-module $M \otimes B$ becomes a $G_A(B)$-module as follows: for $g \in G_A(B)$ the action of $g$ on $M \otimes B$ is given by

$$M \otimes B \xrightarrow{\psi \otimes I} M \otimes A \otimes B \xrightarrow{I \otimes g \otimes I} M \otimes B \otimes B \xrightarrow{I \otimes \mathrm{mult}} M \otimes B.$$

So a comodule is just an affine $G_A$-module (functor).

In [7, p. 287, p. 290] we call a Hopf algebra or coalgebra co-semi-simple if it is the sum of its simple subcoalgebras. This sum is necessarily direct, [7, p. 166, 8.0.6]. A coalgebra being co-semi-simple is equivalent to every comodule of the coalgebra being completely reducible, [7, p. 288, 14.0.1]. A Hopf algebra $A$ being co-semi-simple is equivalent to there being a (unique) coalgebra $C \subset A$ with $A = k \oplus C$, [7, p. 293, 14.0.3, c]. Using this characterization Larson has observed [3, p. 9, Lemma 1.3] that a Hopf algebra being co-semi-simple is unchanged by field extension.

For a commutative Hopf algebra $A$ to be co-semi-simple it is necessary and sufficient for all affine $G_A$-modules to be completely reducible.

**1.1 Theorem.** *Let $B$ be a Hopf algebra with subHopf algebra $A$. If $C$ is a subcoalgebra of $B$ with $B=A\oplus C$ then $AC\subset C\supset CA$.*

**Proof.** We shall use the "$\rightharpoonup$" action of $B^*$ on $B$, [6, p. 328, 2.1] and the "$\leftharpoonup$" action of $B$ on $B^*$ [6, p. 328, 2.3]. For $f\in B^*$, $b, \beta\in B$ the actions are defined by

$$<f\leftharpoonup b, \beta>\equiv<f, \beta S(b)>$$

$$f\rightharpoonup b\equiv\sum_{(b)} b_{(1)}<f, b_{(2)}>.$$

Let $A^{\perp}=\{f\in B^*\,|<f, A>=0\}$. We show

1.2 $$A^{\perp}\leftharpoonup A\subset A^{\perp},$$

1.3 $$A^{\perp}\rightharpoonup B\subset C.$$

If $f\in A^{\perp}$, $a\in A$ then

$$<f\leftharpoonup a, A>=<f, AS(a)>=0$$

since $A$ is a subHopf algebra and $S(a)\in A$. This establishes 1.2.

If $b\in B$ write $b=a+c$ with $a\in A$, $c\in C$. Then

$$f\rightharpoonup b=\sum_{(a)} a_{(1)}<f, a_{(2)}>+\sum_{(c)} c_{(1)}<f, c_{(2)}>.$$

Since $\Delta(a)\in A\otimes A$ and $\Delta(c)\in C\otimes C$ we have 1.3.

We can find $\xi\in B^*$ where $\xi\in A^{\perp}$ and $\xi|C$ is the counit of $C$ because $B=A\oplus C$. Then for $c\in C$, $c=\xi\rightharpoonup c$. If $a\in A$

$$ca=(\xi\rightharpoonup c)a=\sum_{(a)}(\xi\leftharpoonup a_{(2)})\rightharpoonup(ca_{(1)})$$

by [6, p. 328, 2.5]. Say $\Delta(a)=\sum a_i\otimes a_i'$ with $\{a_i\}\cup\{a_i'\}\subset A$. Then

$$ca=\sum_i(\xi\leftharpoonup a_i')\rightharpoonup(ca_i).$$

By 1.2 $\{\xi\leftharpoonup a_i'\}\subset A^{\perp}$ and then by 1.3 $\{(\xi\leftharpoonup a_i')\rightharpoonup(ca_i)\}\subset C$. Thus $ca\in C$.

The mirror proof shows $AC \subset C$. q.e.d.

If $P$ is the projection from $B$ to $A$ with kernel $C$ then 1.1 shows that $P$ is an $A$-bimodule map. In these terms the theorem can be dualized.

Suppose $\pi: A \to B$ is a Hopf algebra map. Then

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{I \otimes \pi} A \otimes B$$

and

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\pi \otimes I} B \otimes A$$

give $A$ right and left $B$-comodule structures (resp.).

**1.4. Theorem.** *Suppose $\pi: A \to B$ is a surjective Hopf algebra map and $\boldsymbol{I} \subset A$ is an ideal where $A = \boldsymbol{I} \oplus \operatorname{Ker} \pi$. If $Q: B \to A$ is the splitting of $A \xrightarrow{\pi} B \to 0$ with* $\operatorname{Im} Q = \boldsymbol{I}$ *then $Q$ is a right and left $B$-comodule map.*

Since the proof of 1.1 is not so easily dualized we include a proof of 1.4.

**Proof.** Since $A = \boldsymbol{I} \oplus \operatorname{Ker} \pi$ is a direct sum of ideals we can write $1 = f + e$ with $f$ and $e$ orthogonal idempotents and $f \in \boldsymbol{I}$, $e \in \operatorname{Ker} \pi$. We have

1.5 $$(I \otimes \pi)\, \Delta \operatorname{Ker} \pi \subset \operatorname{Ker} \pi \otimes B$$

since if $x \in \operatorname{Ker} \pi$

$$(\pi \otimes I)(I \otimes \pi)\Delta x = (\pi \otimes \pi)\Delta x = \Delta \pi x = 0$$

because $\pi$ is a coalgebra map. Thus we can write

1.6 $$(I \otimes \pi)\Delta e = \sum e_i \otimes b_i, \text{ with } \{e_i\} \subset \operatorname{Ker} \pi, \{b_i\} \subset B.$$

Next we show

1.7 $$(I\otimes\pi)\Delta(\boldsymbol{I})\subset\boldsymbol{I}\otimes B.$$

Since $A\xrightarrow{M}A$, $a\to ae$ has kernel $\boldsymbol{I}$ it suffices to show that $(M\otimes I)(I\otimes\pi)\Delta\boldsymbol{I}=0$. For $a\in\boldsymbol{I}$,

$$\begin{aligned}(M\otimes\pi)\Delta a&=\sum_{(a)}a_{(1)}e\otimes\pi(a_{(2)})\\&=\sum_{(a),(e)}a_{(1)}e_{(1)}\otimes\pi(a_{(2)}e_{(2)}S(e_{(3)}))\\&=\sum_{(a),(e)}a_{(1)}e_{(1)}\otimes\pi(a_{(2)}e_{(2)})\pi S(e_{(3)})\\&=\sum_{(e)}[(I\otimes\pi)\Delta ae_{(1)}][1\otimes S\pi(e_{(2)})].\end{aligned}$$

We are using that $\pi$ is an algebra map and $\pi S=S\pi$. By 1.6 the above equals

$$\sum_i[(I\otimes\pi)\Delta ae_i][1\otimes S(b_i)]$$

which is zero since $\{e_i\}\subset\operatorname{Ker}\pi$ implies $ae_i=0$ for all $i$.

By 1.5 and 1.7 we see that $A=\boldsymbol{I}\oplus\operatorname{Ker}\pi$ is a direct sum of right $B$-comodules. Thus the splitting map $Q$ is a right $B$-comodule map.

By the mirror proof $Q$ is a left $B$-comodule map. q.e.d.

We return to the setting of 1.1 where $B$ is a Hopf algebra with subHopf algebra $A$. Let $A^+=\operatorname{Ker}\varepsilon_A$, the augmentation ideal of $A$. The two-sided ideal in $B$ generated by $A^+$ is a Hopf ideal [7, p. 87] and [7, p. 88, Exercise] so that $B/BA^+B$ has a quotient Hopf algebra structure, [7, p. 87]. Under suitable "normality" conditions we have that

1.8 $$BA^+=BA^+B \quad\text{or}\quad A^+B=BA^+B.$$

For example if $B$ is a group algebra and $A$ is the subgroup algebra of a normal subgroup. Or if $B$ is commutative 1.8 will hold.

**1.9 Corollary.** *Suppose $B$ is a Hopf algebra with subHopf* *algebra $A$ and* 1.8 *holds. Then $B/BA^{+}B$ is a co-semi-simple Hopf algebra.*

**Proof.** Say $BA^{+}=BA^{+}B$. Then by 1.1 $BA^{+}=A^{+}\oplus CA^{+}$. Thus if $\pi: B\rightarrow B/BA^{+}$ is the natural Hopf algebra map we see that

$$B/BA^{+}=\pi(A)\oplus\pi(C).$$

$\pi(A)=k$ and $\pi(C)$ is a subcoalgebra so that by [6, p. 333, 3.2] or [7, p. 293, 14.0.3, $c$] the Hopf algebra $B/BA^{+}$ is co-semi-simple. q.e.d.

In a co-semi-simple coalgebra any subcoalgebra has a coalgebra complement. Thus we have

**1.10 Corollary.** *Suppose $B$ is a commutative Hopf algebra with subHopf algebra $A$. If $B$ is co-semi-simple then $B/BA^{+}$ is also.*

On the group scheme level we have the following interpretation of 1.10. The inclusion $A\hookrightarrow B$ gives an epimorphism $G_A \twoheadleftarrow G_B$ with kernel $G_{B/BA^{+}}$. Thus $G_{B/BA^{+}}$ is a normal subgroup scheme of $G_B$. By 1.10 we have that all affine $G_B$-modules are completely reducible implies that all affine $G_{B/BA^{+}}$-modules are completely reducible. We shall show in a subsequent paper that all closed normal subgroup schemes of $G_B$ are of the form $G_{B/BA^{+}}$ for suitable $A\subset B$. (Of course, $A$ is the subalgebra of "invariants" of the normal subgroup scheme.) Then 1.10 takes on the classical form: "a normal (closed) subgroup scheme of a fully reducible affine group scheme is fully reducible."

**1.11 Example.** Suppose $B$ is a commutative Hopf algebra in characteristic $p>0$. Let $B^{(p^n)}=\{b^{p^n}\mid b\in B\}$. This is a subHopf algebra of $B$ if $k$ is perfect. If $kB^{(p^n)}$ is the $k$ span of $B^{(p^n)}$ then $kB^{(p^n)}$ is always a subHopf algebra. By 1.10 we see that $B/BkB^{(p^n)+}$ is co-semi-simple if $B$ is co-semi-simple.

Let $B^{+(p^n)}=\{b^{p^n}\mid b\in B^{+}\}$ then the ideal $BkB^{(p^n)+}$ is equal to $BB^{+(p^n)}$ and so

1.12 $B/BB^{+(p^n)}$ is a co-semi-simple Hopf algebra if $B$ is.

## §2. The intersection theorem.

Example 1.11 will provide one of the key steps in proving our main theorem. In order to use the example we will need the fact that $B$ can be approximated by $\{B/BB^{+(p^n)}\}$, i.e. the intersection of the ideals $BB^{+(p^n)}$ is zero, (under suitable conditions). This will follow from our intersection theorem.

For a commutative algebra $A$ over the ground field $k$ let $\mathrm{Sep}\,A$ (or $\mathrm{Sep}_k A$) denote the subalgebra of $A$ consisting of elements which satisfy a non-zero *separable* polynomial in $k[X]$.

We shall use the following standard facts about Sep:

2.1 If $K$ is a field extension of $k$ then

$$(\mathrm{Sep}_k A)\otimes K=\mathrm{Sep}_K(A\otimes K).$$

2.2 If $k$ is algebraically closed then $\mathrm{Sep}\,A$ is the span of the idempotents of $A$.

2.3 If $A$ is finitely generated then $\mathrm{Sep}\,A$ is finite dimensional.

2.4 If $\boldsymbol{I}$ is an ideal in $A$ consisting of nilpotent elements and $\pi: A \to A/\boldsymbol{I}$ is the natural algebra map then $\pi$ restricted to $\mathrm{Sep}\,A$ carries $\mathrm{Sep}\,A$ isomorphically onto $\mathrm{Sep}\,(A/\boldsymbol{I})$.

2.5 If $B$ is a subalgebra of $A$ then $\mathrm{Sep}\,B=B\cap\mathrm{Sep}\,A$.

Suppose $A$ is a commutative Hopf algebra over a perfect field $k$, let $\mathcal{N}$ be the ideal of all nilpotent elements in $A$ so that $A/\mathcal{N}$ is a reduced algebra. Since $k$ is perfect $A/\mathcal{N}\otimes A/\mathcal{N}$ is reduced and $\mathcal{N}$ lies in the kernel of the composite

$$A\xrightarrow{\Delta}A\otimes A\to A/\mathcal{N}\otimes A/\mathcal{N}.$$

Thus $\Delta\mathcal{N}\subset A\otimes\mathcal{N}+\mathcal{N}\otimes A$. Clearly $\mathcal{N}\subset\mathrm{Ker}\,\varepsilon$ and $S(\mathcal{N})\subset\mathcal{N}$, so that $\mathcal{N}$ is a Hopf ideal and $A/\mathcal{N}$ has a natural Hopf algebra structure where

$\pi: A \to A/\mathcal{N}$ is a Hopf algebra map.

**2.6 Lemma.** *If $A$ is a commutative Hopf algebra and $V$ is a finite dimensional subspace of $A$ or $V$ is a finite subset of $A$ then the subHopf algebra of $A$ generated by $V$ is finitely generated as an algebra.*

**Proof.** Let $C$ be the subcoalgebra of $A$ generated by $V$. By [7, p. 47, 2.2.2] $C$ is finite dimensional. Since $A$ is commutative $SS=I$, [7, p. 74, 4.0.1]. Thus if $D=C+S(C)$ then $D$ is a finite dimensional subcoalgebra of $A$ containing $V$ and $S(D)\subset D$. ($D$ is a subcoalgebra since $S$ is a coalgebra antimorphism, [7, p. 74, 4.0.1].) If $B$ is the subalgebra of $A$ generated by $D$ then $B$ is the subHopf algebra generated by $V$ and $B$ is finitely generated as an algebra. q.e.d.

In the above proof we used the fact that $A$ was commutative to insure $SS=I$. If we do not assume that $A$ is commutative but that for some $n$, $S^n=I$ then the conclusion remains valid. The proof is changed only in that $D$ is defined to be $C+S(C)+\cdots+S^{n-1}(C)$.

**2.7 Lemma.** *If $A$ is a commutative Hopf algebra over a perfect field $k$ and* $\operatorname{Sep} A=k$ *then $A/\mathcal{N}$ is a domain.*

**Proof.** By 2.4 $\operatorname{Sep} A/\mathcal{N}=k$ and so it suffices to prove: if $A$ is a commutative reduced Hopf algebra over a perfect field $k$ then $A$ is a domain. Since $k$ is perfect if we tensor $A$ with the algebraic closure of $k$ what we get is still reduced. Thus by 2.1 we may assume that $k$ is algebraically closed.

To show that $A$ is a domain it suffices to show that for any two elements of $A$ the subHopf algebra which they generate is a domain. Such subHopf algebras are finitely generated as algebras by 2.6 and their Sep is $k$ by 2.5. Thus we may also assume $A$ is finitely generated.

Now $k$ is algebraically closed and $A$ is a finitely generated reduced

Hopf algebra and so is the ring of regular functions on a classical affine algebraic group. Such a group is the disjoint union of finitely many Zariski *irreducible* components. Thus $A$ is the direct sum of a finite number of domain $k$ algebras. $\operatorname{Sep} A=k$ implies the number is one and $A$ is a domain. q.e.d.

For a two-sided ideal $J$ in an algebra $A$ let $J^0$ denote $A$ and let $J^\infty$ denote $\bigcap_n J^n$.

**2.8 Lemma.** *Suppose $A$ is a Hopf algebra containing two-sided ideals $J_1$, $J_2$ and $J_3$. Let $\mathcal{M}=\operatorname{Ker}\varepsilon$.*

a. *If $\Delta J_1\subset J_2\otimes A+A\otimes J_3$ and $J_2^\infty\subset\mathcal{M}$ then $J_1^\infty\subset J_3^\infty$.*

b. *$J_1^\infty\subset\mathcal{M}$ if and only if $J_1^\infty\subset\mathcal{M}^\infty$.*

**Proof.** By the hypothesis of a. we see that

$$\Delta J_1^{2n}\subset\sum_{i=0}^{2n}J_2^{2n-i}\otimes J_3^i\subset J_2^n\otimes A+A\otimes J_3^n.$$

Taking intersection yields

$$\Delta J_1^\infty\subset J_2^\infty\otimes A+A\otimes J_3^\infty. \tag{2.9}$$

Since $\varepsilon$ is the counit applying $\varepsilon\otimes I$ to 2.9 shows that $J_1^\infty\subset J_3^\infty$, which proves a.

Since $\varepsilon$ is the counit

$$\Delta J_1\subset J_1\otimes A+A\otimes\mathcal{M}.$$

Thus if $J_1^\infty\subset\mathcal{M}$ by a. it follows that $J_1^\infty\subset\mathcal{M}^\infty$.

Clearly if $J_1^\infty\subset\mathcal{M}^\infty$ then $J_1^\infty\subset\mathcal{M}$. q.e.d.

**2.9 Lemma.** *If $A$ is a Noetherian commutative algebra and $0\neq a\in A$ then there is a maximal ideal $\mathcal{T}$ of $A$ with $a\notin\mathcal{T}^\infty$. Thus*

$$\bigcap_{\substack{\text{maximal}\\ \text{ideals } \mathcal{T}}}\mathcal{T}^\infty=0.$$

**Proof.** Let $I=\{b\in A\mid ba=0\}$ and let $\mathscr{T}$ be a maximal ideal of $A$ containing $I$. If $a\in\mathscr{T}^{\infty}$ then by the Krull intersection theorem there is $x\in\mathscr{T}$ with $xa=a$. Thus $x-1\in I\subset\mathscr{T}$ a contradiction. q.e.d.

**2.10 Theorem.** *Let $A$ be a finitely generated commutative Hopf algebra with* $\operatorname{Sep}A=k$. *Then for any proper ideal $\boldsymbol{I}$ of $A$, $\boldsymbol{I}^{\infty}=0$.*

**Proof.** Let $\bar{k}$ be the algebraic closure of $k$. By 2.1 $\operatorname{Sep}(A\otimes\bar{k})=\bar{k}$; also, $(\boldsymbol{I}\otimes\bar{k})^{n}=\boldsymbol{I}^{n}\otimes\bar{k}$. Thus it suffices to prove the theorem for $A\otimes\bar{k}$ and we add the assumption that $k$ is algebraically closed.

We show that for each maximal ideal $\mathscr{T}$ of $A$, $\mathscr{T}^{\infty}=0$. By 2.7 $A/\mathscr{N}$ is a domain. Let $\pi: A\rightarrow A/\mathscr{N}$ be the natural map. Since $\mathscr{T}$ is a maximal ideal $\mathscr{N}\subset\mathscr{T}$ and $\pi(\mathscr{T})$ is a proper ideal in $A/\mathscr{N}$. By the Krull intersection theorem—since $A/\mathscr{N}$ is a domain—$\pi(\mathscr{T})^{\infty}=0$. Thus $\pi(\mathscr{T}^{\infty})=0$ and

2.11
$$\mathscr{T}^{\infty}\subset\mathscr{N}\subset\mathscr{M}\equiv\operatorname{Ker}\varepsilon.$$

By 2.8, b,

$$\mathscr{T}^{\infty}\subset\mathscr{M}^{\infty}.$$

Since $A$ is finitely generated and $k$ algebraically closed by the Hilbert nullstellensatz there is an algebra map $\sigma: A\rightarrow k$ with $\operatorname{Ker}\sigma=\mathscr{T}$. If we use $\sigma^{-1}$ to denote $\sigma S$ then an easy calculation shows that

$$\varepsilon=(\sigma^{-1}\otimes\sigma)\Delta: A\rightarrow k\otimes k=k.$$

Thus if $\mathscr{J}=\operatorname{Ker}\sigma^{-1}$ we have

$$\Delta(\mathscr{M})\subset\mathscr{J}\otimes A+A\otimes\mathscr{T}.$$

By what we have just shown for $\mathscr{T}$, 2.11, we know that $\mathscr{J}^{\infty}\subset\mathscr{M}$. By 2.8, a

$$\mathscr{M}^{\infty}\subset\mathscr{T}^{\infty}.$$

Thus $\mathscr{M}^{\infty}=\mathscr{T}^{\infty}$ for each maximal ideal $\mathscr{T}$. By the Hilbert basis theorem and 2.9 $0=\mathscr{M}^{\infty}=\mathscr{T}^{\infty}$. q.e.d.

The assumption of $A$ being a Hopf algebra is necessary in 2.10. Otherwise we have the following simple counter-example.

$$A=k[X, Y]/<XY-Y, Y^{2}>.$$

Let $\bar{Y}$ denote the coset of $Y$ and $\bar{X}$ the coset of $X$. Then $\mathscr{N}$ is generated by $\bar{Y}$ and $A/\mathscr{N}$ is isomorphic to a polynomial ring in one variable. Thus by 2.4, $\operatorname{Sep} A=k$. But $\bar{X}\bar{Y}=\bar{Y}$ and the ideal $\mathscr{T}$ in $A$ generated by $\bar{X}$ and $\bar{Y}$ is maximal with $\bar{Y}\in\mathscr{T}^{\infty}$.

**2.12 Corollary.** *If $B$ is a finitely generated commutative Hopf algebra with* $\operatorname{Sep} B=k$ *then*

$$\bigcap_{n} BB^{+(p^{n})}=0.$$

**Proof.** $B^{+(p^{n})}\subset B^{+p^{n}}=\mathscr{M}^{p^{n}}$ and so 2.10 implies the result. q.e.d.

## §3. Structure of finite dimensional commutative local co-semi-simple Hopf algebras.

**3.1 Example.** Suppose $G$ is a finite abelian $p$-group and $k$ is a field of characteristic $p>0$. The group algebra $kG$ is co-semi-simple since it is the direct sum of the simple coalgebras $\{kg\}_{g\in G}$. The Hopf algebra $kG$ is commutative since $G$ is abelian and $kG^{+}$ consists of nilpotent elements because $G$ is a $p$-group. Thus $kG$ is local.

Shortly we show that 3.1 gives the only example of a finite dimensional commutative local co-semi-simple Hopf algebra when the ground field is algebraically closed, and of positive characteristic. (Since commutative Hopf algebras in characteristic zero are reduced the assumption of positive characteristic is not necessary.)

**3.2 Lemma.** *A finite dimensional commutative Hopf algebra $A$ is local if and only if* $\operatorname{Sep} A=k$.

**Proof.** If $A$ is local then $\mathscr{M}=\operatorname{Ker}\varepsilon$ must be the maximal ideal. By finite dimensionality $\mathscr{M}$ consists of nilpotent elements so that $k=\operatorname{Sep}k=\operatorname{Sep}A/\mathscr{M}=\operatorname{Sep}A$, the last equality by 2.4.

Since $A$ is finite dimensional and commutative it is the direct sum of finitely many primary algebras. If $\operatorname{Sep}A=k$ there is just one, i.e. $A$ is primary, hence local. q.e.d.

In a Hopf algebra $A$ a non-zero element $g$ with $\Delta g=g\otimes g$ is called group-like. One easily checks that a product of group-likes is group-like and for a group-like $g$ the inverse is $S(g)$. Furthermore, since $S$ is a coalgebra antimorphism $S(g)$ is again group-like. Let $G(A)$ denote the set of group-like elements of $A$. By [7, p. 55, 3.2.1] the elements of $G(A)$ are linearly independent and their span in $A$ is (isomorphic to) the group algebra $kG(A)$ as a Hopf algebra.

**3.3 Lemma.** *Suppose $k$ is algebraically closed and $A$ is a cocommutative Hopf algebra.*

a. *$A$ is co-semi-simple if and only if $A$ is the group algebra $kG(A)$.*

b. *If the characteristic is zero and $G$ is an abelian group then* $\operatorname{Sep}kG=k$ *if and only if $G$ has no torsion.*

c. *If the characteristic is $p>0$ and $G$ is an abelian group then* $\operatorname{Sep}kG=k$ *if and only if the torsion elements of $G$ all have $p$-power torsion.*

**Proof.** By [7, p. 158, 8.0.1, a] simple coalgebras are finite dimensional so they are the duals of finite dimensional simple algebras. A cocommutative simple coalgebra is the dual of a finite dimensional commutative simple algebra; i.e. a finite field extension. $k$ is algebraically closed so these are all 1-dimensional. By [7, p. 158, 8.0.1., e] a 1-dimensional coalgebra contains a group-like. If $A$ is co-semi-simple it is the sum of its simple subcoalgebras and so spanned by group-likes. Thus $A=kG(A)$ by the remarks preceding the lemma. The converse of a. is clear.

b. and c. are easy to prove and we omit the proofs. q.e.d.

In the proof of Theorem 3.11 we shall have to discuss finite dimensional Hopf algebras $U$ which are cocommutative and with unique simple subcoalgebra $k$. These Hopf algebras are studied extensively in [5] and [7]. A coalgebra with unique simple subcoalgebra is called irreducible. If a Hopf algebra is irreducible the unique simple subcoalgebra must be the simple subcoalgebra $k$. Here are some facts we shall need.

3.4 If $U$ is an irreducible bialgebra then $U$ is a Hopf algebra [7, p. 193, 9.2.2, 3]. (A bialgebra is a Hopf algebra which does not necessarily have an antipode.)

3.5 If $V$ is a subbialgebra of an irreducible Hopf algebra $U$ then $V$ is a subHopf algebra. This is true because $V$ is irreducible so has an antipode by 3.4. The antipode is the restriction of the antipode of $U$ by [7, p. 81, 4.0.4].

A sequence of elements $\{v_i\}_{i=0}$ in a Hopf algebra is called a sequence of divided powers if $v_0=1$ and

$$\Delta v_n=\sum_{i=0}^{n} v_i \otimes v_{n-i}, \text{ each } n.$$

By induction they are easily shown to be linearly independent if $v_1 \neq 0$. In a Hopf algebra $U$, $P(U)$ denotes the space $\{u \in U \mid \Delta u=1\otimes u+u\otimes 1\}$, and the elements in $P(U)$ are called primitive. In a sequence of divided powers $\{v_i\}_{i=0}$ the element $v_1$ is primitive.

3.6 If $U$ is a finite dimensional cocommutative irreducible Hopf algebra and $k$ is a perfect field of characteristic $p>0$ then there is a number $n\,(=\dim P(U))$ where for each $1\leq j\leq n$ there is a sequence of divided powers $\{{}_ju_i\}_{i=0}^{p^{n_i}-1}$ with $\{{}_ju_1\}_{j=1}^{n}$ a basis of $P(U)$ and the set of products $\{{}_1u_{e_1}\cdots {}_nu_{e_n} \mid e_i<p^{n_i}, \text{ for } i=1,\cdots n\}$ is a basis for $U$, [5, p. 521, Theorems 2 and 3].

We would like to mention here that Theorem 2 in [5] is incorrect as stated. It should read: "Assume $p>0$, $k$ perfect and ${}^1x$ is primitive. ${}^1x$ has coheight $n$ if and only if there is a sequence of divided powers ${}^0x, {}^1x, \dots {}^{p^{n+1}-1}x$, for $n=0, 1, \cdots$. If there is an infinite sequence of divided powers ${}^0x, {}^1x, {}^2x, \dots$ then ${}^1x$ has infinite coheight. If ${}^1x$ has infinite coheight *and the Hopf algebra has finite dimensional primitives* then there is an infinite divided power sequence ${}^0x, {}^1x, {}^2x, \dots$." The proof of Theorem 2 in [5] proves all but the last statement which is not needed for our present purposes. A proof of the last statement will appear in a subsequent paper.

3.7 The dimension of a finite dimensional cocommutative irreducible Hopf algebra over a perfect field $k$ of characteristic $p>0$ is a power of $p$. This follows from counting the basis in 3.6.

At two points in the proof of Theorem 3.11 we shall have to show that certain subHopf algebras are central. We will do this by using the adjoint or *inner* action of a Hopf algebra on itself. For a Hopf algebra $H$ containing elements $g$, $h$ define

$$h^g=\sum_{(g)} g_{(1)}hS(g_{(2)}).$$

By [1, p. 207, 1.7.2] this action makes $H$ a left $H$-module algebra. An algebra $A$ which is an $H$-module is called an $H$-module algebra if

3.8 $$h\cdot 1=\varepsilon(h)1 \qquad \text{all } h\in H,$$

$$h\cdot(ab)=\sum_{(h)}(h_{(1)}\cdot a)(h_{(2)}\cdot b) \qquad 1, a, b\in A.$$

3.9 If $U$ is an irreducible Hopf algebra and $A$ is a commutative $U$-module algebra where $A=\operatorname{Sep} A$ then the action of $U$ on $A$ is trivial, i.e. $u\cdot a=\varepsilon(u)a$ for all $u\in U$, $a\in A$. This is true because by [7, p. 193, 9.2.2], [7, p. 200, 10.0.1] and [7, p. 201, 10.0.2] $U$ has a filtration $k=U_0\subset U_1\subset\cdots$ where $\cup U_i=U$, $U_1=k\oplus P(U)$ and for $u\in U_n^+=U_n\cap\operatorname{Ker}\varepsilon$

3.10 $$\Delta(u)=1\otimes u+u\otimes 1+y \text{ where } y\in U_{n-1}^+\otimes U_{n-1}^+.$$

By definition of $P(U)$ and 3.8 the elements of $P(U)$ act as derivations on $A$. Since a commutative separable algebra has no non-zero derivations the elements of $P(U)$ act as zero on $A$. Thus $U_1^+$ acts as zero on $A$. Say by induction $U_{n-1}^+$ acts as zero on $A$. By 3.10, 3.8 and the induction hypothesis the elements of $U_n^+$ act as derivations, hence as zero on $A$. This proves 3.9.

**3.11 Theorem.** *Suppose $k$ is a field of characteristic $p>0$ and $A$ is a commutative finite dimensional local co-semi-simple Hopf algebra, then $A$ is cocommutative. If $k$ is algebraically closed then $A$ is the group algebra of an abelian $p$-group.*

**Proof.** By [3, p. 9, Lemma 1.3] $A$ tensored with the algebraic closure is co-semi-simple. Since $\mathscr{M}=\operatorname{Ker}\varepsilon$ consists of nilpotent elements $A$ tensored with the algebraic closure is still local. Thus we may assume $k$ is algebraically closed in proving the first statement. The second statement then follows by 3.2 and 3.3.

Since $A$ is finite dimensional $A^*\equiv U$ is a finite dimensional cocommutative Hopf algebra. $\mathscr{M}$ is the unique maximal ideal of $A$ implies that $k$ is the unique minimal or simple subcoalgebra of $U$. Thus $U$ is irreducible. Proving $A$ is cocommutative is equivalent to proving that $U$ is commutative.

The Hopf algebra injection $A^{(p^m)}\to A$ induces a Hopf algebra surjection

$$U=A^*\xrightarrow{\pi}A^{(p^m)*}.$$

The Hopf kernel of $\pi$ is defined in [7, p. 312] as $U_{[m]}=\{u\in U\,|\,(I\otimes\pi)\Delta u=u\otimes 1\}$ and is shown to be a subHopf algebra in [7, p. 312, 16.1.1]. It is easily shown that

3.12 $$U_{[m]}=\{u\in U\,|\,\langle u,\ AA^{+(p^m)}\rangle=0\}=AA^{+(p^m)\perp}$$

and that for the inner action of a cocommutative Hopf algebra on itself (defined between 3.7 and 3.8)

3.13 $$\Delta h^g = \sum_{(g),(h)} h_{(1)}^{g_{(1)}} \otimes h_{(2)}^{g_{(2)}}$$

for all $h$ and $g$ in the Hopf algebra. Thus for $u \in U$, $v \in U_{[m]}$

$$(I \otimes \pi) \Delta v^u = \sum_{(u),(v)} v_{(1)}^{u_{(1)}} \otimes \pi(v_{(2)}^{u_{(2)}})$$

$$= \sum_{(u),(v)} u_{(1)}^{v_{(1)}} \otimes \pi(u_{(2)})^{\pi(v_{(2)})} = \sum_{(v)} u^{v_{(1)}} \otimes 1^{\pi(v_{(2)})}$$

$$= \sum_{(v)} u^{v_{(1)}} \otimes \varepsilon(v_{(2)}) 1 = u^v \otimes 1.$$

Thus $U_{[m]}$ is a submodule under the inner action.

By 3.12 $U_{[m]}$ is naturally isomorphic as a Hopf algebra to $(A/AA^{+(p^m)})^*$. By 1.12 $A/AA^{+(p^m)}$ is co-semi-simple and thus $U_{[m]}$ is semi-simple.

We now proceed by induction on $\dim U$ to show that $U$ is commutative. By the induction assumption we can assume that $U_{[m]}$ is commutative if $U_{[m]} \subsetneqq U$.

Structure of $U_{[1]}$.

By 3.12 $P(U) \subset P(U_{[1]})$ since derivations vanish on $p^{th}$ powers. The opposite inclusion is clear and so $P(U) = P(U_{[1]})$. If $U_{[1]} \subsetneqq U$ we have that $U_{[1]}$ is commutative by the induction assumption.

If $U_{[1]} = U$ then by 3.12 the elements in the augmentation ideal of $A$ raised to the $p^{th}$ power are zero. Thus by [5, p. 519, Lemma 3 and second line above Lemma 4] $U = U_{[1]}$ is the restricted universal enveloping algebra of $P(U)$. Since $U = U_{[1]}$ is semi-simple Hochschild's result [2, p. 603, Theorem] implies that $U = U_{[1]}$ is commutative *and we are done.*

So $U_{[1]}$ is commutative. By 3.12 $U_{[1]} = (A/AA^{+(p)})^*$. Since $k$ is algebraically closed 3.2 and 3.3 imply that $A/AA^{+(p)}$ is the group algebra of an abelian $p$-group. Since every element in the augmentation ideal of $A/AA^{+(p)}$ has $p^{th}$ power zero, the abelian $p$-group is $Z/pZ \otimes \cdots \otimes Z/pZ$ (finitely many times.) Thus

$$A/AA^{+(p)}\cong\bigotimes_{i=1}^{r}k(\boldsymbol{Z}/p\boldsymbol{Z}).$$

This shows that $U_{[1]}$ is the tensor product of subHopf algebras $\{V_i\}_{i=1}^{r}$ where for each $i$.

$V_i$ is generated by $v_i$ subject to the relation

3.15 $$v_i^p=v_i,$$

$$\Delta v_i=v_i\otimes 1+1\otimes v_i,\quad \varepsilon(v_i)=0,\quad S(v_i)=-v_i.$$

In each $V_i$ the element $v_i$ is a basis for $P(V_i)$ so by [7, p. 222, 11.0.7]

3.16 $\{v_i\}_{i=1}^{r}$ is a basis for $P(U_{[1]})=P(U)$.

Next we show that $U_{[m]}$ is central in $U$ if $U_{[m]}\subsetneqq U$. By the induction assumption $U_{[m]}$ is a commutative semi-simple algebra. Since $k$ is algebraically closed $U_{[m]}$ is isomorphic to the sum of finitely many copies of $k$. Thus $U_{[m]}=\operatorname{Sep}U_{[m]}$. We have shown that $U_{[m]}$ is a submodule under the inner action so that $U_{[m]}$ is a $U$-module algebra under the inner action. By 3.9 the inner action of $U$ on $U_{[m]}$ is trivial thus for $u\in U$, $v\in U_{[m]}$

$$uv=\sum_{(u)}(v^{u_{(1)}})\,u_{(2)}=\sum_{(u)}\varepsilon(u_{(1)})\,vu_{(2)}=vu$$

and $U_{[m]}$ is central in $U$.

Now suppose $\dim P(U)>1$. The ideal $UV_i^+$ is two-sided because $V_i$ is central in $U$ and is a Hopf ideal by [7, p. 87], so that $W_i=U/UV_i^+$ is a quotient Hopf algebra. Thus $W_i$ is semi-simple, cocommutative, irreducible [7, p. 168, 8.0.8, d] and of lower dimension than $U$. By the induction hypothesis $W_i$ is commutative.

Let $\pi\colon U\to W_i$ be the natural map and let $C$ denote the "commutator map"

$$C\colon U\otimes U\to U,\ u\otimes v\to\sum_{(u),(v)}u_{(1)}v_{(1)}S(u_{(2)})S(v_{(2)}).$$

Since $U$ is cocommutative $C$ is a coalgebra map and $\operatorname{Im} C$ is a subcoalgebra of $U$. Since $W_i$ is commutative the composite $\pi C$ is equal to $\varepsilon \otimes \varepsilon$. Thus for $x \in \operatorname{Im} C$, $\pi(x) = \varepsilon(x)$ and since $\operatorname{Im} C$ is a subcoalgebra $(\pi \otimes I) \Delta x = 1 \otimes x$.

By [7, p. 304, (16.0.1,d) and p. 309, (16.0.3)] $V_i = \{u \in U \mid (\pi \otimes I) \Delta u = 1 \otimes u\}$ and thus $\operatorname{Im} C \subset V_i$. Since $V_1 \otimes \cdots \otimes V_r = U_{[1]}$ and $r > 1$, (by 3.16), we have that $\operatorname{Im} C \subset V_1 \cap V_2 = k$. Thus $C = \varepsilon \otimes \varepsilon$, because $\varepsilon C = \varepsilon \otimes \varepsilon$ and $\operatorname{Im} C \subset k$ implies $\varepsilon C = C$. For $u, v \in U$ we calculate

$$uv = \sum_{(u),(v)} C(u_{(1)} \otimes v_{(1)}) v_{(2)} u_{(2)}$$

$$= \sum_{(u),(v)} \varepsilon(u_{(1)}) \varepsilon(v_{(1)}) v_{(2)} u_{(2)} = vu$$

and $U$ is commutative *and we are done.*

We now consider the case when $\dim P(U) \leq 1$. By 3.6 if $\dim P(U) = 0$ then $U = k$ *and we are done.*

Say $\dim P(U) = 1$. By 3.6 $U$ has a basis consisting of a sequence of divided powers $\{d_i\}_{i=0}^{p^s-1}$. If $\{x^i\}_{i=0}^{p^s-1}$ is a dual basis for $A$ then $\varepsilon(x^i) = \delta_{io}$, $1 = x^o$, $x^i x^j = x^{i+j}$ so that $A$ is generated by $x^1$ subject to the relation $(x^1)^{p^s} = 0$. The ideal $AA^{+(p^{s-1})}$ has a basis $\{x^{p^{s-1}}, \cdots x^{p^s-1}\}$ so that by 3.12, $U_{[s-1]}$ has a basis consisting of $\{d_i\}_{i=0}^{p^{s-1}-1}$. Thus $U_{[s-1]} \subsetneqq U$ and $\dim U_{[s-1]} = p^{s-1}$. Since $U_{[s-1]}$ is central in $U$ the subalgebra $W$ generated by $U_{[s-1]} \oplus kd_{p^{s-1}}$ is commutative. Since $U_{[s-1]} \oplus kd_{p^{s-1}}$ is a subcoalgebra of $U$, $W$ is a subbialgebra. By 3.4 $W$ is a Hopf algebra and thus by 3.7 the dimension of $W$ is a power of $p$. We have

$$p^{s-1} = \dim U_{[s-1]} < \dim W \leq \dim U = p^s$$

so that $W$ must equal $U$ and $U$ is commutative. q.e.d.

## §4. The main theorem.

**4.1 Theorem.** *Suppose $k$ is a field of characteristic $p > 0$ and $A$ is a commutative co-semi-simple Hopf algebra over $k$ where* $\operatorname{Sep} A = k$.

*Then $A$ is cocommutative. If $k$ is algebraically closed then $A$ is (isomorphic to) the group algebra of an abelian group $G$ where the torsion elements of $G$ have p-power torsion.*

**Proof.** It suffices to prove the first statement; then the second will follow from 3.3.

By 2.6 we may assume that $A$ is finitely generated. Let $a \in A$ and $C$ be the subcoalgebra of $A$ generated by $a$. By [7, p. 46, 2.2.1] $C$ is finite dimensional. Thus by 2.12 there is an integer $n$ with $C \cap AA^{+(p^n)} = 0$.

Let $\pi: A \rightarrow A/AA^{+(p^n)}$ be the natural Hopf algebra map. By 1.12 $A/AA^{+(p^n)}$ is co-semi-simple. Since $A$ is finitely generated $A/AA^{+(p^n)}$ satisfies the hypothesis of 3.11 and therefore must be cocommutative. Since $\pi | C$ is injective, $C$ must be cocommutative. q.e.d.

CORNELL UNIVERSITY
ITHACA, N.Y. 14850

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*Added in proof*: Nagata's results in [4] characterize reduced affine algebraic group schemes in positive characteristic which are fully reducible. Recently Nagata's characterization has been extended to fully reducible affine algebraic group schemes which are not necessarily reduced, [Groupes Algebriques, M. Demazure and P. Gabriel, North Holland, 1970, p. 509, theorem 3.6]. This generalization of Nagata's characterization leads to an alternative method of proof of our theorem 4.1.