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On the complex bordism of complexes with few cells

By

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Introduction

Our immediate motivation for this paper was a wish to produce a proof entirely within the framework of complex bordism of a crucial lemma found in [11; 5.10]. We explain in section 4 how this follows from our principal result.

We aim here to show that for a certain type of complex, X, having four cells, $\mathrm{hom.dim.}_{\Omega_*^U}\Omega_*^U(X)\leq 2$. Specifically, stable homotopy classes $[f]\in\pi_{2n-1}^S$, $[g]\in\pi_{2m-1}^S$, together with an integer q, are chosen so that the Toda bracket $<q, [g], [f]>$ is defined and congruent to 0. We then form a complex, in stable form,

$$Y=S^0\cup_g e^{2m}\cup_F e^{2m+2n}$$

using a coextension F of f. We find that the vanishing of the bracket implies the existence of a stable map $Q: Y\to S^0$ having degree q in dimension 0. This leads us to the final complex

$$X=S^0\cup_Q C(Y)=S^0\cup e^1\cup e^{2m+1}\cup e^{2m+2n+1}.$$

We prove that if q is odd then $\mathrm{hom.dim.}_{\Omega_*^U}\Omega_*^U(X)\leq 2$.

From a broader viewpoint we are seeking some techniques which can yield an upper bound on the value of the homological dimension of a bordism module. This appears to us now as the key problem in the study of the relation of stable homotopy to complex bordism. Our present approach differs quite sharply from the arguments in [11]. First, we avoid the formation of Toda brackets in $\Omega_*^{U,fr}$ and also, we make no appeal to Adams' formula for $e_C<q, [g], [f]>$. Although section 5 of this note is concerned with technical calculations it should be possible to expand our approach into a considerably more general theorem as suggested in the closing paragraph of section 6.

The paper is divided into sections as follows:

1. Some Invariants
2. On Two and Three Cell Complexes
3. On Three and Four Cell Complexes
4. More on Three and Four Cell Complexes
5. Some Characteristic Number Arguments
6. Still More on Three and Four Cell Complexes.

§1. Some Invariants

Suppose that X is a finite complex. There are the natural maps (see [4] particularly for notations)

$$\mu: \Omega_*^U(X) \to H_*(X; Z)$$

$$\eta: k_*(X) \to H_*(X; Z)$$

which upon application of Serre's mod $\mathscr{C}$ theory are seen to have finite cokernels. Therefore

$$\omega_i = \text{order of coker } \{\Omega_i^U(X) \to H_i(X; Z)\}$$

$$\kappa_i = \text{order of coker } \{k_i(X) \to H_i(X; Z)\}$$

are well defined integers. It is convenient to define

$$\tilde{\omega}_i = \omega_i - 1, \qquad \tilde{\kappa}_i = \kappa_i - 1$$

and introduce the invariants

$$\left.\begin{aligned} \tilde{\kappa}(X) &= \sum_i \tilde{\kappa}_i x^i \\ \tilde{\omega}(X) &= \sum_i \tilde{\omega}_i x^i \end{aligned}\right\} \in Z[x]$$

of the complex X. Concerning these invariants we have:

Theorem 1.1: *Let X be a finite complex. Then*

$$\tilde{\kappa}(X)=\tilde{\omega}(X) \textit{ iff } \text{hom.dim.}_{\Omega_*^U}\Omega_*^U(X)\leq 2.$$

Proof: From the commutative diagram

$$\begin{tikzcd}
\Omega_*^U(X) \arrow[dr, "\mu"] \arrow[d, "\zeta"] & \\
k_*(X) \arrow[r, "\eta"'] & H_*(X; Z)
\end{tikzcd}$$

we see that $\tilde{\kappa}(X)=\tilde{\omega}(X)$ *iff* $\operatorname{Im}\mu=\operatorname{Im}\eta$.

Suppose now that $\text{hom.dim.}_{\Omega_*^U}\Omega_*^U(X)\leqq 2$. Then [4; 10.6]

$$\zeta\colon \Omega_*^U(X)\rightarrow k_*(X)$$

is epic. Therefore $\operatorname{Im}\mu=\operatorname{Im}\eta$ and hence $\tilde{\kappa}(X)=\tilde{\omega}(X)$.

Next suppose conversely that $\tilde{\kappa}(X)=\tilde{\omega}(X)$. Then $\operatorname{Im}\mu=\operatorname{Im}\eta$. Recall that there is an exact sequence [9] $0\rightarrow Z\otimes_{Z[t]}k_*(X)\rightarrow H_*(X; A)$ $\rightarrow \operatorname{Tor}_{1,*}^{Z[t]}(Z, k_*(X))\rightarrow 0$. Thus we have a diagram

$$\begin{tikzcd}
Z\otimes_{\Omega_*^U}\Omega_*^U(X) \arrow[dr, "\tilde{\mu}"] \arrow[d, "\tilde{\xi}"] & \\
Z\otimes_{Z[t]}k_*(X) \arrow[r, "\tilde{\eta}"'] & \operatorname{Im}\mu
\end{tikzcd}$$

wherein $\tilde{\mu}$ is epic and $\tilde{\eta}$ is iso. Hence of course

$$\tilde{\xi}\colon Z\otimes_{\Omega_*^U}\Omega_U^*(X)\rightarrow Z\otimes_{Z[t]}k_*(X)$$

is also epic. Therefore

$$\zeta\colon \Omega_*^U(X)\rightarrow k_*(X)$$

maps a set of Ω_*^U-generators for $\Omega_*^U(X)$ onto a set of $Z[t]$ generators for $k_*(X)$, and hence

$$\zeta\colon \Omega_*^U(X)\rightarrow k_*(X)$$

is epic. Therefore by [8] $\text{hom.dim.}_{\Omega_*^U}\Omega_*^U(X)\leq 2$. $**$

Theorem 1.2: *Suppose that X is a finite complex. Then*

$$\tilde{\kappa}(X)=0 \text{ iff } \tilde{\omega}(X)=0 \text{ iff } \text{hom.dim.}_{\Omega_*^U}\Omega_*^U(X)\leq 1.$$

Proof: Observe that $\tilde{\kappa}(X)=0$ *iff*

$$\eta: k_*(X)\to H_*(X; Z)$$

is epic. Similarly $\tilde{\omega}(X)=0$ *iff*

$$\mu: \Omega_*^U(X)\to H_*(X; Z)$$

is epic. The result now follows from [8; Corollary 5]. **

Suppose now that X is a "small" CW-complex, that is, is composed of only a few cells. It then makes sense to ask for computations of $\tilde{\kappa}(X)$ and $\tilde{\omega}(X)$ in terms of the attachment data for X. In this way we can hope to relate $\tilde{\kappa}(X)$ and $\tilde{\omega}(X)$ to other invariants of the attachment data. The next section illustrates what we have in mind.

§2. On Two and Three Cell Complexes

Suppose given $[f]\in\pi_{2n-1}^s$. Form the stable complex

$$Y(f)=S^0\cup_f e^{2n}.$$

Let now q be an integer such that $q[f]=0\in\pi_{2n-1}^s$. We may thus form an extension

$$Q=\text{ext}(q):\quad Y(f)\to S^0$$

of the map

$$q: S^0\to S^0$$

of degree q. Let

$$X(f)=S^0\cup_{\text{ext}(q)}CY(f)$$

be the mapping cone of ext (q). Thus we have a cofibration

$$Y(f)\xrightarrow{Q} S^0\xrightarrow{P} X(f).$$

Note that

$$X(f)=S^0\cup_q e^1\cup_F e^{2n+1}$$

where

$$F\colon S^{2n} \to S^0 \cup_q e^1$$

is a suitable coextension of f.

Denote by $\sigma \in k_0(X(f))$ the canonical class and by $A(\sigma) \subset Z[t]$ the annihilator ideal of σ. We propose to show how $\tilde{k}(X(f))$ is related first of all to $A(\sigma)$ and secondly to $e_C(f)$. (See [2] [15] for information about the invariant) $e_C(f)$.)

Convention: We will fix throughout the remainder of this section the notations of the preceeding discussion.

Let us begin by choosing generators for $k_*(Y(f))$. To this end observe that we have a cofibration

$$S^{2n-1} \xrightarrow[f]{} S^0 \xrightarrow[c]{} Y(f).$$

Note that

$$f_*\colon \tilde{k}_*(S^{2n-1}) \to \tilde{k}_*(S^0)$$

is the zero map, so we obtain an exact sequence

$$0 \to \tilde{k}_*(S^0) \xrightarrow[c*]{} \tilde{k}_*(Y(f)) \xrightarrow[\partial*]{} \tilde{k}_*(S^{2n-1}) \to 0.$$

Denote by $i_k \in k_k(S^k)$ the canonical generator. We may then choose classes

$$\alpha_0 \in \tilde{k}_0(Y(f))\colon \alpha_0 = c_*(i_0)$$

$$\alpha_{2n} \in \tilde{k}_{2n}(Y(f))\colon \partial_* \alpha_{2n} = i_{2n-1}$$

which generate $k_*(Y(f))$ as a $Z[t]$-module. Observe that α_{2n} is unique only up to an element of the form $t^n \alpha_0 \in k_{2n}(Y(f))$.

Consider next the cofibration

$$Y(f) \xrightarrow{Q} S^0 \xrightarrow{P} X(f).$$

We find

$$Q_*(\alpha_0) = q i_0$$

$$Q_*(\alpha_{2n}) = rt^n i_0$$

for a suitable integer r. Note that r is uniquely determined mod q. The following is now clear:

Proposition 2.1: *With the notations preceeding we have*

$$A(\sigma) = (q, r \cdot t^n).$$

Proof: We have an exact triangle

$$\begin{tikzcd} \tilde{k}_*(Y(f)) \arrow[rr, "Q*"] & & \tilde{k}_*(S^0) \arrow[dl, "P*"] \\ & \tilde{k}_*(X(f)) \arrow[ul] & \end{tikzcd}$$

from which it is clear that $\operatorname{Im} Q_* = \ker P_*$. But as $P_*(i_0) = \alpha$ we have $\ker P_* = A(\sigma)$ and the result follows. $**$

Theorem 2.2: *With the notations preceeding, we have*

$$e_C(f) = +r/q \in Q/Z.$$

Proof: Let us begin with a description of $e_C(f)$ suitable to our purposes. Let

$$\mathrm{ch}: k_*(\ ) \to H_*(\ ; Q)$$

be the homology Chern character. Denote by $e_k \in H_k(S^k; Z)$ a homology generator, and by e_k^* its dual cohomology generator. Returning to the cofibration

$$S^{2n-1} \xrightarrow[f]{} S^0 \xrightarrow[c]{} Y(f)$$

we see that we may choose unique classes

$$a_0 \in \tilde{H}_0(Y(f); Z): c_* e_0 = a_0$$

$$a_{2n} \in \tilde{H}_{2n}(Y(f); Z): \partial_* a_{2n} = e_{2n-1}.$$

The homology of $Y(f)$ is free abelian and these classes are generators.

Then we have

$$\mathrm{ch}\,\alpha_0 = a_0$$

$$\mathrm{ch}\,\alpha_{2n} = \lambda a_0 + a_{2n}: \qquad \lambda \in Q.$$

As usual, one finds λ is unique in Q/Z, i.e., its residue mod Z is independent of the choice of α_0, α_{2n}, a_0, a_{2n}. According to [7] the Spanier-Whitehead dual of $Y(f)$ is again $Y(f)$. Thus applying Spanier-Whitehead duality to the above construction and recalling the relation of $k^*(\ )$ to $K^*(\ )$ (see for example [4; §10]) we find from [2] [15] that $\lambda = e_C(f) \in Q/Z$.

Introduce now the diagram of cofibrations

$$\begin{array}{ccccc}
S^{2n-1} & \longrightarrow & E^{2n} & \longrightarrow & S^{2n} \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle *} & & {\scriptstyle F}\downarrow \\
S^0 & \xrightarrow{q} & S^0 & \longrightarrow & S^0 \cup_q e^1 \\
\downarrow{\scriptstyle c} & & \downarrow{\scriptstyle id} & & {\scriptstyle T}\downarrow \\
Y(f) & \xrightarrow{Q} & S^0 & \xrightarrow{P} & X(f).
\end{array}$$

Applying the functor $\tilde{k}_*(\ )$ to this diagram we obtain

$$\begin{array}{ccccccc}
 & & \cdots\tilde{k}_{2n}(S^{2n-1}) & \longrightarrow & \tilde{k}_{2n}(E^{2n}) & \longrightarrow & \tilde{k}_{2n}(S^{2n})\cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \cdots\tilde{k}_{2n}(S^0) & \longrightarrow & \tilde{k}_{2n}(S^0) & \longrightarrow & \tilde{k}_{2n}(S^0 \cup_q e^1)\cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots\tilde{k}_{2n+1}(X(f)) & \xrightarrow{\Delta *} & \tilde{k}_{2n}(Y(f)) & \xrightarrow{Q*} & \tilde{k}_{2n}(S^0) & \longrightarrow & \tilde{k}_{2n}(X(f))\cdots \\
 & & \downarrow{\scriptstyle \partial *} & & \downarrow & & \downarrow \\
 & & k_{2n-1}(S^{2n-1}) & & \vdots & & \vdots
\end{array}$$

which is seen to be isomorphic to the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & Z & & \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & 0 \longrightarrow Zt^n i_0 & & \xrightarrow{\quad q \quad} & & Zt^n i_0 & \longrightarrow & Z_q & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle c*} & & & & \downarrow{\scriptstyle id} & & \downarrow & & \\
0 \longrightarrow Z\gamma_{2n+1} & \xrightarrow{\Delta *} & Zt^n\alpha_0 \oplus Z\alpha_{2n} & & \xrightarrow{P*} & & Zt^n i_0 & \longrightarrow & Z_{(q,r)}t^n & & \\
 & & \downarrow{\scriptstyle \partial_*} & & & & \downarrow & & \downarrow & & \\
 & & Zi_{2n} & & & & 0 & & 0 & & \\
 & & \downarrow & & & & & & & & \\
 & & 0 & & & & & & & &
\end{array}$$

The following formulas are clear

$$P_* t^n \alpha_0 = q t^n i_0$$

$$P_* \alpha_{2n} = r t^n i_0$$

and hence

$$\Delta_* \gamma_{2n+1} = A t^n \alpha_0 + B \alpha_{2n}$$

where

$$Aq + Br = 0.$$

Thus if we write $\mathrm{ch}_i(x)$ for the component of degree i of $\mathrm{ch}(x)$ we find

$$\begin{aligned} 0 = \Delta_* 0 = \Delta_* \mathrm{ch}_1(\gamma_{2n+1}) &= \mathrm{ch}_0(A t^n \alpha_0 + B \alpha_{2n}) \\ &= A a_0 + B e_C(f) a_0 \end{aligned}$$

because $H_1(X(f); Z) = 0$. Hence

$$A + B e_C(f) = 0$$

i.e.,

$$e_C(f) = -A/B.$$

Recalling that $Aq + Br = 0$ we find that $A/B = -r/q$ and the result follows upon substitution. $**$

Remark: Suppose given the space $X(f)$, but not necessarily the cofibrations required to describe its attachment data. Then on general grounds we find that $A(\sigma) = (q, st^n)$. However s is unique only up to a unit in Z_q. At any rate $q/(q, s) \in Z$ is uniquely determined by $\sigma \in k_0(X(f))$. We then have:

Corollary 2.3: *With the notations preceeding the order of $e_C(f)$ in Q/Z is $q/(q, s)$.* $**$

A more succinct way to state the preceeding corollary is to consider the annihilator ideal $\bar{A}(\sigma) \subset Z$ of the canonical class $\sigma \in K_0(X(f))$, regarding $K_*(X(f))$ as Z_2-graded. One easily sees by localization and degrading [4; §10] that $\bar{A}(\sigma)$ is generated by (q, r). Thus we have:

Corollary 2.4: *Let*

$$X(f)=S^0\cup_q e^1\cup_F e^{2n+1}$$

be the stable complex where q is a positive integer and

$$F\colon S^{2n}\to S^0\cup_q e^1$$

is a coextension of $[f]\in\pi^s_{2n-1}$. Let $\sigma\in\tilde{K}_0(X(f))$ denote the canonical class and $\bar{A}(\sigma)\subset Z$ its annihilator ideal. Then $e_C(f)=i/q$ where i is the index of $\bar{A}(\sigma)$ in Z. **

Return now to the complex $X(f)$ and the invariant $\tilde{\kappa}(X(f))$, which we will write $\tilde{\kappa}(f)$ for short.

Proposition 2.5: $\tilde{\kappa}(f)=(|e_C(f)|-1)\,x^{2n+1}$, *where $|e_C(f)|$ denotes the order of $e_C(f)$ in Q/Z.*

Proof: Recall that

$$A(\sigma)=(q, rt^n),$$

where $\sigma\in\tilde{k}_0(X(f))$ is the canonical class. Consider the exact sequence [9]

$$\cdots\tilde{k}_{2n+1}(X(f))\xrightarrow{\eta}\tilde{H}_{2n+1}(X(f);Z)\xrightarrow{\Delta}\tilde{k}_{2n-2}(X(f))\xrightarrow{m_t}\tilde{k}_{2n}(X(f)).$$

Clearly

$$\operatorname{coker}_{2n+1}\eta=\operatorname{Im}_{2n+1}\Delta=\ker_{2n-2}m_t$$

while

$\ker_{2n-2}m_t=$subgroup of $Z_q[t]$ generated by rt^{n-1}. Therefore $|\ker_{2n-2}m_t|=q/(q, r)$, where $|\quad|$ denotes order, and the result follows from (2.3). **

Theorem 2.6: *With the notations preceeding we have*

$$e_C(f)=0 \Leftrightarrow \text{hom.dim.}_{\Omega_*^U}\Omega_*^U(X(f)) \leq 1.$$

Proof: Suppose that

$$\text{hom.dim.}_{\Omega_*^U}\Omega_*^U(X(f)) \leq 1.$$

Then [4; 3.11]

$$\mu: \Omega_*^U(X(f)) \to H_*(X(f); Z)$$

is epic. From the commutative diagram

$$\begin{tikzcd} \Omega_*^U(X(f)) \arrow[d, "\zeta"] \arrow[dr, "\mu"] & \\ k_*(X(f)) \arrow[r, "\eta"'] & H_*(X(f); Z) \end{tikzcd}$$

we therefore find

$$\eta: k_*(X(f)) \to H_*(X(f); Z)$$

is epic. Henec $\tilde{\kappa}(f)=0$ and therefore by (2.5) $e_C(f)=0 \in Q/Z$.

Conversely, suppose $e_C(f)=0 \in Q/Z$. Then $\tilde{\kappa}(f)=0$ by (2.5), and the result follows from (1.2). **

The preceeding discussion should be compared to [5; §7] where (2.6) was originally obtained by employing complex cobordism only. For reference in future sections we summarize briefly the connection between $k_*(\ )$ and $\Omega_*^U(\ )$ as applied to the study of $X(f)$. Let $\alpha \in \tilde{\Omega}_0^U(X(f))$ denote the canonical class and $A(\alpha) \subset \Omega_*^U$ its annihilator ideal. Then $A(\alpha)=(q, [M])$ where $[M] \in \Omega_{2n}^U$ may be determined uniquely modulo $q\Omega_{2n}^U$. According to [4; 5.10] $\text{hom.dim.}_{\Omega_*^U}\Omega_*^U(X(f)) \leq 2$. Therefore by [4; §12] the natural mapping

$$\zeta: A(\alpha) \to A(\sigma)$$

is epic. Hence we may assume that $Td[M^{2n}]=r$ and thus we obtain [5; 7.2]

Corollary 2.7: *With the notations preceeding,*

$$e_C(f)=\text{Td}[M^{2n}]/q. \quad **$$

Finally we note that as hom.dim.$_{\Omega^U_*}\Omega^U_*(X(f))\leq 2$ the map

$$\zeta\colon \Omega^*_U(X(f))\to k_*(X(f))$$

is epic. Therefore the commutative diagram

$$\begin{tikzcd} \Omega^U_*(X(f)) \arrow[d, "\zeta"'] \arrow[dr] & \\ k_*(X(f)) \arrow[r] & H_*(X; Z) \end{tikzcd}$$

yields [5; §7]

Corollary 2.8: $\tilde{\omega}(f)=(|e_C(f)|-1)x^{2n+1}$. **

§3. On Three and Four Cell Complexes

In this section we will commence our study of the complex bordism of three and four cell complexes and its relation to the (stable) attachment data of such complexes. Our first task will be to describe the attachment data in the manner most suited to our study.

We will suppose given homotopy classes $[f]\in\pi^s_{2n-1}$, $[g]\in\pi^s_{2m-1}$ and an integer q such that

$$q[g]=0=[g][f]$$
$$0\in\langle q, [g], [f]\rangle.$$

We may then form the complexes

$$Y(g)=S^0\cup_g e^{2m}$$
$$Y(g, f)=S^0\cup_g e^{2m}\cup_F e^{2m+2n}$$

where

$$F\colon S^{2m+2n-1}\to Y(g)$$

is a coextension of f. (Such a coextension exists since $[g][f]=0$). We let

$$Q\colon Y(g)\to S^0$$

be an extension of the map of degree q on the bottom spheres (recall

$q[g]=0)$. Then

$$QF\colon S^{2m+2n-1}\to Y(g)\to S^0$$

is a representative of the Toda bracket $<q, [g], [f]>$ and hence we may assume that choices have been made such that QF is null homotopic. Thus we may construct a mapping

$$H\colon Y(g, f)\to S^0$$

of degree q on the bottom sphere. We let $X(g, f)$ be the mapping cone of H. The complex $X(g, f)$ will be the main object of study, and its attachment data consists of q, $[g]$, $[f]$ and the various conditions needed to form the extensions and coextensions to manufacture $X(g, f)$ from this deta. The following cofibration sequences

$$Y(g, f)\xrightarrow[H]{} S^0\xrightarrow[L]{} X(g, f)$$

$$S^0\hookrightarrow Y(g, f)\longrightarrow \Sigma^{2m}Y(f)$$

will also prove useful. Note also that

$$X(g, f)=S^0\cup_q e^1\cup_G e^{2m+1}\cup_F e^{2m+2n+1},$$

and therefore

$$\tilde{H}_i(X(g, f); Z)=\begin{cases} Z_q\colon & i=0\\ Z\ \colon & i=2m+1,\ 2n+1\\ 0\ \colon & \text{otherwise.}\end{cases}$$

Let us proceed by choosing generators for $\tilde{\Omega}^U_*(Y(g, f))$ as an Ω^U_*-module. From the cofibrations

$$S^{2m-1}\longrightarrow S^0\xrightarrow{c} Y(g)$$

$$S^{2m+2n-1}\longrightarrow Y(g)\xrightarrow{d} Y(g, f)$$

we find that it is possible to choose classes $\gamma_0\in\tilde{\Omega}^U_0(Y(g, f))$, $\gamma_{2m}\in\Omega^U_{2m}(Y(g, f)$ and $\gamma_{2m+2n}\in\Omega^U_{2m+2n}(Y(g, f))$ such that

$$\gamma_0=c_*\sigma_0$$

$$\gamma_{2m} = d_* \gamma'_{2m}$$

$$\partial_* \gamma_{2m+2n} = \sigma_{2m+2n-1}$$

where $\sigma_i \in \Omega_i^U(S^i)$ denotes the canonical generator and $\gamma'_{2m} \in \Omega_*^U(Y(g))$ a class such that $\partial_* \gamma'_{2m} = \sigma_{2m-1} \in \tilde{\Omega}_{2m-1}^U(S^{2m-1})$. Note that γ_{2m} is uniquely determined up to an element of the form $[A^{2m}]\gamma_0$, $[A^{2m}] \in \Omega_{2m}^U$ while γ_{2m+2n} is uniquely determined up to an element of the form $[B^{2m+2n}]\gamma_0 + [C^{2n}]\gamma_{2m}$, $[B^{2m+2n}] \in \Omega_{2m+2n}^U$, $[C^{2n}] \in \Omega_{2n}^U$. These classes freely generate $\tilde{\Omega}_*^U(Y(g, f))$ over Ω_*^U.

Notations: Throughout the remainder of our discussion of three and four cell complexes we will fix the notations preceeding. In addition we will denote by $\alpha = [S^0, L] \in \tilde{\Omega}_0^U(X(g, f))$ the canonical class. The annihilator ideal of α is denoted by $A(\alpha)$. Similarly we have the canonical class $\sigma = \zeta(\alpha) \in \tilde{k}_0(X(g, f))$ and its annihilator ideal $A(\sigma) \subset Z[t]$.

Proposition 3.1: *With the notations preceeding we have*

$$A(\alpha) = (q, [V_g], [V])$$

where $[V_g] \in \Omega_{2m}^U$ is uniquely determined in $\Omega_*^U/(q)$ and $[V] \in \Omega_{2m+2n}^U$ is uniquely determined in $\Omega_*^U/(q, [V_g])$.

Proof: Note that

$$A(\alpha) = \ker L_* = \operatorname{Im} H_*.$$

Recall that $\tilde{\Omega}_*^U(Y(g, f))$ is freely generated by the classes γ_0, γ_{2m}, γ_{2m+2n} which have the required uniqueness properties. **

In a similar manner we also obtain:

Proposition 3.2: *With the notations preceeding we have*

$$A(\sigma) = (q, at^m, bt^{n+m})$$

where $a \in Z$ is uniquely determined in Z_q and b is uniquely determined in $Z_{(q,a)}$. **

Remark: Note that in view of our discussion of how our generators for $\tilde{\Omega}_*^U(Y(g, f))$ were chosen and the work of section 2 we may assume that $a = \mathrm{Td}[V_g]$ and $e_{\boldsymbol{C}}(g) = a/q$.

Definition: With the notations preceeding let

$$|e_{\Omega}(g, f)| = \text{order of } [V] \text{ in } \Omega_*^U/(q, [V_g]),$$

and

$$|e(g, f)| = \text{order of } b \text{ in } Z_{(q,a)}.$$

Proposition 3.3: *The natural map*

$$A(\alpha) \xrightarrow{\zeta} A(\sigma)$$

is epic.

Proof: Consider the cofibration

$$Y(g, f) \xrightarrow[H]{} S^0 \xrightarrow[L]{} X(g, f).$$

Recalling that $A(\alpha) = \mathrm{Im}\, H_*$ and similarly for $A(\sigma)$, we obtain the diagram

$$\begin{array}{ccccc} \tilde{\Omega}_*^U(Y(g, f)) & \rightarrow & A(\alpha) & \rightarrow & 0 \\ \downarrow \zeta & & \downarrow & & \\ \tilde{k}_*(X(g, f)) & \rightarrow & A(\sigma) & \rightarrow & 0. \end{array}$$

Since $H_*(Y(g, f))$ is a free Z-module, the map

$$\zeta \colon \tilde{\Omega}_*^U(Y(g, f)) \rightarrow \tilde{k}_*(Y(g, f))$$

is epic [4; 10.1] and the result follows from commutativity. **

Corollary 3.4: *With the notations preceeding we may assume $b = \mathrm{Td}[V]$.* **

Corollary 3.5: $|e(g, f)| = $ *order* $\mathrm{Td}[V]$ *in* $Z_{(q,a)}$. $**$

Theorem 3.6: *With the notations preceeding we have*

$$\tilde{\omega}(X(g, f)) = (|e_C(g)| - 1)\, x^{2m+1} + (|e_\Omega(g, f)| - 1)\, x^{2m+2n+1}$$

and

$$\tilde{\kappa}(X(g, f)) = (|e_C(g)| - 1)\, x^{2m+1} + (|e(g, f)| - 1)\, x^{2m+2n+1}.$$

Proof: Introduce again the cofibration

$$Y(g, f) \xrightarrow[H]{} S^0 \xrightarrow[L]{} X(g, f).$$

In a standard manner we obtain the cofibration

$$S^0 \xrightarrow[L]{} X(g, f) \xrightarrow[D]{} \sum Y(g, f).$$

Thus we may regard $\sum Y(g, f)$ as obtained from $X(g, f)$ by attaching a cell along a map that represents $\alpha \in \tilde{\Omega}_0^U(X(g, f))$ as a spherical bordism class. Thus we may apply [4; 12.3] to conclude

$$\operatorname{coker}\{\tilde{\Omega}^U_{2m+2n+1}(X(g, f)) \to H_{2m+2n+1}(X(g, f); Z)\}$$
$$= [Z \otimes_{\Omega_*^U} A(\alpha)]_{2m+2n}.$$

(Recall that $\tilde{H}_*(\sum(Y(g, f)); Z)$ is a free Z-module and [4; 3.11].) Next note that $[Z \otimes_{\Omega_*^U} A(\alpha)]_{2m+2n}$ is cyclic with generator $[V]$ which has order $|e_\Omega(g, f)|$. If we now note that the $2m+2n$ skeleton of $X(g, f)$ is $X(g)$ we may combine our above observation with (2.8) to obtain $\tilde{\omega}(X(g, f)) = (|e_C(g)| - 1)\, x^{2m+1} + (|e_\Omega(g, f)| - 1)\, x^{2m+2n+1}$. The $k_*(\ )$ theory result is completely analogous and its proof is omitted. $**$

Corollary 3.7: *With the notations preceeding we have*

$$|e(g, f)| = |e_\Omega(g, f)| \text{ iff } \operatorname{hom.dim.}_{\Omega_*^U} \Omega_*^U(X(g, f)) \leq 2.$$

Proof: This is immediate from (3.6) and (1.1). $**$

Thus, unravelling some of our notation, we have arrived at the following criteria for $\Omega_*^U(X(g, f))$ to have homological dimension at most 2 as an Ω_*^U-module.

Corollary 3.8: *With the notations preceeding,* $\text{hom.dim.}_{\Omega_*^U}\Omega_*^U(X(g, f))\leq 2$ *iff the order of* $\text{Td}[V]$ *in* $Z_{(q,a)}$ *is exactly equal to the order of* $[V]$ *in* $\Omega_*^U/(q, [V_g])$. **

It is clear that our next task must be to investigate the problem of if and when $\text{hom.dim.}_{\Omega_*^U}\Omega_*^U(X(g, f))$ can exceed 2. It is evident from [4; 5.10] and (2.6) that we must at least have $e_C(g)\neq 0\in Q/Z$. It is perhaps worthwhile to record this observation.

Proposition 3.9: *With the notations preceeding, suppose that* $e_C(g)=0$. *Then* $\text{hom.dim.}_{\Omega_*^U}\Omega_*^U(X(g, f))\leq 2$. **

We return now to the question of if and when $\text{hom.dim.}_{\Omega_*^U}\Omega_*^U(X(g, f))$ can exceed 2. We will present a semi-complete answer to this question that leans heavily on the criteria of (3.8) and characteristic number arguments. The following section is devoted to a reduction of (3.8) to a characteristic number criterion. The ensuing characteristic number arguments are quite involved and occupy a section of their own. In section 6 we will return to apply them to the study of $\Omega_*^U(X(g, f))$. It is perhaps therefore best to close this section with an example borrowed form [5; 8.1] in the hopes that it sheds light on the general case.

Example: We choose $[g]=\eta\neq 0\in\pi_1^s$, $[f]=\nu\in\pi_3^s$ where ν is the map of Hopf invariant one mod 2, and $q=2$. Observe that

$$2\eta=0=\eta\nu$$

$$<2, \eta, \nu>\subset\pi_5^s=0.$$

Thus we may form the complex $X(\eta, \nu)$. With $\alpha\in\Omega_0^U(X(\eta, \nu))$ the canonical class it was shown in [5; 8.1] that $A(\alpha)=(2, [CP(1)], [V^6])$

where $[V^6]\in \Omega_6^U$ is a Milnor manifold for the prime 2. In [5] we then went on to show that $\text{hom.dim.}_{\Omega_*^U}\Omega_*^U(X(\eta, \nu))\geq 3$. In terms of our present discussion this may be seen as follows: By replacing $[V^6]$ by $[V^6]-\text{Td}[V^6][\boldsymbol{CP}(1)]^3$ we may assume that $\text{Td}[V^6]=0$, and hence by (3.5) that $|e(\eta, \nu)|=0$. On the other hand $[V^6]$ is an acceptable polynomial generator for $\Omega_*^U/(2)$ in degree 6. Thus $|e_\Omega(\eta, \nu)|=2$. Hence $\text{hom.dim.}_{\Omega_*^U}\Omega_*^U(X(\eta, \nu))>2$ by (3.7). Finally observe that

$$X(\eta, \nu)=X(\eta)\cup_{\bar{\nu}}e^7$$

where

$$\bar{\nu}: S^6\to X(\eta)$$

is a suitable coextension of ν and represents the bordism element $[V^6]\alpha$, which has order 2, as a spherical class. Since 2 is a prime we may apply [4; 5.12] to conclude that $\text{hom.dim.}_{\Omega_*^U}\Omega_*^U(X(\eta, \nu))=3$ exactly.

§4. More on Three and Four Cell Complexes

We shall continue to employ the notations of section 3. We recall the two basic cofibrations

$$Y(g, f)\xrightarrow[H]{} S^0\xrightarrow[L]{} X(g, f)$$

$$S^0 \longrightarrow Y(g, f)\xrightarrow[C]{}\Sigma^{2m}(Y(f)).$$

We have the generators $\gamma_0, \gamma_{2m}, \gamma_{2m+2n}\in\tilde{\Omega}_*^U(Y(g, f))$ with

$$H_*\gamma_0=q\sigma_0$$

$$H_*\gamma_{2m}=[V_g]\sigma_0: \qquad [V_g]\in\Omega_{2m}^U$$

$$H_*\gamma_{2m+2n}=[V]\sigma_0: \quad [V]\in\Omega_{2m+2n}^U.$$

Of course $A(\alpha)=(q, [V_g], [V])$ where $\alpha\in\tilde{\Omega}_0^U(X(g, f))$ is the canonical class. Our main concern in this section is to better pin down the manifold V via characteristic numbers. It is convenient to introduce another manifold $[V_f]\in\Omega_{2n}^U$ as follows. Choose an integer q' such that

$q'[f]=0\in\pi^s_{2n-1}$ and let

$$Q'\colon Y(f)\to S^0$$

be an extension of the map of degree q on the bottom sphere. We then have the composite

$$D\colon Y(g,f)\xrightarrow[C]{}\Sigma^{2m}Y(f)\xrightarrow[\Sigma^{2m}Q']{}S^{2m}.$$

Note that

$$D_*\gamma_0=0$$

$$D_*\gamma_{2m}=q'\sigma_{2m}$$

$$D_*\gamma_{2m+2n}=[V_f]\sigma_{2m}$$

where $[V_f]\in\Omega^U_{2n}$ is well determined modulo $q'\Omega^U_*$.

For our study q, $[V_g]$, $[V]$ and $[V_f]$ are critically related. The exact value of q' is not immediately relevant.

Notation: Denote by

$$\Phi\colon\Omega^U_*\to\Omega^{U,fr}_*$$

the natural forgetful map.

Proposition 4.1: *With the notations preceeding suppose that q' is odd. Then*

$$[V_f]=q'[\theta]\ \mathrm{mod}\ \textit{torsion}$$

for some (U, fr)-manifold $[\theta]$.

Proof: Observe that there is a cofibration

$$Y(f)\xrightarrow[Q']{}S^0\xrightarrow[P']{}X(f)$$

and that $A(\beta)=(q',[V_f])$ where $\beta\in\tilde{\Omega}^U_0(X(f))$ is the canonical class $[S^0,P]$ Hence by [12; A.2]

$$s_\omega(\gamma)[V_f]\equiv 0\ \mathrm{mod}\ q'$$

for all non-empty partitions ω. An application of the main result of [10] now completes the proof. **

The following constitutes our main technical result.

Theorem 4.2: *With the notations preceeding we have the following K-theory number congruences*

$$s_\omega(\gamma)[V] \equiv \mathrm{Td}[V_g]s_\omega(\gamma)[0] \mod q$$

for all non-empty partitions ω.

Proof: We have the diagram

$$\begin{array}{l} Y(g, f) \xrightarrow[H]{} S^0 \\ {\scriptstyle D}\downarrow \\ S^{2m} \end{array}$$

from which we obtain

$$H_*\gamma_{2m+2n} = [V]\sigma_0$$

$$D_*\gamma_{2m+2n} = [V_f]\sigma_{2m}.$$

Represent γ_{2m+2n} by

$$\varphi: L^{2m+2n} \to Y(g, f).$$

Then we have [5; §2] the following formulas

$$s_\omega(\gamma)[V] = <\varphi^! H^!(\eta-1)s_\omega(\gamma)[L^{2m+2n}], [L^{2m+2n}]>$$

$$s_\omega(\gamma)[V_f] = <\varphi^! D^!(\eta-1)s_\omega(\gamma)[L^{2m+2n}], [L^{2m+2n}]>$$

where $\eta \in K(S^{2k})$ is the canonical Bott generator. Now we must recall that we may choose classes

$$a_0 \in \tilde{H}_0(Y(g, f); Z)$$

$$a_{2m} \in \tilde{H}_{2m}(Y(g, f); Z)$$

$$a_{2m+2n} \in \tilde{H}_{2m+2n}(Y(g, f); Z)$$

such that

$$\operatorname{ch} H^!(\eta-1)=q g_0+\operatorname{Td}[V_g]\, g_{2m}+\operatorname{Td}[V]\, g_{2m+2n}.$$

We now apply the RR theorem [14] to convert a K-theory computation to a cohomology computation and we find

$$\begin{aligned} s_\omega(\gamma)[V] &= <\operatorname{ch}(\varphi^! H^!(\eta-1)\, s_\omega(\gamma)[L])\operatorname{Td}[L],\, [L]> \\ &= <\varphi^*(q g_0+\operatorname{Td}[V_g]g_{2m}+\operatorname{Td}[V]g_{2m+2n})\operatorname{ch} s_\omega(\gamma)(L)\operatorname{Td}(L),\, [L]> \\ &\quad \operatorname{Td}[V_g]<\varphi^* g_{2m}\operatorname{ch} s_\omega(\gamma)(L)\operatorname{Td}(L),\, [L]> \end{aligned}$$

for dimensional reasons provided that $\omega\neq\emptyset$. On the other hand

$$\begin{aligned} s_\omega(\gamma)[V_f] &= <\operatorname{ch}(\varphi^! D^!(\eta-1)\, s_\omega(\gamma)(L))\operatorname{Td}(L),\, [L]> \\ &= <\varphi^*(q' g_{2m}+\operatorname{Td}[V_f]g_{2m+2n})\operatorname{ch} s_\omega(\gamma)(L)\operatorname{Td}(L),\, [L]> \\ &= q'<\varphi^* g_{2m}\operatorname{ch} s_\omega(\gamma)(L)\operatorname{Td}(L),\, [L]> \end{aligned}$$

again provided $\omega\neq\emptyset$. Thus we find

$$(*) \qquad q' s_\omega(\gamma)[V]\equiv\operatorname{Td}[V_g]s_\omega(\gamma)[V_f] \mod q.$$

By (4.1)

$$\Phi[V_f]=q'[\theta] \text{ mod torsion}$$

and hence

$$s_\omega(\gamma)[\theta]=\frac{s_\omega(\gamma)[V_f]}{q'}$$

provided $\omega\neq\emptyset$. Upon substituting this back into $(*)$ the result follows. **

Although it is somewhat premature at this point to do so, we can handle a class of spaces $X(g, f)$ at this time simply with the aid of (4.2). We do so in the expectation that the technical difficulties of the general case will become apparent and our main line of argument can be discerned through the tedious calculations of the next section.

Theorem 4.3: *In the notations preceeding suppose given a space $X(g, f)$ where $e_C(g)=1/p$ for some odd prime p. Then* $\operatorname{hom.dim.}_{\Omega^U_*}\Omega^U_*(X(g, f))\leq 2$.

Proof: By localization techniques (see §6) we may assume that $q=p^t$ for some integer t. Then according to (2.7) $\mathrm{Td}[V_g]=p^{t-1}$. By appealing to [12; A.7] we find that $m\equiv 0 \bmod p-1$ and we may assume that $[V_g]=p^{t-1}[CP(p-1)]^{m/p-1}$.

Let p^{ε} denote the order of $\mathrm{Td}[V]$ in $Z_{p^{t-1}}$. We must show that

$$p^{\varepsilon}[V]\in(p^t, [V_g])$$

for then (3.8) will yield the desired conclusion. According to the definition of ε we have

$$\mathrm{Td}[V]\equiv 0 \bmod p^{t-1-\varepsilon}.$$

As $e_C(g)\in Q/Z$ has odd order it follows from (3.9) and localization that we may assume $[f]\in\pi^s_{2n-1}$ has odd order. Applying (4.2) we obtain the basic set of equations

(A) $$s_\omega(\gamma)[V]\equiv p^{t-1}s_\omega(\gamma)[\theta] \bmod p^t : \omega\neq\emptyset$$

where $[\theta]\in\Omega^{U,fr}_{2n}$.

Next we apply the results of [10] to find a closed manifold C^{2n}, and integers a, b, such that

(B) $$2^a[\theta]=b[E^{2n}]+\Phi[C^{2n}]$$

where $[E^{2n}]$ denotes the (U, fr) bordism class of the $2n$-cell with the special (U, fr) structure described in [10]. Thus we may rewrite our basic equations as

(C) $$2^a s_\omega(\gamma)[V]\equiv bp^{t-1}s_\omega(\gamma)[E^{2n}]+p^{t-1}s_\omega(\gamma)[C^{2n}] \bmod p^t$$

for all non-empty partitions ω. Next recall that

(D) $$\mathrm{Td}[V]\equiv 0 \bmod p^{t-1-\varepsilon}.$$

Combining these two facts we find that

(E) $$s_\omega(\gamma)[V]\equiv 0 \bmod p^{t-1-\varepsilon}$$

for all partitions. Hence by the Stong [13]-Hattori [6] theorem

(F) $$[V]=p^{t-1-\varepsilon}[\bar{V}]$$

for a suitable closed U-manifold $\bar{V}$. Thus our basic equations may be written as

(G) $$p^{\varepsilon}[V]=p^{t-1}[\bar{V}]$$

and hence our basic equations lead to

(H) $$2^{a}s_{\omega}(\gamma)[\bar{V}]\equiv bs_{\omega}(\gamma)[E^{2n}]+s_{\omega}(\gamma)[C] \mod p:\ \omega\neq\emptyset$$

Now let c be chosen such that $c2^{a}\equiv 1 \mod p$, and let

(I) $$[\hat{V}]=[\bar{V}]-c[C^{2n}][CP(p-1)]^{m/p-1}.$$

Observe that by the product formula for K-numbers

(J) $$\begin{aligned} s_{\omega}(\sigma)[C^{2n}][CP(p-1)]^{m/p-1} &= s_{\omega}(\gamma)[C^{2n}] \\ &+\sum_{\substack{\omega'\omega'' \\ \omega''\neq\phi}} s_{\omega'}(\gamma)[C^{2n}]s_{\omega''}(\gamma)[CP(p-1)]^{m/p-1} \\ &\equiv s_{\omega}(\gamma)[C^{2n}] \mod p:\ \omega\neq\emptyset \end{aligned}$$

since

(K) $$s_{\omega}(\gamma)[CP(p-1)]^{s}\equiv 0 \mod p:\ \omega\neq\emptyset$$

for any positive integer s. Thus we find

(L) $$2^{a}s_{\omega}(\gamma)[V]\equiv bs_{\omega}(\gamma)[E^{2n}] \mod p:\ \omega\neq\emptyset.$$

Now note according to [13] [14; pp. 121-124]

(M) $$s_{(p-1)}(\gamma)[M]\equiv 0 \mod p$$

for any closed U-manifold M, while according to [10]

(N) $$s_{p-1}(\gamma)[E^{2n}]\equiv\pm 1 \mod p.$$

Thus putting $\omega=(p-1)$ into (L) and using (M), (N) we obtain

(O) $$0\equiv 2^{a}s_{(p-1)}(\gamma)[V]\equiv bs_{(p-1)}(\gamma)[E^{2n}]\equiv b \mod p.$$

Therefore

(P) $$b\equiv 0 \mod p.$$

Hence putting (P) into (L) leads to

(Q) $$s_\omega(\gamma)[\hat{V}]\equiv 0 \mod p:\ \omega\neq\emptyset.$$

Therefore by [11; 5.6]

(R) $$[V]=\begin{cases} p[A] & : \ m+n\not\equiv 0 \mod p-1, \\ p[A]+\mathrm{Td}[V][CP(p-1)]^{\frac{m+n}{p-1}}: & m+n\equiv 0 \mod p-1. \end{cases}$$

Recalling (G) and (I) and $m\equiv 0 \mod p-1$ we find

(S) $$p^\varepsilon[V]=\begin{cases} p^t[A] & : \ n\not\equiv 0 \mod p-1 \\ p^t[A]+p^{t-1}(c[C^{2n}]+[CP(p-1)]^{n/p-1})[CP(p-1)]^{m/p-1} & \\ & : \ n\equiv 0 \mod p-1 \end{cases}$$

and so $p^\varepsilon[V]\in(p^t, p^{t-1}[CP(p-1)]^{m/p-1})$ in all cases, as was to be shown. **

Notation: Let $\alpha_1\in\pi^s_{2p-3}$, p an odd prime, be the class introduced by Toda [15] [16] of order p and Hopf invariant 1 mod p. Following [11] write $V(1/2)$ for the stable complex $Y(\alpha_1)$. Denote by $\gamma\in\tilde{\Omega}_0(V(1/2))$ the canonical class.

Corollary 4.4: *With the notations preceeding suppose that* $[M]\in\Omega^U_{2k}$, $k>0$, *and*

$$[M]\gamma\in\mathrm{Im}\{\tilde{\Omega}^{fr}_*(V(1/2))\to\tilde{\Omega}^U_*(V(1/2))\}.$$

Then $[M]\gamma=0\in\tilde{\Omega}^U_*(V(1/2))$.

Proof: Represent $[M]$ by

$$F: S^{2k}\to V(1/2).$$

Observe by [12; A.5] that we may assume $k>2p-2$. Write $k=2n+(2p-2)$ and introduce

$$\lambda: S^{2n+(2p-2)}\xrightarrow[F]{} V(1/2)\xrightarrow[\text{collapse}]{} S^{2p-1}.$$

Then we readily see that

$$X=V(1/2)\cup_F e^{2k+1}=X(\alpha_1,\ \lambda).$$

Therefore by (4.3) $\mathrm{hom.bim.}_{\Omega_*^U}\Omega_*^U(X)\leq 2$. Let us write $\bar{\gamma}\in\tilde{\Omega}_0^U(X)$ for the canonical class. Clearly

$$A(\bar{\gamma})=(p,\ [CP(p-1)],\ [M]).$$

According to [4; 5.3] the girth of the ideal $A(\bar{\gamma})$ is at most 2. Since $(p, [CP(p-1)])$ is a prime ideal in Ω_*^U this implies that $[M]\in(p, [CP(p-1)])=A(\gamma)$ and the result follows. **

Note that (4.4) provides an alternate proof of the key result [11; 5.10] without (direct) recourse to Toda brackets and the numerous unpleasant cases and computations of [11; §5]. It was primarily to accomplish just this that the present study was undertaken.

It is to be hoped that the reader will bear the outline of the preceeding arguments in mind through the tedious characteristic number arguments of the next section.

§5. Some Characteristic Number Arguments

Our objective in this section is to provide the technical results concerning characteristic numbers that are needed to complete our study. It is convenient to state the main result now, although its proof requires many preliminary maneuvers.

Theorem 5.1: *Let p be an odd prime and r, s, i, j integers satisfying the following conditions*

$$0\leq s\leq r,\ i,\ j\geq 0$$

$$i,\ j\equiv 0 \mod p-1$$

$$s\leq \nu_p(i)$$

where $\nu_p(i)$ denotes the power of p in i. Suppose that $[V]\in\Omega_{2(i+j)}^U$ and $[\theta]\in\Omega_{2j}^{U;fr}$ are such that

$$s_\omega(\gamma)[V]\equiv p^{r-s}s_\omega(\gamma)[\theta] \mod p^{r+1},$$

for all non-empty partitions ω.

Then the order of $[V]$ *in* $\Omega_*^U/(p^{r+1}, p^{r-s}[\boldsymbol{C}P(p-1)]^{i/p-1})$ *is equal to the order of* $\mathrm{Td}[V]$ *in* $Z_{p^{r-s}}$.

The proof of this theorem will occupy the entire section. As we shall employ the results of [10] at several key points in the argument we begin by recalling these results in the form most convenient to our needs.

Recollections on the structure of $\Omega_*^{U,\mathrm{fr}}$ /Torsion: Let

$$\sigma\colon (E^{2n}, S^{2n-1}) \to (BU, *)$$

be a map representing $1 \in \pi_{2n}(BU, *)$. Then σ induces a complex bundle ξ over E^{2n} with a compatible framing on S^{2n-1}. Clearly regarded simply as a bundle over E^{2n}, ξ is trivial. Thus we may regard ξ as providing (E^{2n}, S^{2n-1}) with the structure of a (U, fr)-manifold. Denote by $[\sum^{2n}] \in \Omega_{2n}^{U,\mathrm{fr}}$ the bordism class of (E^{2n}, S^{2n-1}) with the (U, fr) structure ξ. (This is the same manifold denoted by $[E^{2n}]$ in the previous section.) According to [10] we then have:

Theorem: *There is a basis for* $[\Omega_{2n}^{U,\mathrm{fr}}/\mathrm{Torsion}] \otimes Z[1/2]$ *consisting of* $[\sum^{2n}]$ *and closed manifolds.* $**$.

The K-theory characteristic numbers of $[\sum^{2n}]$ were computed as a key step in the proof of the preceeding theorem. Among these results [10] we have:

Propositon: *Let p be an odd prime, n a positive integer with* $n \equiv 0 \mod p-1$. *Then* $s_{(p-1)}(\gamma)[\sum^{2n}] \equiv (-1)^{n+1} \mod p$. $**$

This should be contrasted with the result of Stong [14; pp. 121–124] that for a closed U-manifold $[C]$, $s_{p-1}(\gamma)[C] \equiv 0 \mod p$ regardless of the dimension of $[C]$. (Of course this may also be deduced from the two results of [10] preceeding.)

Combining the preceeding Proposition with [11; 5.7] and [12;

Appendix] we find:

Proposition: *Let p be an odd prime and n a positive integer satisfying $n \equiv 0 \bmod p-1$. Then there exists integers $q'(n)$, $q''(n)$ relatively prime to p such that $p^{1+\nu_p(n)}q'(n)[\Sigma^{2n}]$ is (U, fr) cobordant to $q''(n)[CP(p-1)]^{n/p-1}$.* **

Corollary: *Denote by $\Phi: \Omega_*^U \to \Omega_*^{U,\mathrm{fr}}$ the standard forgetful homomorphism. Then $\Phi([CP(p-1)]^m)$ is divisable by $p^{1+\nu_p(m)}$ and no higher power of p.* **

We are now prepared to begin the proof of Theorem 5.1.

Proof of Theorem 5.1. It is convenient to divide the proof into two cases, depending on $s \leq \nu_p(j)$ or $s > \nu_p(j)$. The first of these is by far the easier case and so we will begin with it.

Case I: $s \leq \nu_p(j)$.

It is then clear that $s \leq \min\{\nu_p(i), \nu_p(j)\}$ and hence in view of [10] the following lemma is clear.

Lemma 5.2: *The manifolds $[CP(p-1)]^{i/p-1}$, $[CP(p-1)]^{j/p-1}$, $[CP(p-1)]^{i+j/p-1}$ are all divisable by p^{s+1} in $\Omega_*^{U,\mathrm{fr}}$.*

Proof: One has merely to note that

$$s+1 \leq \min\{\nu_p(i), \nu_p(j)\} + 1 \leq \nu_p(i+j)+1 \leq \begin{cases} \nu_p(i)+1 \\ \nu_p(j)+1 \end{cases}$$

and apply [10]. **

Since we are ignoring the prime 2 we find according to [10] that we may write

$$[\theta] = m[\Sigma^{2j}] + \Phi[C^{2j}]$$

where C^{2j} is a closed U-manifold.

Lemma 5.3: *With the notations preceeding we may assume that* $\mathrm{Td}[C^{2j}]=0$.

Proof: Let us set

$$[\bar{C}^{2j}]=[C^{2j}]-\mathrm{Td}[C^{2j}][CP(p-1)]^{j/p-1}$$

Note that $\mathrm{Td}[\bar{C}^{2j}]=0$. Next observe that as $[CP(p-1)]^{j/p-1}$ is divisible by p^{s+1} in $\Omega_*^{U,\mathrm{fr}}$ we must have [10]

$$s_\omega(\gamma)[CP(p-1)]^{j/p-1}\equiv 0 \mod p^{s+1}$$

for all non-empty partitions. Thus we have

$$s_\omega(\gamma)[\bar{C}^{2j}]\equiv s_\omega(\gamma)[C^{2j}] \mod p^{s+1}: \qquad \omega\neq\emptyset$$

and hence

$$\begin{aligned} s_\omega(\gamma)[V] &\equiv p^{r-s}ms_\omega(\gamma)[\Sigma^{2j}]+p^{r-s}s_\omega(\gamma)[C^{2j}] \mod p^{r+1} \\ &\equiv p^{r-s}ms_\omega(\gamma)[\Sigma^{2j}]+p^{r-s}s_\omega(\gamma)[\bar{C}^{2j}] \mod p^{r+1} \end{aligned}$$

as required. **

Henceforth we will therefore assume that $[C^{2j}]=[\bar{C}^{2j}]$, that is, that $\mathrm{Td}[C^{2j}]=0$.

Lemma 5.4: *Let*

$$[W]=[V]-p^{r-s}[CP(p-1)]^{i/p-1}[C^{2j}].$$

Then

$$s_\omega(\gamma)[W]\equiv p^{r-s}ms_\omega(\gamma)[\Sigma^{2j}] \mod p^{r+1}: \qquad \omega\neq\emptyset.$$

Proof: By the product formula for K-theory numbers we obtain

$$\begin{aligned} s_\omega(\gamma)[CP(p-1)]^{i/p-1}[C^{2j}] &= \mathrm{Td}[CP(p-1)]^{i/p-1}s_\omega(\gamma)[C^{2j}] \\ &+\sum_{\substack{\omega'\omega''=\omega \\ \omega'\neq\phi}} s_{\omega'}(\gamma)[CP(p-1)]^{i/p-1}s_{\omega''}(\gamma)[C^{2j}] \\ &+s_\omega(\gamma)[CP(p-1)]^{i/p-1}\mathrm{Td}[C^{2j}]. \end{aligned}$$

From (5.2) it follows that

$$s_\alpha(\gamma)[CP(p-1)]^{i/p-1}\equiv 0 \mod p^{s+1}: \qquad \alpha\neq\emptyset$$

and from (5.3) that

$$\mathrm{Td}[C^{2j}]=0.$$

Thus we have for $\omega\neq\emptyset$

$$s_\omega(\gamma)[CP(p-1)]^{i/p-1}[C^{2j}]\equiv s_\omega(\gamma)[C^{2j}] \mod p^{s+1},$$

and therefore for all $\omega\neq\emptyset$

$$p^{r-s}s_\omega(\gamma)[[CP(p-1)]^{i/p-1}[C^{2j}]]\equiv p^{r-s}s_\omega(\gamma)[C^{2j}] \mod p^{r+1}.$$

Return now to the basic equation

$$s_\omega(\gamma)[V]\equiv p^{r-s}ms_\omega(\gamma)[\Sigma^{2j}]+p^{r-s}s_\omega(\gamma)[C^{2j}] \mod p^{r+1}$$

valid for all non-empty partitions ω. According to our preceeding discussion we thus find

$$s_\omega(\gamma)[V]\equiv p^{r-s}ms_\omega(\gamma)[\Sigma^{2j}]$$
$$+p^{r-s}s_\omega(\gamma)[[CP(p-1)]^{i/p-1}[C^{2j}]] \mod p^{r+1}$$

and hence for $\omega\neq\emptyset$

$$p^{r-s}ms_\omega(\gamma)[\Sigma^{2j}]\equiv s_\omega(\gamma)[[V]-p^{r-s}[CP(p-1)^{i/p-1}][C^{2j}]] \mod p^{r+1}$$
$$\equiv s_\omega(\gamma)[W] \mod p^{r+1}$$

as was to be shown. **

Lemma 5.5: $\mathrm{Td}[W]=\mathrm{Td}[V]$.

Proof: Immediate from the fact that $\mathrm{Td}[C^{2j}]=0$. **

Let p^l be the order of $\mathrm{Td}[W]$ in $Z_{p^{r-s}}$. Note that $0\leq l\leq r-s$ and that $p^l\,\mathrm{Td}[W]=p^{r-s}N$ for some integer N.

Lemma 5.6: *Let*

$$[X]=p^l[W]-p^{r-s}N[CP(p-1)]^{i+j/p-1}.$$

Then

$$\mathrm{Td}[X]=0$$

and for all non-empty partitions

$$s_\omega(\gamma)[X]\equiv p^{r-s+l}ms_\omega(\gamma)[\Sigma^{2j}] \mod p^{r+1}.$$

Proof: First of all we have

$$\mathrm{Td}[X]=p^l\,\mathrm{Td}[W]-p^{r-s}N=0.$$

Next observe that for all non-empty partitions

$$s_\omega(\gamma)[CP(p-1)]^{i+j/p-1}\equiv 0 \mod p^{s+1}$$

by (5.2) and [10]. Therefore

$$s_\omega(\gamma)[X]=p^l s_\omega(\gamma)[W]\equiv p^{r-s+l}ms_\omega(\gamma)[\Sigma^{2j}] \mod p^{r+1},$$

by (5.4) and the result follows. **

Lemma 5.7: *Let p be an odd prime, n a positive integer satisfying $n\equiv 0 \bmod p-1$. Suppose that B is a closed U-manifold satisfying*

$$s_\omega(\gamma)[B]\equiv qs_\omega(\gamma)[\Sigma^{2n}] \mod p^t: \qquad \omega\neq\emptyset$$

for some integer q and non-negative integer t. Then $q\equiv 0 \bmod p$. If moreover $\mathrm{Td}[B]=0$ *then* $q\equiv 0 \mod p^t$.

Proof: Let $\omega=(p-1)$. Then we have

$$0\equiv s_{(p-1)}(\gamma)[B]=q(-1)^{n+1} \mod p$$

which shows that $q\equiv 0 \mod p$. Thus

$$s_\omega(\gamma)[B]\equiv 0 \mod p: \qquad \omega\neq\emptyset.$$

If now $\mathrm{Td}[B]=0$ then we may conclude by the Stong [13]-Hattori [6] theorem that $[B]=p[B']$. Moreover if $t>1$ and we write $q=pq'$ we have

$$s_\omega(\gamma)[B']\equiv q's_\omega(\gamma)[\Sigma^{2n}] \mod p^{t-1}$$

for all non-empty partitions ω, and as $\mathrm{Td}[B']=0$ we may repeat the above procedure, etc. **

Lemma 5.8: $[X]\in p^{r+1}\Omega^U_{2(i+j)}$.

Proof: According to (5.6) we have

$$\mathrm{Td}[X]=0$$

$$s_\omega(\gamma)[X]\equiv p^{r-s+l}ms_\omega(\gamma)[\Sigma^{2j}] \mod p^{r+1}: \qquad \omega\neq\emptyset.$$

Suppose first that $r+1\leq r-s+l$. Then clearly

$$s_\omega(\gamma)[X]\equiv 0 \mod p^{r+1}: \qquad \text{all } \omega.$$

Hence by the Stong-Hattori theorem we have $[X]\in p^{r+1}\Omega^U_{2(i+j)}$ as required. On the other hand suppose that $r+1\geq r-s+l$. Then first of all

$$s_\omega(\gamma)[X]\equiv 0 \mod p^{r-s+l}: \qquad \text{all } \omega.$$

Hence by the Stong-Hattori theorem there exists a closed U-manifold Y such that

$$[X]=p^{r-s+l}[Y].$$

Of course

$$0=\mathrm{Td}[X]=p^{r-s+l}\mathrm{Td}[Y]$$

and so $\mathrm{Td}[Y]=0$. Moreover the equations

$$s_\omega(\gamma)[X]\equiv p^{r-s+l}ms_\omega(\gamma)[\Sigma^{2j}] \mod p^{r+1}: \qquad \omega\neq\emptyset$$

imply that for all non-empty partitions

$$s_\omega(\gamma)[Y]\equiv ms_\omega(\gamma)[\Sigma^{2j}] \mod p^{s-l+1}.$$

According to (5.7) we then have

$$m\equiv 0 \mod p^{s-l+1}.$$

Therefore

$$s_\omega(\gamma)[Y]\equiv 0 \mod p^{s-l+1}: \qquad \omega\neq\emptyset$$

and since $\mathrm{Td}[Y]=0$ we may apply the Stong-Hattori theorem to conclude

$$[Y]=p^{s-l+1}[Z]$$

for some closed U-manifold Z. Thus

$$[X]=p^{r-s+l}[Y]=p^{r-s+l}p^{s-l+1}[Z]$$

and hence $[X]\in p^{r+1}\Omega^U_{2(i+j)}$ in this case also. **

We are now able to complete the proof of (5.1) Case I as follows. According to (5.5) the order of $\mathrm{Td}[V]$ in $Z_{p^{r-s}}$ is p^l. We must therefore show that $p^l\mathrm{Td}[V]\in(p^{r+1}, p^{r-s}[\boldsymbol{CP}(p-1)]^{i/p-1})$. According to (5.4) we have

$$[V]\equiv[W] \mod (p^{r+1}, p^{r-s}[\boldsymbol{CP}(p-1)]^{i/p-1})$$

and so it will suffice to show that $p^l[W]\in(p^{r+1}, p^{r-s}[\boldsymbol{CP}(p-1)]^{i/p-1})$. According to (5.6)

$$p^l[W]\equiv[X] \mod (p^{r+1}, p^{r-s}[\boldsymbol{CP}(p-1)]^{i/p-1})$$

and hence it will suffice to show that $[X]\in(p^{r+1}, p^{r-s}[\boldsymbol{CP}(p-1)]^{i/p-1})$. However this last inclusion is clearly a consequence of (5.8). Thus we have completed the proof of (5.1) under the assumption that $s\leq\nu_p(j)$. We are now left to deal with the case $s>\nu_p(j)$.

Case II: $s>\nu_p(j)$.

As in the preceeding case we shall require numerous preliminary facts and figures. Our starting point will again be the basic equations

$$s_\omega(\gamma)[V]\equiv mp^{r-s}s_\omega(\gamma)[\Sigma^{2j}]+p^{r-s}s_\omega(\gamma)[C^{2j}] \mod p^{r+1},$$

for all non-empty partitions, obtained by writing

$$[\theta]=m[\Sigma^{2j}]+\Phi[C^{2j}]$$

with the aid of [10].

Lemma 5.9: *Let*

$$[W]=[V]-p^{r-s}[CP(p-1)]^{i/p-1}[C^{2j}].$$

Then

$$s_\omega(\gamma)[W] \equiv p^{r-s} m s_\omega(\gamma)[\Sigma^{2j}] \mod p^{r+1}.$$

Proof: First of all recall that $s \leq \nu_p(i)$. Therefore

$$s_\omega(\gamma)[\boldsymbol{CP}(p-1)]^{i/p-1} \equiv 0 \mod p^{s+1}$$

for all non-empty partitions ω. Hence by the product formula for K-theory numbers

$$s_\omega(\gamma)[[\boldsymbol{CP}(p-1)]^{i/p-1}[C^{2j}]] \equiv s_\omega(\gamma)[C^{2j}] \mod p^{s+1}.$$

Thus we find for any non-empty partition

$$\begin{aligned} s_\omega(\gamma)[W] &= s_\omega(\gamma)[V] - p^{r-s} s_\omega(\gamma)[[CP(p-1)]^{i/p-1}[C^{2j}]] \\ &\equiv s_\omega(\gamma)[V] - p^{r-s} s_\omega(\gamma)[C^{2j}] \mod p^{r+1} \\ &\equiv p^{r-s} m s_\omega(\gamma)[\Sigma^{2j}] + p^{r-s} s_\omega(\gamma)[C^{2j}] \\ &\qquad - p^{r-s} s_\omega(\gamma)[C^{2j}] \mod p^{r+1} \\ &\equiv p^{r-s} m s_\omega(\gamma)[\Sigma^{2j}] \mod p^{r+1} \end{aligned}$$

as required. **

Let t be the smallest integer such that $p^t m \operatorname{Td}[\Sigma^{2j}]$ has denominator prime to p. Thus there is an integer $q \neq 0$, relatively prime to p, such that

$$q p^t m [\Sigma^{2j}] = q' [\boldsymbol{CP}(p-1)]^{j/p-1}$$

for some integer q' relatively prime to p.

Lemma 5.10: $s_\omega(\gamma)[CP(p-1)]^{\frac{i+j}{p-1}} \equiv s_\omega(\gamma)[CP(p-1)]^{j/p-1} \mod p^{s+1}$ *for all non-empty partitions.*

Proof: By the product formula for K-theory numbers we have

$$\begin{aligned} &s_\omega(\gamma)[\boldsymbol{CP}(p-1)]^{i+j/p-1} \\ &\quad = \sum_{\omega'\omega''=\omega} s_{\omega'}(\gamma)[\boldsymbol{CP}(p-1)]^{i/p-1} s_{\omega''}(\gamma)[\boldsymbol{CP}(p-1)]^{j/p-1}. \end{aligned}$$

Now recall that as $s \leq \nu_p(\mathrm{i})$

$$s_\omega(\gamma)[CP(p-1)]^{i/p-1} \equiv 0 \mod p^{s+1}: \qquad \omega' \neq \emptyset.$$

Substituting and recalling $\mathrm{Td}[CP(p-1)]^{i/p-1}=1$ now yields the result. **

Lemma 5.11: *Let*

$$[D]=qp^t[W]-q'p^{r-s}[CP(p-1)]^{i+j/p-1}.$$

Then

$$s_\omega(\gamma)[D] \equiv 0 \mod p^{r+1}: \qquad \omega \neq \emptyset.$$

Proof: By direct computation (5.9) and (5.10) we obtain

$$\begin{aligned} s_\omega(\gamma)[D] &= qp^t s_\omega(\gamma)[W]-q'p^{r-s}s_\omega(\gamma)[CP(p-1)]^{i+j/p-1} \\ &\equiv qp^t p^{r-s} m s_\omega(\gamma)[\Sigma^{2j}]-q'p^{r-s}s_\omega(\gamma)[CP(p-1)]^{j/p-1} \mod p^{r+1} \\ &\equiv p^{r-s}(qp^t s_\omega(\gamma)[\Sigma^{2j}]-q's_\omega(\gamma)[CP(p-1)]^{j/p-1}) \mod p^{r+1} \\ &\equiv p^{r-s}(0) \equiv 0 \mod p^{r+1} \end{aligned}$$

as required. **

Lemma 5.12: $p^t[V] \in (p^{r+1}, p^{r-s}[CP(p-1)]^{i/p-1})$

Proof: Since

$$[V] \equiv [W] \mod (p^{r+1}, p^{r-s}[CP(p-1)]^{i/p-1})$$

it will suffice to show that $p^t[W] \in (p^{r+1}, p^{r-s}[CP(p-1)]^{i/p-1})$. Since q is relatively prime to p it is equivalent to show $qp^t[W] \in (p^{r+1}, p^{r-s}[CP(p-1)]^{i/p-1}])$ and hence in view of (5.11) that $[D] \in (p^{r+1}, p^{r-s}[CP(p-1)]^{i/p-1})$. However according to (5.11) and [12; A.5, A.6] we have

$$[D]=p^{r+1}[A]+\mathrm{Td}[D][CP(p-1)]^{i+j/p-1}$$

and

$$\mathrm{Td}[D] \equiv 0 \mod p^{r+1-1-\nu_p(i+j)}.$$

Now recall that we are assuming

$$\nu_p(j) < s \leq \nu_p(\mathrm{i}).$$

Therefore

$$\nu_p(i+j) = \nu_p(j) < s.$$

Hence

$$r - \nu_p(i+j) > r - s.$$

Thus we may write

$$\mathrm{Td}[D] = p^{r-s} d$$

and so

$$[D] = p^{r+1}[A] + dp^{r-s}[CP(p-1)]^{i+j/p-1}.$$

Thus by inspection $[D] \in (p^{r+1}, p^{r-s}[CP(p-1)]^{i/p-1})$. **

Lemma 5.13: $p^t \mathrm{Td}[V] \equiv 0 \mod p^{r-s}$.

Proof: According to (5.12) we may write

$$[V] = p^{r+1}[A] + p^{r-s}[B][CP(p-1)]^{i/p-1}$$

and so

$$\mathrm{Td}[V] = p^{r+1}\mathrm{Td}[A] + p^{r-s}\mathrm{Td}[B]$$

from which the result follows. **

Thus if we denote by p^l, $0 \leq l \leq r-s$, the order of $\mathrm{Td}[V]$ in $Z_{p^{r-s}}$ we have $l \leq t$. Thus the proof of (5.1) Case II will be completed (in view of (5.12)) if we can show $l \geq t$. That is it will suffice to show that there exists an integer q, prime to p, such that $qp^l m \,\mathrm{Td}[\Sigma^{2j}] \in Z$.

Lemma 5.14: *The order of* $\mathrm{Td}[W]$ *in* $Z_{p^{r-s}}$ *is equal to* p^l.

Proof: This follows from the definition of $[W]$ given in (5.9). **

Lemma 5.15: *$p^l[W]=p^{r-s}[B]$ for some closed U-manifold B of dimension $2i+2j$.*

Proof: According to (5.9)

$$p^l s_\omega(\gamma)[W] \equiv p^l p^{r-s} m s_\omega(\gamma)[\Sigma^{2j}] \mod p^{r+1},$$

for all non-empty partitions ω. By definition of l

$$p^l \mathrm{Td}[W] \equiv 0 \mod p^{r-s}.$$

Therefore

$$p^l s_\omega(\gamma)[W] \equiv 0 \mod p^{r-s}$$

for all partitions ω and thus an application of the Stong-Hattori theorem is all that is required to complete the proof. **

Lemma 5.16: *Let p be an odd prime, a, b, positive integers satisfying a, $b \equiv 0 \mod p-1$. Suppose that $[A] \in \Omega_{2a}^U$ satisfies*

$$s_\omega(\gamma)[A] \equiv q s_\omega(\gamma)[\Sigma^{2b}] \mod p^t: \qquad \omega \neq \emptyset$$

for some integers q and t with $t \leq 1+\nu_p(a)$. Then $q \equiv 0 \mod p^t$.

Proof: Recall that [10] [1; 2.6] (or [12; Appendix])

$$s_\omega(\gamma)[CP(p-1)]^{a/p-1} \equiv 0 \mod p^{1+\nu_p(a)}: \qquad \omega \neq \emptyset.$$

Thus of course

$$s_\omega(\gamma)[\boldsymbol{CP}(p-1)]^{a/p-1} \equiv 0 \mod p^t: \qquad \omega \neq \emptyset.$$

Let

$$[\tilde{A}] = [A] - \mathrm{Td}[A][\boldsymbol{CP}(p-1)]^{a/p-1}.$$

Observe that

$$s_\omega(\gamma)[\tilde{A}]\equiv s_\omega(\gamma)[A]\equiv qs_\omega(\gamma)[\Sigma^{2b}] \mod p^t: \qquad \omega\neq\emptyset.$$

On the other hand $\mathrm{Td}[\tilde{A}]=0$. An application of (5.7) completes the proof. **

Lemma 5.17: *There exists an integer q relatively prime to p such that $qp^l m\,\mathrm{Td}[\Sigma^{2j}]\in Z$.*

Proof: Combining (5.10) and (5.15) we obtain for non-empty partitions

$$p^{r-s}s_\omega(\gamma)[B]\equiv p^l p^{r-s}ms_\omega(\gamma)[\Sigma^{2j}] \mod p^{r+1}.$$

Therefore

$$s_\omega(\gamma)[B]\equiv p^l ms_\omega(\gamma)[\Sigma^{2j}] \mod p^{s+1}.$$

Recall that as we are assuming $\nu_p(j)<s$ we must have $\nu_p(i+j)=\nu_p(j)<s$. Hence of course

$$s_\omega(\gamma)[B]\equiv p^l ms_\omega(\gamma)[\Sigma^{2j}] \mod p^{\nu_p(i+j)+1}.$$

Applying (5.16) we obtain

$$p^l m\equiv 0 \mod p^{1+\nu_p(i+j)},$$

and since $\nu_p(i+j)=\nu_p(i)$

$$p^l m\equiv 0 \mod p^{1+\nu_p(i)}.$$

Therefore according to [10] and [1; 2.6] we find that $p^l m\,\mathrm{Td}[\Sigma^{2j}]$ has denominator prime to p which implies the desired conclusion. **

Thus we have $l\geq t$. Hence as noted prior to (5.14) we have completed the proof of (5.1). **

To complete this section we shall discuss the computation of the order of $\mathrm{Td}[V]$ in $Z_{p^{r-s}}$.

We have employed the homomorphism $\mathrm{Td}\colon \Omega_{2n}^{U,\mathrm{fr}}\to Q$ in the following context. For a fixed odd prime, p, let $Q_{(p)}\subset Q$ be the subring of rationals with denominator prime to p. There is induced by composition the homomorphism $\mathrm{td}\colon \Omega_{2n}^{U,\mathrm{fr}}\to Q/Q_{(p)}$. It is a corollary of the results proved in [10] and discussed at the beginning of this section that the image of $\mathrm{td}: \Omega_{2n}^{U,\mathrm{fr}} \to Q/Q_{(p)}$ is a cyclic group generated by $\mathrm{td}[\Sigma^{2n}]$ and having, if $n \equiv 0 \mod p-1$, order $p^{\nu_p(n+1)}$.

Corollary 5.18: *Under the hypothesis of* (5.1), *Case II, the order of* $\mathrm{Td}[V]$ *in* $Z_{p^{r-s}}$ *is equal to the order of* $\mathrm{td}(\theta)$ *in* $Q/Q_{(p)}$.

Our proof of (5.1) in Case II consisted of showing that p^l is the least power of p for which there is an integer q, prime to p, with $qp^l m \,\mathrm{Td}[\Sigma^{2j}] \in Z$. Since $\mathrm{Td}(\theta) = m\,\mathrm{Td}(\Sigma^{2j})$ in $Q/Q_{(p)}$ the corollary follows.

In Case I the situation is not as simple. It will be recalled, (5.4), that in Case 1 we replace $[V]$ with $[W]$ so that $\mathrm{Td}[V] = \mathrm{Td}[W]$, $[V] \equiv [W]$ modulo the ideal $(p^{r+1}, p^{r-s}[CP(p-1)]^{i/p-1})$ and

$$s_\omega(\gamma)[W] \equiv p^{r-s} m s_\omega(\gamma)[\Sigma^{2j}] \mod p^{r+1}$$

for all $\omega \neq \emptyset$. In particular, $s_\omega(\gamma)[W] \equiv 0 \mod p^{r-s}$ for all $\omega \neq \emptyset$. According to [10] again there is, therefore $\theta' \in \Omega_{2(i+j)}^{U,\mathrm{fr}}$ for which $\Phi[W] = p^{r-s}\theta'$. Surely the order of $\mathrm{Td}[W] = \mathrm{Td}[V] \in Z_{p^{r-s}}$ is then equal to the order of $\mathrm{td}(\theta') \in Q/Q_{(p)}$. In addition

$$s_\omega(\gamma)\,\theta' = m s_\omega(\gamma)[\Sigma^{2j}] \mod p^{s+1}$$

for all $\omega \neq \emptyset$. Since we are neglecting 2 we may as before write

$$\theta' = \Phi[D^{2(i+j)}] + k[\Sigma^{2(i+j)}].$$

As always it is convenient to have $\mathrm{Td}[D^{2(i+j)}] = 0$. Since we are in Case 1 then as noted in (5.2) $\Phi[CP(p-1)]^{i+j/p-1} \in p^{s+1}\Omega_{2(i+j)}^{U,\mathrm{fr}}$. Thus $[D^{2(i+j)}]$ may be replaced by

$$[D^{2(i+j)}] - \mathrm{Td}[D^{2(i+j)}][CP(p-1)]^{i+j/p-1}$$

therefore without loss of the congruence modulo p^{s+1}, $\mathrm{Td}[D^{2(i+j)}] = 0$.

Lemma 5.19: *If* $t \leq s+1$ *then* $m \equiv 0$ *modulo* p^t *if and only if* $k \equiv 0 \mod p^t$.

Proof: The congruence, for $\omega \neq \emptyset$,

$s_\omega(\gamma)[D^{2(i+j)}]+ks_\omega(\gamma)[\Sigma^{2(i+j)}] \equiv ms_\omega(\gamma)[\Sigma^{2j}] \mod p^{s+1}$

yields upon specialization to $\omega=(p-1)$ the relation

$$ks_{p-1}(\gamma)[\Sigma^{2(i+j)}]=ms_{p-1}(\gamma)[\Sigma^{2j}] \mod p$$

since $s_{p-1}[D^{2(i+j)}] \equiv 0 \mod p$.

As $s_{p-1}(\gamma)[\Sigma^{2n}] \equiv (-1)^{n+1} \mod p$ when $n \equiv 0 \mod p-1$ we may conclude that $k \equiv 0 \mod p$ if and only if $m \equiv 0 \mod p$. Suppose $k=pk_1$, $m=pm_1$ then for $\omega \neq \emptyset$

$$s_\omega(\gamma)[D^{2(i+j)}]=p(m_1 s_\omega(\gamma)[\Sigma^{2j}]-k_1 s_\omega(\gamma)[\Sigma^{2(i+j)}]) \mod p^{s+1}.$$

Because $\mathrm{Td}[D^{2(i+j)}]=0$ we then see from the Stong-Hattori theorem that there is a $[D_1^{2(i+j)}] \in \Omega^U_{2(i+j)}$ with $\mathrm{Td}[D_1^{2(i+j)}]=0$ and $p[D_1^{2(i+j)}]=[D^{2(i+j)}]$. Of course this immediately yields

$$s_\omega(\gamma)[D_1^{2(j+i)}]+k_1 s_\omega(\gamma)[\Sigma^{2(i+j)}] \equiv m_1 s_\omega(\gamma)[\Sigma^{2j}] \mod p^s.$$

We may proceed inductively to establish the lemma as long as $t \leq s+1$. **

Lemma 5.20: *If* $\nu_p(m)<s+1$ *then* $\nu_p(k)=\nu_p(m)$.

Proof: Since $m \equiv 0 \mod p^{\nu_p(m)}$ and $\nu_p(m)<s+1$ it follows from (5.11) that $k \equiv 0 \mod p^{\nu_p(m)}$ also. If $k \equiv 0 \mod p^{\nu_p(m)+1}$ however we see that $m \equiv 0 \mod p^{\nu_p(m)+1}$, which is a contradiction of the definition of $\nu_p(m)$. **

Now the value of $\nu_p(m)$, or $\nu_p(k)$, determines the order of $m\,\mathrm{td}[\Sigma^{2j}]$, or $\mathrm{td}(\theta')=k\,\mathrm{td}[\Sigma^{2(i+j)}]$, in $Q/Q_{(p)}$. In other words, if $m=p^{\nu_p(m)}q$, $(q,p)=1$ then $p^{\nu_p(m)}q\,\mathrm{td}[\Sigma^{2j}]$ has order $p^{\nu_p(j)-\nu_p(m)+1}$. Similarly $\mathrm{td}(\theta')$ has order $p^{\nu_p(i+j)-\nu_p(k)+1}$,

Corollary 5.21: *Under the hypothesis of* (5.11) *Case* I, *if the order of* $\mathrm{td}(\theta) \in Q/Q_{(p)}$ *is at least* $p^{\nu_p(j)-s+1}$ *then the order of* $\mathrm{Td}[V]$ *in* $Z_{p^{r-s}}$ *is*

$$p^{\nu_p(i+j)-\nu_p(j)}\mathrm{ord}(\mathrm{td}(\theta)). \quad **$$

Since $\operatorname{ord}(\operatorname{td}(\theta))=m\operatorname{td}][\Sigma^{2j}])$ is at least $p^{\nu_p(j)-s+1}$ it follows that $\nu_p(m)<s+1$ and thus by (5.20), $\nu_p(k)=\nu_p(m)$. Hence $\operatorname{ord}(\operatorname{td}(\theta'))$ $=k\operatorname{td}[\Sigma^{2(i+j)}]) = p^{\nu_p(i+j)-\nu_p(m)+1} = p^{\nu_p(i+j)-\nu_p(j)}p^{\nu_p(j)-m+1} = p^{\nu_p(i+j)-\nu_p(j)}$ $\operatorname{ord}(\operatorname{td}(\theta))$. As we noted at the beginning the order of $\operatorname{Td}[V]$ in $Z_{p^{r-s}}$ is equal to the order of $\operatorname{td}(\theta')$ in $Q/Q_{(p)}$.

If the order of $\operatorname{td}(\theta)$ is less than $p^{\nu_p(j)-s+1}$ in this case we can only conclude that $\nu_p(k)\geq s+1$ so that the order of $\operatorname{Td}[V]$ will not exceed $p^{\nu_p(i+j)-s}$.

§6. Still More on Three and Four Cell Complexes

We now have available the tools to settle the problem undertaken at the end of section 3. Our semi-complete solution may be stated as follows:

Theorem 6.1: *Suppose given homotopy classes* $[f]\in\pi^s_{2n-1}$, $[g]\in\pi^s_{2m-1}$ *and an integer* q *such that*

$$q[g]=0=[g][f]$$

$$0\in\langle q, [g], [f]\rangle.$$

Form the complex $X(g, f)$ *as in section* 3. *If* q *is odd then* $\operatorname{hom.dim.}_{\Omega^U_*}\Omega^U_*(X(g, f))\leq 2$.

Proof: Let p be a prime and denote by $Z[1/p']$ the subgroup of the rational numbers Q with denominators prime to p. Let

$$\Omega^U_*(\ )\otimes Z[1/p']: X \rightsquigarrow \Omega^U_*(X)\otimes_Z Z[1/p']$$

denote the bordism functor with coefficients in $Z[1/p']$. This localized theory has coefficient ring $\Omega^U_*\otimes Z[1/p']$. Elementary facts [3] about localizations show that

$$\operatorname{hom.dim.}_{\Omega^U_*}\Omega^U_*(X(g, f))\leq 2 \Leftrightarrow$$

$$\operatorname{hom.dim.}_{\Omega^U_*\otimes Z[1/p']}\Omega^U_*(X(g, f))\otimes Z[1/p']\leq 2 \quad \text{for all primes } p.$$

Suppose that q is relatively prime to p. Then

$$Z_q \otimes Z[1/p'] = 0$$

and one sees

$$\begin{aligned}\Omega_*^U(X(g, f)) \otimes Z[1/p'] &= \Omega_*^U(\Sigma^{2m+1} Y(f)) \otimes Z[1/p'] \\ &\cong \Omega_*^U(S^{2m+1} \vee S^{2m+2n+1}) \otimes Z[1/p']\end{aligned}$$

as $\Omega_*^U \otimes Z[1/p']$-modules. Hence

$$\text{hom.dim.}_{\Omega_*^U \otimes Z[1/p']} \Omega_*^U(X(g, f)) \otimes Z[1/p'] = 0 : (p, q) = 1.$$

Suppose next that $(p, q) = p^{r+1}$, $r \geq 0$. Then p is odd. Let g_p and f_p denote the p-components of g and f. Observe that we may form the complex $X(g_p, f_p)$ using p^{r+1} for q. Observe that

$$\tilde{\Omega}_*^U(X(g, f)) \otimes Z[1/p'] \cong \tilde{\Omega}_*^U(X(g_p, f_p)) \otimes Z[1/p']$$

as $\Omega_*^U \otimes Z[1/p']$-modules. Hence it will suffice to show that

$$\text{hom.dim.}_{\Omega_*^U Z[1/p']} \Omega_*^U(X(g_p, f_p)) \otimes Z[1/p'] \leq 2.$$

Since $Z[1/p']$ is a flat Z-module it is sufficient to show

$$\text{hom.dim.}_{\Omega_*^U} \Omega_*^U(X(g_p, f_p)) \leq 2.$$

In view of (3.9) it is therefore sufficient to consider the case $e_C(g_p) \neq 0$. Let us assume that $e_C(g_p) = 1/p^{s+1}$, which we may do without loss. In the notations of section 3 we obtain from (4.2) the equations (recall remark following (3.2))

$$s_\omega(\gamma)[V] \equiv p^{r-s} s_\omega(\gamma)[\theta] \mod p^{r+1}$$

for some (U, fr) manifold θ of dimension $2n$. Assume that $n \equiv 0 \bmod p-1$. As $e_C(g_p) \neq 0$ it follows that $m \equiv 0 \bmod p-1$ also. Hence according to (5.1) the order of $[V]$ in $\Omega_*^U/(p^{r+1}, p^{r-s}[CP(p-1)]^{m/p-1})$ is exactly equal to the order of $\text{Td}[V]$ in $Z_{p^{r-s}}$. An appeal to [12; Appendix] shows that we may assume without loss of generality that $[V_{g_p}] = p^{r-s}[CP(p-1)]^{m/p-1}$. Thus the criteria of (3.8) applies to show $\text{hom.dim.}_{\Omega_*^U} \Omega_*^U(X(g_p, f_p)) \leq 2$ as required.

It remains finally to consider the case $n \not\equiv 0 \bmod p-1$. According to (4.1) and [12; A.6] (recall $q' = p^t$) it follows that we may assume

θ is a *closed* U manifold. Note that $s \leq \nu_p(m)$ and hence

$$s_\omega(\gamma)[CP(p-1)]^{m/p-1} \equiv 0 \mod p^{s+1}: \omega \neq \emptyset.$$

Thus by the product formula for K-theory numbers

$$p^{r-s} s_\omega(\gamma)[\theta][CP(p-1)]^{m/p-1} \equiv p^{r-s} s_\omega(\gamma)[\theta] \mod p^{r+1}$$

for all non-empty partitions. (Recall θ is closed). Hence

$$s_\omega(\gamma)[[V]-p^{r-s}[\theta][CP(p-1)]^{m/p-1}] \equiv 0 \mod p^{r+1}: \quad \omega \neq \emptyset$$

so by [12; A.7]

$$[V]-p^{r-s}[\theta][CP(p-1)]^{m/p-1} = p^{r+1}[A].$$

(Note $n+m \not\equiv 0 \mod p-1$.) Thus

$$[V] \in (p^{r+1}, p^{r-s}[CP(p-1)]^{m/p-1})$$

and an application of (3.8) yields that $\text{hom.dim.}_{\Omega_*^U} \Omega_*^U(X(g_p, f_p)) \leq 2$ in this case also. **

With the aid of (6.1) and the type of argument employed in (4.4) it is possible to describe

$$\text{Im}\{\tilde{\Omega}_*^{fr}(X(g)) \to \tilde{\Omega}_*^U(X(g))\}.$$

The details are left to the reader.

The preceeding result suggest that one try to study complexes of the form

$$X = S^0 \cup_q e^1 \cup e^{2n_1+1} \cup e^{2n_2+1} \cdots \cup e^{2n_k+1}$$

where q is odd and $n_1 < n_2 < \cdots < n_k$ and try to show $\text{hom.dim.}_{\Omega_*^U} \Omega_*^U(X)$ ≤ 2. We have no idea how to deduce such a general result with our present techniques.

Added in Proof. Some results on the annihilator ideal of the canonical class in $\Omega_0^U(X)$ may be found in a publication of the second author that is to appear in the Indiana Journal of Math.

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