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One-parameter Subgroups and a Lie Subgroup of an Infinite Dimensional Rotation Group.

By

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Summary

Let $\mathscr{S}_r$ be the real topological vector space of real-valued rapidly decreasing functions and let $\mathcal{O}(\mathscr{S}_r)$ be the group of rotations of $\mathscr{S}_r$. Then every one-parameter subgroup of $\mathcal{O}(\mathscr{S}_r)$ induces a flow in $\mathscr{S}_r^*$, the conjugate space of $\mathscr{S}_r$ with the Gaussian White Noise as an invariant measure.

The author constructed a group of functions which is isomorphic to a subgroup of $\mathcal{O}(\mathscr{S}_r)$ and some of its one-parameter subgroups.

But the problem whether it contains sufficiently many one-parameter subgroups has been a problem. In Part I of the present paper, we answer this problem affirmatively by constructing two classes of one-parameter subgroups in a concrete way.

In Part II, we construct an infinite dimensional Lie subgroup of $\mathcal{O}(\mathscr{S}_r)$ and the corresponding Lie algebra. Namely, we construct a topological subgroup $\mathfrak{G}$ of $\mathcal{O}(\mathscr{S}_r)$ which is coordinated by the nuclear space $\mathscr{S}_r$ and the algebra $\mathfrak{a}$ of generators of one-parameter subgroups of $\mathfrak{G}$ which is closed under the commutation. Furthermore, we establish the exponential map from $\mathfrak{a}$ into $\mathfrak{G}$ and prove continuity.

CONTENTS

Introduction.

Let $\mathscr{S}$ be the complex topological vector space of all rapidly decreasing C^∞-functions on the real line, the topology of which is defined by countable number of norms:

$$\|\xi\|_{k,p}=\sup_{-\infty<x<+\infty}|x^p\xi^{(k)}(x)|<+\infty, \qquad (k, p=0, 1, 2, 3, \cdots,),$$

and let $\mathscr{S}_r$ be the real topological vector space consisting of all real-valued functions in $\mathscr{S}$.

We say that a linear homeomorphism $\mathfrak{g}$ from $\mathscr{S}$ onto itself is a rotation of $\mathscr{S}$ if

$$(1) \qquad \int_{-\infty}^{+\infty}|\mathfrak{g}\xi(x)|^2\,dx=\int_{-\infty}^{+\infty}|\xi(x)|^2\,dx$$

for every $\xi(x)$ in $\mathscr{S}$. We also define a rotation of $\mathscr{S}_r$ in the same manner. Let $U(\mathscr{S})$ and $\mathcal{O}(\mathscr{S}_r)$ be the group of all rotations of $\mathscr{S}$ and $\mathscr{S}_r$, respectively.

Let $\mathfrak{F}$ be the Fourier transform defined by

$$(\mathfrak{F}\xi)(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\xi(x)e^{-i\lambda x}\,dx$$

for $\xi(x)$ in $\mathscr{S}$. Then $\mathfrak{F}$ belongs to $U(\mathscr{S})$.

Let $\varphi(x)$ be a complex-valued locally square-summable function on the real line and let $f_\varphi(x)$ be the absolutely continuous function defined by

(2) $$f_\varphi(x)=\int_0^x |\varphi(y)|^2 dy, \qquad -\infty<x<+\infty.$$

We further assume the following two conditions:

(A.1) $$\begin{cases} \lim\limits_{x\to+\infty} f_\varphi(x)=+\infty \\ \lim\limits_{x\to-\infty} f_\varphi(x)=-\infty \end{cases}$$

(A.2) $\varphi(x)\neq 0, \qquad a.e.$

Then we can define a transformation $\mathfrak{g}[\varphi]$ of a function $\xi(x)$ by

(3) $$(\mathfrak{g}[\varphi]\xi)(x)=\varphi(x)\xi(f_\varphi(x)).$$

Obviously $\mathfrak{g}[\varphi]$ satisfies (1).

Let $\mathcal{U}_{\mathscr{S}}$ be the collection of locally square-summable functions $\varphi(x)$ which satisfy (A.1) and (A.2) and for which $\mathfrak{g}[\varphi]$ belong to $\mathcal{U}(\mathscr{S})$.

The author determined the class of functions $\mathcal{U}_{\mathscr{S}}$ explicitly as follows.

Theorem A. (H. Sato [1], Theorem 1). *A function $\varphi(x)$ belongs to $\mathcal{U}_{\mathscr{S}}$ if and only if it satisfies the following four conditions.*

(S.1) $\varphi(x)$ *is a C^∞-function.*

(S.2) $\varphi(x)\neq 0, \qquad -\infty<x<+\infty.$

(S.3) *For arbitrary non-negative integers k, p, there exists a positive number $\gamma=\gamma(k, p)$ such that*

$$\lim_{|x|\to+\infty}\frac{|\varphi^{(k)}(x)x^p|}{|f_\varphi(x)|^\gamma}=0.$$

(S.4) *For every non-negative integer p there exists a positive number $\rho=\rho(p)$ such that*

$$\lim_{|x|\to+\infty} \frac{|f_\varphi(x)|^p}{|x|^p|\varphi(x)|}=0.$$

We say that a function $\varphi(x)$ is slowly increasing if it is a C^∞-function and for every non-negative integer k, there exists a non-negative integer $p=p(k)$ such that

$$\|\varphi\|_{k,-p}=\sup_{-\infty<x<+\infty}\frac{|\varphi^{(k)}(x)|}{1+|x|^p}<+\infty. \tag{4}$$

As a corollary of Theorem A, we can easily prove the following theorem. (see H. Sato [1], Theorem 2).

Theorem B. *Let $\varphi(x)$ be a function on the real line such that*

$$\inf_x|\varphi(x)|>0.$$

Then $\varphi(x)$ belongs to $\mathcal{U}_{\mathcal{S}}$ if and only if $\varphi(x)$ is a slowly increasing function.

On the other hand, $\mathcal{U}_{\mathcal{S}}$ is a group with respect to the operation $\otimes$ defined by

$$(\varphi\otimes\psi)(x)=\varphi(x)\psi(f_\varphi(x)) \tag{5}$$

for every φ, ψ in $\mathcal{U}_{\mathcal{S}}$ and the map $\mathfrak{g}: \varphi\to\mathfrak{g}[\varphi]$ is a group isomorphism of $\mathcal{U}_{\mathcal{S}}$ onto a subgroup of $\boldsymbol{U}(\mathcal{S})$. (H. Sato [1]). In particular, define a subgroup $\mathcal{U}^r_{\mathcal{S}}$ of $\mathcal{U}_{\mathcal{S}}$ by

$$\mathcal{U}^r_{\mathcal{S}}=\{\varphi(x)\in\mathcal{U}_{\mathcal{S}}: \textit{real-valued.}\} \tag{6}$$

Then g is a group isomorphism of $\mathcal{U}^r_{\mathcal{S}}$ onto a subgroup of $\mathcal{O}(\mathcal{S}_r)$. Moreover define a subgroup $\mathcal{U}^h_{\mathcal{S}}$ of $\mathcal{U}_{\mathcal{S}}$ by

$$\mathcal{U}^h_{\mathcal{S}}=\{\varphi(x)\in\mathcal{U}_{\mathcal{S}}: \varphi(x)=\overline{\varphi(-x)}\} \tag{7}$$

and define for every φ in $\mathcal{U}^h_{\mathcal{S}}$

$$\tilde{\mathfrak{g}}[\varphi]=\mathfrak{F}^{-1}\mathfrak{g}[\varphi]\mathfrak{F}. \tag{8}$$

Then $\tilde{\mathfrak{g}}$ is a group isomorphism of $\mathcal{U}^h_{\mathcal{S}}$ onto a subgroup of $\mathcal{O}(\mathcal{S}_r)$.

Therefore, every one-parameter subgroup of $\mathcal{U}^r_{\mathcal{S}}$ or $\mathcal{U}^h_{\mathcal{S}}$ corresponds to a one-parameter subgroup of $\mathcal{O}(\mathcal{S}_r)$ through the map $\mathfrak{g}$ or $\tilde{\mathfrak{g}}$, respectively.

In Part I we give two classes of one-parameter subgroups of $\mathcal{O}(\mathcal{S}_r)$ by constructing those of $\mathcal{U}^r_{\mathcal{S}}$ and $\mathcal{U}^h_{\mathcal{S}}$.

In Section 1 we define the velocity function $h(t, x: f)$ corresponding to a bounded continuous function f and prove several properties of $h(t, x: f)$.

With these notations, we prove the following theorem in Section 2.

Theorem 1. *Let f be a real valued, bounded and slowly increasing function and put*

$$(9) \qquad \varphi_t(x) = \exp \frac{1}{2} \int_0^t f(h(r, x: f))\, dr, \qquad -\infty < t,\ x < +\infty.$$

Then $\{\varphi_t\}_{-\infty<t<+\infty}$ forms a one-parameter subgroup of $\mathcal{U}^r_{\mathcal{S}}$. Consequently, $\{\mathfrak{g}[\varphi_t]\}_{-\infty<t<+\infty}$ is a one-parameter subgroup of $\mathcal{O}(\mathcal{S}_r)$.

In Section 3, we prove

Theorem 2. *Let f be a real bounded even and slowly increasing function and let g be a real odd slowly increasing function. Then $\{\varphi_t\}_{-\infty<t<+\infty}$ defined by*

$$(10) \qquad \varphi_t(x) = \exp \int_0^t \frac{1}{2} f(h(r, x: f)) + i g(h(r, x: f))\, dr, \qquad -\infty < t,\ x < +\infty,$$

forms a one-parameter subgroup of $\mathcal{U}^h_{\mathcal{S}}$. Consequently $\{\tilde{\mathfrak{g}}[\varphi_t]\}_{-\infty<t<+\infty}$ is a one-parameter subgroup of $\mathcal{O}(\mathcal{S}_r)$.

These theorems show that $\mathcal{O}(\mathcal{S}_r)$ contains sufficiently rich family of one-parameter subgroups.

Utilising the results in Part I, we construct, in Part II, an infinite dimensional Lie subgroup $\mathfrak{G}$ of $\mathcal{O}(\mathcal{S}_r)$ which is coordinated by $\mathcal{S}_r$, and construct a Lie algebra $\mathfrak{a}$ consisting of generators of one-parameter subgroups of $\mathfrak{G}$, which is a topological vector space isomorphic to $\mathcal{S}_r$, and is closed under the commutation. Furthermore, we succeed in establishing the exponential map from $\mathfrak{a}$ into $\mathfrak{G}$ and proving its continuity.

T. Hida, I. Kubo, H. Nomoto and H. Yoshizawa [2] have constructed a three dimensional Lie group which induces flows on the invariant measure space of the White Noise, and the associated Lie algebra. In this paper we construct an infinite dimensional Lie subgroup of $\mathcal{O}(\mathcal{S}_r)$.

To begin with, we define an infinite dimensional Lie group without differentiable structure, its one-parameter subgroup and an infinite dimensional Lie algebra.

Let $\mathfrak{X}$ be a locally convex Hausdorff topological vector space.

Definition 1. An *$\mathfrak{X}$-Lie group* $\mathfrak{G}$ is a topological group with a homeomorphism ϕ from $\mathfrak{G}$ onto an open subset of $\mathfrak{X}$. We say $\mathfrak{X}$ is the *coordinate space* of $\mathfrak{G}$ and ϕ is the *coordinate function*.

Let $\mathfrak{G}$ be an $\mathfrak{X}$-Lie group with the coordinate function ϕ.

Definition 2. A subset $\{\mathfrak{g}_t\}_{-\infty<t<+\infty}$ of the $\mathfrak{X}$-Lie group $\mathfrak{G}$ is a *one-parameter subgroup* of $\mathfrak{G}$ if it satisfies the following two conditions:

(P.1) $$\begin{cases} \mathfrak{g}_s\mathfrak{g}_t=\mathfrak{g}_{s+t}, & -\infty<t,\ s<+\infty, \\ \mathfrak{g}_0=\mathbf{I} \end{cases}$$

where $\mathbf{I}$ is the unity of $\mathfrak{G}$;

(P.2) $\phi(\mathfrak{g}_t)$ is continuous and contiously differentiable in t with respect to the topology of $\mathfrak{X}$.

Definition 3. We call $\mathfrak{a}$ a *Lie algebra* if it is a locally convex Hausdorff topological vector space and there is given a rule of bilinear composition $(X, Y) \to [X, Y]$ in $\mathfrak{a}$ which is continuous and satisfies

$$[X, X]=0,$$

$$[X, [Y, Z]]+[Y, [Z, X]]+[Z, [X, Y]]=0,$$

for every X, Y and Z in $\mathfrak{a}$.

We start with defining a subgroup $\mathfrak{G}$ of $\mathcal{O}(\mathcal{S}_r)$ in the following manner. For every u in $\mathcal{S}_r$ define a transformation $\hat{\mathfrak{g}}[u]$ on $\mathcal{S}_r$ by

$$(11) \qquad (\hat{\mathfrak{g}}[u]\xi)(x)=\left(\mathfrak{g}\left[\exp\frac{1}{2}u\right]\xi\right)(x) = \exp\frac{1}{2}u(x)\,\xi(\hat{f}_u(x)),$$

where

$$(12) \qquad \hat{f}_u(x)=f_{\exp\frac{1}{2}u}(x)=\int_0^x \exp u(y)\,dy, \qquad -\infty<x<+\infty.$$

Then, since we have

$$0<\inf_x \exp\frac{1}{2}u(x)<\sup_x \exp\frac{1}{2}u(x)<+\infty,$$

by Theorem B, exp u belongs to $\mathcal{U}^r_{\mathcal{S}}$ and consequently, $\hat{\mathfrak{g}}[u]$ belongs to $\mathcal{O}(\mathcal{S}_r)$ for every u in $\mathcal{S}_r$. Put

$$\mathfrak{G}=\{\hat{\mathfrak{g}}[u]:\ u\in\mathcal{S}_r\}$$

and let ϕ be a map from $\mathfrak{G}$ onto $\mathcal{S}_r$ defined by

$$(13) \qquad \phi(\mathfrak{g}[u])=u.$$

We shall show that ϕ is one to one and introduce a topology in $\mathfrak{G}$ in which $\mathfrak{G}$ and $\mathcal{S}_r$ are homeomorphic. Then we show that $\mathfrak{G}$ is an $\mathcal{S}_r$-Lie group with the coordinate function ϕ (Theorem 4).

Let $\{\mathfrak{g}_t\}$ be a one-parameter subgroup of $\mathfrak{G}$. Then as shown in [2], the generator of $\mathfrak{g}_t$ is given by

$$(14) \qquad X(f)=\frac{1}{2}fI+F_f\frac{d}{dx},$$

where

$$f = \frac{d}{dt}\phi(\mathfrak{g}_t)|_{t=0} \tag{15}$$

which belongs to $\mathscr{S}_r$ according to (P.2) and the completeness of $\mathscr{S}_r$, and

$$F_f(x) = \int_0^x f(y)\,dy, \qquad -\infty < x < +\infty. \tag{16}$$

Put

$$\mathfrak{a} = \{X(f) : f \in \mathscr{S}_r\}.$$

Then $\mathfrak{a}$ is a vector space isomorphic to $\mathscr{S}_r$. Therefore introducing the topology of $\mathscr{S}_r$ in $\mathfrak{a}$, we prove that $\mathfrak{a}$ is a Lie algebra with the commutator

$$[X(f), X(g)] = X(f)X(g) - X(g)X(f) \tag{17}$$

(Theorem 5).

Finally, for every $X(f)$ in $\mathfrak{a}$, we establish the exponential $\operatorname{Exp} tX(f)$ which is a one-parameter subgroup of $\mathfrak{G}$ with generator $X(f)$ (Theorem 6). In particular, put $t=1$. Then we prove that $\operatorname{Exp} X(f) = \operatorname{Exp} 1 \cdot X(f)$ is a continuous map from $\mathfrak{a}$ into $\mathfrak{G}$ (Theorem 7).

Part I. One-parameter subgroups of the group $\mathcal{O}(\mathscr{S}_r)$.

§1. Velocity Function

In this section we define the *velocity function* and prove several properties of it for later use.

Let f be a real bounded continuous function on the real line and put

$$\Lambda_0 = \{x \in (-\infty, +\infty);\ F_f(x) = 0\},$$

where

(16) $$F_f(x)=\int_0^x f(y)dy, \qquad -\infty<x<+\infty.$$

Let Λ^i, Λ^a and Λ be the set of isolated points, accumulation points and the complement, of Λ_0, respectively. Since Λ_0 is a closed subset of the real line, Λ is open and is an at most countable union of disjoint open intervals, say,

(1.1) $$\Lambda=\bigcup_{n\in \mathrm{N}_f}(\alpha_n, \beta_n)$$

where N_f is a subset of all natural numbers. 0 always belongs to Λ_0.

At first we define a function on each interval (α_n, β_n) by

(1.2) $$\eta_n(x)=\int_{\gamma_n}^x \frac{dy}{F_f(y)}, \qquad \alpha_n<x<\beta_n,$$

where

(1.3) $$\gamma_n=\begin{cases} \alpha_n+1, & \text{if } \beta_n=+\infty, \\ \beta_n-1, & \text{if } \alpha_n=-\infty, \\ \dfrac{1}{2}(\alpha_n+\beta_n), & \textit{otherwise.} \end{cases}$$

The sign of $F_f(x)$ is unchanged in each interval (α_n, β_n). Let us assume that $F_f(x)$ is positive in the interval (α_n, β_n). Then we have

$$\lim_{x\uparrow\beta_n}\eta_n(x)=+\infty,$$

$$\lim_{x\downarrow\alpha_n}\eta_n(x)=-\infty.$$

For, if β_n is finite, we have

$$\eta_n(x)=\int_{\gamma_n}^x \frac{dy}{F_f(y)}$$

$$\geqslant \frac{1}{\|f\|_{00}}\int_{\gamma_n}^x \frac{dy}{\beta_n-y}\uparrow+\infty,$$

as x converges to β_n from below, and if β_n is infinite, we have

$$\eta_n(x)\geqslant\frac{1}{\|f_n\|_{00}}\int_{\gamma_n}^x \frac{dy}{y}\uparrow+\infty$$

as x diverges to infinite, because of the facts that

$$|F_f(x)|=|F_f(x)-F_f(\beta_n)|$$
$$\leqslant\int_x^{\beta_n}|f(y)|dy$$
$$\leqslant\|f\|_{00}(\beta_n-x),$$

and that

$$|F_f(x)|\leqslant\|f\|_{00}|x|,$$

respectively, where $\|f\|_{00}=\sup_x|f(x)|$. The second equality is also proved in the same manner. Since $\eta_n(x)$ is monotone increasing, η_n maps the interval (α_n, β_n) onto $(-\infty, +\infty)$ homeomorphically and the inverse function η_n^{-1} is well-defined, which maps $(-\infty, +\infty)$ onto (α_n, β_n).

Define a function $h(t, x: f)$ on $(-\infty, +\infty)\times\Lambda$ by

(1.4) $$h(t, x: f)=\eta_n^{-1}(\eta_n(x)+t), \qquad -\infty<t<+\infty, \quad \alpha_n<x<\beta_n.$$

It is evident that

(1.5) $$\alpha_n<h(t, x: f)<\beta_n, \qquad -\infty<t<+\infty, \quad \alpha_n<x<\beta_n.$$

On $(-\infty, +\infty)\times\Lambda_0$ define $h(t, x: f)$ by

(1.6) $$h(t, x: f)=x, \qquad -\infty<t<+\infty, \quad x\in\Lambda_0.$$

We have thus defined a function $h(t, x: f)$ on $(-\infty, +\infty)\times(-\infty, +\infty)$.

Remark. $h(t, x: f)$ is determined independently of the choice of $\{\gamma_n\}$. In fact, assume that x is in Λ, say, in an interval (α_n, β_n). Let γ_n' be an arbitrary number in the interval and put

$$\eta_n'(x)=\int_{\gamma_n'}^{x}\frac{dy}{F_f(y)}, \qquad \alpha_n<x<\beta_n.$$

Then we have

$$\eta_n(x)=\eta_n'(x)+\int_{\gamma_n}^{\gamma_n'}\frac{dy}{F_f(y)}.$$

For every x in (α_n, β_n) and every t, put

$$z=h(t, x: f)=\eta_n^{-1}(\eta_n(x)+t).$$

Then we have

$$\eta_n(z)=\eta_n(x)+t,$$

$$\eta_n(z)-\int_{\gamma_n'}^{\gamma_n}\frac{dy}{F_f(y)}=\eta_n(x)+t-\int_{\gamma_n'}^{\gamma_n}\frac{dy}{F_f(y)}$$

and therefore

$$\eta_n'(z)=\eta_n'(x)+t.$$

Thus we have

$$z=h(t, x: f)=\eta_n'^{-1}(\eta_n'(x)+t).$$

The above remark shows that $h(t, x: f)$ is determined uniquely once a function f is assigned. We call $h(t, x: f)$ *the velocity function corresponding to* f.

Proposition 1. *If f is a bounded continuous function on the real line, then the velocity function $h(t, x: f)$ is continuous in (t, x) and we have*

(1.7) $$h(t, h(s, x: f): f)=h(t+s, x: f), \qquad -\infty<t, s, x<+\infty.$$

Proof. By definition, it is obvious that $h(t, x: f)$ is continuous at (t_0, x_0) if x_0 is in the interior of Λ_0.

Assume that x_0 is in Λ_0. If there exists a sequence $\{x_\nu\}$ in Λ_0 which converges to x_0 from above, then, observing (1.5), we have

$$\lim_{\substack{t\to t_0\\ x\downarrow x_0}} h(t,\, x: f)=\lim_{\substack{t\to t_0\\ \nu\to+\infty}} h(t,\, x_\nu: f)$$

$$=\lim_{\nu\to+\infty} x_\nu = x_0$$

$$=h(t_0,\, x_0: f).$$

If there is no such sequence in Λ_0, then there is an interval (α_n, β_n) such that $x_0=\alpha_n$. In this case, it is not difficult to show that

$$\lim_{\substack{t\to t_0\\ x\downarrow x_0}} h(t,\, x: f)=\lim_{\substack{t\to t_0\\ x\downarrow x_0}} \eta_n^{-1}(\eta_n(x)+t)$$

$$=\alpha_n=x_0=h(t_0,\, x_0: f).$$

Therefore we have for every (t_0, x_0)

$$\lim_{\substack{t\to t_0\\ x\downarrow x_0}} h(t_0,\, x_0: f)=h(t_0,\, x_0: f).$$

Similarly we can prove that

$$\lim_{\substack{t\to t_0\\ x\leqslant x_0\\ x\to x_0}} h(t,\, x: f)=h(t_0,\, x_0: f).$$

Thus we have proved that

$$\lim_{\substack{t\to t_0\\ x\to x_0}} h(t,\, x: f)=h(t_0,\, x_0: f).$$

The formula (1.7) is easily proved by the definition (1.4) and (1.6).

Proposition 2. *If f is a bounded continuous and continuously differentiable function, then $h(t, x: f)$ is continuously differentiable in x and satisfies*

$$\frac{\partial h(t,\, x: f)}{\partial x}=\exp\int_0^t f(h(r,\, x: f))\, dr. \tag{1.8}$$

Before proving the proposition, we prove two lemmas.

Lemma 1. *If f is a bounded continuous and continuously differentiable function, then $f(x)$ vanishes identically in Λ^a.*

Proof. For every x in Λ^a, there exists a sequence $\{x_\nu\}$ in Λ_0 which converges to x monotonously. Without loss of generality, we assume it converges to x from below. Since $F_f(x_1)=F_f(x_2)=\cdots=F_f(x_\nu)=\cdots=0$ and $f(x)=\frac{d}{dx}F_f(x)$ is continuous, there exists a number x'_ν in each interval $(x_\nu, x_{\nu+1})$ such that $f(x'_\nu)=0$, $\nu=1, 2, 3, \cdots$. The sequence $\{x'_\nu\}$ converges to x together with $\{x_\nu\}$ and we have

$$f(x)=\lim_{\nu\to+\infty} f(x'_\nu)=0.$$

Lemma 2. *Assume that x is in Λ, say, $x\in(\alpha_n, \beta_n)$. Then we have*

$$\exp\int_0^t f(h(r, x: f))\,dr=\frac{F_f[\eta_n^{-1}(\eta_n(x)+t]}{F_f(x)} \tag{1.9}$$

Proof. By a certain transformation of variables in the integration, we have the following evaluations.

$$\begin{aligned}&\int_0^t f(h(r, x: f))\,dr\\ &=\int_0^t f(\eta_n^{-1}(\eta_n(x)+r))\,dr\\ &=\int_{\eta_n(x)}^{\eta_n(x)+t} f(\eta_n^{-1}(u))\,du\\ &=\int_x^{\eta_n^{-1}(\eta_n(x)+t)}\frac{f(y)}{F_f(y)}\,dy\\ &=\log\frac{F_f[\eta_n^{-1}(\eta_n(x)+t)]}{F_f(x)}.\end{aligned}$$

Proof of Proposition 2. Put

$$h'(t, x: f)=\exp\int_0^t f(h(r, x: f))\,dr. \tag{1.10}$$

Then, since f and $h(r, x: f)$ are continuous by Proposition 1, $h'(t, x: f)$ is continuous in (t, x). Therefore, in order to prove the proposition, it is sufficient to show

$$h(t, x : f)=\int_0^x h'(t,\ y : f)\,dy \tag{1.11}$$

$$=\int_0^x \exp\int_0^t f(h(r,\ y : f))\,dr\,dy.$$

Assume x is positive and in Λ, say, $x\in(\alpha_n, \beta_n)$. Observing that by Lemma 1, $h'(t, x : f)=1$ for every x in Λ^a and that Λ^i is a null set and 0 is in Λ_0, we have

$$\int_0^x h'(t,\ y : f)\,dy$$

$$=\Big\{\int_{[0,x]\cap\Lambda_0}+\int_{[0,x]\cap\Lambda}\Big\}h'(t,\ y : f)\,dy$$

$$=\int_{[0,x]\cap\Lambda^a}dy+\sum_{(\alpha_m,\beta_m)\subset[0,x]}\int_{\alpha_m}^{\beta_m}h'(t,\ y : f)\,dy$$

$$+\int_{\alpha_n}^x h'(t,\ y : f)\,dy.$$

By Lemma 2 and by a certain transformation of variables in the integration we have

$$\int_{\alpha_n}^x h'(t,\ y : f)\,dy$$

$$=\int_{\alpha_n}^x \exp\int_0^t f(h(r,\ y : f))\,dr\,dy$$

$$=\int_{\alpha_n}^x \frac{F_f[\eta_n^{-1}(\eta_n(y)+t)]}{F_f(y)}\,dy$$

$$=\eta_n^{-1}(\eta_n(x)+t)-\eta_n^{-1}(\eta_n(\alpha_n+0)+t)$$

$$=h(t, x : f)-\alpha_n.$$

Similarly we have

$$\int_{\alpha_m}^{\beta_m}h'(t,\ y : f)\,dy=\beta_m-\alpha_m,$$

if α_m and β_m are finite and consequently we have

$$\int_0^x h'(t,\ y: f)\,dy$$

$$= \int_{[0,x]\cap\Lambda^a} dy + \sum_{(\alpha_m,\beta_m)\subset[0,x]} (\beta_m-\alpha_m)+h(t,\ x: f)-\alpha_n$$

$$= \int_{[0,x]\cap\Lambda_0} dy + \sum_{(\alpha_m,\beta_m)\subset[0,x]} \int_{\alpha_m}^{\beta_m} dy+h(t,\ x: f)-\alpha_n$$

$$=\int_0^{\alpha_n} dy+h(t,\ x: f)-\alpha_n$$

$$=h(t,\ x: f).$$

We can prove (1.11) by the same way when x is positive and in Λ_0, and when x is non-positive. Thus we have proved the proposition.

From the proof of the above proposition, we have the following lemma.

Lemma 3. *If f is a bounded continuous function, then we have*

(1.12) $$h(0,\ x: f)=x,$$

(1.13) $$|h(t,\ x: f)-h(s,\ x: f)|$$

$$\leqslant |t-s|\,|x|\,\|f\|_{00}\exp[\max(|t|,\ |s|)\|f\|_{00}],$$

$$-\infty<t,\ s,\ x<+\infty$$

where

$$\|f\|_{00}=\sup_x|f(x)|$$

and

(1.14) $$K(t,f)^{-1}|x|\leqslant|h(t,\ x: f)|\leqslant K(t,f)|x|,$$

$$-\infty<t,\ x<+\infty$$

where

$$K(t,f)=\exp(|t|\,\|f\|_{00}).$$

Proof. (1.12) and (1.14) are easily proved by (1.11).

(1.13) is shown by the mean value theorem with respect to the argument t and by (1.11).

§2. One-Parameter Subgroups of the Group $\mathcal{U}^r_{\mathcal{S}}$.

In this section, we discuss how to construct one-paramer subgroups of $\mathcal{U}^r_{\mathcal{S}}$.

Let $\{\varphi_t\}_{-\infty<t<+\infty}$ be a one-parameter subgroup of $\mathcal{U}^r_{\mathcal{S}}$. Then we have

$$\text{(2.1)}\qquad \begin{cases} \varphi_s \otimes \varphi_t = \varphi_{s+t}, \quad -\infty < s,\ t < +\infty, \\ \varphi_0 = 1 \end{cases}$$

or equivalently

$$\text{(2.2)}\qquad \begin{cases} \varphi_s(x)\varphi_t\left(\int_0^x \varphi_s(y)^2 dy\right) = \varphi_{s+t}(x), \qquad -\infty < s,\ t < +\infty, \\ \varphi_0(x) \equiv 1. \end{cases}$$

We assume that $\varphi_t(x)$ is continuous in (t, x) and continuously differentiable in t. Since φ_t is in $\mathcal{U}^r_{\mathcal{S}}$, $\varphi_t(x)$ never vanishes, and observing that $\varphi_0(x)\equiv 1$ and that the continuity of $\varphi_t(x)$ in t, we know that $\varphi_t(x)$ is always positive.

Put

$$\text{(2.3)}\qquad u(t, x) = \log \varphi_t(x)^2, \qquad -\infty < t,\ x < +\infty.$$

Then (2.2) is equivalent to

$$\text{(2.4)}\qquad \begin{cases} u(s, x) + u\left(t, \int_0^x \exp u(s, y)\, dy\right) = u(s+t, x), \\ \qquad\qquad -\infty < s,\ t,\ x < +\infty, \\ u(0, x) \equiv 0 \end{cases}$$

Differentiate both sides of (2.4) in s. Then we have

$$\frac{\partial u(s, x)}{\partial s} + \frac{\partial u(t, X)}{\partial X}\bigg|_{X=\int_0^x \exp u(s,y) dy} \int_0^x \frac{\partial u(s, y)}{\partial s} \exp u(s, y)\, dy$$

$$=\frac{u(s+t, x)}{s}.$$

Evaluating at $s=0$ in the above equality. Then, considering $u(0, x)\equiv 0$, we have

$$f(x)+F_f(x)\frac{\partial u(t, x)}{\partial x}=\frac{\partial u(t, x)}{\partial t}, \tag{2.5}$$

where

$$f(x)=\left.\frac{\partial u(s, x)}{\partial s}\right|_{s=0}, \tag{2.6}$$

$$F_f(x)=\int_0^x f(y)\,dy.$$

Conversely, assume that $f(x)$ is a bounded continuous and continuously differentiable function on the real line. Then, by the theory of partial differential equation (see for example I.G. Petrovskii [6]), the equation (2.5) has a unique solution under the initial condition

$$u(0, x)=0. \tag{2.7}$$

In fact, using the velocity function, we construct the solution explicity as follows.

Proposition 3. *For any bounded continuous and continuously differentiable function* f,

$$u(t, x)=\int_0^t f(h(s, x:f))\,ds \tag{2.8}$$

is the unique solution of the equation (2.5) *under the initial condition* (2.7). *It is defined for all* (t, x) *in* $(-\infty, +\infty)\times(-\infty, +\infty)$ *and satisfies* (2.6).

Proof. The differentiablity of $u(t, x)$ is obvious from that of f and $h(t, x:f)$. In fact we have

$$\frac{\partial u}{\partial t}=f(h(t, x:f))$$

$$\frac{\partial u}{\partial x}=\int_0^t f'(h(s, x:f))\frac{\partial h(s, x:f)}{\partial x}ds$$

$$=\int_0^t f'(h(s, x:f))\exp\int_0^s f(h(r, x:f))\,dr\,ds.$$

Assume x is in Λ, say, $x\in(\alpha_n, \beta_n)$. Then by Lemma 2, we have

$$F_f(x)\frac{\partial u}{\partial x}$$

$$=\int_0^t f'(\eta_n^{-1}(\eta_n(x)+s))F_f[\eta_n^{-1}(\eta_n(x)+s)]\,ds$$

$$=f[\eta_n^{-1}(\eta_n(x)+s)]\,|_{s=0}^{t}$$

$$=f[\eta_n^{-1}(\eta_n(x)+t)]-f(x)$$

$$=f(h(t, x:f))-f(x)$$

$$=\frac{\partial u}{\partial t}-f(x).$$

Therefore (2.5) is true.

Assume x is in Λ_0. Then we have $F_f(x)=0$ and $h(t, x:f)=x$. Therefore (2.5) is true.

By the definition (2.8), (2.7) is obvious and by (1.12), (2.6) is also true.

Thus we have proved the Proposition.

Lemma 4. *Let f be a bounded and slowly increasing function. Then the function $u(t, x)$ defined by (2.8) is also bounded and slowly increasing in x uniformly for t in every finite interval. In other words, $u(t, x)$ is arbitrary times continuously differentiable, and for every positive number* T *and for every non-negative integer k, there exists a non-negative integer p such that*

$$\sup_{-T\leq t\leq T}\ \sup_{-\infty<x<+\infty}\frac{|u^{(k)}(t, x)|}{1+|x|^p}<+\infty \tag{2.9}$$

where

$$u^{(k)}(t,\ x)=\frac{\partial^{k}}{\partial x^{k}}\ u(t,\ x).$$

Proof. We prove the lemma by mathematical induction with respect to k.

In case of $k=0$, it is obvious that

$$u(t,\ x)=\int_{0}^{t}f(h(r,\ x:f))\,dr \tag{2.8}$$

is continuous in x since $h(t,\ x:f)$ and f are continuous. Moreover for every positive number T we have

$$\sup_{-T\leqslant t\leqslant T}\ \sup_{-\infty<x<+\infty}|u(t,\ x)|$$
$$\leqslant T\sup|f(x)|<+\infty.$$

Assume that the lemma is true in case of $k=0,\ 1,\ 2,\ \cdots,\ n$.

Differentiating both sides of (2.8) and using Proposition 2, we have

$$\begin{aligned} u^{(1)}(t,\ x)&=\frac{\partial}{\partial x}\ u(t,\ x) \\ &=\int_{0}^{t}f'(h(r,\ x:f))\frac{\partial}{\partial x}\ h(r,\ x:f)\,dr \\ &=\int_{0}^{t}f'(h(r,\ x:f))\exp u(r,\ x)\,dr. \end{aligned} \tag{2.10}$$

Differentiate again both sides of (2.10). Then it is not difficult to show

$$u^{(n)}(t,\ x)=\sum_{\nu=1}^{n}\int_{0}^{t}Q_{n\nu}[u]f^{(\nu)}(h(r,\ x:f))\,dr \tag{2.11}$$

where $Q_{n\nu}[u]$, $\nu=1,2,3,\cdots,n$, is a polynomial in $\exp u(r,\ x)$, $u^{(1)}(r,\ x)$, $u^{(2)}(r,\ x)$, $\cdots$, $u^{(n-1)}(r,\ x)$. Therefore $u^{(n)}(r,\ x)$ is again continuously differentiable, so that we have

$$u^{(n+1)}(t,\ x)=\sum_{\nu=1}^{n+1}\int_{0}^{t}Q_{n+1,\nu}[u]f^{(\nu)}(h(r,\ x:f))\,dr. \tag{2.12}$$

Since $Q_{n+1,\nu}[u]$, $\nu=1,\ 2,\ 3,\ \cdots,\ n+1$ is a polynomial in $\exp u(r,\ x)$,

$u^{(1)}(r, x)$, $u^{(2)}(r, x)$, $\cdots$, $u^{(n)}(r, x)$ and since they are divergent at most in a polynomial order uniformly for r in $[-T, T]$, we have only to show that $f^{(\nu)}(h(r, x: f))$, $\nu=1, 2, 3, \cdots$, is divergent at most in a polynomial order uniformly for r in $[-T, T]$. Since f is slowly increasing, for every non-negative integer ν, there exists a non-negative integer q such that

$$\|f\|_{\nu,-q}=\sup_x \frac{|f^{(\nu)}(x)|}{1+|x|^q}<+\infty.$$

Observing Lemma 3, we have

$$\sup_{-T\leqslant r\leqslant T}\ \sup_{-\infty<x<+\infty} \frac{|f^{(\nu)}(h(r, x: f))|}{1+|x|^q}$$

$$=\sup_{-T\leqslant r\leqslant T}\ \sup_x \frac{|f^{(\nu)}(h(r, x: f))|}{1+|h(r, x: f)|^q}\ \frac{1+|h(r, x: f)|^q}{1+|x|^q}$$

$$\leqslant \|f\|_{\nu,-q} \exp[T \sup_x |f(x)|]<+\infty.$$

Thus we have proved the lemma.

Summing up Proposition 3 and Lemma 4, we have the following theorem.

Theorem 1. *Let f be a real-valued bounded and slowly increasing function and put*

$$(9)\qquad \varphi_t(x)=\exp \frac{1}{2} \int_0^t f(h(r, x: f))\, dr, \qquad -\infty<t,\ x<+\infty.$$

Then $\{\varphi_t\}_{-\infty<t<+\infty}$ is a one-parameter subgroup of $\mathcal{U}^r_{\mathcal{S}}$ and consequently $\{\mathfrak{g}[\varphi_t]\}_{-\infty<t<+\infty}$ is a one-parameter subgroup of $\mathcal{O}(\mathcal{S}_r)$.

Proof. At first we show that $\varphi_t(x)$ belongs to $\mathcal{U}^r_{\mathcal{S}}$ for every fixed t. Since f is bounded and slowly increasing, by Lemma 4,

$$u(t, x)=\int_0^t f(h(r, x: f))\, dr$$

is also bounded and slowly increasing in x. Consequently, it is easy to show that $\varphi_t(x)=\exp\frac{1}{2}u(t, x)$ is slowly increasing in x. Moreover, since we have

$$0<\exp[-\frac{1}{2}|t|\,\|f\|_{00}]\leqslant\varphi_t(x)$$

$$\leqslant\exp\frac{1}{2}|t|\,\|f\|_{00}<+\infty,$$

by Theorem B, $\varphi_t(x)$ belongs to $\mathcal{U}^r_{\mathcal{G}}$ for every fixed t.

Secondly we show (2.2). Observing

$$f_{\varphi_s}(x)=\int_0^x\varphi_s(y)^2\,dy \tag{2.13}$$

$$=\int_0^x\exp\int_0^s f(h(r, y: f))\,dr\,dy$$

$$=h(s, x: f)$$

and Proposition 1, we have

$$\varphi_s(x)\,\varphi_t\Big(\int_0^x\varphi_s(y)^2dy\Big)$$

$$=\varphi_s(x)\,\varphi_t(h(s, x: f))$$

$$=\varphi_s(x)\exp\frac{1}{2}\int_0^t f(h(r, h(s, x: f): f))\,dr$$

$$=\exp\frac{1}{2}\int_0^s f(h(r, x: f))\,dr\exp\frac{1}{2}\int_s^{s+t} f(h(r, x: f))\,dr$$

$$=\exp\frac{1}{2}\int_0^{t+s} f(h(r, x: f))\,dr$$

$$=\varphi_{s+t}(x), \qquad -\infty<s,\ t,\ x<+\infty.$$

Thus we have proved the theorem.

§3. One-Parameter Subgroups of the Group $\mathcal{U}^h_{\mathcal{G}}$.

In this section, we give a method of constructing one-parameter subgroups of $\mathcal{U}^h_{\mathcal{G}}$.

Let $\{\varphi_t\}_{-\infty<t<+\infty}$ be a one-parameter subgroup of $\mathcal{U}^h_{\mathcal{G}}$. Then we have

$$\begin{cases}\varphi_s \otimes \varphi_t = \varphi_{s+t}, & -\infty < s,\ t < +\infty, \\ \varphi_0 = 1. \end{cases} \tag{3.1}$$

According to Lemma 7 and the following remark of H. Sato [1], (3.1) is equivalent to

$$\begin{cases}\varphi_s^+ \otimes \varphi_t^+ = \varphi_{s+t}^+, & -\infty < s,\ t < +\infty \\ \varphi_0^+ = 1 \end{cases} \tag{3.2}$$

and

$$\begin{cases}\varphi_s^e(x)\,\varphi_t^e(f_{\varphi_s^+}(x)) = \varphi_{s+t}^e(x), & -\infty < s,\ t,\ x < +\infty, \\ \varphi_0^e(x) \equiv 1, \end{cases} \tag{3.3}$$

where

$$\varphi_t^+(x) = |\varphi_t(x)|,\ \varphi_t^e(x) = \frac{\varphi_t(x)}{|\varphi_t(x)|}.$$

Define two subgroups of $\mathcal{U}_{\mathcal{S}}^h$ by

$$\mathcal{U}_{\mathcal{S}}^{h+} = \{\varphi(x) \in \mathcal{U}_{\mathcal{S}}^r \cap \mathcal{U}_{\mathcal{S}}^h;\ \varphi(x) > 0\}$$

$$\mathcal{U}_{\mathcal{S}}^{he} = \{\varphi(x) \in \mathcal{U}_{\mathcal{S}}^h :\ |\varphi(x)| \equiv 1\}.$$

Then $\mathcal{U}_{\mathcal{S}}^{h+}$ is also a subgroup of $\mathcal{U}_{\mathcal{S}}^r$. Therefore, in order to construct a one-parameter subgroup of $\mathcal{U}_{\mathcal{S}}^h$, we may first construct that of $\mathcal{U}_{\mathcal{S}}^{h+}$ and then sovle the equation (3.3) in $\mathcal{U}_{\mathcal{S}}^{he}$.

We make use of the method given in Section 2.

Lemma 5. *Let f be a bounded and slowly increasing function. Then $\{\varphi_t^+\}_{-\infty<t<+\infty}$ defined by*

$$\varphi_t^+(x) = \exp\frac{1}{2}\int_0^t f(h(r,\ x : f))\,dr, \qquad -\infty < t,\ x < +\infty \tag{3.4}$$

is a one-parameter subgroup of $\mathcal{U}_{\mathcal{S}}^{h+}$ if and only if f is an even function, that is,

$$f(x) = f(-x), \qquad -\infty < x < +\infty. \tag{3.5}$$

Proof. Assume that $\{\varphi_t^+\}_{-\infty<t<+\infty}$ defined by (3.4) is a one-parameter subgroup of $\mathcal{U}^{h+}_{\mathcal{S}}$. Then, since φ_t^+ belongs to $\mathcal{U}^{h+}_{\mathcal{S}}$ for every t, we have

$$\varphi_t^+(x)=\varphi_t^+(-x) \tag{3.6}$$

or equivalently

$$\int_0^t f(h(r,\ x:f))\,dr=\int_0^t f(h(r,\ -x:f))\,dr. \tag{3.7}$$

Differentiating both sides of (3.7) in t and evaluating at $t=0$, we have (3.5) by Lemma 3.

Conversely, assume (3.5) is true. Then, since by Theorem 1, $\{\varphi_t^+\}_{-\infty<t<+\infty}$ given by (3.4) is a one-parameter subgroup of $\mathcal{U}^{r}_{\mathcal{S}}$, we have only to prove (3.7) for every t.

In order to prove (3.7), it is sufficient to show

$$h(t,\ x:f)=-h(t,\ -x:f), \qquad -\infty<t,\ x<+\infty. \tag{3.8}$$

From (3.5) we have

$$F_f(x)=\int_0^x f(y)\,dy=-F_f(-x), \qquad -\infty<x<+\infty.$$

Hence the collection Λ_0 of all null points of $F_f(x)$ and its complement

$$\Lambda=\bigcup_{n\in \mathrm{N}_f}(\alpha_n,\ \beta_n) \tag{1.1}$$

are symmetric sets. Therefore, for x in Λ_0, $-x$ is also in Λ_0 and (3.8) is trivial since by the definition (1.6) we have

$$h(t,\ x:f)=x=-(-x)=-h(t,\ -x:f).$$

Let x be in Λ, say, $x\in(\alpha_n,\ \beta_n)$. Then, since Λ is symmetric, there exists a number n' in N_f such that $\alpha_{n'}=-\beta_n$ and $\beta_{n'}=-\alpha_n$. Furthermore by (1.3) we have $\gamma_{n'}=-\gamma_n$ and have

$$\eta_n(x)=\eta_{n'}(-x), \qquad \alpha_n<x<\beta_n \tag{3.9}$$

since we have

$$\eta_n(x)=\int_{\gamma_n}^{x}\frac{dy}{F_f(y)}=\int_{-\gamma_n}^{-x}\frac{dy}{F_f(y)}$$

$$=\int_{\gamma_{n'}}^{-x}\frac{dy}{F_f(y)}=\eta_{n'}(-x), \qquad \alpha_n<x<\beta_n.$$

Put

$$z=h(t,\ x:f)=\eta_n^{-1}(\eta_n(x)+t).$$

Then we have

$$\eta_n(z)=\eta_n(x)+t$$

and observing (3.9), we have

$$\eta_{n'}(-z)=\eta_{n'}(-x)+t$$

Consequently it holds that

$$-z=-h(t,\ x:f)$$

$$=\eta_{n'}^{-1}(\eta_{n'}(x)+t)=h(t,\ -x:f).$$

Thus we have proved the lemma.

Let f be a real-valued, bounded, even and slowly increasing function and let $\{\varphi_t^+\}_{-\infty<t<+\infty}$ be a one-parameter subgroup of $\mathcal{U}_{\mathcal{Y}}^{h+}$ defined by (3.4). Then we solve the equation (3.3) as follows.
Put

$$\text{(3.10)} \qquad v(t,\ x)=\frac{1}{i}\log\varphi_t^e(x),\ -\infty<t,\ x<+\infty,$$

and assume that $v(t, x)$ is continuous in (t, x) and continuously differentiable in t. Then, since φ_t^e belongs to $\mathcal{U}_{\mathcal{Y}}^{he}$ for every t, $|\varphi_t^e(x)|\equiv 1$ and $v(t, x)$ is real-valued. Furthermore, observing (2.13), we have from (3.3)

$$\text{(3.11)} \qquad \begin{cases} v(s,\ x)+v(t,\ h(s,\ x:f))=v(s+t,\ x), \\ \qquad\qquad -\infty<t,\ s,\ x<+\infty, \\ v(0,\ x)\equiv 0. \end{cases}$$

Differentiating both sides of (3.11) in s and evaluating at $s=0$, we have a partial differential equation

$$(3.12) \qquad \begin{cases} g(x)+F_f(x)\dfrac{\partial v(t, x)}{\partial x}=\dfrac{\partial v(t, x)}{\partial t}, & -\infty<t,\ x<+\infty, \\ v(0, x)\equiv 0, \end{cases}$$

where

$$(3.13) \qquad g(x)=\frac{\partial v(s, x)}{\partial s}\bigg|_{s=0},$$

and

$$F_f(x)=\frac{\partial h(s, x: f)}{\partial s}\bigg|_{s=0}=\int_0^x f(y)\,dy.$$

Conversely, assume that g is a real continuously differentiable function. Then (3.12) has a unique solution and it is given by:

Proposition 4. *Let f be a real-valued, bounded and continuously differentiable function and let g be a real-valued, continuously differentiable function. Then*

$$(3.14) \qquad v(t, x)=\int_0^t g(h(r, x: f))\,dr,$$

is the unique solution of (3.12) *defined for all* (t, x).

Lemma 6. *Let f be a real-valued, bounded and slowly increasing function and let g be a real slowly increasing function. Then the function $v(t, x)$ defined by* (3.14) *is slowly increasing for every fixed t.*

Lemma 7. *Let f be a real-valued, bounded, even and slowly increasing function and let g be a real-valued, slowly increasing function. Then $\{\varphi_t^e\}_{-\infty<t<+\infty}$ given by*

$$(3.15) \qquad \varphi_t^e(x)=\exp i\int_0^t g(h(r, x: f))\,dr, \qquad -\infty<t,\ x<+\infty$$

is a solution of (3.3) *if and only if g is an odd function, that is,*

$$(3.16) \qquad g(-x)=-g(x).$$

Proposition 4, Lemma 6 and Lemma 7 are proved in the same manner as Proposition 3, Lemma 4 and Lemma 5, respectively.

Summing up the above, we have the following theorem.

Theorem 2. *Let f be a real bounded even and slowly increasing function and let g be a real odd slowly increasing function. Then $\{\varphi_t\}_{-\infty<t<+\infty}$ defined by*

$$\varphi_t(x)=\exp\int_0^t \frac{1}{2} f(h(r, x: f))+ig(h(r, x: f))\,dr, \tag{10}$$

$$-\infty<t,\ x<+\infty,$$

forms a one-parameter subgroup of $\mathscr{U}^h_{\mathscr{S}}$. Consequently $\{\mathfrak{g}[\varphi_t]\}_{-\infty<t<+\infty}$ is a one-parameter subgroup of $\mathcal{O}(\mathscr{S}_r)$.

Part II. An infinite dimensional Lie subgroup of the group $\mathcal{O}(\mathscr{S}_r)$.

§4. Subgroup $\mathfrak{G}$ and Group $\mathfrak{S}$.

In this section, we define a product operation $\odot$ in $\mathscr{S}_r$ and show that $\mathscr{S}_r$ forms a group with respect to the operation $\odot$. We denote this group by $\mathfrak{S}$. Then we show that $\mathfrak{G}$ is algebraically isomorphic to $\mathfrak{S}$ and consequently $\mathfrak{G}$ is a subgroup of $\mathcal{O}(\mathscr{S}_r)$. At first we define a product operation $\odot$ on $\mathscr{S}_r$ by

$$(u\odot v)(x)=u(x)+v(\hat{f}_u(x)), \qquad u, v\in\mathscr{S}_r, \tag{4.0}$$

where $\hat{f}_u(x)$ is defined by

$$\hat{f}_u(x)=\int_0^x \exp u(y)\,dy, \qquad -\infty<x<+\infty. \tag{12}$$

Since $\exp u(x)$ never vanishes $\hat{f}_u$ maps the real axis onto itself homeomorphically, and moreover, it is easy to prove the following inequalities;

$$K(u)^{-1}|x|\leqslant|\hat{f}_u(x)|\leqslant K(u)|x|, \qquad -\infty<x<+\infty, \tag{4.1}$$

where

$$K(u)=\exp\|u\|_{00}=\exp[\sup_x|u(x)|]. \tag{4.2}$$

Before proving that $\mathscr{S}_r$ is a group with respect to the operation $\odot$, we prepare the following lemmas.

Lemma 8. *Let u be a $k-1$ times continuously differentiable function and v be a k times continuously differentiable function where k is an arbitrary positive integer. Then we can find polynomials $P_{k\nu}[u]$ in* $\exp u$, u', u'', $\cdots$, $u^{(k-1)}$; $\nu=0, 1, 2, \cdots, k$, *the expressions of which are independently of the choice of functions u and v in such a way that*

$$\frac{d^k}{dx^k}v(\hat{f}_u(x))=\sum_{\nu=0}^{k}P_{k\nu}[u]v^{(\nu)}(\hat{f}_u(x)), \tag{4.3}$$

where

$$v^{(\nu)}(\hat{f}_u(x))=\frac{d^\nu}{dX^\nu}v(X)|_{X=\hat{f}_u(x)}, \qquad \nu=0, 1, 2, 3, \cdots, k.$$

Specifically, we have

$$P_{kk}[u]=\exp ku(x). \tag{4.4}$$

Proof. We prove the lemma by mathematical induction with respect to k.

In case $k=1$, it is evident that

$$\frac{d}{dx}v(\hat{f}_u(x))=\exp u(x)\,v'(\hat{f}_u(x)).$$

Assume that the lemma is true in case $k=n$. Then we have

$$\frac{d^n}{dx^n}v(\hat{f}_u(x))=\sum_{\nu=0}^{n}P_{n\nu}[u]v^{(\nu)}(\hat{f}_u(x)).$$

Differentiate both sides of the above equality. Then we have

$$\frac{d^{n+1}}{dx^{n+1}}v(\hat{f}_u(x))$$

$$= \sum_{\nu=0}^{n} \left\{ \frac{d}{dx} P_{n\nu}[u] \right\} v^{(\nu)}(\hat{f}_u(x)) + P_{n\nu}[u] \exp u(x)\, v^{(\nu+1)}(f_u(x))$$

$$= \sum_{\nu=0}^{n+1} P_{n+1,\nu}[u]\, v^{(\nu)}(\hat{f}_u(x)),$$

where

$$P_{n+1,\nu}[u] = \frac{d}{dx} P_{n\nu}[u] + \exp u(x)\, P_{n,\nu-1}[u],$$

$$\nu = 0, 1, 2, \cdots, n+1.$$

Considering that $P_{n\nu}[u]$, $\nu = 0, 1, 2, \cdots, n$, are polynomials in $\exp u$, u', u'', $\cdots$, $u^{(n-1)}$, we can easily prove that $P_{n+1,\nu}[u]$, $\nu = 0, 1, 2, \cdots, n+1$, are again polynomials in $\exp u$, u', u'', $\cdots$, $u^{(n)}$. The other assertions are also easily proved.

Lemma 9. *For every u and v in $\mathscr{S}_r$,*

$$v \circ \hat{f}_u(x) = v(\hat{f}_u(x))$$

belongs to $\mathscr{S}_r$ and the transformation from v to $v \circ \hat{f}_u$ is continuous in the topology of $\mathscr{S}_r$ uniformly in u on any bounded subset V of $\mathscr{S}_r$.

Proof. To prove the lemma, we show that $\|v \circ f_u\|_{kp}$ is finite for every non-negative integers k and p. In fact, considering Lemma 8, we have for any non-negative integer k

$$\frac{d^k}{dx^k} v \circ \hat{f}_u(x) = \sum_{\nu=0}^{k} P_{k\nu}[u]\, v^{(\nu)}(\hat{f}_u(x)).$$

Since $P_{k\nu}[u]$: $\nu = 0, 1, 2, \cdots, k$, is a polynomial in $\exp u$, u', $\cdots$, $u^{(k-1)}$, and they are uniformly bounded on V, we have for every non-negative integer p

$$\|v \circ f_u\|_{kp} = \sup_x \left| x^p \frac{d^k}{dx^k} v(f_u(x)) \right|$$

$$\leqslant \sum_{\nu=0}^{k} M_{k\nu} \sup_x | x^p v^{(\nu)}(\hat{f}_u(x)) |$$

$$\leqslant \sum_{\nu=0}^{k} M_{k\nu} \sup_{x\neq 0} \frac{|x|^p}{|\hat{f}_u(x)|^p} |f_u(x)^p v^{(\nu)}(f_u(x))|$$

$$\leqslant \sum_{\nu=0}^{k} M_{k\nu} K(u)^p \sup_{y} |y^p v^{(\nu)}(y)|$$

$$\leqslant M^p \sum_{\nu=0}^{k} M_{k\nu} \|v\|_{\nu p} < +\infty$$

where

$$M_{k\nu} = \sup_{\substack{u\in V \\ -\infty<x<+\infty}} |P_{k\nu}[u]|, \qquad \nu=0, 1, \cdots, k,$$

and

$$M = \sup_{u\in V} K(u).$$

The above inequality together linearity proves the uniform continuity of the transformation.

Lemma 10. *For every u in $\mathscr{S}_r$, $v(x)=u\circ f_u^{-1}(x)$ belongs to $\mathscr{S}_r$.*

Proof. It is sufficient to show that $\|v\|_{kp}$ is finite for every non-negative integers k and p. We prove it by mathematical induction with respect to k.

In case $k=0$, considering that $u(x)=v\circ f_u(x)$ belongs to $\mathscr{S}_r$, we have for every non-negative integer p,

$$\|v\|_{0p} = \sup_x |x^p v(x)|$$

$$= \sup_x |\hat{f}_u(x)^p u(x)|$$

$$= \sup_{x\neq 0} \left| \frac{\hat{f}_u(x)^p}{x^p} x^p u(x) \right|$$

$$\leqslant K(u)^p \|u\|_{0p} < +\infty.$$

Assume that the $\|v\|_{kp}$, $k=0, 1, 2, \cdots, n-1$, $p=0, 1, 2, 3, \cdots$, are all finite.

Then by Lemma 8 we have

$$u^{(n)}(x)=\sum_{\nu=0}^{n-1}P_{n\nu}[u]\,v^{(\nu)}(\hat{f}_u(x))+\exp nu(x)\,v^{(n)}(\hat{f}_u(x))$$

and therefore

$$v^{(n)}(\hat{f}_u(x))=\exp(-nu)\{u^{(n)}(x)-\sum_{\nu=0}^{n-1}P_{n\nu}[u]\,v^{(\nu)}(\hat{f}_u(x))\}. \tag{4.5}$$

Since $\exp u(x)$, $u'(x)$, $u''(x)$, $\cdots$, $u^{(n-1)}(x)$ are uniformly bounded on the real line, we have

$$\begin{aligned}\|v\|_{np}&\leqslant K(u)^n\sup_x|x^p v^{(n)}(x)|\\ &\leqslant K(u)^n\sup_x|x^p u^{(n)}(x)|\\ &\quad+K(u)^n\sum_{\nu=0}^{n-1}M_{n\nu}\sup_x|x^p v^{(\nu)}(\hat{f}_u(x))|\\ &\leqslant K(u)^n\|u\|_{np}+K(u)^n\sum_{\nu=0}^{n-1}M_{n\nu}\sup_{x\neq 0}\left|\frac{x^p}{\hat{f}_u(x)^p}\hat{f}_u(x)^p v^{(\nu)}(\hat{f}_u(x))\right|\\ &\leqslant K(u)^n\|u\|_{np}+K(u)^{n+p}\sum_{\nu=0}^{n-1}M_{n\nu}\sup_y|y^p v^{(\nu)}(y)|\\ &\leqslant K(u)^n\|u\|_{np}+K(u)^{n+p}\sum_{\nu=0}^{n-1}M_{n\nu}\|v\|_{\nu p}<+\infty,\end{aligned}$$

where

$$M_{n\nu}=\sup_x|P_{n\nu}[u]|,\qquad \nu=0,1,2,\cdots,n-1.$$

This completes the proof of the lemma.

Proposition 5. *$\mathscr{S}_r$ forms a group with respect to the operation $\odot$.*

Proof. Let u, v and w be arbitrary elements of $\mathscr{S}_r$. By Lemma 9, it is obvious that $u\odot v=u+v\circ\hat{f}_u$ belongs to $\mathscr{S}_r$. The associative law $(u\odot v)\odot w=u\odot(v\odot w)$ is true since $\hat{f}_v\circ f_u=\hat{f}_{u\odot v}$ holds. The unit element is the null function and, finally, the inverse element u^{-1} is defined by

(4.6) $$u^{-1}(x) = -u(\hat{f}_u^{-1}(x)).$$

In fact we have

$$(u \odot u^{-1})(x) = u(x) - u \circ \hat{f}_u^{-1} \circ \hat{f}_u(x)$$
$$= u(x) - u(x) = 0,$$

and by Lemma 10, u^{-1} belongs to $\mathscr{S}_r$.

Thus we have proved the proposition.

We denote the group $\mathscr{S}_r$ with the operation $\odot$ by $\mathfrak{S}$. The following proposition clarifies the relation between $\mathfrak{G}$ and $\mathfrak{S}$.

Proposition 6. *The map ϕ defined in* (13) *is a group isomorphism from $\mathfrak{G}$ onto $\mathfrak{S}$.*

Proof. By the definition of $\mathfrak{G}$ and by a slight modification of H. Sato [1], Lemma 6, it is easy to show that ϕ maps $\mathfrak{G}$ onto $\mathfrak{S}$ in a one-to-one manner.

Therefore, we have only to prove that ϕ^{-1} is a group homomorphism. For every u and v in $\mathfrak{S}$, we have

$$(\phi^{-1}(u)\phi^{-1}(v)\xi)(x)$$
$$= (\mathfrak{g}[u]\mathfrak{g}[v]\xi)(x)$$
$$= \left(\mathfrak{g}[u]\left(\exp\frac{1}{2}v\right)\xi \circ \hat{f}_v\right)(x)$$
$$= \exp\frac{1}{2}u\left(\exp\frac{1}{2}v \circ \hat{f}_u\right)\{\xi \circ \hat{f}_v \circ \hat{f}_u(x)\}$$
$$= \exp\frac{1}{2}u \odot v\{\xi \circ \hat{f}_{u \odot v}(x)\}$$
$$= (\mathfrak{g}[u \odot v]\xi)(x)$$
$$= (\phi^{-1}(u \odot v)\xi)(x)$$

for every $\xi(x)$ in $\mathscr{S}_r$. This result proves

(4.7) $$\phi^{-1}(u)\phi^{-1}(v) = \phi^{-1}(u \odot v)$$

or equivalently

$$\hat{\mathfrak{g}}[u]\hat{\mathfrak{g}}[v]=\hat{\mathfrak{g}}[u\odot v] \tag{4.8}$$

for every u and v in $\mathfrak{S}$.

According to the above propositions, we have:

Theorem 3. *$\mathfrak{G}$ is a subgroup of $\mathcal{O}(\mathscr{S}_r)$ and algebraically isomorphic to the group $\mathfrak{S}$ through the map ϕ.*

§5. Topological Group $\mathfrak{G}$

In this section we introduce a topology in $\mathfrak{G}$ so that $\mathfrak{G}$ is a topological group and consequently an $\mathscr{S}_r$-Lie group.

To begin with, observing that $\mathfrak{G}$ and $\mathscr{S}_r$ are isomorphic algebraically and $\mathfrak{S}$ coincides with $\mathscr{S}_r$ as a set, we introduce such a topology in $\mathfrak{G}$ that $\mathfrak{G}$ and $\mathscr{S}_r$ are homeomorphic through the map ϕ.

To show that $\mathfrak{G}$ is a topological group with respect to this topology, we must show that

$$\phi(\phi^{-1}(u)\phi^{-1}(v)):\ \mathscr{S}_r\times\mathscr{S}_r\rightarrow\mathscr{S}_r$$

and

$$\phi(\phi^{-1}(u)^{-1}):\ \mathscr{S}_r\rightarrow\mathscr{S}_r$$

are continuous maps. We know that

$$\phi(\phi^{-1}(u)\phi^{-1}(v))=u\odot v$$

and

$$\phi(\phi^{-1}(u)^{-1})=u^{-1}.$$

Therefore, in order to prove that $\mathfrak{G}$ is a topological group, it is sufficient to show that $\mathfrak{S}$ is a topological group in the topology of $\mathscr{S}_r$.

We start with proving the following lemma.

Lemma 11. *For every fixed v in $\mathscr{S}_r$, the transformation from u to $v\circ\hat{f}_u$, which maps $\mathscr{S}_r$ to $\mathscr{S}_r$, is continuous in the topology of $\mathscr{S}_r$.*

Proof. Since $\mathscr{S}_r$ is a metrizable space, it is sufficient to show that $v\circ\hat{f}_{u_n}$ converges to $v\circ\hat{f}_u$ if u_n converges to u_0 in $\mathscr{S}_r$.

Let $\{u_n\}$ be a sequence in $\mathscr{S}_r$ which converges to u_0 and let k and p be arbitrary non-negative integers. Then by Lemma 8, we have

$$\begin{aligned}
&\|v\circ\hat{f}_{u_n}-v\circ\hat{f}_{u_0}\|_{kp}\\
&=\sup_x|x^p\sum_{\nu=0}^{k}\{P_{k\nu}[u_n]v^{(\nu)}(\hat{f}_{u_n}(x))-P_{k\nu}[u_0]v^{(\nu)}(\hat{f}_{u_0}(x))\}|\\
&\leqslant\sup_x|x^p\sum_{\nu=0}^{k}\{P_{k\nu}[u_n]-P_{k\nu}[u_0]\}v^{(\nu)}(\hat{f}_{u_n}(x))|\\
&\quad+\sup_x|x^p\sum_{\nu=0}^{k}P_{k\nu}[u_0]\{v^{(\nu)}(\hat{f}_{u_n}(x))-v^{(\nu)}(\hat{f}_{u_0}(x))\}|\\
&\leqslant\sum_{\nu=0}^{k}K(u_n)^p\|v\|_{\nu p}\sup_x|P_{k\nu}[u_n]-P_{k\nu}[u_0]|\\
&\quad+\sum_{\nu=0}^{k}\sup_x|P_{k\nu}[u_0]|\sup_x|x^p\{v^{(\nu)}(\hat{f}_{u_n}(x)-v^{(\nu)}(\hat{f}_{u_0}(x))\}\|.
\end{aligned}$$

The first term of the right side converges to zero as $n\to+\infty$ since all the sequences $u_n(x)$, $u'_n(x)$, $\cdots$, $u_n{}^{(k-1)}(x)$ converge to u_0 uniformly on the real line.

Before proving that the second term also converges to zero, we remark that there exists a positive constant K such that

$$K^{-1}|x|\leqslant|\hat{f}_{u_n}(x)|\leqslant K|x|,\qquad -\infty<x<+\infty \tag{5.1}$$

uniformly in n. In fact K is given by

$$K=\exp\sup_n\sup_x|u_n(x)|$$

which is finite since $\{u_n\}$ is a bounded set in $\mathscr{S}_r$.

For every non-negative integer ν and for any positive number ε, there exists a positive constant R such that

$$\sup_{|x|>R}|x^p v^{(\nu)}(x)|<\varepsilon(3K^p)^{-1},$$

since v belongs to $\mathscr{S}_r$.

By (5.1), $|x|>KR$ implies $|\hat{f}_{u_n}(x)|>R$ for every n, therefore we have

$$\sup_{|x|>KR}|x^p\{v^{(\nu)}(\hat{f}_{u_n}(x))-v^{(\nu)}(f_{u_0}(x))\}|$$

$$\leqslant K^p \sup_{|x|>KR}|f_{u_n}(x)^p v^{(\nu)}(f_{u_n}(x))|$$

$$+K^p \sup_{|x|>KR}|\hat{f}_{u_0}(x)^p v^{(\nu)}(f_{u_0}(x))|$$

$$<\frac{2}{3}\varepsilon.$$

On the other hand, it is easy to show that

$$|v^{(\nu)}(x)-v^{(\nu)}(y)|\leqslant\|v\|_{\nu+1,0}|x-y|, \qquad -\infty<x,\ y<+\infty,$$

and that

$$|\hat{f}_{u_n}(x)-\acute{f}_{u_0}(x)|\leqslant K|x|\,\|u-u_0\|_{00}$$

If n is sufficiently large that

$$\|u_n-u_0\|_{00}<[3K^{p+2}R^{p+1}\|v\|_{\nu+1,0}]^{-1}\varepsilon$$

then we have

$$\sup_{|x|\leqslant KR}|x^p v^{(\nu)}(\acute{f}_{u_n}(x))-x^p v^{(\nu)}(\hat{f}_{u_0}(x))|$$

$$\leqslant (KR)^p \sup_{|x|<KR}|v^{(\nu)}(\hat{f}_{u_n}(x))-v^{(\nu)}(\hat{f}_{u_0}(x))|$$

$$\leqslant (KR)^p\|v\|_{\nu+1,0}K^2R\|u_n-u_0\|_{00}<\frac{1}{3}\varepsilon.$$

Summing up the above estimations, we have

$$\sup_x|x^p\{v^{(\nu)}(\hat{f}_{u_n}(x))-v^{(\nu)}(\hat{f}_{u_0}(x))\}|$$

$$\leqslant \sup_{|x|>KR}|x^p\{v^{(\nu)}(\hat{f}_{u_n}(x))-v^{(\nu)}(\hat{f}_{u_0}(x))\}|$$

$$+\sup_{|x|\leqslant KR}|x^p\{v^{(\nu)}(\hat{f}_{u_n}(x))-v^{(\nu)}(\hat{f}_{u_0}(x))\}|$$

$$<\frac{2}{3}\varepsilon+\frac{1}{3}\varepsilon=\varepsilon,$$

for sufficiently large n. Thus the lemma is proved.

Proposition 7. *$\mathfrak{S}$ is a topological group with respect to the product $\odot$.*

Proof. *Continuity of the product operation.*

Let $\{u_n\}$ and $\{v_n\}$ be sequences in $\mathfrak{S}$ which converges to u_0 and v_0, respectively. Then for every non-negative integers k and p we have

$$\begin{aligned} &\|u_n \odot v_n - u_0 \odot v_0\|_{kp} \\ &\quad = \|(u_n + v_n \circ \hat{f}_{u_n}) - (u_0 + v_0 \circ \hat{f}_{u_0})\|_{kp} \\ &\quad \leqslant \|u_n - u_0\|_{kp} + \|v_n \circ \hat{f}_{u_n} - v_0 \circ \hat{f}_{u_n}\|_{kp} \\ &\qquad + \|v_0 \circ \hat{f}_{u_n} - v_0 \circ \hat{f}_{u_0}\|_{kp}. \end{aligned}$$

By assumption, the first term of the right side in the last inequality converges to 0 as $n \to +\infty$, and by Lemma 11 the third term converges to 0. Since $\{u_n\}$ is a bounded set, by Lemma 9 the second term converges to 0. Therefore the product operation is continuous.

Continuity of the inverse operation.

Let $\{u_n\}$ be a sequence in $\mathfrak{S}$ which converges to u_0. To prove the continuity of the inverse operation, it is sufficient to prove that

$$\lim_{n \to +\infty} u_n^{-1} \odot u_0 = 0. \tag{5.2}$$

In fact, by the continuity of the product operation, $u_n^{-1} = u_n^{-1} \odot u_0 \odot u_0^{-1}$ converges to $0 \odot u_0^{-1} = u_0^{-1}$ if $u_n^{-1} \odot u_0$ converges to 0.

Since it is easy to show that

$$\hat{f}_u^{-1}(x) = \hat{f}_{u^{-1}}(x) \tag{5.3}$$

for every u in $\mathfrak{S}$, we have

$$\begin{aligned} w_n(x) &= (u_n^{-1} \odot u_0)(x) \\ &= -u_n(\hat{f}_{u_n}^{-1}(x)) + u_0(\hat{f}_{u_n}^{-1}(x)). \end{aligned}$$

In order to prove (5.2), we show that $\|w_n\|_{kp}$ converges to 0 for every non-negative integers k and p by mathematical induction with respect to k.

In case $k=0$, we have for every non-negative integer p

$$\begin{aligned}\|w_n\|_{0p}&=\sup_x|x^p w_n(x)|\\&=\sup_x|x^p\{u_0(\hat{f}_{u_n}^{-1}(x))-u_n(\hat{f}_{u_n}^{-1}(x))\}|\\&=\sup_x|\hat{f}_{u_n}(x)^p\{u_0(x)-u_n(x)\}|\\&\leqslant\sup_{x\neq 0}\left|\frac{\hat{f}_{u_n}(x)^p}{x^p}x^p\{u_0(x)-u_n(x)\}\right|\\&=K^p\|u_0-u_n\|_{0p}\to 0 \qquad \text{as } n\to+\infty,\end{aligned}$$

where

$$K=\sup_n K(u_n). \tag{5.4}$$

Assume that $\|w_n\|_{kp}$ converges to 0 for $k=0, 1, 2, \cdots, m$, $p=0, 1, 2, \cdots$.

Then by the equality

$$w_n(\hat{f}_{u_n}(x))=-u_n(x)+u_0(x)$$

and by Lemma 8 we have

$$\begin{aligned}&-u_n^{(m)}(x)+u_0^{(m)}(x)\\&\quad=\sum_{\nu=0}^{m-1}P_{m\nu}[u_n]w_n^{(\nu)}(\hat{f}_{u_n}(x))+\exp mu_n w_n^{(m)}(\hat{f}_{u_n}(x))\end{aligned}$$

and therefore

$$\begin{aligned}w_n^{(m)}(x)&=\exp(-mu_n\circ\hat{f}_{u_n}^{-1}(x))\{-u_n^{(m)}(\hat{f}_{u_n}^{-1}(x))+u_0^{(m)}(\hat{f}_{u_n}^{-}(x))\}\\&\quad-\exp(-mu_n\circ\hat{f}_{u_n}^{-1}(x)\sum_{\nu=0}^{m-1}P_{m\nu}[u_n\hat{f}_{u_n}^{-1}]w_n^{(\nu)}(\hat{f}_{u_n}(x)).\end{aligned}$$

Considering that $P_{m\nu}[u_n\circ\hat{f}_{u_n}^{-1}(x)]$, $\nu=0, 1, 2, \cdots, m-1$ are polynomials in $\exp u_n\circ f_{u_n}^{-1}$, $u'_n\circ f_{u_n}^{-1}$, $\cdots$, $u_0^{(m-1)}\circ f_{u_n}^{-1}$ and that they are uniformly bounded in x and n, we have for every non-negative integer p

$$\|w_n\|_{mp}=\sup_x|x^p w_n^{(m)}(x)|$$

$$\leqslant K^m \sup_x |x^p\{u_n^{(m)}(\hat{f}_{u_n}^{-1}(x)) - u_0^{(m)}(\hat{f}_{u_n}^{-1}(x))\}|$$

$$+K^m \sum_{\nu=0}^{m-1} M_{m\nu} \sup_x |x^p w_n^{(\nu)}(x)|$$

$$\leqslant K^{m+p}\|u_n - u_0\|_{mp} + K^m \sum_{\nu=0}^{m-1} M_{m\nu}\|w_n\|_{\nu p}$$

$$\to 0 \text{ as } n \to +\infty,$$

where K is given by (5.4) and

$$M_{m\nu} = \sup_n \sup_x |P_{m\nu}[u_n]|, \qquad \nu = 0, 1, 2, \cdots, m-1.$$

Thus we have proved the proposition.

Theorem 4. *$\mathfrak{G}$ is a complete, separable, metrizable and arcwise connected topological group, and consequently, it is an $\mathscr{S}_r$-Lie group with the coordinate function ϕ.*

Proof. $\mathfrak{G}$ is a complete, separable, metrizable and arcwise connected topological space since $\mathscr{S}_r$, which is homeomorphic to $\mathfrak{G}$, has these properties, and it is an $\mathscr{S}_r$-Lie group since $\mathfrak{S}$ is a topological group in the topology of $\mathscr{S}_r$.

§6. Lie algebra $\mathfrak{a}$.

In this section, we determine the generator of a one-parameter subgroup of $\mathfrak{G}$ and show that the space of all generators forms a Lie algebra with respect to the commutator.

Let $\{\mathfrak{g}_t\}_{-\infty<t<+\infty}$ be a one-parameter subgroup of $\mathfrak{G}$ and put $u(t, x) = \phi(\mathfrak{g}_t)(x)$. Then for every $\xi(x)$ in $\mathscr{S}_r$ we have

$$(\mathfrak{g}_t \xi)(x) = \left\{\exp \frac{1}{2} u(t, x)\right\} \xi\left(\int_0^x \exp u(y)\, dy\right). \tag{6.1}$$

Differentiating the right side of (6.1) in t at $t=0$, and marking that $u(0, x) \equiv 0$, we have the generator of $\{\mathfrak{g}_t\}$ in the form

(6.2)
$$\frac{d}{dt}(\mathfrak{g}_t\xi(x))|_{t=0}$$
$$=\frac{1}{2}f(x)\xi(x)+F_f(x)\frac{d}{dx}\xi(x)$$
$$=(X(f)\xi)(x),$$

where

(15)
$$f(x)=\frac{\partial u(t, x)}{\partial t}\bigg|_{t=0}=\frac{d}{dt}\phi(\mathfrak{g}_t)(x)\bigg|_{t=0},$$

(16)
$$F_f(x)=\int_0^x f(y)\,dy,$$

and where

(14)
$$X(f)=\frac{1}{2}fI+F_f\frac{d}{dx}$$

and by the condition (P.2) of the definition of a one-parameter subgroup of $\mathfrak{G}$ in Section 1, f belongs to $\mathscr{S}_r$.

Let $\mathfrak{a}$ be the collection of all such operators defined in (14) for f in $\mathscr{S}_r$, that is,

$$\mathfrak{a}=\{X(f): f\in\mathscr{S}_r\}.$$

Then obviously $\mathfrak{a}$ is a linear space isomorphic to $\mathscr{S}_r$. We introduce a topology in $\mathfrak{a}$ such that $\mathfrak{a}$ and $\mathscr{S}_r$ are homeomorphic through the isomorphism $X(f)\rightarrow f$. Since $\mathfrak{a}$ and $\mathscr{S}_r$ are isomorphic not only algebraically but also topologically, $\mathfrak{a}$ is a topological vector space.

Proposition 8. *$\mathfrak{a}$ is closed with respect to the commutator*

$$[X(f), X(g)]=X(f)X(g)-X(g)X(f)$$

which is continuous in $\mathfrak{a}$.

Proof. For every f and g in $\mathscr{S}_r$, we have

$$X(f)X(g)=\frac{1}{4}fgI+\frac{1}{2}F_fg'I+\frac{1}{2}F_fg\frac{d}{dx}$$

$$+\frac{1}{2}fF_g\frac{d}{dx}+F_fg\frac{d}{dx}+F_fF_g\frac{d^2}{dx^2},$$

$$X(gXf(f)=\frac{1}{4}gfI+\frac{1}{2}F_gf'I+\frac{1}{2}F_gf\frac{d}{dx}$$

$$+\frac{1}{2}gF_f\frac{d}{dx}+F_gf\frac{d}{dx}+F_gF_f\frac{d^2}{dx^2},$$

and therefore

$$[X(f), X(g)]$$

$$=\frac{1}{2}(F_fg'-F_gf')I+(F_fg-F_gf)\frac{d}{dx}$$

$$=X(F_fg'-F_gf').$$

Since $\mathfrak{a}$ and $\mathcal{S}_r$ are isomorphic, in order to prove the proposition, it is sufficient to prove that

$$[f, g]=F_fg'-F_gf' \tag{6.3}$$

belongs to $\mathcal{S}_r$ and continuous in $\mathcal{S}_r\times\mathcal{S}_r$. To prove it, we have only to show that the map $(f, g)\to F_fg'$ is a continuos map from $\mathcal{S}_r\times\mathcal{S}_r$ to $\mathcal{S}_r$

Let f aud g be in $\mathcal{S}_r$. Then for every non-negative integers k and p, we have

$$\sup_x|F_f(x)|=\sup_x\left|\int_0^x f(y)\,dy\right|$$

$$\leqslant\sup_x(1+x^2)|f(x)|\sup_{x'}\left|\int_0^{x'}\frac{dy}{1+y^2}\right|$$

$$\leqslant\pi\{\|f\|_{00}+\|f\|_{02}\}.$$

Since F_fg' is bilinear, we show the continuity at (0, 0). In fact we have

$$\|F_fg'\|_{kp}\leqslant\sum_{r=0}^{k}\binom{k}{r}\sup_x\left|x^p\frac{d^{k-r}}{dx^{k-r}}F_f(x)\,g^{(r+1)}(x)\right|$$

$$\leqslant\sum_{r=0}^{k-1}\binom{k}{r}\sup_x|x^pf^{(k-r-1)}(x)g^{(r+1)}(x)|$$

$$+\sup_x |x^p F_f(x) g^{(k+1)}(x)|$$

$$\leqslant \sum_{r=0}^{k-1} \binom{k}{r} \|f\|_{k-r-1,0} \|g\|_{r+1,0}$$

$$+\pi(\|f\|_{00}+\|f\|_{02})\|g\|_{k+1,p}.$$

Thus we have proved the proposition.

Theorem 5. $\mathfrak{a}$ *is a Lie algebra with respect to the commutator.*

Remark. $\mathscr{S}_r$ is also a Lie algebra with the bracket defined in (6.3), which is isomorphic to $\mathfrak{a}$ as a Lie algebra.

§7. Exponential map.

In this section we establish the exponential map from the Lie algebra $\mathfrak{a}$ into the $\mathscr{S}_r$-Lie group $\mathfrak{G}$ and prove its continuity. First we prove the following lemma.

Lemma 12. *Let f be a function in $\mathscr{S}_r$. Then the function $u(t, x)$ defined by*

$$u(t, x)=\int_0^x f(h(r, x: f))\, dr, \qquad -\infty<t, x<+\infty \tag{7.1}$$

is also a real-valued, rapidly decreasing function in x uniformly in every finite t-interval. In other words, $u(t, x)$ is a C^∞-function in x and for every positive number T and for every non-negative integers k and p we have

$$\sup_{-T\leqslant t\leqslant T} \sup_x |x^p u^{(k)}(t, x)|<+\infty$$

where $u^{(k)}(t, x)=\dfrac{\partial k}{\partial x^k} u(t, x)$. Moreover, the $\mathscr{S}_r$-valued function $u_t=u(t, \cdot)$ is continuous in t.

Proof. Let f be a function in $\mathscr{S}_r$ and $u(t, x)$ be a function defined by (7.1). Then the first part of the lemma is proved in the same manner as Lemma 4. Therefore, we have only to prove the second part of the lemma.

We show for every non-negative integers k and p

$$\lim_{t\to s}\|u(t,\cdot)-u(s,\cdot)\|_{kp}=0, \qquad -\infty<s<+\infty, \tag{7.2}$$

by mathematical induction with respect to k.

In case $k=0$, by Lemma 3 we have for every non-negative integer p

$$\begin{aligned}
&\|u(t,\cdot)-u(s,\cdot)\|_{0p}\\
&=\sup_x\left|x^p\int_s^t f(h(r,x:f))\,dr\right|\\
&=\sup_{x\neq 0}\left|\int_s^t \frac{x^p}{h(r,x:f)^p}h(r,x:f)^p f(h(r,x:f))\,dr\right|\\
&\leqslant \exp[p\cdot\max(|t|,|s|)\|f\|_{00}]\|f\|_{0p}|t-s|.
\end{aligned}$$

The right side in the last inequality converges to 0 as t converges to s.

Assume that $\|u(t,\cdot)-u(s,\cdot)\|_{kp}$ converges to 0 for $k=0,1,2,\cdots,n$, $p=0,1,2,\cdots$. Then by (7.1) and (1.11) we have

$$h(t,x:f)=\int_0^x \exp u(t,y)\,dy$$

and Lemma 8 is applicable. Hence we have

$$\begin{aligned}
&\frac{\partial^n}{\partial x^n}\{u(t,x)-u(s,x)\}\\
&=\frac{\partial^n}{\partial x^n}\int_s^t f(h(r,x:f))\,dr\\
&=\sum_{\nu=0}^n\int_s^t P_{n\nu}[u(r,x)]f^{(\nu)}(h(r,x:f)\,dr.
\end{aligned}$$

Observing $P_{n\nu}[u(r,x)]$, $\nu=0,1,2,\cdots,n$ are polynomials in $\exp u(r,x)$, $\frac{\partial}{\partial x}u(r,x),\cdots,\frac{\partial^{n-1}}{\partial x^{n-1}}u(r,x)$ and, by assumption, they are bounded uniformly in r, we have by Lemma 3 for every non-negative integer p

$$\begin{aligned}
&\|u(t, \cdot)-u(s, \cdot)\|_{np} \\
&=\sup_x\left|x^p \frac{\partial^n}{\partial x^n}\{u(t, x)-u(s, x)\}\right| \\
&\leqslant \sum_{\nu=0}^{u} M'_{n\nu}\left|\int_s^t \sup_{x\neq 0}\left|\frac{x^p}{h(r, x: f)^p} h(r, x: f)^p f^{( )}(h(r, x: f))\right| dr\right| \\
&\leqslant \sum_{\nu=0}^{n} M'_{n\nu} \exp[p\cdot\max(|t|, |s|)\|f\|_{00}]\|f\|_{\nu p}|t-s|,
\end{aligned}$$

where

$$M'_{n\nu}=\sup_{\substack{s\leqslant r\leqslant t\\ x}}|P_{n\nu}[u(r, x)]|, \qquad \nu=0, 1, 2, \cdots, n.$$

The right side converges to 0 as t converges to s.

Thus we have proved the lemma.

Now we define the exponential map. For every $X(f)$ in $\mathfrak{a}$ we define

$$\text{(7.3)} \quad \operatorname{Exp} t\cdot X(f)=\hat{\mathfrak{g}}[u(t, \cdot)]=\mathfrak{g}\left[\exp\frac{1}{2}u(t, \cdot)\right], \qquad -\infty<t<+\infty$$

where $u(t, x)$ is given in (7.1) and call it the *exponential of* $X(f)$.

Theorem 6. *For every* $X(f)$ *in* $\mathfrak{a}$, $\operatorname{Exp} t\cdot X(f)$ *is a one-parameter subgroup of the* $\mathscr{S}_r$*-Lie group* $\mathfrak{G}$.

Proof. Let $X(f)$ be in $\mathfrak{a}$ where f is a function in $\mathscr{S}_r$. Then by Lemma 12, $u(t, x)$ defined by (7.1) is a function in $\mathscr{S}_r$ for every fixed t and consequently $\operatorname{Exp} t\cdot X(f)$ is in $\mathfrak{G}$ for every fixed t. Furthermore, by Theorem 1 $\left\{\exp\frac{1}{2}u(t, \cdot)\right\}_{-\infty<t<+\infty}$ forms a one-parameter subgroup of the group $\mathscr{U}^r_{\mathscr{S}}$ and therefore (P.1) is true.

The continuity of $\operatorname{Exp} t\cdot X(f)$ in t is derived from Lemma 12.

We prove the continuously differentiablity of $u(t, x)=\phi$ (Exp $t\cdot X(f))(x)$ in the topology of $\mathscr{S}_r$.

Observing that $u(t, x)$ is the solution of the differential equation (2.5) by Proposition 3, and that $F_f(x)$, together with all its derivatives, is a bounded function, we can easily show that

$$\frac{\partial}{\partial t} u(t, x) = F_f(x) \frac{\partial}{\partial x} u(t, x) + f(x)$$

is an $\mathscr{S}_r$-valued continuous function in t, since $u(t, x)$, therefore $\frac{\partial}{\partial x} u(t, x)$, is continuous in t.

On the other hand, by the mean value theorem we have for every t and s

$$\frac{1}{t-s}\{u(t, x) - u(s, x)\} - \frac{\partial}{\partial s} u(s, x)$$

$$= \frac{\partial}{\partial t} u(t, x)\Big|_{t=\tau} - \frac{\partial}{\partial t} u(t, x)\Big|_{t=s}$$

where τ is a number between t and s. By the continuity of $\frac{\partial}{\partial t} u(t, x)$, the right side converges to 0 in the topology of $\mathscr{S}_r$ as t converges to s.

We have thus proved the theorem.

In particular, we write simply $\operatorname{Exp} X(f)$ instead of $\operatorname{Exp} 1 \cdot X(f)$. Then we have the following proposition and lemma.

Proposition 9. *For every $X(f)$ in $\mathfrak{a}$, we have*

$$\operatorname{Exp} t \cdot X(f) = \operatorname{Exp} t X(f), \qquad -\infty < t < +\infty.$$

Proof. To prove the proposition, it is sufficient to show that

$$(7.4) \qquad \int_0^t f(h(s, x: f))\, ds = \int_0^1 t f(h(s, x: tf))\, ds, \qquad -\infty < t, x < +\infty,$$

for every f in $\mathscr{S}_r$.

In case $t=0$, (7.4) is trivially true.

We assume $t \neq 0$.

Since we have $tF_f(x) = F_{tf}(x)$, F_f and F_{tf} vanishes in the same set Λ_0. For every x in Λ_0, we have

$$h(s, x: f) = x = h(s, x: tf), \qquad -\infty < t, s < +\infty,$$

and consequently we have

$$\int_0^t f(h(s,\ x:f))\,ds=\int_0^t f(x)\,ds$$

$$=tf(x)=t\int_0^1 f(x)\,ds$$

$$=\int_0^1 tf(h(s,\ x:\ tf))\,ds.$$

For every x in $\Lambda=\Lambda_0^c$, it is not difficult to show that

$$h(s,\ x:f)=h\left(\frac{s}{t},\ x:\ tf\right), \qquad -\infty<s<+\infty.$$

Consequently we have

$$\int_0^t f(h(s,\ x:f))\,ds$$

$$=\int_0^t f\left(h\left(\frac{s}{t},\ x:\ tf\right)\right)ds$$

$$=\int_0^1 tf(h(s,\ x:\ tf))\,ds.$$

We have thus proved the proposition.

Hence we write simply $\mathrm{Exp}\,tX(f)$ instead of $\mathrm{Exp}\,t\cdot X(f)$.

Lemma 13. *If $X(f)$ converges to $X(f_0)$ in $\mathfrak{a}$, then $\phi(\mathrm{Exp}\,tX(f))(x)$ converges to $\phi(\mathrm{Exp}\,tX(f_0))(x)$ uniformly in t of $[0, 1]$, that is,*

$$\lim_{f\to f_0}\sup_{0\leqslant t\leqslant 1}\sup_x|\phi(\mathrm{Exp}\,tX(f))(x)-\phi(\mathrm{Exp}\,tX(f_0))(x)|=0.$$

Proof. Let $u(t,x)=\phi(\mathrm{Exp}\,tX(f))(x)$ and $u_0(t,x)=\phi(\mathrm{Exp}\,tX(f_0))(x)$. Then they are the solution of equations

$$\frac{\partial u}{\partial t}=F_f\frac{\partial u}{\partial x}+f,$$

$$\frac{\partial u_0}{\partial t}=F_{f_0}\frac{\partial u_0}{\partial x}+f_0,$$

under the initial condition $u(0, x)=u_0(0, x)=0$, respectively.

Put $v=u-u_0$. Then v is the solution of the equation

$$\frac{\partial v}{\partial t}=F_f\frac{\partial v}{\partial x}+(F_f-F_{f_0})\frac{\partial u_0}{\partial x}+f-f_0$$

under the initial condition $v(0, x)=0$. Therefore, by Haar's inequality (S. Mizohata [7]), there is a constant c such that

$$\begin{aligned}&\sup_{\substack{x\\0\leqslant t\leqslant 1}}|\phi(\operatorname{Exp} tX(f))(x)-\phi(\operatorname{Exp} tX(f_0))(x)|\\&\quad=\sup_{\substack{x\\0\leqslant t\leqslant 1}}|u(t, x)-u_0(t, x)|\\&\quad=\sup_{\substack{x\\0\leqslant t\leqslant 1}}|v(t, x)|\\&\quad\leqslant c\sup_{\substack{x\\0\leqslant t\leqslant 1}}|F_f(x)-F_{f_0}(x)|\left|\frac{\partial u_0}{\partial x}(t, x)\right|\\&\qquad+\sup_x|f(x)-f_0(x)|.\end{aligned}$$

Utilising the estimation in the proof of Proposition 8, we have

$$\begin{aligned}&\sup_{\substack{x\\0\leqslant t\leqslant 1}}|\phi(\operatorname{Exp} tX(f))(x)-\phi(\operatorname{Exp} tX(f_0))(x)|\\&\quad\leqslant\pi c'\{\|f-f_0\|_{00}+\|f-f_2\|_{00}\}+\|f-f_0\|_{00}\end{aligned}$$

where c' is a constant.

This completed the proof.

Theorem 7. $\operatorname{Exp} X(f)$ *maps a neighborhood of* 0 *in* $\mathfrak{a}$ *into a neighborhood of* $\mathbf{I}$ *in* $\mathfrak{G}$ *continuously.*

Proof. We have only to show that the mapping $X(f)\to\operatorname{Exp} X(f)$ is continuous. Let $\{X(f_\nu)\}_{\nu=1}^{+\infty}$ be a sequence in $\mathfrak{a}$ which converges to $X(f_0)$. Since $\mathfrak{a}$ can be identified with $\mathscr{S}_r$, $\{f_\nu\}_{\nu=1}^{+\infty}$ is a sequence in $\mathscr{S}_r$ which converges to f_0 and $\{f_\nu\}_{\nu=1}^{+\infty}$ is a bounded set in $\mathscr{S}_r$.

We show that for every non-negative integers k and p

$$\lim_{\nu\to+\infty}\|\phi(\operatorname{Exp}X(f_\nu))-\phi(\operatorname{Exp}X(f_0))\|_{kp}=0 \tag{7.5}$$

by mathematical induction with respect to k.

Put

$$u_\nu(x)=\phi(\operatorname{Exp}X(f_\nu))(x), \qquad \nu=0, 1, 2, 3, \cdots.$$

Then by definition, we have

$$u_\nu(x)=\int_0^1 f_\nu(h_\nu(s, x))\,ds, \qquad \nu=0, 1, 2, 3, \cdots,$$

where

$$h_\nu(s, x)=h(s, x: f_\nu), \qquad \nu=0, 1, 2, \cdots.$$

In case of $k=0$. Let ε be any positive number and p be any non-negative integer. Since f_0 is in $\mathscr{S}_r$, there is a positive number R such that

$$\sup_{|x|>R}|x^p f_0(x)|<\frac{1}{3}\varepsilon.$$

Observing that $\{f_\nu\}$ is a bounded set in $\mathscr{S}_r$, put

$$K=\exp\sup_\nu\sup_x|f_\nu(x)|.$$

Then by (1.11) we have

$$K^{-1}|x|\leqslant|h_\nu(s, x)|\leqslant K|x|$$

for any ν and s in $[0, 1]$. Therefore we have

$$\sup_{|x|\geqslant KR}\int_0^1|x^p f_\nu(h_\nu(s, x))|\,ds<(3K^p)^{-1}\varepsilon$$

for any ν.

On the other hand, by (1.11), we have

$$\sup_{\substack{|x|\leqslant KR\\0\leqslant s\leqslant1}}|h_\nu(s, x)-h_0(s, x)|$$

$$\leqslant\sup_{\substack{|x|\leqslant KR\\0\leqslant s\leqslant1}}\left|\int_0^x \exp u_\nu(s, y)-\exp u_0(s, y)\,dy\right|$$

$$\leqslant K \sup_{\substack{|x| \leqslant KR \\ 0 \leqslant s \leqslant 1}} |x| \, |u_\nu(s, x) - u_0(s, x)|$$

$$\leqslant K^2 R \sup_{\substack{x \\ 0 \leqslant s \leqslant 1}} |u_\nu(s, x) - u_0(s, x)|$$

where $u_\nu(s, x) = \phi(\operatorname{Exp} sX(f_\nu))(x)$, $\nu = 0, 1, 2, \cdots$.
By Lemma 13 the right side converges to 0 as f_ν converges to f_0.

Consequently we have

$$\begin{aligned}
&\|u_\nu - u_0\|_{0p} \\
&= \sup_x \left| x^p \int_0^1 f_\nu(h_\nu(s, x)) - f_0(h_0(s, x)) \, ds \right| \\
&\leqslant \sup_x \left| x^p \int_0^1 f_\nu(h_\nu(s, x)) - f_0(h(s, x)) \, ds \right| \\
&\quad + \sup_x \left| x^p \int_0^1 f_0(h_\nu(s, x)) - f_0(h_0(s, x)) \, ds \right| \\
&\leqslant K^p \sup_x \int_0^1 |h(s, x)^p| \{ |f_\nu(h_\nu(s, x) - f_0(h_\nu(s, x))| \} \, ds \\
&\quad + K^p \sup_{|x| > KR} \int_0^1 |h(s, x)^p| \{ |f_0(h_\nu(s, x))| + |f_0(h_0(s, x))| \} \, ds \\
&\quad + \sup_{|x| \leqslant KR} |x|^p \int_0^1 |f_0(h_\nu(s, x)) - f_0(h_0(s, x))| \, ds \\
&\leqslant K^p \|f_\nu - f_0\|_{0p} + \frac{2}{3} \varepsilon \\
&\quad + (KR)^p \sup_x \int_0^1 |f_0(h_\nu(s, x)) - f_0(h_0(s, x))| \, ds \\
&\leqslant K^p \|f_\nu - f_0\|_{0p} + \frac{2}{3} \varepsilon \\
&\quad + K^{p+2} R^{p+1} \|f_0\|_{1,0} \sup_{\substack{x \\ 0 \leqslant s \leqslant 1}} |u_\nu(s, x) - u_0(s, x)|
\end{aligned}$$

Observing Lemma 13, the right side becomes less than ε as ν diverges. Since ε is arbitrary, (7.5) is proved in case of $k = 0$.

Assume that (7.5) is true for $k = 0, 1, 2, \cdots, m-1$. Then by Lemma 8 we have

$$u_\nu^{(m)}(x)=\int_0^1\sum_{\nu=0}^{m}P_{m\lambda}[u]f_\nu^{(\lambda)}(h_\nu(s,\ x))\,ds, \qquad \nu=0,\ 1,\ 2,\ \cdots.$$

Therefore, considering $P_{m\lambda}[u]$, $\lambda=0,\ 1,\ \cdots,\ m$ are polynomials in $\exp u$, u', u'', $\cdots$, $u^{(m-1)}$, it is sufficient to show

$$\lim_{\nu\to+\infty}\left\|\int_0^1 f_\nu^{(\lambda)}(h_\nu(s,\ \cdot)-f_0^{(\lambda)}(h_0(s,\ \cdot))\,ds\right\|_{mp}=0, \tag{7.6}$$

$\lambda=0,\ 1,\ 2,\ \cdots,\ m,\ p=0,\ 1,\ 2,\ 3,\ \cdots.$
But it is not difficult to show (7.6) by a slight modification of the proof in case of $k=0$.

We have thus proved the theorem.

Remark. We have established the exponential map $\mathrm{Exp}\,X(f)$ which maps a neighborhood of 0 of $\mathfrak{a}$ *into* a neighborhood of **I** of $\mathfrak{G}$. But the problem whether $\mathrm{Exp}\,X(f)$ maps a neighborhood of 0 of $\mathfrak{a}$ *onto* a neighborhood of **I** of $\mathfrak{G}$ is still open.

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