J. Math. Kyoto Univ. (JMKYAZ)
11-2 (1971) 197-251

Algebra of stable homotopy of Z_p-spaces and applications

By

Hirosi TODA

(Received September 1, 1970)

Introduction

The stable homotopy classes of maps: $\sum^{t+n}X \to \sum^n Y$ will be denoted by $\pi_t^S(X; Y)$. When $X=Y$, $\mathscr{A}_*(X)=\sum \mathscr{A}_t(X)$, $\mathscr{A}_t(X)=\pi_t^S(X; X)$ forms a graded ring with the composition as the multiplication. p denotes an odd prime and $M=S^n \cup e^{n+1}$ a Moore space of type (Z_p, n). We call a space X a Z_p-space if $\mathscr{A}_*(X)$ is an algebra over Z_p or equivalently $M \wedge X$ is the same homotopy type of $\sum^n X \vee \sum^{n+1} X$ (n: large). Then $\pi_t^S(M\wedge X; M\wedge Y)$ is decomposed into $\pi_{t+1}^S(X; Y) \oplus \pi_t^S(X; Y) \oplus \pi_t^S(X; Y) \oplus \pi_{t-1}^S(X; Y)$. For given $\gamma \in \pi_t^S(X; Y)$, the smash product $1_M \wedge \gamma$ is decomposed to $\theta(\gamma) \oplus \gamma \oplus \gamma \oplus 0$, and we have a linear map

$$\theta: \pi_t^S(X; Y) \to \pi_{t+1}^S(X; Y)$$

This θ is a derivation:

$$\theta(\gamma\gamma')=\theta(\gamma)\cdot\gamma'+(-1)^{\deg\gamma}\gamma\theta(\gamma')$$

and if the spaces satisfy a sort of associativity then θ is a differential $\theta\theta=0$ (Theorem 2.2).

On the other hand, for a given $\xi \in \mathscr{A}_t(M)$ the decomposition $\xi \wedge 1_X = \lambda_X(\xi) \oplus$(other terms) defines a linear map

$$\lambda_X: \mathscr{A}_t(M) \to \mathscr{A}_{t+1}(X).$$

The basic property of this operation is the following commutation low (Theorem 2.4):

$$\lambda_Y(\xi)\gamma-(-1)^{(t+1)k}\gamma\lambda_X(\xi)=(-1)^{tk}\theta(\gamma)\lambda_X(\delta\xi)-\lambda_Y(\xi\delta)\theta(\gamma)$$

where $\gamma\in\pi_k^S(X;Y)$, $\xi\in\mathscr{A}_t(M)$ and δ is a generator of $\mathscr{A}_{-1}(M)\approx Z_p$ given by a smashing map. Note that $\lambda_X(\delta)=1_X$ and $\lambda_X(1_M)=0$.

We shall show how these operations are applied to determine multiplicative structure of $\mathscr{A}_*(M)$ and $\mathscr{A}_*(V(1))$, where $V(k)$, $0\leq k\leq 3$, is a Z_p-space (spectrum) given in [14]. For $V(0)=M$ we have the equality (Theorem 2.6), $\theta=D$ for D in [3],

$$\lambda_M(\xi)=-\theta(\xi),\quad \xi\in\mathscr{A}_t(M),$$

from which several relations in $\mathscr{A}_*(M)$ follow. For example, a generator α of $\mathscr{A}_q(M)$, $q=2(p-1)$, whose mapping cone is $V(1)$, satisfies $\theta(\alpha)=0$ and $(\alpha\delta-\delta\alpha)\xi=(-1)^{\deg\xi}\xi(\alpha\delta-\delta\alpha)$, in particular Yamamoto's relation $\alpha^2\delta-2\alpha\delta\alpha+\delta\alpha^2=0$ follows.

In the sections 3 and 4 we shall determine the structure of the algebra $\mathscr{A}_*(V(1))$ for degree less than $(p^2-1)q-5$, where we assume $p\geq 5$ since we need the existence of a generator $\beta\in\mathscr{A}_{pq+q}(V(1))$ whose mapping cone is $V(2)$. Let $i_1\colon M\to V(1)$ and $\pi_1\colon V(1)\to\sum^{q+1}M$ be the natural maps and put $\beta_{(s)}=\pi_1\beta^s i_1\in\mathscr{A}_{(sp+s-1)q-1}(M)$. For the above range of degrees, the algebra $\mathscr{A}_*(V(1))$ is generated by seven elements $\delta_0=i_1\delta\pi_1\in\mathscr{A}_{-q-2}$, $\delta_1=i_1\pi_1\in\mathscr{A}_{-q-1}$, $\alpha''\in\mathscr{A}_{q-2}$, $\alpha'=\lambda_{V(1)}(\delta\alpha\delta)\in\mathscr{A}_{q-1}$, $\beta'=\lambda_{V(1)}(\delta\beta_{(1)}\delta)\in\mathscr{A}_{pq-2}$, β and $\beta''\in\mathscr{A}_{(p+2)q-3}$, where $\theta(\delta_0)=-\delta_1$, $\theta(\alpha'')=\alpha'$, $\theta(\beta'')=\beta\alpha''-\alpha''\beta$ and $\theta(\delta_1)=\theta(\alpha')=\theta(\beta')=\theta(\beta)=0$. An additive basis for $\mathscr{A}_*(V(1))$ is given in Theorem 3.6. In the section 4 we determine a generating system of relations, among them the following are useful and analogy of the Yamamoto's relation:

$$\beta^2\delta_1-2\beta\delta_1\beta+\delta_1\beta^2=0,\quad \beta^2\alpha''-2\beta\alpha''\beta+\alpha''\beta^2=0,$$

$$\beta^3\delta_0-3\beta^2\delta_0\beta+3\beta\delta_0\beta^2-\delta_0\beta^3=0.$$

In the first half of the section 5, we reprove Yamamoto's result on $\mathscr{A}_*(M)$ and generalize his relations (Theorems 5.1, 5.2). Let $i\colon S^n\to M$ and $\pi\colon M\to S^{n+1}$ be the natural maps and put $(\delta=i\pi)$

$$\alpha_r = \pi\alpha^r i \in G_{rq-1} \quad \text{and} \quad \beta_s = \pi\beta_{(s)} i \in G_{(sp+s-1)q-2}.$$

α_r is detected by Adams invariant, and recently L. Smith has proved the non-trivialtiy of β_s for general $s \geq 1$. For the elements β_s, we have the equality

$$t(r+s-t)\beta_r\beta_s = rs\beta_t\beta_{r+s-t}$$

and hence every monomial $\prod_{i=1}^{k}\beta_{s_i}$ is a multiple of β_{pt}, $\beta_t\beta^{k-1}$ or $\beta_{pt-1}\beta_2\beta_1^{k-2}$ (Theorem 5.3).

In the second half of the section 5, we consider a class $\gamma_{[1]} \in \mathscr{A}_{p^2q-1}(V(1))$ whose mapping cone is $V\left(2\frac{1}{2}\right)/V(1)$ and put $\gamma_{(1)} = \pi_1\gamma_{[1]}i_1 \in \mathscr{A}_{(p^2-1)q-2}(M)$ and $\gamma_1 = \pi\gamma_{(1)}i \in G_{(p^2-1)q-3}$. Then we have (Theorem 5.5)

$$\gamma_{(1)} \equiv (\beta_{(1)}\delta + \delta\beta_{(1)})^p \qquad (\mathrm{mod}\ \beta_{(p-1)}\delta\alpha)$$

and this implies (Theorem 5.8)

$$\beta_s\beta_1^p = 0 \quad \text{and} \quad \beta_s^{p+1} = 0 \qquad \text{for } s \geq 2.$$

The non-triviality of γ_1 (a multiple of $\alpha_1\beta_{p-1}$) is an open question, we have however

$$\alpha_1\beta_{p-1}\beta_s = 0 \qquad \text{for } s \geq 3 \text{ if } \gamma_1 \neq 0.$$

The section 6 is a deep discussion for the case $p=3$, whence $V(2)$ and β do not exist, M and $V(1)$ are not associative and the products $\alpha''\alpha''$, $\alpha'\alpha''$ are non-trivial (trivial if $p \geq 5$). So the structures of $\mathscr{A}_*(V(1))$ and $\mathscr{A}_*(M)$ are quite different from the case $p \geq 5$. Some of Yamamoto's relations in [15] fail for the case $p=3$, and the corrected values will be given in Theorem 6.8.

In the last section we shall prove $\beta''\beta'' \neq 0$ for $p=5$ ($\beta''\beta''=0$ for $p \geq 7$), the only particular property of the case $p=5$.

1. Smash Products in Stable Homotopy.

In this paper, all spaces, maps and homotopies are base pointed. $[X, Y]$ denote the set of the homotopy classes of the maps $f: X \to Y$, in which we use the same symbol $f\in[X, Y]$ for the homotopy class of f. $X\wedge X'=X\times X'/X\vee X'$ and $f\wedge f'\colon X\wedge X'\to Y\wedge Y'$ are the smash products of spaces and maps. Denote by

$$T=T_{X,X'}\colon X\wedge X'\to X'\wedge X$$

and

$$1=1_X\colon X\to X$$

the map switching the factors and the identity map respectively. S^n denotes the unit n-sphere with the identification $S^m\wedge S^n=S^{m+n}$. We write simply

$$1_n=1_{S^n},\quad T_{X,n}=T_{X,S^n},\quad T_{n,X}=T_{S^n,X},\quad T_{m,n}=T_{S^m,S^n}.$$

The n-fold iterated suspensions of a space X and a map f are defined by

$$\Sigma^n X=S^n\wedge X \quad\text{and}\quad \Sigma^n f=1_n\wedge f.$$

We identify $S^0\wedge X$ and $X\wedge S^0$ with X naturally. Whenever considering triple smash products $(X\wedge X')\wedge X''$ and $X\wedge(X'\wedge X'')$, two of the spaces X, X', X'' are compact, and the triple smash products are identified. The composition of maps $f\colon X\to Y$ and $g\colon Y\to Z$ is denoted by $gf\colon X\to Z$. Then the following equalities hold.

(1.1) $$1_Y f=f1_X=f,\qquad h(gf)=(hg)f,$$
$$1_0\wedge f=f\wedge 1_0=f,\qquad (f\wedge f')\wedge f''=f\wedge(f'\wedge f''),$$
$$(gf)\wedge(g'f')=(g\wedge g')(f\wedge f'),$$
$$\Sigma^m(\Sigma^n f)=\Sigma^{m+n}f,\qquad \Sigma^m(gf)=(\Sigma^m g)(\Sigma^m f),$$
$$T_{0,X}=T_{X,0}=1_X,\qquad T_{Y,Y'}(f\wedge f')=(f'\wedge f)T_{X,X'}$$
and $$T_{m,n}=(-1)^{mn}\ \textit{in}\ [S^{m+n}, S^{m+n}].$$

Apparently, the composition, the smash product and Σ^n are compatible with the homotopy, and (1.1) holds for the homotopy classes of the maps.

For each integer k we define *the k-th stable homotopy groups* of the spaces X, Y by

$$\pi_k^S(X;\ Y)=\lim_n [\Sigma^{n+k}X,\ \Sigma^n Y] \qquad (n+k,\ n\geq 0)$$

where the limit is taken over the suspension Σ. $\pi_k^S(X;\ Y)$ is an abelian group. We define *t-suspension isomorphism*

$$\Sigma^t:\ \pi_k^S(X;\ Y)\cong \pi_k^S(\Sigma^t X;\ \Sigma^t Y) \tag{1.2}$$

by associating to each $f\in[\Sigma^{n+k}X,\ \Sigma^n Y]=[S^{n+k}\wedge X,\ S^n\wedge Y]$ the class

$$(T_{t,n}\wedge 1_Y)\Sigma^t f(T_{n+k,t}\wedge 1_X)\in[\Sigma^{n+k}(\Sigma^t X),\ \Sigma^n(\Sigma^t Y)],$$

that is

(1.2)′ $\Sigma^t\{f\}=(-1)^{tk}\{f\}$ *for the classes* $\{\ \}$ *of the same* $f\in[\Sigma^{n+k+t}X,\ \Sigma^{n+t}X]$.

Sometimes we use the same notation $f\in\pi_0^S(X;\ Y)$ for the limit $\{f\}$ of $f\in[X;\ Y]$, e.g., $T_{X,Y}\in\pi_0^S(X\wedge Y;\ Y\wedge X)$, $1_X\in\pi_0^S(X;\ Y)$.

The *product* (composition)

$$\pi_h^S(Y;\ Z)\otimes\pi_k^S(X;\ Y)\rightarrow\pi_{h+k}^S(X;\ Y)$$

is defined by

$$\{g\}\{f\}=\{g(\Sigma^{m+h-n}f)\}$$

for $g\in[\Sigma^{m+h}Y,\ \Sigma^m Z]$, $f\in[\Sigma^{n+k}X,\ \Sigma^n Y]$, $m+h\geq n$.

This product is well-defined, *bilinear*, *associative* and has the units 1_X. We write

$$\pi_*^S(X;\ Y)=\sum_k\pi_k^S(X;\ Y),$$

$$\mathscr{A}_*(X)=\sum_k\mathscr{A}_k(X),\ \mathscr{A}_k(X)=\pi_k^S(X;\ Y).$$

Then $\mathscr{A}_*(X)$ is a *graded ring* and $\pi_*^S(X;\ Y)$ is a left $\mathscr{A}_*(Y)$-right $\mathscr{A}_*(X)$ module.

The homomorphisms

$$g_*=\beta_*\colon \pi_k^S(X;\ Y)\to\pi_{k+h}^S(X;\ Z)$$

and

$$f^*=\alpha^*\colon \pi_h^S(Y;\ Z)\to\pi_{k+h}^S(X;\ Z)$$

are defined by $\beta_*(\alpha)=\alpha^*(\beta)=\beta\alpha$ for $\beta=\{g\}$ and $\alpha=\{f\}$.

Next we shall define a *smash product*.

$$\wedge\colon \pi_h^S(X;\ Y)\otimes\pi_k^S(X';\ Y')\to\pi_{h+k}^S(X\wedge X';\ Y\wedge Y').$$

Let $f\in[\Sigma^{l+h}X,\ \Sigma^l Y]$ and $f'\in[\Sigma^{m+k}X',\ \Sigma^m Y']$ be representatives of $\alpha\in\pi_h^S(X;\ Y)$ and $\alpha'\in\pi_k^S(X';\ Y')$ and put

$$\begin{aligned} f*f'&=(1_l\wedge T_{Y,m}\wedge 1_{Y'})(f\wedge f')(1_{l+h}\wedge T_{m+k,X}\wedge 1_{X'})\\ &:\Sigma^{l+m+h+k}(X\wedge X')=S^{l+h}\wedge S^{m+k}\wedge X\wedge X'\to S^{l+h}\wedge X\wedge S^{m+k}\wedge X^l\\ &\to S^l\wedge Y\wedge S^m\wedge Y'\to S^l\wedge S^m\wedge Y\wedge Y'=\Sigma^{l+m}(Y\wedge Y'). \end{aligned}$$

Then $\alpha\wedge\alpha'$ is the class $\{(-1)^{mh}(f*f')\}$.

Theorem 1.1. *The smash product $\wedge$ is well defined and bilinear. The following formulas hold.*

(1.3) $$(\beta\alpha)\wedge(\beta'\alpha')=(-1)^{\deg\alpha\deg\beta'}(\beta\wedge\beta')(\alpha\wedge\alpha'),$$

(1.4) $$T_{Y,Y'}(\alpha\wedge\alpha')=(-1)^{\deg\alpha\deg\alpha'}(\alpha'\wedge\alpha)T_{X,X'},$$

(1.5) $$1_t\wedge\alpha=\Sigma^t\alpha,$$

(1.6) $$(\alpha\wedge\alpha')\wedge\alpha''=\alpha\wedge(\alpha'\wedge\alpha'').$$

Proof. By (1.1), we have $(\Sigma f)*f'=\Sigma(f*f')$ and $f*(\Sigma f')=(T_{1,l}\wedge 1)\Sigma(f*f')(T_{l+h,1}\wedge 1)=(-1)^h\Sigma(f*f')$. Thus the definition of $\alpha\wedge\alpha'$ is compatible with the suspension, and the smash product is well defined. Next the equalities

$$(g(\Sigma^h f))*(g'(\Sigma^{h'}f'))=(g*g')((\Sigma^h f)*(\Sigma^{h'}f'))$$

$$(T_{l,m}\wedge T_{Y,Y'})(f*f')=(f'*f)(T_{l+h,m+k}\wedge T_{X,X'})$$

and

$$(f*f')*f''=f*(f'*f'')$$

are verified without difficulties. Then (1.3), (1.4) and (1.6) follow. (1.5) follows from the definition of $\sum^t$.

As a corollary we have the following.

(1.7) $\alpha\wedge\alpha'=(\alpha\wedge 1_{Y'})(1_X\wedge\alpha')=(-1)^{\deg\alpha\deg\alpha'}(1_Y\wedge\alpha')(\alpha\wedge 1_{X'})$.

(1.8) $(\sum^t\alpha)\wedge\alpha'=\sum^t(\alpha\wedge\alpha'),\quad \sum^t(\beta\alpha)=(\sum^t\beta)(\sum^t\alpha)$.

Also we remark

(1.9) $1_0\wedge\alpha=\alpha\wedge 1_0=\alpha$ *and* $T_{0,X}=T_{X,0}=1_X$.

Let $I=[0, 1]$ be the unit interval with the base point (1), and let

$$\psi\colon I\to S^1$$

be a mapping of degree 1 which identify (0) and (1) to the base point. The cone over X is defined by $CX=I\wedge X$. Consider the mapping cone of a map $f\in[X, Y]$: $C_f=Y\cup_f CX$, and consider the cofibering

$$Y\xrightarrow{i}C_f\xrightarrow{\pi}\sum X$$

where π is induced by $\psi\wedge 1_X\colon CX\to\sum X$. Then there exists a homeomorphism

$$h\colon \sum^n C_f\to C_{\sum^n f}$$

such that the diagram

$$\text{(1.10)}\qquad \begin{array}{ccccc} \sum^n Y & \xrightarrow{\sum^n i} & \sum^n C_f & \xrightarrow{\sum^n\pi} & \sum^{n+1}X=S^n\wedge S^1\wedge X \\ \downarrow 1 & & \downarrow h & & \downarrow T_{n,1}\wedge 1_X \\ \sum^n Y & \xrightarrow{i} & C_{\sum^n f} & \xrightarrow{\pi} & \sum^{n+1}X=S^1\wedge S^n\wedge X \end{array}$$

commutes. For $1=1_Z$ we can identify $C_f\wedge Z$ with $C_{f\wedge 1}$ then we have a cofibering

$$X\wedge Z\xrightarrow{i\wedge 1}C_{f\wedge 1}=C_f\wedge Z\xrightarrow{\pi\wedge 1}\sum Y\wedge Z.$$

Consider a representative $f\in[\sum^{n+k}X, \sum^n Y]$ of an element $\alpha\in\pi_k^S(X; Y)$, and the mapping cone $C_f=\sum^n Y\cup_f C\sum^{n+k}X$ and the cofibering $\sum^n Y\xrightarrow{i}C_f\xrightarrow{\pi}\sum^{n+k+1}X$. Then

$$i \in \pi_n^S(Y;\, C_f) \quad \text{and} \quad \pi \in \pi_{-k-n-1}^S(C_f;\, X),$$

and we have the following two sequences.

(1.11) $\xrightarrow{\pi_*} \pi_h^S(W;\, X) \xrightarrow{\alpha_*} \pi_{h+k}^S(W;\, Y) \xrightarrow{i_*} \pi_{h+k+n}^S(W;\, C_f) \xrightarrow{\pi_*} \cdots$

(1.11)* $\xrightarrow{i^*} \pi_h^S(Y;\, Z) \xrightarrow{\alpha^*} \pi_{h+k}^S(X;\, Z) \xrightarrow{\pi^*} \pi_{h-n-1}^S(C_f;\, Z) \xrightarrow{i^*} \cdots$

As is well known

(1.12) *the above sequences* (1.11) *and* (1.11)* *are exact if each spaces are finite CW-complexes.*

Let $M = M_p = S^1 \cup_p e^2$ be a Moore space of type $(Z_p, 1)$, i.e. a mapping cone of a map $p = p \cdot 1_1 \in \mathscr{A}_0(S^0)$, where p is a prime. We have a cofibering sequence

$$S^1 \xrightarrow{p} S^1 \xrightarrow{i} M_p \xrightarrow{\pi} S^2 \xrightarrow{Sp} \cdots$$

where $$i \in \pi_1^S(S^0;\, M_p),\ \pi \in \pi_{-2}^S(M_p;\, S^0).$$

Lemma 1.2. *The following four conditions are equivalent.*

(i) $p \cdot 1_X = 0$ in $\mathscr{A}_0(X)$.

(ii) $\mathscr{A}_*(X)$ *is an algebra over the field* Z_p.

(iii) $\pi_*^S(X;\, Y)$ *and* $\pi_*^S(Y;\, X)$ *are* Z_p*-modules for any* Y.

(iv) *There exists elements* $\mu_X \in \pi_{-1}^S(M_p \wedge X;\, X)$ *and* $\varphi_X \in \pi_2^S(X;\, M_p \wedge X)$ *which satisfy the following*

$$\mu_X \varphi_X = 0, \qquad \mu_X(i \wedge 1_X) = (\pi \wedge 1_X)\varphi_X = 1_X$$

(1.13) *and*

$$(i \wedge 1_X)\mu_X + \varphi_X(\pi \wedge 1_X) = 1_{M \wedge X}.$$

Proof. Since 1_X is the unit, (i), (ii) and (iii) are equivalent. (iv) implies $p \cdot 1_X = p(i \wedge 1_X)\mu_X = ((pi) \wedge 1_X)\mu_X = 0$ and (i). Assume that (i) holds, then for sufficiently large n, $p \cdot 1 : \sum^{n+1} X \to \sum^{n+1} X$ is homotopic to zero. By use of the homotopy we have a homotopy equivalence $h' : C_{p \cdot 1} = \sum^{n+1} X \cup_{p \cdot 1} C \sum^{n+1} X \to C_0 = \sum^{n+1} X \vee \sum^{n+2} X$ such that the diagram

$$\begin{array}{ccccc}\sum^{n+1}X & \xrightarrow{i} & \sum^{n+1}X\cup_{p\cdot 1}C\sum^{n+1}X & \xrightarrow{\pi} & \sum^{n+2}X \\ \downarrow 1 & & \downarrow h' & & \downarrow 1 \\ \sum^{n+1}X & \xrightarrow{i_1} & \sum^{n+1}X\vee\sum^{n+2}X & \xrightarrow{\pi_2} & \sum^{n+2}X\end{array}$$

homotopy commutes. $M_p\wedge X$ is a mapping cone of $f=p\cdot 1_{\sum X}$, then $p\cdot 1=\sum^n f$ and we have a homeomorphism $h:\sum^n(M_p\wedge X)\rightarrow C_{p\cdot 1}$ of (1.10). Put $h_0=h'h$, then we have the following homotopy commutative diagram

$$\begin{array}{ccccc}\sum^{n+1}X & \xrightarrow{\Sigma^n(i\wedge 1_X)} & \sum^n(M_p\wedge X) & \xrightarrow{\Sigma^n(\pi\wedge 1_X)} & \sum^{n+2}X \\ \downarrow 1 & & \downarrow h_0 & & \downarrow (-1)^n \\ \sum^{n+1}X & \xrightarrow{i_1} & \sum^{n+1}X\vee\sum^{n+2}X & \xrightarrow{\pi_2} & \sum^{n+2}X,\end{array}$$

where the lower sequence is the natural cofibering. We also have another natural cofibering

$$\sum^{n+2}X\xrightarrow{i_2}\sum^{n+1}X\vee\sum^{n+2}X\xrightarrow{\pi_1}\sum^{n+1}X,$$

then the equalities (in homotopy classes)

$$\pi_1 i_1=\sum^{n+1}1_X,\quad \pi_2 i_2=\sum^{n+2}1_X,\quad i_1\pi_1+i_2\pi_2=1$$

hold. For a homotopy inverse $\overline{h_0}$ of h_0, put $\mu=\pi_1 h_0$ and $\varphi=(-1)^n\overline{h_0}i_2$. Then $\pi_1 i_2=0$ and the above equalities imply

$$\mu\varphi=0,\quad \mu\sum^n(i\wedge 1_X)=\sum^{n+1}1_X,\quad \sum^n(\pi\wedge 1_X)\varphi=\sum^{n+2}1_X$$

and

$$\sum^n(i\wedge 1_X)\mu+\varphi\sum^n(\pi\wedge 1_X)=1.$$

Let μ_X and φ_X be the limits of μ and φ respectively, then (1.13) holds. Q.E.D.

The condition (i) is to say that the stable order of X is p (or possibly 1) in the sense of [9]. It is well known that $p\cdot 1_{M_p}=0$ iff $p>2$. In the following we always assume that p is an odd prime. Put

$$\delta=i\pi\in\mathscr{A}_{-1}(M_p).$$

Lemma 1.3. *Let $M=M_p$ and $T=T_{M,M}\in\mathscr{A}_0(M\wedge M)$, then the* *following relations hold* [2].

(i) $T=-(i\wedge 1_M)\mu_M+\varphi_M(\pi\wedge 1_M)+\varphi_M\delta\mu_M$,

(ii) $1_M\wedge i=-(i\wedge 1_M)+\varphi_M\delta$, $1_M\wedge\pi=\pi\wedge 1_M+\delta\mu_M$,

(iii) $\mu_M T=-\mu_M$, $T\varphi_M=\varphi_M$.

Proof: Since p is an odd prime, $\mathscr{A}_1(M_p)=0$ and $\mathscr{A}_0\ (M_p)\cong Z_p$ is generated by $1=1_M$. Then $\mu T\varphi=0$, $\mu T(i\wedge 1)=x\cdot 1$ and $(\pi\wedge 1)T\varphi=y\cdot 1$ for some x, $y\in Z_p$. Using (1.7), (1.4), (1.9), (1.13),

$$x\cdot i=\mu T(i\wedge 1)(1_0\wedge i)=\mu T(i\wedge i)=-\mu(i\wedge i)T_{0,0}$$
$$=-\mu(i\wedge 1)(1_0\wedge i)=-i.$$

and similarly $y\cdot\pi=(\pi\wedge\pi)T\varphi=(\pi\wedge\pi)\varphi=\pi$. Thus $x=-1$, $y=1$, and $\mu T(i\wedge 1)=-1$, $(\pi\wedge 1)T\varphi=1$. Then $-\mu=\mu T(i\wedge 1)\mu=\mu T$ and $\varphi=\varphi(\pi\wedge 1)T\varphi=T\varphi$ by (1.13), and (iii) is proved. Next

$$(\pi\wedge 1)T(i\wedge 1)=(\pi\wedge 1)(1\wedge i)T_{0,M}=(1_0\wedge i)(\pi\wedge 1_0)=i\pi=\delta.$$

Then we have (i):

$$T=\{(i\wedge 1)\mu+\varphi(\pi\wedge 1)\}T\{(i\wedge 1)\mu+\varphi(\pi\wedge 1)\}$$
$$=(i\wedge 1)\mu T(i\wedge 1)\mu+\varphi(\pi\wedge 1)T(i\wedge 1)\mu+\varphi(\pi\wedge 1)T\varphi(\pi\wedge 1)$$
$$=-(i\wedge 1)\mu+\varphi\delta\mu+\varphi(\pi\wedge 1),$$

and (ii): $1\wedge i=(1\wedge i)T_{0,M}=T(i\wedge 1)$

$$=\{-(i\wedge 1)+\varphi\delta\}\mu(i\wedge 1)+\varphi(\pi i\wedge 1)$$
$$=-(i\wedge 1)+\varphi\delta,$$
$$1\wedge\pi=(\pi\wedge 1)T=\pi\wedge 1+\delta\mu.$$

Q.E.D.

Remark 1.4. *Let* (μ_X, φ_X) *and* (μ'_X, φ'_X) *both satisfy* (1.13). *Then* $\mu_X=\mu'_X$ *iff* $\varphi_X=\varphi'_X$.

For, if $\mu_X=\mu'_X$ then $\varphi'_X=\{(i\wedge 1)\mu_X+\varphi_X(\pi\wedge 1_X)\}\varphi'_X=\varphi_X$, and conversely.

2. Operations θ and λ_X in Z_p-Spaces.

Definition. A space X which is equipped two classes $\mu_X \in \pi_{-1}^S(M_p\wedge X;X)$ and $\varphi_X\in\pi_2^S(X;M_p\wedge X)$ satisfying the equalities of (1.13) is called as a *Z_p-space*. A map (class) $\gamma\in\pi_k^S(X;Y)$ is called a *Z_p-map* if it satisfies

$$(-1)^k\gamma\mu_X=\mu_Y(1_M\wedge\gamma) \quad\text{and}\quad \varphi_Y\gamma=(1_M\wedge\gamma)\varphi_X.$$

A Z_p-space (X, μ_X, φ_X) is called *associative* if

$$\mu_X(1_M\wedge\mu_X)=-\mu_X(\mu_M\wedge 1_X),\quad (1_M\wedge\varphi_X)\varphi_X=(\varphi_M\wedge 1_X)\varphi_X.$$

For the examples of Z_p-spaces in this paper, the elements μ_X and φ_X will be unique by the following

Proposition 2.1. *Let (X, μ_X, φ_X) be a Z_p-space such that X is a finite CW-complex. If $\mathscr{A}_1(X)=0$ then μ_X and φ_X are unique and there exists uniquely an element $\alpha_X\in\mathscr{A}_2(X)$ such that*

$$\mu_X(1_M\wedge\mu_X)+\mu_X(\mu_M\wedge 1_X)=\alpha_X(\pi\wedge\pi\wedge 1_X)$$

and

$$(1_M\wedge\varphi_X)\varphi_X-(\varphi_M\wedge 1_X)\varphi_X=(i\wedge i\wedge 1_X)\alpha_X.$$

In particular, X is associative if $\mathscr{A}_1(X)=\mathscr{A}_2(X)=0$.

Proof. Consider the exact sequence (1.11)* for the case $X=Y=Z$, $n=1$, $\alpha=p\cdot 1_X$, then the condition $\mathscr{A}_1(X)=0$ implies that $(i\wedge 1_X)^*$: $\pi_{-1}^S(M_p\wedge X;X)\to\mathscr{A}_0(X)$ is a monomorphism. Since μ_X satisfies $(i\wedge 1_X)^*\mu_X=\mu_X(i\wedge 1_X)=1_X$, it follows the uniqueness of μ_X. The uniqueness of φ_X is proved similarly.

By Lemma 1.3, (ii),

$$\begin{aligned}\mu_X(1_M\wedge\mu_X)(i\wedge i\wedge 1_X)&=\mu_X(1_M\wedge\mu_X)(i\wedge 1_M\wedge 1_X)(i\wedge 1_X)\\&=\mu_X(1_M\wedge\mu_X)(-1_M\wedge i\wedge 1_X+\varphi_M\delta\wedge 1_X)(i\wedge 1_X)\end{aligned}$$

$$=-\mu_X(i\wedge 1_X)+\mu_X(1_M\wedge\mu_X)(\varphi_M\delta i\wedge 1_X)=-1_X$$

and

$$\mu_X(\mu_M\wedge 1_X)(i\wedge i\wedge 1_X)=\mu_X(\mu_M(i\wedge 1_M)\wedge 1_X)(i\wedge 1_X)=1_X.$$

Thus $\mu_X(1_M\wedge\mu_X)+\mu_X(\mu_M\wedge 1_X)$ is in the kernel of

$$(i\wedge 1_X)^*(i\wedge 1_M\wedge 1_X)^*: \pi_{-2}^S(M\wedge M\wedge X; X)\rightarrow\pi_{-1}^S(M\wedge X; X)\rightarrow\mathscr{A}_0(X).$$

As above $(i\wedge 1_X)^*$ is a monomorphism. Also $\mathrm{Ker}(i\wedge 1_M\wedge 1_X)^*=(\pi\wedge 1_M\wedge 1_X)^*\pi_0^S(M_p\wedge X; X)$ and $(\pi\wedge 1_X)^*: \mathscr{A}_2(X)\rightarrow\pi_0^S(M_p\wedge X; X)$ is an epimorphism. Thus there exists an element $\alpha_X\in\mathscr{A}_2(X)$ such that

$$\mu_X(1_M\wedge\mu_X)+\mu_X(\mu_M\wedge 1_X)=\alpha_X(\pi\wedge 1_X)(\pi\wedge 1_M\wedge 1_X)=\alpha_X(\pi\wedge\pi\wedge 1_X).$$

Since $(\pi\wedge\pi\wedge 1_X)(\varphi_M\wedge 1_X)\varphi_X=1_X$, $\alpha_X(\pi\wedge\pi\wedge 1_X)=\alpha'_X(\pi\wedge\pi\wedge 1_X)$ implies $\alpha_X=\alpha'_X$. Thus α_X is unique.

Next by use of (1.13) and Lemma 1.3,

$$\begin{aligned}
&(i\wedge i\wedge 1_X)\alpha_X(\pi\wedge\pi\wedge 1_X)\\
&=(i\wedge 1_M\wedge 1_X)(i\wedge 1_X)\mu_X(1_M\wedge\mu_X+\mu_M\wedge 1_X)\\
&=(i\wedge 1_M\wedge 1_X)(1_M\wedge 1_X-\varphi_X(\pi\wedge 1_X))(1_M\wedge\mu_X+\mu_M\wedge 1_X)\\
&=(-1_M\wedge i\wedge 1_X+\varphi_M\delta\wedge 1_X)(1_M\wedge\mu_X)+(i\wedge 1_M)\mu_M\wedge 1_X\\
&\quad -(i\wedge\varphi_X)(\pi\wedge\mu_X+\pi\mu_M\wedge 1_X)\\
&=-1_M\wedge(1_M\wedge 1_X-\varphi_X(\pi\wedge 1_X))+\varphi_M\delta\wedge\mu_X\\
&\quad +(1_M\wedge 1_M-\varphi_M(\pi\wedge 1_M))\wedge 1_X-\delta\wedge\varphi_X\mu_X-\delta\mu_M\wedge\varphi_X\\
&=1_M\wedge\varphi_X(\pi\wedge 1_X)+\varphi_M\delta\wedge\mu_X-\varphi_M(\pi\wedge 1_M)\wedge 1_X\\
&\quad -\delta\wedge\varphi_X\mu_X-\delta\mu_M\wedge\varphi_X
\end{aligned}$$

and

$$\begin{aligned}
&(1_M\wedge\varphi_X-\varphi_M\wedge 1_X)\varphi_X(\pi\wedge\pi\wedge 1_X)\\
&=(1_M\wedge\varphi_X-\varphi_M\wedge 1_X)(1_M\wedge 1_X-(i\wedge 1_X)\mu_X)(\pi\wedge 1_M\wedge 1_X)\\
&=(1_M\wedge\varphi_X)(1_M\wedge\pi\wedge 1_X-\delta\mu_M\wedge 1_X)-(i\wedge\varphi_X)(\pi\wedge\mu_X)
\end{aligned}$$

$$-\varphi_M(\pi\wedge 1_M)\wedge 1_X+(\varphi_M i\wedge 1_X)(\pi\wedge\mu_X)$$

$$=1_M\wedge\varphi_X(\pi\wedge 1_X)-\delta\mu_M\wedge\varphi_X-\delta\wedge\varphi_X\mu_X-\varphi_M(\pi\wedge 1_M)\wedge 1_X$$

$$+\varphi_M\delta\wedge\mu_X$$

$$=(i\wedge i\wedge 1_X)\alpha_X(\pi\wedge\pi\wedge 1_X).$$

Since $(\pi\wedge\pi\wedge 1_X)(\varphi_M\wedge 1_X)\varphi_X=1_X$, we have $(1_M\wedge\varphi_X-\varphi_M\wedge 1_X)\varphi_X=(i\wedge i\wedge 1_X)\alpha_X$.

The following (2.2) is directly verified from Theorem 1.1 and (1.9).

(2.2) *Let (X, μ_X, φ_X) be a Z_p-space and let X' be an arbitrary compact space, then*

$$(X\wedge X',\ \mu_{X\wedge X'}=\mu_X\wedge 1_{X'},\quad \varphi_{X\wedge X'}=\varphi_X\wedge 1_{X'}),$$

$$(X'\wedge X,\ \mu_{X'\wedge X}=(1_{X'}\wedge\mu_X)(T_{M,X'}\wedge 1_X),$$

$$\varphi_{X'\wedge X}=(T_{X',M}\wedge 1_X)(1_{X'}\wedge\varphi_X))$$

and

$$(\textstyle\sum^t X,\ \mu_{\sum^t X}=\sum^t\mu_X(T_{M,t}\wedge 1_X),\quad \varphi_{\sum^t X}=(T_{t,M}\wedge 1_X)\sum^t\varphi_X)$$

are Z_p-spaces.

We have easily

(2.3) *Let $\alpha\in\pi_k^S(X;\ Y)$ be a Z_p-map and let $\alpha'\in\pi_h^S(X';\ Y')$ be an arbitrary element, then by use of the above Z_p-structures, we have that $\alpha\wedge\alpha'$, $\alpha'\wedge\alpha$ and $\sum^t\alpha$ are Z_p-maps. In particular, $\alpha'\wedge 1_X$ is a Z_p-map.*

Let X and Y be Z_p-spaces, then we define a homomorphism

$$\theta\colon \pi_k^S(X;\ Y)\to\pi_{k+1}^S(X;\ Y)$$

by the formula

$$\theta(\gamma)=\mu_Y(1_M\wedge\gamma)\varphi_X\qquad\text{for } \gamma\in\pi_k^S(X;\ Y).$$

This operation has the following property

Theorem 2.2.

(i) *θ is derivative*: $\theta(\gamma\gamma')=\theta(\gamma)\gamma'+(-1)^{\deg\gamma}\gamma\theta(\gamma')$.

(ii) $\theta(\gamma)=0$ *iff* γ *is a* Z_p*-map.*

(iii) $\theta(\beta\wedge\gamma)=(-1)^{\deg\beta}\beta\wedge\theta(\gamma)$, *in particular*, $\theta(\sum^t\gamma)=\sum^t\theta(\gamma)$.

(iv) *If* X *and* Y *are associative then* $\theta\theta(\gamma)=0$.

Proof. Using Theorem 1.1, (1.13) and (1.9) we have the following.

$$\begin{aligned}\theta(\gamma\gamma')&=\mu_Y(1_M\wedge\gamma\gamma')\varphi_W=\mu_Y(1_M\wedge\gamma)(1_M\wedge\gamma')\varphi_W\\&=\mu_Y(1_M\wedge\gamma)\{(i\wedge 1_X)\mu_X+\varphi_X(\pi\wedge 1_X)\}(1_M\wedge\gamma')\varphi_W\\&=(-1)^{\deg\gamma}\mu_Y(i\wedge 1_Y)(1_0\wedge\gamma)\theta(\gamma')+\theta(\gamma)(1_0\wedge\gamma')(\pi\wedge 1_W)\varphi_W\\&=\theta(\gamma)\gamma'+(-1)^{\deg\gamma}\gamma\theta(\gamma').\end{aligned}$$

$\theta(\gamma)=\mu_Y(1_M\wedge\gamma)\varphi_X=\mu_Y\varphi_Y\gamma=0$ for a Z_p-map γ, and conversely if $\theta(\gamma)=0$ then

$$\begin{aligned}\mu_Y(1_M\wedge\gamma)&=\mu_Y(1_M\wedge\gamma)\{\varphi_X(\pi\wedge 1_X)+(i\wedge 1_X)\mu_X\}\\&=\theta(\gamma)(\pi\wedge 1_X)+(-1)^{\deg\gamma}\mu_Y(i\wedge 1_Y)(1_0\wedge\gamma)\mu_X\\&=(-1)^{\deg\gamma}\gamma\mu_X\end{aligned}$$

and

$$\begin{aligned}(1_M\wedge\gamma)\varphi_X&=\{\varphi_Y(\pi_X\wedge 1_Y)+(i\wedge 1_Y)\mu_Y\}(1_M\wedge\gamma)\varphi_X\\&=\varphi_Y\gamma.\end{aligned}$$

$$\begin{aligned}\theta(\beta\wedge\gamma)&=\mu_{Y'\wedge Y}(1_M\wedge\beta\wedge\gamma)\varphi_{X'\wedge X}\\&=(1_{Y'}\wedge\mu_Y)(T_{M,Y'}\wedge 1_Y)(1_M\wedge\beta\wedge\gamma)(T_{X',M}\wedge 1_X)(1_{X'}\wedge\varphi)\\&=(1_{Y'}\wedge\mu_Y)(\beta\wedge 1_M\wedge\gamma)(1_{X'}\wedge\varphi_X)\\&=(-1)^{\deg\beta}\beta\wedge\theta(\gamma).\end{aligned}$$

By the associativity of X and Y,

$$\begin{aligned}\theta\theta(\gamma)&=\mu_Y(1_M\wedge\mu_Y)(1_M\wedge 1_M\wedge\gamma)(1_M\wedge\varphi_X)\varphi_X\\&=-\mu_Y(\mu_M\wedge 1_Y)(1_M\wedge 1_M\wedge\gamma)(\varphi_M\wedge 1_X)\varphi_X\end{aligned}$$

$$= -\mu_Y(\mu_M \varphi_M \wedge \gamma)\varphi_X = 0. \qquad \text{q.e.d.}$$

Remark that

(2.4) $1_M \wedge \gamma = (i \wedge 1_Y)\gamma\mu_X + \varphi_Y\gamma(\pi \wedge 1_X) + (i \wedge 1_Y)\theta(\gamma)(\pi \wedge 1_X)$ *holds and this characterizes* $\theta(\gamma)$.

The following lemma may be used to show the triviality of the derivation θ.

Lemma 2.3. *Let X and Y be finite CW-complexes and Z_p-spaces and let $C = C_f = \sum^n Y \cup_f C\sum^{n+k} X$ be a mapping cone of a representative $f \in [\sum^{n+k} X, \sum^n Y]$ of $\gamma \in \pi_k^S(X; Y)$, then*

$$p \cdot 1_C = i\theta(\gamma)\pi \text{ in } \mathscr{A}_0(C_f).$$

Thus C_f is a Z_p-space if $\theta(\gamma) = 0$. Further assume $\pi_{-k}^S(Y; X) = \mathscr{A}_1(X) = \mathscr{A}_1(Y) = 0$, then $\theta(\gamma) = 0$ iff C_f is a Z_p-space.

Proof. First assume that n is sufficiently large so that there exist maps $\mu_W \in [M_p \wedge W, \sum W]$ and $\varphi_W \in [\sum^2 W, M_p \wedge W]$ for $W = \sum^n Y, \sum^{n+k} X$ satisfying (1.13). For $s \in I$, $w \in \sum W$, we represent by (s, w) the corresponding points of $C\sum W = I \wedge \sum W$ and of $M_p \wedge W = \sum W \cup_{p \cdot 1} C\sum W$, in latter case $(0, w) = (p \wedge 1_W)(w)$. Then $(s, w) \to \mu_W(s, w)$ defines a null homotopy

$$\bar{\mu}_W : I \times \sum W \to \sum W$$

of $p \wedge 1_W = p \cdot 1_{\sum W}$. As we define a map

$$\varphi'_W : \sum^2 W \to M_p \wedge W$$

by putting $\varphi'_W(\psi(s), w) = (2s-1, W)$ for $s \geq \frac{1}{2}$ and $\varphi'_W(\psi(s), w) = \bar{\mu}_W(1-2s, w) \in \sum W \subset M_p \wedge W$, then it is easily seen that in homotopy classes $(\pi \wedge 1_W)\varphi'_W = 1_W$ and $\mu_W \varphi'_W = 0$. By (1.13)

$$\varphi'_W = \{(i \wedge 1_W)\mu_W + \varphi_W(\pi \wedge 1_W)\}\varphi'_W = \varphi_W.$$

Thus $i\theta(\gamma)\pi$ is represented by $\sum i\mu_{Y'}(1_M \wedge f)\varphi'_{X'}\sum\pi \in [\sum C_f, \sum C_f]$, where $X'=\sum^{n+k+1}X$, $Y'=\sum^{n+1}Y$.

Next consider a homotopy $g_t: C_{\Sigma f}\to C_{\Sigma f}=\sum^{n+1}Y\cup_{\Sigma f}C\sum^{n+k+1}X$ given by the formulas ($t, s\in I$, $x\in\sum^{n+k+1}X=X'$, $y\in\sum^{n+1}Y=Y'$)

$$g_t(y)=\bar{\mu}_{Y'}(t, y)$$

$$g_t(s, x)=\begin{cases}\bar{\mu}_{Y'}(t-2s, \sum f(x)), & 0\leq s\leq t/2,\\ (\sum f)\,\bar{\mu}_{X'}(2s-t, x), & t/2\leq s\leq t,\\ ((s-t)/(1-t), \bar{\mu}_{X'}(t, x)), & t\leq s\leq 1.\end{cases}$$

Then g_t is well defined (e.g., $g_t(t/2, x)=(p\wedge 1_Y)\sum f(x)=\sum f(p\wedge 1_{X'})$), $g_0=h(p\wedge 1_C)h^{-1}$ for the homeomorphism $h: \sum C_f\to C_{\Sigma f}$ and $g_1=i\mu_{Y'}(1_M\wedge f)(-\varphi_{X'})\pi$ for the natural maps $\sum^{n+1}Y\xrightarrow{i}C_{\Sigma f}\xrightarrow{\pi}\sum^{n+k+1}X$. By the commutativity of (1.10) it follows $p\cdot 1_C(=p\wedge 1_C)=\sum i\mu_{Y'}(1_M\wedge f)\varphi_{X'}\sum\pi=i\theta(\gamma)\pi$ in $\mathscr{A}_0(C_f)$.

When n is smaller, consider $\sum^{2N}f$ for sufficiently large N then we have the same relation.

Next assume $\pi^S_{-k}(Y; X)=\mathscr{A}_1(X)=\mathscr{A}_1(Y)=0$ and that C_f is a Z_p-space. Then $i\theta(\gamma)\pi=p\cdot 1_C=0$, and $\theta(\gamma)$ is a kernel of

$$i_*\pi^*: \pi^S_{k+1}(X; Y)\to\pi^S_{-n}(C_f; Y)\to\mathscr{A}_0(C_f).$$

By the exactness of (1.11), $\operatorname{Ker} i_*$ is an image of $\pi^S_{-n-k}(C_f; X)$. By (1.11)*, $\mathscr{A}_1(X)\to\pi^S_{-n-k}(C_f; X)\to\pi^S_{-k}(Y; X)$ and $\mathscr{A}_1(Y)\to\operatorname{Ker}\pi^*\to 0$ are exact. It follows that $\operatorname{Ker} i_*=\operatorname{Ker}\pi^*=0$ and $\theta(\gamma)\in\operatorname{Ker}(i_*\pi^*)=0$.

q.e.d.

As an analogy of θ, for each Z_p-space X we define a linear map

$$\lambda=\lambda_X: \mathscr{A}_t(M_p)\to\mathscr{A}_{t+1}(X)$$

by the formula

$$\lambda_X(\xi)=\mu_X(\xi\wedge 1_X)\varphi_X \qquad \text{for } \xi\in\mathscr{A}_t(M_p).$$

Recall $\delta=i\pi\in\mathscr{A}_{-1}(M_p)$. Obviously

(2.5) $$\delta i=\pi\delta=\delta\delta=0.$$

Theorem 2.4.

(i) $\lambda_X(\xi\xi')=\lambda_X(\xi)\lambda_X(\delta\xi')+\lambda_X(\xi\delta)\lambda_X(\xi')$.

(ii) $\lambda_{X'\wedge X}(\xi)=1_{X'}\wedge\lambda_X(\xi)$, *in particular* $\lambda_{\Sigma^t X}(\xi)=\sum^t\lambda_X(\xi)$.

(iii) *For* $\gamma\in\pi_k^S(X;\ Y)$ *and* $\xi\in\mathscr{A}_t(M_p)$,

$$\lambda_Y(\xi)\gamma+\lambda_Y(\xi\delta)\theta(\gamma)=(-1)^{(t+1)k}\gamma\lambda_X(\xi)+(-1)^{tk}\theta(\gamma)\lambda_X(\delta\xi).$$

(iv) $\lambda_X(\delta\xi\delta)=(\pi\xi i)\wedge 1_X$, $\lambda_X(\delta)=1_X$ *and* $\lambda_X(1_M)=0$.

(v) $\theta(\lambda_X(\xi))=\lambda_X(\theta(\xi))=0$ *if* X *is associative.*

Proof. By use of (1.13), Theorem 1.1 and (1.9) we have

$$\begin{aligned}\lambda(\xi\xi')&=\mu(\xi\wedge 1)\{(i\wedge 1)\mu+\varphi(\pi\wedge 1)\}(\xi'\wedge 1)\varphi\\&=\mu(\xi i\wedge 1)(\pi\wedge 1)\varphi\lambda(\xi')+\lambda(\xi)\mu(i\wedge 1)(\pi\xi'\wedge 1)\varphi\\&=\lambda(\xi\delta)\lambda(\xi')+\lambda(\xi)\lambda(\delta\xi'),\end{aligned}$$

$$\begin{aligned}\lambda_{X'\wedge X}(\xi)&=(1_{X'}\wedge\mu)(T_{M,Y'}\wedge 1)(\xi\wedge 1_{X'}\wedge 1)(T_{X',M}\wedge 1)(1_{X'}\wedge\varphi)\\&=(1_{X'}\wedge\mu)(1_{X'}\wedge\xi\wedge 1)(1_{X'}\wedge\varphi)=1_{X'}\wedge\lambda(\xi),\end{aligned}$$

$$\begin{aligned}&\lambda_Y(\xi)\gamma+\lambda_Y(\xi\delta)\theta(\gamma)\\&=\mu_Y(\xi\wedge 1_Y)\varphi_Y\gamma+\mu_Y(\xi i\pi\wedge 1_Y)\varphi_Y\mu_Y(1_M\wedge\gamma)\varphi_X\\&=\mu_Y(\xi\wedge 1_Y)\varphi_Y(1_0\wedge\gamma)(\pi\wedge 1_X)\varphi_X\\&\qquad+\mu_Y(\xi\wedge 1_Y)(i\wedge 1_Y)\mu_Y(1_M\wedge\gamma)\varphi_X\\&=\mu_Y(\xi\wedge 1_Y)\{\varphi_Y(\pi\wedge 1_Y)+(i\wedge 1_Y)\mu_Y\}(1_M\wedge\gamma)\varphi_X\\&=\mu_Y(\xi\wedge 1_Y)(1_M\wedge\gamma)\varphi_X=(-1)^{tk}\mu_Y(1_M\wedge\gamma)(\xi\wedge 1_X)\varphi_X\\&=(-1)^{tk}\mu_Y(1_M\wedge\gamma)\{(i\wedge 1_X)\mu_X+\varphi_X(\pi\wedge 1_X)\}(\xi\wedge 1_X)\varphi_X\\&=(-1)^{(t+1)k}\mu_Y(i\wedge 1_Y)(1_0\wedge\gamma)\lambda_X(\xi)\\&\qquad+(-1)^{tk}\theta(\gamma)\mu_X(i\wedge 1_X)(\pi\xi\wedge 1_X)\varphi_X\\&=(-1)^{(t+1)k}\gamma\lambda_X(\xi)+(-1)^{tk}\theta(\gamma)\lambda_X(\delta\xi),\end{aligned}$$

$$\begin{aligned}\lambda_X(\delta\xi\delta)&=\lambda_X(i\pi\xi i\pi)=\mu_X(i\wedge 1_X)(\pi\xi i\wedge 1_X)(\pi\wedge 1_X)\varphi_X\\&=\pi\xi i\wedge 1_X,\end{aligned}$$

$$\lambda_X(\delta)=\mu_X(i\wedge 1_X)(\pi\wedge 1_X)\varphi_X=1_X,$$

$$\lambda_X(1_M)=\mu_X(1_M\wedge 1_X)\varphi_X=\mu_X\varphi_X=0,$$

and by the associativity and by use of next Theorem 2.6,

$$\begin{aligned}\theta(\lambda_X(\xi))&=\mu_X(1_M\wedge\mu_X)(1_M\wedge\xi\wedge 1_X)(1_M\wedge\varphi_X)\varphi_X\\&=-\mu_X(\mu_M\wedge 1_X)(1_M\wedge\xi\wedge 1_X)(\varphi_M\wedge 1_X)\varphi_X\\&=-\lambda_X(\theta(\xi))=\lambda_X(\lambda_M(\xi))\\&=\mu_X(\mu_M\wedge 1_X)(\xi\wedge 1_M\wedge 1_X)(\varphi_M\wedge 1_X)\varphi_X\\&=\mu_X(1_M\wedge\mu_X)(\xi\wedge 1_M\wedge 1_X)(1_M\wedge\varphi_X)\varphi_X\\&=\mu_X(\xi\wedge\mu_X\varphi_X)\varphi_X=0.\end{aligned}$$

q.e.d.

Corollary. 2.5. *If* $\theta(\gamma)=0$ *or if* $\xi\delta=\delta\xi=0$ (e.g. $\xi=\delta\eta\delta$) *then the following commutativity holds*:

$$\lambda_Y(\xi)\gamma=(-1)^{(\deg\xi+1)\deg\gamma}\gamma\lambda_X(\xi).$$

In the remaining part of this section we consider the case $X=M_p$ (p: odd prime). M_p is a Z_p-space and both of θ and λ_M are defined on $\mathscr{A}_t(M_p)$, and our θ coincides with D of [3].

Theorem 2.6. *For* $\xi\in\mathscr{A}_t(M_p)$ *we have*

$$\lambda_M(\xi)=-\theta(\xi).$$

Proof. By Lemma 1.3, (iii),

$$\begin{aligned}\lambda_M(\xi)&=\mu_M(\xi\wedge 1_M)\varphi_M=-\mu_M T(\xi\wedge 1_M)T\varphi_M\\&=-\mu_M(1_M\wedge\xi)\varphi_M=-\theta(\xi).\end{aligned}$$

Corollary 2.7. *Let* $\xi\in\mathscr{A}_t(M_p)$ *and* $\eta\in\mathscr{A}_s(M_p)$ *then the following equalities hold.*

(2.6) $\quad\theta(\delta)=-1,$

(2.7) $\lambda_M(\delta\xi\delta)=(\pi\xi i)\wedge 1_M=\xi\delta-(-1)^t\delta\xi+\delta\theta(\xi)\delta$,

(2.8) $\lambda_M(\delta\xi)=\xi+\delta\theta(\xi)$, $\lambda_M(\xi\delta)=(-1)^t\xi-\theta(\xi)\delta$,

(2.9) $$\theta(\xi)\eta+(-1)^{t+1}\xi\theta(\eta)+\theta(\xi)\delta\theta(\eta)$$
$$=(-1)^{ts+1}\{\theta(\eta)\xi+(-1)^{s+1}\eta\theta(\xi)+\theta(\eta)\delta\theta(\xi)\},$$

(2.10) $$\xi\eta-(-1)^{ts}\eta\xi$$
$$=(-1)^{ts}\eta\delta\theta(\xi)-\delta\theta(\xi)\eta+(-1)^t\delta\xi\theta(\eta)-\xi\delta\theta(\eta)-\delta\theta(\xi)\delta\theta(\eta)$$
$$=-\theta(\xi)\delta\eta+(-1)^{ts+t+1}\eta\theta(\xi)\delta+(-1)^{ts+t+s}\theta(\eta)\xi\delta$$
$$+(-1)^{ts+s+1}\theta(\eta)\delta\xi+(-1)^{ts}\theta(\eta)\delta\theta(\xi)\delta.$$

(2.11) $$\xi\delta\eta-(-1)^t\delta\xi\eta-(-1)^{st+s}\eta\xi\delta+(-1)^{st+s+t}\eta\delta\xi$$
$$=(-1)^{st+s}\eta\delta\theta(\xi)\delta-\delta\theta(\xi)\delta\eta.$$

Proof. $\theta(\delta)=-\lambda_M(\delta)=-1$.

$$\lambda_M(\delta\xi\delta)=-\theta(\delta\xi\delta)=-\theta(\delta)\xi\delta+\delta\theta(\xi)\delta+(-1)^t\delta\xi\theta(\delta)$$
$$=\xi\delta-(-1)^t\delta\xi+\delta\theta(\xi)\delta,$$
$$\lambda_M(\delta\xi)=-\theta(\delta\xi)=-\theta(\delta)\xi+\delta\theta(\xi)=\xi+\delta\theta(\xi),$$
$$\lambda_M(\xi\delta)=-\theta(\xi\delta)=-\theta(\xi)\delta+(-1)^t\xi$$

(2.9) follows from (iii) of Theorem 2.4 of the case $\gamma=\eta$. Similarly the formulas of (2.10) and (2.11) are obtained by replacing ξ by $\delta\xi$, $\xi\delta$ and $\delta\xi\delta$ respectively. More discussions can be seen in [3].

3. Z_p-Spectrum $V(k)$.

The spectra handled in this paper will be suspension spectra $X=\{X_n, j_n: \sum X_n\to X_{n+1}\}$ satisfying

(3.1) *for sufficiently large* n, $X_{n+1}=\sum X_n$, $j_n=1$ *and* X_n *are* $(n-1)$-*connected finite CW-complexes.*

Then for sufficiently large n

$$\sum:\pi_k^S(X_n;\ Y_n)\to\pi_k^S(X_{n+1};\ Y_{n+1})$$

is an isomorphism, and we put

$$\pi_k(X;\ Y)=\lim \pi_k^S(X_n;\ Y_n).$$

Notations in the section 1 such as π_*, $\mathscr{A}_k$, $\mathscr{A}_*$ are used also for spectra. The composition product in π_*^S induces a product

$$\pi_h(Y;\ Z)\otimes\pi_k(X;\ Y)\to\pi_{h+k}(X;\ Z)$$

by virtue of (1.8), and the new product is bilinear, associative and has the units 1_X the limit of 1_{X_n}.

A Z_p-spectrum X is a spectrum satisfying (3.1) and having Z_p-spaces $(X_n, \mu_{X_n}, \varphi_{X_n})$ in which $\mu_{X_{n+1}}=\mu_{\Sigma X_n}$, $\varphi_{X_{n+1}}=\varphi_{\Sigma X_n}$ as in (2.2). Then the operations

$$\theta: \pi_k(X;\ Y)\to\pi_{k+1}(X;\ Y)$$

and

$$\lambda_X: \mathscr{A}_t(M_p)\to\mathscr{A}_{t+1}(X)$$

are defined as the limits of θ in π_k^S and λ_{X_n} by virtue of Theorem 2.2, (iii) and Theorem 2.4, (ii). Apparently

(3.2) *The formulas in the previous section valid for spectra if* $\wedge$ *and* Σ^t *are not contained.*

$S=\{S^n, j_n=1_{n+1}\}$ is the sphere spectrum. We denote

$$\pi_k(X)=\pi_k(S;\ X) \quad \text{and} \quad G_k=\pi_k(S).$$

$M=\{\Sigma^{n-1}M_p, j_n=1(n\geq 1)\}$ is the Moore spectrum which is a Z_p-spectrum (p: odd prime). We may identify

$$\mathscr{A}_t(M)=\mathscr{A}_t(M_p).$$

Now consider spectra $V(k)=\{V(k)_n\}$ given in [14] having $H^*(V(k);Z_p)\cong E(Q_0, Q_1, \cdots, Q_k)$. $V(0)=M$ and $V(k)$ is, if it exists, given by a mapping cone of an element in $\mathscr{A}_{2p^k-2}(V(k-1))$, $k\geq 1$. The existence of $V(k)$ is assured [14; Theorem 1.1] for $k=1$, $p\geq 3$, for $k=2$, $p\geq 5$ and for $k=3$, $p\geq 7$.

Until the end of section 5, we assume $p\geq 5$, so $V(1)$, $V(2)$ exist.

The case $p=3$ will be discussed in section 6. For sufficiently large n we have the following cofiberings:

$$S^n \xrightarrow{p} S^n \xrightarrow{i} V(0)_n \xrightarrow{\pi} S^{n+1} \to \cdots,$$

$$\sum^q V(0)_n \xrightarrow{\alpha} V(0)_n \xrightarrow{i_1} V(1)_n \xrightarrow{\pi_1} \sum^{q+1} V(0)_n \to \cdots,$$

$$\sum^{pq+q} V(1)_n \xrightarrow{\beta} V(1)_n \xrightarrow{i_2} V(2)_n \xrightarrow{\pi_2} \sum^{pq+q+1} V(1)_n \to \cdots,$$

where $q=2(p-1)$. As the limits of these maps we have

$$i \in \pi_0(M), \quad \pi \in \pi_{-1}(M; S), \quad \alpha \in \mathscr{A}_q(M),$$

$$i_1 \in \pi_0(M, V(1)), \quad \pi_1 \in \pi_{-q-1}(V(1), M),$$

$$\beta \in \mathscr{A}_{pq+q}(V(1)), \quad i_2 \in \pi_0(V(1), V(2))$$

and

$$\pi_2 \in \pi_{-pq-q-1}(V(2), V(1)).$$

The following relations are obvious.

(3.3) $$\pi i = 0,$$

$$i_1 \alpha = \pi_1 i_1 = \alpha \pi_1 = 0,$$

$$i_2 \beta = \pi_2 i_2 = \beta \pi_2 = 0.$$

Theorem 4.4 of [14] shows

(3.4). *$V(1)$ and $V(2)$ are Z_p-spectra.*

The above cofiberings induce exact sequence of the types (1.11) and (1.11)* and they are translated to exact sequences in spectra:

(3.5) $$0 \to G_k \otimes Z_p \xrightarrow{i_*} \pi_k(M) \xrightarrow{\pi_*} \mathrm{Tor}(G_{k-1}, Z_p) \to 0$$

(3.5)* $$0 \to \pi_{k+1}(X) \otimes Z_p \xrightarrow{\pi^*} \pi_k(M; X) \xrightarrow{i^*} \mathrm{Tor}(\pi_k(X), Z_p) \to 0$$

(3.6) $$\xrightarrow{\alpha_*} \pi_k(W; M) \xrightarrow{i_{1*}} \pi_k(W; V(1)) \xrightarrow{\pi_{1*}} \pi_{k-q-1}(W; M) \xrightarrow{\alpha_*} \cdots$$

(3.6)* $$\xrightarrow{\alpha^*} \pi_{k+q+1}(M; X) \xrightarrow{\pi_1^*} \pi_k(V(1); X) \xrightarrow{i_1^*} \pi_k(M; X) \xrightarrow{\alpha^*} \cdots$$

For example, $\pi_k(M)=0$ for $k<q-1$ and $k \neq 0$, and we have $\mathscr{A}_1(M) = \mathscr{A}_1(V(1)) = \pi_1(M; V(1)) = \pi_{-q}(V(1); M) = 0$. Thus it follows from Lemma 2.3

(3.7) $$\theta(\alpha)=\theta(\beta)=\theta(i_1)=\theta(\pi_1)=0.$$

Put

$$\beta_{(s)}=\pi_1\beta^s i_1 \in \mathscr{A}_{(ps+s-1)q-1}(M)$$

then since θ is derivative

(3.7)′ $$\theta(\beta_{(s)})=0.$$

By (2.7)

(3.8) $$\lambda_M(\delta\alpha\delta)=\alpha\delta-\delta\alpha \quad \textit{and} \quad \lambda_M(\delta\beta_{(s)}\delta)=\beta_{(s)}\delta+\delta\beta_{(s)}$$

and by Corollary 2.5,

(3.8)′ *for any* $\xi\in\mathscr{A}_t(M)$, $(\alpha\delta-\delta\alpha)\xi=(-1)^t\xi(\alpha\delta-\delta\alpha)$ *and* $(\beta_{(s)}\delta+\delta\beta_{(s)})\xi=\xi(\beta_{(s)}\delta+\delta\beta_{(s)})$.

In particular

(3.8)″ $$\alpha^2\delta+\delta\alpha^2=2\alpha\delta\alpha.$$

From now we shall compute $\mathscr{A}_*(V(1))$ up to some range. We put

$$\alpha_1=\pi\alpha i\in G_{q-1},$$

$$\beta_s=\pi\beta_{(s)}i\in G_{(sp+s-1)q-2},$$

$$\alpha'=\lambda_{V(1)}(\delta\alpha\delta)\,(=\alpha_1\wedge 1_{V(1)})\in\mathscr{A}_{q-1}(V(1))$$

and

$$\beta'=\lambda_{V(1)}(\delta\beta_{(1)}\delta)\,(=\beta_1\wedge 1_{V(1)})\in\mathscr{A}_{pq-2}(V(1)).$$

Then the following commutativity follows from Corollary 2.5, (3.8) and (1.7), (1.9).

(3.9) $$\alpha'\xi=(-1)^t\xi\alpha',\quad \beta'\xi=\xi\beta' \qquad \textit{for any } \xi\in\mathscr{A}_t(V(1)),$$

$$\alpha'\xi'=(-1)^t\xi'(\alpha\delta-\delta\alpha),\quad \beta'\xi'=\xi'(\beta_{(1)}\delta+\delta\beta_{(1)})$$

$$\textit{for any } \xi'\in\pi_t(M;\,V(1))$$

and

$$\alpha'\xi''=(-1)^t\xi''\alpha_1,\quad \beta'\xi''=\xi''\beta_1\qquad \textit{for any } \xi''\in\pi_t(V(1)).$$

Lemma 3.1. *There exists an element α'' of $\mathscr{A}_{q-2}(V(1))$ such that $\alpha''i_1=\alpha'i_1\delta$.*

Proof. First we show

$$(3.10)\qquad \delta\alpha\delta\alpha=\alpha\delta\alpha\delta.$$

For, $\delta\alpha\delta\alpha=\frac{1}{2}\delta(\alpha^2\delta+\delta\alpha^2)=\frac{1}{2}(\alpha^2\delta+\delta\alpha^2)\delta=\alpha\delta\alpha\delta$ by (3.8)$''$. Then, by (3.9) and (3.3)

$$\alpha^*(\alpha'i_1\delta)=\alpha'i_1\delta\alpha=i_1(\alpha\delta-\delta\alpha)\delta\alpha=i_1\delta\alpha\delta\alpha=i_1\alpha\delta\alpha\delta=0.$$

By the exactness of (3.6)*, the existence of α'' is proved.

We use the following notations:

$$i_0=i_1i\in\pi_0(V(1)),\quad \pi_0=\pi\pi_1\in\pi_{-q-2}(V(1);\,S),$$
$$\delta_1=i_1\pi_1\in\mathscr{A}_{q-1}(V(1)),\quad \delta_0=i_0\pi_0=i_1\delta\pi_1\in\mathscr{A}_{-q-2}(V(1)).$$

Theorem 3.2. *For* $\deg<p^2q-3$

$$\pi_*(V(1))=P(\beta,\,\beta')\otimes\{i_0,\,\alpha'i_0,\,\delta_1\beta i_0,\,\alpha''\beta i_0,\,\delta_0\beta^2i_0,\,\delta_0\beta^2\alpha'i_0\}.$$

Proof. By Theorem 5.2 of [14], for $\deg<p^2q-3$

$$\pi_*(V(1))\cong P(\beta)\otimes A\otimes P(\beta_1),$$

where A is spanned by non-zero elements $\iota\in\pi_0$, $\alpha_1\in\pi_{q-1}$, $\tilde{\beta}_1\in\pi_{pq-1}$, $g_0\in\pi_{(p+2)q-2}$, $\beta_2\in\pi_{(2p+1)q-2}$ and $\beta_2\alpha_1$. Here β is just same as our β, $\tilde{\beta}_1=\beta_{(1)}i$ and β_1 is same as ours up to non-zero coefficient by (5.4) of [14]. ι corresponds to 1_0 or i_0. α_1 is detected by $h_0(\mathscr{P}^1)$, and it coincides with ours up to non-zero coefficient since the mapping cone of $\alpha_1=\pi\alpha i$ is $V\left(\frac{1}{2}\right)/S$ in which $\mathscr{P}^1\neq 0$. We shall show

(*) *$\alpha''\beta i_0$ and $\delta_0\beta^2i_0$ are non-trivial.*

Then we may take $A=\{i_0,\,i_0\alpha_1,\,i_1\beta_{(1)}i,\,\alpha''\beta i_0,\,\delta_0\beta^2i_0,\,\delta_0\beta^2i_0\alpha_1\}$. By

(3.9), $\xi\beta_1=\beta'\xi$ for any ξ, $i_0\alpha_1=\alpha' i_0$, $\delta_0\beta^2 i_0\alpha_1=\delta_0\beta^2\alpha' i_0$ and $i_1\beta_{(1)}i=i_1\pi_1\beta i_1 i=\delta_1\beta i_0$. Thus it is sufficient to prove (*).

Assume that $\alpha''\beta i_0=0$. Then a representative α'': $V(1)_{n+q-2}\to V(1)_n$ of α'' can be extended over a map A: $C_{\beta i_0}=V(1)_{n+q-2}\cup e^{n+(p+2)q}\to V(1)_n$, where $C_{\beta i_0}$ is the $n+(p+2)q$ skeleton of $V(2)_{n+q-2}$. Consider a mapping cone $C_A=V(1)_n\cup e^{n+q-1}\cup e^{n+q}\cup e^{n+2q}\cup e^{n+2q+1}\cup e^{n+(p+2)q}$ of A, then $\mathscr{P}^p\mathscr{P}^1 e^{n+q}=\mathscr{P}^p\mathscr{P}^1\Delta e^{n+q-1}=Q_2 e^{n+q-1}=e^{n+(p+2)q}$ by the cohomology structure of $V(2)$. Next $V(1)_n\cup e^{n+q-1}\cup e^{n+q}$ is the mapping cone of $\alpha'' i_1$ and $\alpha'' i_1=\alpha' i_1\delta=i_1\delta(\alpha\delta-\delta\alpha)=i_1\delta\alpha\delta=i_0\alpha_1\pi$ by Lemma 3.1 and (3.9). Since α_1 is detected by $\mathscr{P}^1$, $\mathscr{P}^1 e^n=e^{n+q}$. Thus $\mathscr{P}^p\mathscr{P}^1\mathscr{P}^1 e^n=e^{n+(p+2)q}\neq 0$ in C_A. But this contradicts to Adem relation $\mathscr{P}^p\mathscr{P}^1\mathscr{P}^1=\mathscr{P}^1(2\mathscr{P}^p\mathscr{P}^1-\mathscr{P}^{p+1})$ since there is no cell of dimension $n+(p+1)q$. Thus $\alpha''\beta i_0\neq 0$.

$\delta_0\beta^2 i_0\neq 0$ is proved similarly by assuming $(\delta_0\beta)(\beta i_0)=0$ and by constructing a complex

$$V(1)_n\cup e^{n+pq-1}\cup e^{n+pq}\cup e^{n+(p+1)q}\cup e^{n+(p+1)q+1}\cup e^{n+(2p+1)q}$$

in which $\mathscr{P}^p\mathscr{P}^1\mathscr{P}^p e^n\neq 0$ contradicting to Adem relation $\mathscr{P}^p\mathscr{P}^1\mathscr{P}^p=\mathscr{P}^{2p}\mathscr{P}^1+\mathscr{P}^1\mathscr{P}^{2p}$. q.e.d.

Since $\pi_*(V(1))$ is a Z_p-module (Lemma 1.2), (3.5)* implies a splitting

$$0\to\pi_{k+1}(V(1))\xrightarrow{\pi^*}\pi_k(M;\ V(1))\xrightarrow{i^*}\pi_k(V(1))\to 0.$$

Obviously $i^*(\eta i_1)=\eta i_0$ and $\pi^* i^*(\xi)=\xi\delta$, so we have

Corollary 3.3. *For degree* $<p^2q-4$

$$\pi_*(M;\ V(1))=P(\beta,\ \beta')\otimes\{i_1,\ \alpha' i_1,\ \delta_1\beta i_1,\ \alpha''\beta i_1,\ \delta_0\beta^2 i_1,\ \delta_0\beta^2\alpha' i_1\}\otimes E(\delta).$$

Lemma 3.4. $\beta' i_0=\delta_0\beta i_0$ *and* $\beta' i_1=\delta_0\beta i_1+\delta_1\beta i_1\delta$.

Proof. By use of (3.9)

$$\beta' i_0 = i_0 \beta_1 = i_0 \pi_0 \beta i_0 = \delta_0 \beta i_0$$

and

$$\beta' i_1 = i_1(\beta_{(1)}\delta + \delta\beta_{(1)}) = i_1 \pi_1 \beta i_1 \delta + i_1 \delta \pi_1 \beta i_1$$
$$= \delta_1 \beta i_1 \delta + \delta_0 \beta i_1.$$

Lemma 3.5. *There exists an element β'' of $\mathscr{A}_{(p+2)q-3}(V(1))$ such that $\beta'' i_1 = \alpha'' \beta i_1 \delta$.*

For, $i_1^*: \mathscr{A}_{(p+2)q-3}(V(1)) \to \pi_{(p+2)q-3}(M; V(1))$ is an epimorphism since $\pi_{(p+3)q-3}(M; V(1)) = 0$ by Corollary 3.3.

Finally we compute $\mathscr{A}_*(V(1))$.

Theorem 3.6. $(p \geq 5)$. *For degree $<(p^2-1)q-5$*

$$\mathscr{A}_*(V(1)) = P(\beta, \beta') \otimes \{1, \alpha', \delta_1\beta, \alpha''\beta, \delta_0\beta^2, \delta_0\beta^2\alpha'\} \otimes E(\delta_0)$$
$$+ P(\beta, \beta') \otimes \{\delta_1, \alpha'', \delta_1\beta\delta_1, \delta_0\beta, \alpha''\beta\delta_1, \beta'', \delta_0\beta^2\delta_1, \delta_0\beta^2\alpha''\}.$$

Proof. Consider the exact sequence (3.6)* for $X = V(1)$, then we may forget $P(\beta, \beta')$ and it is sufficient to prove the following relations.

$$\left.\begin{aligned} a^*(\eta i_1 \delta) &= \eta i_1 \delta \alpha = -\eta \alpha' i_1 \\ i_1^*(\eta \alpha'') &= \eta \alpha'' i_1 = \eta \alpha' i_1 \delta \end{aligned}\right\} \quad \text{for } \eta = 1,\ \delta_0\beta^2,$$

$$i_1^*(\beta'') = \alpha''\beta i_1 \delta,$$

$$i_1^*(\beta' - \delta_0\beta) = \delta_1 \beta i_1 \delta$$

$$\left.\begin{aligned} i_1^*(\xi) &= \xi i_1 \\ \pi_1^*(\xi i_1 \delta) &= \xi \delta_0 \end{aligned}\right\} \quad \text{for } \xi = 1, \alpha', \delta_1\beta, \alpha''\beta, \delta_0\beta^2, \delta_0\beta^2\alpha',$$

$$\pi_1^*(\eta i_1) = \eta \delta_1 \qquad \text{for } \eta = 1, \delta_1\beta, \alpha''\beta, \delta_0\beta^2.$$

The second, the third and the fourth relations follow from Lemmas 3.1, 3.5 and 3.4. The first formula follows from the following (3.11) and the remaining formulas are obvious.

(3.11) $$\alpha' i_1 = -i_1 \delta\alpha$$

For, $\alpha' i_1 = i_1(\alpha\delta - \delta\alpha) = -i_1\delta\alpha$ by (3.9) and (3.3).

4. Multiplicative Structure of $\mathscr{A}_*(V(1))$.

Theorem 3.6 shows that $\mathscr{A}_*(V(1))$ is multiplicatively generated by δ_0, δ_1, α'', α', β', β, β'' for degree $<(p^2-1)q-5$. The purpose of this section is to give a complete generating system of relations for degree $<(p^2-1)q-5$.

First we recall (3.9).

(4.1). *$\beta'\xi = \xi\beta'$ and $\alpha'\xi = (-1)^{\deg \xi}\xi\alpha'$, in particular $\alpha'\alpha' = 0$.*

Next the following trivialities hold because of the triviality of $\mathscr{A}_k(V(1))$ for the corresponding degrees.

(4.2)
$$\delta_i\delta_j = 0 \qquad (i, j = 0, 1),$$
$$\alpha''\delta_0 = \delta_0\alpha'' = \alpha'\delta_1 = 0,$$
$$\alpha''\alpha'' = \alpha'\alpha'' = 0,$$
$$\beta''\delta_0 = \delta_0\beta'' = \beta''\alpha'' = \alpha''\beta'' = \alpha'\beta'' = 0,$$
$$\alpha''\beta\alpha'' = 0,$$

and

$$\beta''\beta'' = 0 \qquad \text{if } p > 5.$$

By Lemmas 3.1, 3.4 and 3.5 we have

(4.3)
$$\alpha''\delta_1 = \alpha'\delta_0,$$
$$\delta_0\beta\delta_0 = \beta'\delta_0,$$
$$\delta_0\beta\delta_1 = \beta'\delta_1 - \delta_1\beta\delta_0$$

and

$$\beta''\delta_1 = \alpha''\beta\delta_0.$$

We shall determine the derivation θ for the generators.

Theorem 4.1. *$\theta(\delta_0)=-\delta_1$, $\theta(\delta_1)=0$, $\theta(\alpha'')=\alpha'$, $\theta(\alpha')=\theta(\beta')=\theta(\beta)=0$ and $\theta(\beta'')=\beta\alpha''-\alpha''\beta$.*

Proof. By (3.7) and (2.6), $\theta(\delta_0)=\theta(i_1\delta\pi_1)=-i_1\pi_1=-\delta_1$, $\theta(\delta_1)=\theta(i_1\pi_1)=0$. $\theta(\alpha'')\in\mathscr{A}_{q-1}=\{a'\}$ by Theorem 3.6. Put $\theta(\alpha'')=x\alpha'$ then $x\alpha'\delta_0=\theta(\alpha'')\delta_0=\theta(\alpha''\delta_0)-\alpha''\theta(\delta_0)=\alpha''\delta_1=\alpha'\delta_0$ by (4.2) and (4.3). Thus $x=1$ and $\theta(\alpha'')=\alpha'$. By Theorem 2.4, (v) and (3.7) $\theta(\alpha')=\theta(\beta')=\theta(\beta)=0$. $\theta(\beta'')\in\mathscr{A}_{(p+2)q-2}=\{\beta\alpha'', \alpha''\beta\}$ by Theorem 3.6. Put $\theta(\beta'')=x\beta\alpha''+y\alpha''\beta$. By (4.3), $\theta(\beta''\delta_1)=\theta(\alpha''\beta\delta_0)=\theta(\alpha'')\beta\delta_0+\alpha''\beta\theta(\delta_0)=\alpha'\beta\delta_0-\alpha''\beta\delta_1=\beta\alpha'\delta_0-\alpha''\beta\delta_1$, and $\theta(\beta''\delta_1)=\theta(\beta'')\delta_1=x\beta\alpha''\delta_1+y\alpha''\beta\delta_1=x\beta\alpha'\delta_0+y\alpha''\beta\delta_1$. Thus $x=1$, $y=-1$ and $\theta(\beta'')=\beta\alpha''-\alpha''\beta$. q.e.d.

(4.4) $$\delta_1\alpha''=\alpha'\delta_0.$$

For $0=\theta(\delta_0\alpha'')=-\delta_1\alpha''+\delta_0\alpha'=-\delta_1\alpha''+\alpha'\delta_0$ by (4.3) and (4.2).

In order to prove more relations we prepare the following theorem. For the simplicity we write $\lambda_V=\lambda_{V(1)}$.

Theorem 4.2. *Let $\gamma\in\mathscr{A}_t(V(1))$ and put $\gamma_{(1)}=\pi_1\gamma i_1\in\mathscr{A}_{t-q-1}(M)$, then*

$$\lambda_V(\gamma_{(1)}\delta)=\gamma\delta_1-(-1)^t\delta_1\gamma+\varepsilon$$

and

$$\lambda_V(\delta\gamma_{(1)}\delta)=(-1)^t\gamma\delta_0+(-1)^t\delta_0\gamma+\varepsilon'$$

for ε, ε' satisfying

$$\varepsilon\delta_1=(-1)^{t+1}\delta_1\varepsilon=\delta_1\theta(\gamma)\delta_0$$

and

$$\varepsilon'\delta_1=\delta_1\gamma\delta_0+\delta_0\theta(\gamma)\delta_0,\quad \delta_1\varepsilon'=\delta_0\gamma\delta_1+(-1)^t\delta_0\theta(\gamma)\delta_0.$$

In particular, $$\lambda_V(\beta_{(1)}\delta)=\beta\delta_1-\delta_1\beta.$$

Proof. Put $\varepsilon=\lambda_V(\gamma_{(1)}\delta)-\gamma\delta_1+(-1)^t\delta_1\gamma$, then by Corollary 2.5, (3.7) and (2.8)

$$\begin{aligned}\varepsilon\delta_1&=\lambda_V(\gamma_{(1)}\delta)i_1\pi_1+(-1)^t\delta_1\gamma\delta_1\\&=i_1\lambda_M(\gamma_{(1)}\delta)\pi_1+(-1)^t\delta_1\gamma\delta_1\\&=(-1)^{t-q-1}i_1\gamma_{(1)}\pi_1-i_1\theta(\pi_1\gamma i_1)\delta\pi_1+(-1)^t\delta_1\gamma\delta_1\\&=i_1\pi_1\theta(\gamma)i_1\delta\pi_1=\delta_1\theta(\gamma)\delta_0\end{aligned}$$

and

$$\begin{aligned}\delta_1\varepsilon&=i_1\pi_1\lambda_V(\gamma_{(1)}\delta)-\delta_1\gamma\delta_1\\&=(-1)^{t+1}i_1\lambda_M(\gamma_{(1)}\delta)\pi_1-\delta_1\gamma\delta_1=(-1)^{t+1}\delta_1\theta(\gamma)\delta_0.\end{aligned}$$

Similarly by putting $\varepsilon'=\lambda_V(\delta\gamma_{(1)}\delta)-(-1)^t\gamma\delta_0-(-1)^t\delta_0\gamma$

$$\begin{aligned}\varepsilon'\delta_1&=i_1\lambda_M(\delta\gamma_{(1)}\delta)\pi_1-(-1)^t\delta_0\gamma\delta_1\\&=i_1(\gamma_{(1)}\delta-(-1)^{t+1}\delta\gamma_{(1)}+\delta\theta(\gamma_{(1)})\delta)\pi_1-(-1)^t\delta_0\gamma\delta_1\\&=\delta_1\gamma\delta_0+\delta_0\theta(\gamma)\delta_0\end{aligned}$$

and

$$\begin{aligned}\delta_1\varepsilon'&=(-1)^t i_1\lambda_M(\delta\gamma_{(1)}\delta)\pi_1-(-1)^t\delta_1\gamma\delta_0\\&=(-1)^t i_1(\gamma_{(1)}\delta-(-1)^{t+1}\delta\gamma_{(1)}+\delta\theta(\gamma_{(1)})\delta)\pi_1-(-1)^t\delta_1\gamma\delta_0\\&=\delta_0\gamma\delta_1+(-1)^t\delta_0\theta(\gamma)\delta_0.\end{aligned}$$

For the case $\gamma=\beta$, $\varepsilon\in\mathscr{A}_{pq-1}(V(1))=\{\delta_1\beta,\ \beta\delta_1\}$. Put $\varepsilon=x\delta_1\beta+y\beta\delta_1$, then $x\delta_1\beta\delta_1=y\delta_1\beta\delta_1=0$ since $\theta(\beta)=0$. Thus $x=y=0$ and $\lambda_V(\beta_{(1)}\delta)=\beta\delta_1-\delta_1\beta$. q.e.d.

Theorem 4.3. *The following formulas hold.*

(4.5) (i) $\delta_1\beta^2=2\beta\delta_1\beta-\beta^2\delta_1$,

(ii) $\alpha''\beta^2=2\beta\alpha''\beta-\beta^2\alpha''$,

(iii) $\delta_0\beta\alpha''=\beta'\alpha''-\alpha''\beta\delta_0$,

(iv) $\delta_1\beta\alpha''=\alpha''\beta\delta_1$,

(v) $\alpha'\delta_0\beta=\beta'\alpha'-\beta\alpha'\delta_0+2\alpha''\beta\delta_1$,

(vi) $\beta''\beta=\beta\beta''$,

(vii) $\delta_1\beta''=\beta'\alpha''-\alpha''\beta\delta_0$.

Proof. By Corollary 2.5, (3.7) and Theorem 4.2, $(\beta\delta_1-\delta_1\beta)\beta=\beta(\beta\delta_1-\delta_1\beta)$, and (i) follows.

Consider $\delta_0\beta\alpha''\in\mathscr{A}_{(p+1)q-4}=\{\alpha''\beta\delta_0, \beta'\alpha''\}$ and put $\delta_0\beta\alpha''=x\cdot\alpha''\beta\delta_0+y\beta'\alpha''$. Apply δ_1 from the left and from the right, then we have by (4.2), (4.4), (4.3), (4.1)

$$0=\delta_1\delta_0\beta\alpha''=x\cdot\alpha'\delta_0\beta\delta_0+y\beta'\delta_1\alpha''=(x+y)\beta'\alpha'\delta_0$$

and

$$\beta'\alpha'\delta_0=\delta_0\beta\alpha'\delta_0=\delta_0\beta\alpha''\delta_1=x\cdot 0+y\beta'\alpha''\delta_1=y\beta'\alpha'\delta_0.$$

Thus $y=1$, $x=-y=-1$ and (iii) follows.

Consider $\alpha''\beta^2\in\mathscr{A}_{(2p+3)q-2}=\{\beta^2\alpha'', \beta\alpha''\beta\}$ and put $\alpha''\beta^2=x\beta^2\alpha''+y\beta\alpha''\beta$. Then by Theorem 4.1 and (4.1)

$$\beta^2\alpha'=\alpha'\beta^2=\theta(\alpha''\beta^2)=x\beta^2\alpha'+y\beta\alpha'\beta=(x+y)\beta^2\alpha',$$

and

$$\begin{aligned}\delta_0\beta^2\alpha'\delta_0&=\alpha'\delta_0\beta^2\delta_0=\delta_1(\alpha''\beta^2)\delta_0\\&=x\delta_1\beta^2\cdot 0+y\delta_1\beta(\beta'\alpha''-\delta_0\beta\alpha'')\\&=y(\delta_1\beta\delta_0+\delta_0\beta\delta_1)\beta\alpha''-y\delta_1\beta\delta_0\beta\alpha''\\&=\frac{y}{2}\delta_0(\beta^2\delta_1+\delta_1\beta^2)\alpha''=\frac{y}{2}\delta_0\beta^2\alpha'\delta_0\end{aligned}$$

by (4.1)$\sim$(4.4) and (iii). Thus $y=2$ and $x=1-y=-1$, and (ii) is proved.

Next consider $\delta_1\beta\alpha''\in\mathscr{A}_{(p+1)q-3}=\{\alpha''\beta\delta_1, \beta'\alpha', \beta\alpha'\delta_0\}$ and put $\delta_1\beta\alpha''=x\alpha''\beta\delta_1+y\beta'\alpha'+z\beta\alpha'\delta_0$. Apply $\beta\delta_0$ to each term from the right, then

$$\delta_1\beta\alpha''\beta\delta_0=\frac{1}{2}\delta_1(\beta^2\alpha''+\alpha''\beta^2)\delta_0=\frac{1}{2}\alpha'\delta_0\beta^2\delta_0=\frac{1}{2}\delta_0\beta^2\alpha'\delta_0,$$

$$\alpha''\beta\delta_1\beta\delta_0=\frac{1}{2}\alpha''(\beta^2\delta_1+\delta_1\beta^2)\delta_0=\frac{1}{2}\alpha'\delta_0\beta^2\delta_0=\frac{1}{2}\delta_0\beta^2\alpha'\delta_0,$$

$$\beta'\alpha'\beta\delta_0=\beta'\beta\alpha'\delta_0$$

and

$$\beta\alpha'\delta_0\beta\delta_0=\beta\alpha'\beta'\delta_0=\beta'\beta\alpha'\delta_0.$$

Since $\delta_0\beta^2\alpha'\delta_0$ and $\beta'\beta\alpha'\delta_0$ are independent we have $x=1$ and $y+z=0$. Apply δ_0 from the right, then

$$0=\delta_1\beta\alpha''\delta_0=x\cdot 0+y\beta'\alpha'\delta_0+z\cdot 0=y\beta'\alpha'\delta_0.$$

Thus $y=z=0$ and (iv) is proved.

Apply Theorem 2.4, (iii) to $\gamma=\alpha''$ and $\xi=\beta_{(1)}\delta$, then by Theorems 4.1, 4.2 and by $\lambda_V(\delta\beta_{(1)}\delta)=\beta'$ we have

$$(\beta\delta_1-\delta_1\beta)\alpha''=\alpha''(\beta\delta_1-\delta_1\beta)+\alpha'\beta'.$$

Then (v) follows by (4.3), (4.4) and (iv).

Consider $\beta''\beta\in\mathscr{A}_{(2p+3)q-3}=\{\beta\beta''\}$, put $\beta''\beta=x\beta\beta''$ and apply θ, then $(\beta\alpha''-\alpha''\beta)\beta=x\beta(\beta\alpha''-\alpha''\beta)$, i.e., $\alpha''\beta^2=(1+x)\beta\alpha''\beta-x\beta^2\alpha''$. Comparing with (ii), we have $x=1$ and (vi).

Finally, by (4.2) and Theorem 4.1,

$$0=\theta(\delta_0\beta'')=-\delta_1\beta''+\delta_0(\beta\alpha''-\alpha''\beta)=-\delta_1\beta''+\delta_0\beta\alpha''$$

Then (vii) follows from (iii).

Lemma 4.2′. $\lambda_V(\delta\beta_{(2)}\delta)=\beta^2\delta_0-2\beta\delta_0\beta+\delta_0\beta^2+2\beta\beta'$.

Proof. Apply Theorem 4.2 for $\gamma=\beta^2$, then $\gamma_{(1)}=\beta_{(2)}$, $\theta(\beta^2)=0$ and $\lambda_V(\delta\beta_{(2)}\delta)=\beta^2\delta_0+\delta_0\beta^2+\varepsilon'$ for some ε' satisfying $\varepsilon'\delta_1=\delta_1\beta^2\delta_0$ and $\delta_1\varepsilon'=\delta_0\beta^2\delta_1$. Since $\varepsilon'\in\mathscr{A}_{(2p+1)q-2}=\{\beta^2\delta_0,\ \beta\delta_0\beta,\ \delta_0\beta^2,\ \beta\beta'\}$, put $\varepsilon'=x\beta^2\delta_0+y\beta\delta_0\beta+z\delta_0\beta^2+w\beta\beta'$, then

$$\varepsilon'\delta_1=\delta_1\beta^2\delta_0=2(\beta\delta_1\beta-\beta^2\delta_1)\delta_0=2\beta\delta_1\beta\delta_0$$

and

$$\varepsilon'\delta_1=y\beta(\beta'\delta_1-\delta_1\beta\delta_0)+z\delta_0\beta^2\delta_1+w\beta\beta'\delta_1$$

$$=-y\beta\delta_1\beta\delta_0+(y+w)\beta\beta'\delta_1+z\delta_0\beta^2\delta_1.$$

Thus $y=-2$, $w=-y=2$ and $z=0$. Also we have

$$\begin{aligned}\delta_0\beta^2\delta_1=\delta_1\varepsilon'&=x\delta_1\beta^2\delta_0+2(\beta'\delta_1-\delta_1\beta\delta_0)\beta\\&=2x\beta\delta_1\beta\delta_0+2\delta_0\beta\delta_1\beta\\&=2x\beta\delta_1\beta\delta_0+\delta_0\beta^2\delta_1.\end{aligned}$$

Thus $x=0$ and the lemma is proved.

Theorem 4.4. *The following formula holds.*

(4.6) $$\delta_0\beta^3=\beta^3\delta_0-3\beta^2\delta_0\beta+3\beta\delta_0\beta^2.$$

Proof: By Corollary 2.5, $\lambda_V(\delta\beta_{(2)}\delta)=\beta^2\delta_0-2\beta\delta_0\beta+\delta_0\beta^2+2\beta\beta'$ commutes with β. Then the formula follows since $\beta\beta'=\beta'\beta$. q.e.d.

Theorem 4.5. *Up to non-zero coefficient, the following formula holds.*

(4.7) $$\beta''\beta''=(\beta')^2\delta_1\beta\delta_1 \qquad \textit{if } p=5.$$

The proof will be given in the last section.

Theorem 4.6. *For degree$<(p^2-1)q-5$ the multiplicative relations in $\mathscr{A}_*(V(1))$ are generated by the relations* (4.1)$\sim$(4.7).

Proof. It is sufficient to prove that the product of a monomial in Theorem 3.6 with seven generators δ_0, δ_1, α'', α', β', β, β'', from the right, can be written in a linear combination of the monomials in Theorem 3.6 by use of (4.1)$\sim$(4.7). This is obvious for β' by (4.1). Also the product with α' can be reduced to products with other generators by (4.1). Then we forget the term $P(\beta, \beta')$ in Theorem 3.6 and consider twenty monomials there and check the product with δ_0, δ_1, α'', β, β''. Except the relations (4.1)$\sim$(4.7) and relations directly follow from them, the products in question are the following:

(4.8) (i) $\delta_1\beta\delta_0\beta=\beta'\delta_1\beta-\frac{1}{2}\delta_0\beta^2\delta_1$,

(ii) $\delta_1\beta\delta_1\beta=\beta\delta_1\beta\delta_1$

(iii) $\alpha''\beta\delta_0\beta=\beta'\alpha''\beta-\frac{1}{2}\delta_0\beta^2\alpha''$,

(iv) $\alpha''\beta\delta_1\beta=\frac{1}{2}\delta_0\beta^2\alpha'+\beta\alpha''\beta\delta_1-\frac{1}{2}\beta^2\alpha''\delta_1$,

(v) $\delta_0\beta^2\delta_0\beta=\beta\delta_0\beta^2\delta_0-\beta^2\delta_0\beta\delta_0+\beta'\delta_0\beta^2$,

(vi) $\delta_0\beta^2\delta_1\beta=2\beta^2\delta_1\beta\delta_0+2\beta\delta_0\beta^2\delta_1-2\beta^2\beta'\delta_1$,

(vii) $\delta_0\beta^2\alpha'\delta_0\beta=\beta\delta_0\beta^2\alpha'\delta_0-\beta^2\beta'\alpha'\delta_0+\beta'\delta_0\beta^2\alpha'$,

(viii) $\delta_0\beta^2\alpha''\beta=2\beta^2\alpha''\beta\delta_0+2\beta\delta_0\beta^2\alpha''-2\beta^2\beta'\alpha''$,

(ix) $\delta_0\beta^2\alpha'\beta=4\beta^3\delta_0\alpha'-3\beta^2\beta'\alpha'-6\beta^2\alpha''\beta\delta_1+3\beta\delta_0\beta^2\alpha'$,

(x) $\delta_1\beta\delta_1\beta''=\beta'\alpha''\beta\delta_1-\frac{1}{2}\delta_0\beta^2\alpha'\delta_0$,

(xi) $\delta_1\beta\beta''=\frac{1}{2}\delta_0\beta^2\alpha''$,

(xii) $\alpha''\beta\delta_1\beta''=0$,

(xiii) $\delta_0\beta^2\delta_1\beta''=\beta'\delta_0\beta^2\alpha''$.

For example, (i) follows from (4.3) and (4.5), (i). The details are left to the readers.

Proposition 4.7. *The following relations hold:*

(4.9) (i) $\beta^r\delta_1\beta^s=s\beta^{r+s-1}\delta_1\beta+(1-s)\beta^{r+s}\delta_1$
$=r\beta\delta_1\beta^{r+s-1}+(1-r)\delta_1\beta^{r+s}$,

(ii) $\beta^r\alpha''\beta^s=s\beta^{r+s-1}\alpha''\beta+(1-s)\beta^{r+s}\alpha''$
$=r\beta\alpha''\beta^{r+s-1}+(1-r)\alpha''\beta^{r+s}$,

(iii) $\beta^r\delta_0\beta^s=\binom{s-1}{2}\beta^{r+s}\delta_0-s(s-2)\beta^{r+s-1}\delta_0\beta+\binom{s}{2}\beta^{r+s-2}\delta_0\beta^2$

$$=\binom{r-1}{2}\delta_0\beta^{r+s}-r(r-2)\beta\delta_0\beta^{r+s-1}+\binom{r}{2}\beta^2\delta_0\beta^{r+s-2}.$$

Proof. We prove the first equality of (iii), the others are proved similarly. By induction on s, we have using (4.6)

$$\beta^r\delta_0\beta^{s+1}=\binom{s-1}{2}\beta^{r+s}\delta_0\beta-s(s-2)\beta^{r+s-1}\delta_0\beta^2$$

$$+\binom{s}{2}\beta^{r+s-2}(3\beta\delta_0\beta^2-3\beta^2\delta_0\beta+\beta^3\delta_0)$$

$$=\binom{s}{2}\beta^{r+s+1}\delta_0-\left(3\binom{s}{2}-\binom{s-1}{2}\right)\beta^{r+s}\delta_0\beta$$

$$+\left(3\binom{s}{2}-s(s-2)\right)\beta^{r+s-1}\delta_0\beta^2$$

$$=\binom{s}{2}\beta^{r+s+1}\delta_0-(s+1)(s-1)\beta^{r+s}\delta_0\beta+\binom{s+1}{2}\beta^{r+s-1}\delta_0\beta^2.$$

q.e.d.

As is easily seen the relations (4.1)~(4.7) are symmetric. Since every polynomial on the seven generators are written in the form of Theorem 3.6, the same is true for the symmetric form:

Proposition 4.8. *For degree* $<(p^2-1)q-5$

$$\mathscr{A}_*(V(1))=E(\delta_0)\otimes\{1,\alpha',\beta\delta_1,\beta\alpha'',\beta^2\delta_0,\alpha'\beta^2\delta_0\}\otimes P(\beta,\beta')$$

$$+\{\delta_1,\alpha'',\delta_1\beta\delta_1,\beta\delta_0,\delta_1\beta\alpha'',\beta'',\delta_1\beta^2\delta_0,\alpha''\beta^2\delta_0\}\otimes P(\beta,\beta').$$

5. Applications to $\mathscr{A}_*(M)$ and G_*

First recall the following elements:

$$\delta=i\pi\in\mathscr{A}_{-1}(M),\qquad \alpha\in\mathscr{A}_q(M)$$

and

$$\beta_{(s)}=\pi_1\beta^s i_1\in\mathscr{A}_{(qs+s-1)q-1}(M),\qquad (s\geq 1).$$

The following relations contain the relations in [15] for $p\geq 5$.

Theorem 5.1. $(p \geq 5)$. *The following relations hold*:

(5.1) (i) $\delta\delta = \alpha\beta_{(s)} = \beta_{(s)}\alpha = 0$,

(ii) $\delta\alpha^2 = 2\alpha\delta\alpha - \alpha^2\delta$,

(iii) $\alpha\delta\beta_{(s)} = \beta_{(s)}\delta\alpha$,

(iv) *if* $r+s \not\equiv 0 \pmod{p}$, *then* $\beta_{(r)}\beta_{(s)} = 0$,

(v) *if* $r+s \not\equiv 0, 1 \pmod{p}$, *then*

$$\beta_{(r)}\delta\beta_{(s)} = \frac{rs}{r+s-1}\beta_{(r+s-1)}\delta\beta_{(1)},$$

if $r+s \not\equiv 0, 2 \pmod{p}$, *then*

$$\beta_{(r)}\delta\beta_{(s)} = \frac{rs}{r+s-2}\beta_{(r+s-2)}\delta\beta_{(2)}.$$

(vi) $\beta_{(r+s)} \in \pm\{\beta_{(r)}, \alpha, \beta_{(s)}\}$.

(vii) *if* $r+s \not\equiv 0 \pmod{p}$, $\frac{r}{r+s}\beta_{(r+s)} \in \pm\{\beta_{(r)}, \beta_{(s)}, \alpha\}$,

$$\frac{r}{r+s}\beta_{(r+s)} \in \pm\{\alpha, \beta_{(s)}, \beta_{(r)}\}.$$

Proof. By (3.3), $\delta\delta = i\pi\, i\pi = 0$, $\alpha\beta_{(s)} = \alpha\pi_1\beta^s i_1 = 0$ and $\beta_{(s)}\alpha = \pi_1\beta^s i_1\alpha = 0$. Apply (3.8)′ for $\xi = \alpha, \beta_{(s)}$, then we have (ii), (iii). By (4.9), (i) $\beta^r\delta_1\beta^s = s\beta^{r+s-1}\delta_1\beta + (1-s)\beta^{r+s}\delta_1$ and $\delta_1\beta^{r+s} = (r+s)\beta^{r+s-1}\delta_1\beta + (1-r-s)\beta^{r+s}\delta_1$. It follows

(5.2) $$(r+s)\beta^r\delta_1\beta^s = s\delta_1\beta^{r+s} + r\beta^{r+s}\delta_1.$$

If $r+s \not\equiv 0$ then $\beta_{(r)}\beta_{(s)} = \pi_1\beta^r\delta_1\beta^s i_1 = 0$ since $\pi_1\delta_1 = \delta_1 i_1 = 0$. Similarly, from the first formula of (4.9), (iii) and the corresponding formula for $\delta_0\beta^{r+s}$, we have

(5.3) (i) $(r+s)(r+s-1)\beta^r\delta_0\beta^s$

$$= s(s-1)\delta_0\beta^{r+s} - r(s-1)(r+s-1)\beta^{r+s}\delta_0 + rs(r+s)\beta^{r+s-1}\delta_0\beta,$$

(ii) $(r+s)(r+s-2)\beta^r\delta_0\beta^s$

$$=s(s-2)\delta_0\beta^{r+s}-r(s-2)(r+s-2)\beta^{r+s}\delta_0$$
$$+rs(r+s)\beta^{r+s-2}\delta_0\beta^2.$$

Then (v) follows easily. Up to sign, $\pi_1\beta^r$ is an extension of $\beta_{(r)}$ and $\beta^s i_1$ is a coextension of $\beta_{(s)}$. Then (vi) and (vii) follows from $\beta_{(r+s)}=(\pi_1\beta^r)(\beta^s i_1)$, $(r+s)\beta_{(r)}(\pi_1\beta^s)=r\beta_{(r+s)}i_1$ and $(r+s)(\beta^s i_1)\beta_{(r)}=r\pi_1\beta_{(r+s)}$ by use of theorems in Chapter 1 of [10]. q.e.d.

The following supplementary results are also obtained by similar discussions and by the exactness of (3.6), (3.6)*.

(5.4) (i) *In general*, $\beta_{(r)}\beta_{(s)}=s\beta_{(r+s-1)}\beta_{(1)}=r\beta_{(1)}\beta_{(r+s-1)}$ *and* $\beta_{(r)}\delta\beta_{(s)}=-s(s-2)\beta_{(r+s-1)}\delta\beta_{(1)}+\binom{s}{2}\beta_{(r+s-1)}\delta\beta_{(2)}=-r(r-2)\beta_{(1)}\delta\beta_{(r+s-1)}+\binom{r}{2}\beta_{(2)}\delta\beta_{(r+s-1)}$.

(ii) $\beta_{(pt)}$ *and* $\beta_{(r)}\beta_{(s)}$ *are divisible by* α *from the both sides*, i.e., *they are elements of forms* $\alpha\xi=\xi'\alpha$.

For (ii), $i_1\beta_{(pt)}=\delta_1\beta^{pt}\pi_1=\beta^{pt}\delta_1\pi_1=0$ by (5.2) and $i_1\beta_{(r)}\beta_{(s)}=\delta_1\beta^r\delta_1\beta^s\pi_1=\beta^s\delta_1\beta^r\delta_1\pi_1=0$ by the following

(5.5) $$\delta_1\beta^r\delta_1\beta^s=\beta^s\delta_1\beta^r\delta_1.$$

This is true for $r=1$ since $\delta_1\beta\delta_1$ commutes with β by (4.8), (ii). Then by (4.9), (i), $\delta_1\beta^r\delta_1\beta^s=r\beta^{r-1}\delta_1\beta\delta_1\beta^s=r\beta^s\delta_1\beta\delta_1\beta^{r-1}=\beta^s\delta_1\beta^r\delta_1$.

The structure of the algebra $\mathscr{A}_*(M)$ for degree $<p^2q-4$ was determined in [15]. We shall compute this from Corollary 3.3, so here we need not use the results on G_*.

Theorem 5.2. *For degree* $<p^2q-4$,

$$\mathscr{A}_*(M)=P(\alpha)\otimes E(\delta,\alpha\delta-\delta\alpha)$$
$$+E(\delta)\otimes\{\beta_{(s)}\}\otimes P(\delta\beta_{(1)})\otimes E(\delta,\alpha\delta-\delta\alpha).$$

Proof. Let $A=\{1,\alpha',\delta_1\beta,\alpha''\beta,\delta_0\beta^2,\delta_0\beta^2\alpha'\}$. Then Corollary 3.3. states $\pi_*(M;V(1))=P(\beta,\beta')\otimes Ai_1\otimes E(\delta)$. By (3.9) $\pi_*(M;V(1))$

$=P(\beta)\otimes Ai_1\otimes P(\beta_{(1)}\delta+\delta\beta_{(1)})\otimes E(\delta)$. Now consider the exact sequence (3.6):

$$\mathscr{A}_{k-q}(M)\xrightarrow{\alpha_*}\mathscr{A}_k(M)\xrightarrow{i_{1*}}\pi_k(M;\ V(1))\xrightarrow{\pi_{1*}}\mathscr{A}_{k-q-1}(M)\xrightarrow{\alpha_*}$$

and compute $\mathscr{A}_k(M)$ by induction on k. Then it is sufficient to check the correspondence between basic elements. For degree$<pq-3$, $\pi_*(M;\ V(1))=E(\alpha')\otimes\{i_1\}\otimes E(\delta)=\{i_1\}\otimes E(\delta,\ \alpha\delta-\delta\alpha)$ by (3.9). Since $P(\alpha)\xrightarrow{\alpha_*}P(\alpha)\xrightarrow{i_1^*}\{i_1\}$ is a short exact sequence, the first part $P(\alpha)\otimes E(\delta,\ \alpha\delta-\delta\alpha)$ of $\mathscr{A}_*(M)$ and $E(\alpha')\otimes\{i_1\}\otimes E(\delta)$ form a sub exact sequence and we may omit them and consider the remaining parts. Here we need

(5.6) $$\pi_1\alpha'=-\alpha\delta\pi_1 \quad \textit{and} \quad \pi_1\alpha''=-\delta\alpha\delta\pi_1.$$

For $\pi_1\alpha'=-(\alpha\delta-\delta\alpha)\pi_1=-\alpha\delta\pi_1$ by (3.9) and (3.3). Apply the above result on $\mathscr{A}_*(M)$ for deg$<pq-3$ to the exact sequence (3.6)*, then we have that $\pi_{-2}(V(1);\ M)=\{\alpha\delta\pi_1\}\approx Z_p\approx\pi_{-3}(V(1);\ M)=\{\delta\alpha\delta\pi_1\}$. Put $\pi_1\alpha''=x\delta\alpha\delta\pi_1$, then $-\alpha\delta\pi_1=\pi_1\alpha'=\pi_1\theta(\alpha'')=-\theta(\pi_1\alpha'')=-x\theta(\delta\alpha\delta\pi_1)=x(\alpha\delta\pi_1-\delta\alpha\pi_1)=x\alpha\delta\pi_1$. Thus $x=-1$ and (5.6) is proved.

Now in $\mathscr{A}_*(M)$ we can replace $P(\delta\beta_{(1)})$ by $P(\beta_{(1)}\delta+\delta\beta_{(1)})$ since $\beta_{(s)}(\beta_{(1)}\delta+\delta\beta_{(1)})^k=\beta_{(s)}(\delta\beta_{(1)})^k$ by Theorem 5.1 ($s<p$). For the sake of simplicity we put $B=\beta_{(1)}\delta+\delta\beta_{(1)}$ and also we forget the term $E(\delta)$. By (5.1), (i), $\beta_{(s)}(\alpha\delta-\delta\alpha)=-\beta_{(s)}\delta\alpha$. We check the exact sequence replacing $\mathscr{A}_*(M)$ by $E(\delta)\otimes\{\beta_{(s)}\}\otimes\{1,\ \delta\alpha\}\otimes P(B)$ and $\pi_*(M;\ V(1))$ by $P(\beta)\otimes\{i_1,\ \alpha' i_1,\ \delta_1\beta i_1,\ \alpha''\beta i_1,\ \delta_0\beta^2 i_1,\ \delta_0\beta^2\alpha' i_1\}\otimes P(B)-\{i_1,\ \alpha' i_1\}$. The correspondence of basic elements are given as follows ($t\geq 0$):

$$\pi_{1*}(\beta^s i_1 B^t)=\pi_1\beta_s i_1 B^t=\beta_{(s)}B^t \qquad (s\geq 1),$$

$$\pi_{1*}(\beta^s\alpha' i_1 B^t)=\beta_{(s)}(\alpha\delta-\delta\alpha)B^t=-\beta_{(s)}\delta\alpha B^t \qquad (s\geq 1),$$

$$\pi_{1*}((s+1)\beta^s\alpha''\beta i_1 B^t)=\pi_1(s\alpha''\beta^{s+1}+\beta^{s+1}\alpha'')i_1 B^t$$

$$=-s\delta\alpha\delta\beta_{(s+1)}B^t+\beta_{(s+1)}\delta\alpha\delta B^t$$

$$\equiv -s\delta\alpha\delta\beta_{(s+1)}B^t \quad (\mathrm{mod\,Im}\,\delta^*) \qquad (p-1>s\geq 0),$$

$$\alpha_*(\delta\beta_{(s)}B^t)=\alpha\delta\beta_{(s)}B^t=\beta_{(s)}\delta\alpha B^t,$$

$$i_{1*}(\beta_{(s)}B^t)=\delta_1\beta^s i_1 B^t=s\beta^{s-1}\delta_1\beta i_1 B^t \qquad (p>s\geq 1),$$

$$i_{1*}(\delta\beta_{(s)}B^t)\equiv i_{1*}(\delta\beta_{(s)}B^t-(s-2)\beta_{(s)}B^t\delta)$$

$$=\delta_0\beta^s i_1 B^t-(s-2)\delta_1\beta^s i_1\delta B^t$$

$$=\binom{s}{2}\beta^{s-2}\delta_0\beta^2 i_1 B^t-s(s-2)(\beta^{s-1}\delta_0\beta i_1+\beta^{s-1}\delta_1\beta i_1\delta)B^t$$

$$=\binom{s}{2}\beta^{s-2}\delta_0\beta^2 i_1 B^t-s(s-2)\beta^{s-1}i_1 B^{t+1} \qquad (p>s\geq 2),$$

$$i_{1*}(\delta\beta_{(s)}\delta\alpha B^t)=i_1(\delta\beta_{(s)})(\delta\alpha-\alpha\delta)B^t$$

$$=-\alpha' i_{1*}(\delta\beta_{(s)})B^t$$

$$=-\binom{s}{2}\beta^{s-2}\delta_0\beta^2\alpha' i_1 B^t+s(s-2)\beta^{s-1}\alpha' i_1 B^{t+1} \qquad (p>s\geq 2),$$

$$i_{1*}(\delta\beta_{(1)}B^t)\equiv i_{1*}(\delta\beta_{(1)}B^t+\beta_{(1)}B^t\delta)=i_1 B^{t+1},$$

$$i_{1*}(\delta\beta_{(1)}\delta\alpha B^t)=-\alpha' i_1(\delta\beta_{(1)}B^t)=-\alpha' i_1 B^{t+1}.$$

Conseqently we have proved the theorem.

Next recall the elements

$$\beta_s=\pi\beta_{(s)}i=\pi_0\beta^s i_0\in G_{(sp+s-1)q-2}.$$

Recently, the non-triviality of β_s for general $s\geq 1$ has been proved by L. Smith [6]. We also denote

$$\alpha_r=\pi\alpha^r i\in G_{rq-1}.$$

Theorem 5.3. (i) $t(r+s-t)\beta_r\beta_s=rs\beta_t\beta_{r+s-t}$, (ii) *Every monomial* $\beta_{r_1}\cdots\beta_{r_k}$ *is some multiple of* $\beta_{ps}(k=1)$, $\beta_t\beta_1^{k-1}(t\not\equiv 0 \pmod p)$ *or* $\beta_{ps-1}\beta_2\beta_1^{k-2}$ (iii) $\alpha_r\beta_s=0$ *if* $r\geq 2$.

Proof. $\pi\beta_{(r)}\delta\beta_{(s)}i=\pi\beta_{(r)}i\pi\beta_{(s)}i=\beta_r\beta_s$. Then it follows from (5.4), (i)

$$\beta_r\beta_s=-s(s-2)\beta_{r+s-1}\beta_1+\binom{s}{2}\beta_{r+s-2}\beta_2.$$

Put $r=1$, then $\beta_s\beta_1=\beta_1\beta_s=-s(s-2)\beta_s\beta_1+\binom{s}{2}\beta_{s-1}\beta_2$, i.e.,

$$2(t-1)^2\beta_t\beta_1=t(t-1)\beta_{t-1}\beta_2.$$

Thus

$$(5.7)\qquad \beta_r\beta_s=\begin{cases}\dfrac{rs}{r+s-1}\beta_{r+s-1}\beta_1, & \text{if } r+s\not\equiv 1 \pmod p\\[2ex] \dfrac{rs}{2(r+s-2)}\beta_{r+s-2}\beta_2 & \text{if } r+s\not\equiv 2 \pmod p),\end{cases}$$

and (i) follows directly. Repeating (5.7) we have (ii).

From (5.1), (ii), we have $\alpha^r\delta=r\alpha\delta\alpha^{r-1}+(1-r)\delta\alpha^r$, then $\alpha_r\beta_s=\pi\alpha^r i\pi\beta_{(s)}i=\pi\alpha^r\delta\beta_{(s)}i=\pi(r\alpha\delta\alpha^{r-2}+(1-r)\delta\alpha^{r-1})\alpha\beta_{(s)}i=0$, by (5.1), (i), if $r\geqq 2$. q.e.d.

Remark 5.4. *The relations* $\alpha^r\beta_{(s)}=\beta_{(s)}\alpha^r=0$ *and* $\beta_{(s)}\beta_{(t)}=0$ $s+t\not\equiv 0 \pmod p$ *imply*

$$\{\alpha_r, p\iota, \beta_s\}\equiv\{\beta_s, p\iota, \alpha_r\}\equiv 0$$

and

$$\{\beta_s, p\iota, \beta_t\}\equiv 0 \qquad \textit{if } s+t\not\equiv 0 \pmod p).$$

But as is seen in (15.6) and Theorem 15.2 of [12], $\{\beta_1, p\iota, \beta_{p-1}\}\not\equiv 0$. This shows

$$\beta_{(1)}\beta_{(p-1)}\neq 0.$$

From now we consider some application of the complex $V(3)$ and the class

$$\gamma\in\mathscr{A}_{(p^2+p+1)q}(V(2)) \qquad (p>5)$$

of the attaching map of $V(3)=V(2)\cup_\gamma C\sum^{(p^2+p+1)q}V(2)$. For the case $p=5$, the existence of $V(3)$ does not known but $V\left(2\frac{1}{2}\right)$ does exist. So, we can consider

$$\gamma i_2\in\mathscr{A}_{(p^2+p+1)q}(V(1);\ V(2))$$

even for the case $p=5$. Put

$$\gamma_{[1]}=\pi_2(\gamma i_2)\in \mathscr{A}_{p^2q-1}(V(1)).$$

The mapping cone of $\gamma_{[1]}$ is $V\left(2\frac{1}{2}\right)/V(1)$ and in which $\mathscr{P}^{p^2}\neq 0$ for the bottom class. Put

$$\gamma_{(1)}=\pi_1\gamma_{[1]}i_1\in \mathscr{A}_{(p^2-1)q-2}(V(1)).$$

Theorem 5.5. *$\gamma_{(1)}=x((\beta_{(1)}\delta)^p+(\delta\beta_{(1)})^p)+y\beta_{(p-1)}\delta\alpha$ for some integers $x\not\equiv 0 \pmod p$ and y.*

Proof. By Theorem 5.2, $\mathscr{A}_{(p^2-1)q-2}(M)$ is spanned by $(\beta_{(1)}\delta)^p$, $(\delta\beta_{(1)})^p$, $\beta_{(p-1)}\delta\alpha$ and $\alpha^{p^2-2}\delta\alpha\delta$. Put

$$\gamma_{(1)}=x(\beta_{(1)}\delta)^p+x'(\delta\beta_{(1)})^p+y\beta_{(p-1)}\delta\alpha+z\alpha^{p^2-2}\delta\alpha\delta.$$

Theorem 4.4 of [14] says there exists a multiplication $M\wedge V\left(2\frac{1}{2}\right)\rightarrow V\left(2\frac{1}{2}\right)$. This induces a multiplication $M\wedge V\left(2\frac{1}{2}\right)/V(1)\rightarrow V\left(2\frac{1}{2}\right)/V(1)$ where $V\left(2\frac{1}{2}\right)/V(1)$ is a mapping cone of $\gamma_{[1]}=\pi_2(\gamma i_2)$. Thus $V\left(2\frac{1}{2}\right)/V(1)$ is a Z_p-space, and by Lemma 2.3, we have

(5.8) $$\theta(\gamma_{[1]})=0 \quad \textit{hence} \quad \theta(\gamma_{(1)})=0.$$

Since $\beta_{(1)}\beta_{(1)}=0$, $\theta((\beta_{(1)}\delta)^p)=(\beta_{(1)}\delta)^{p-1}\beta_{(1)}$, $\theta(\delta\beta_{(1)})^p=-(\beta_{(1)}\delta)^{p-1}\cdot\beta_{(1)}$, $\theta(\beta_{(p-1)}\delta\alpha)=\beta_{(p-1)}\alpha=0$ and $\theta(\alpha^{p^2-2}\delta\alpha\delta)=-\alpha^{p^2-1}\delta+\alpha^{p^2-2}\delta\alpha$ by (3.7), (3.7)′ and (5.1), (i). Then

$$0=\theta(\gamma_{(1)})=(x-x')(\beta_{(1)}\delta)^{p-1}\beta_{(1)}-z\alpha^{p^2-2}(\alpha\delta-\delta\alpha),$$

and $x=x'$ and $z=0$. Next put

(5.9) $$\gamma'=\gamma_{[1]}+y\beta^{p-1}\alpha'.$$

The element $\beta^{p-1}\alpha'$ is the composition of p elements which induce trivial homomorphisms of the chomology. It follows the functional cohomology operation is trivial for $\beta^{p-1}\alpha'$. So, the cohomology operations for the mapping cones of both of γ' and $\gamma_{[1]}$ are the same. The mapping cone of $\gamma_{[1]}$ is $V\left(2\frac{1}{2}\right)/V(1)$ and its cohomology corresponds to the part $\{Q_2, Q_3\}\otimes E(Q_0, Q_1)$ of $H^*(V(3); Z_p)=E(Q_0, Q_1, Q_2, Q_3)$. Then $Q_3=\mathcal{P}^{p^2}Q_2$. Thus $\mathcal{P}^{p^2}\neq 0$ for the bottom cell of $C_{\gamma'}$. By (5.9),

$$\pi_1\gamma' i_1=\gamma_{(1)}+y\pi_1\beta^{p-1}i_1(\alpha\delta-\delta\alpha)=\gamma_{(1)}-y\beta_{p-1}\delta\alpha$$
$$=x((\beta_{(1)}\delta)^p+(\delta\beta_{(1)})^p).$$

Now assume that $x\equiv 0$, then $\pi_1\gamma' i_1=0$. This shows that the cell corresponding to Q_3 is attached only to the bottom Moore space. Thus we have a subcomplex $S^n\cup e^{n+1}\cup e^{n+p^2q}$ of $C_{\gamma'}$ such that $\mathcal{P}^{p^2}\neq 0$. But since the p-component of G_{p^2q-2} vanishes [11], the attaching map of e^{n+p^2q} deforms into S^n and this contradicts to the triviality of mod p Hopf invariant. Thus $x\not\equiv 0$. q.e.d.

Remark that

(5.10) $$(\beta_{(1)}\delta)^k+(\delta\beta_{(1)})^k=(\beta_{(1)}\delta+\delta\beta_{(1)})^k,$$

(5.11) $$\beta\gamma_{[1]}=\beta\pi_2(\gamma i_2)=0 \qquad (\gamma_{[1]}\beta=0 \text{ if } p>5).$$

Theorem 5.6. *$\alpha\gamma_{(1)}=\gamma_{(1)}\alpha=0$ and $\beta_{(s)}\gamma_{(1)}=0$ for $s\geq 2$. $\beta_{(s)}\delta\gamma_{(1)}=0$ for $s\geq 3$.*

Proof. By (3.3) $\alpha\gamma_{(1)}=\alpha\pi_1\gamma_{[1]}i_1=0$ and $\gamma_{(1)}\alpha=\pi_1\gamma_{[1]}i_1\alpha=0$. By (4.9), (i) and (5.11)

$$\beta_{(s)}\gamma_{(1)}=\pi_1\beta^s\delta_1\gamma_{[1]}i_1=\pi_1(s\beta\delta_1\beta^{s-1}+(1-s)\delta_1\beta^s)\gamma_{[1]}i_1=0$$

if $s\geq 2$. Similarly $\beta_{(s)}\delta\gamma_{(1)}=0$ $(s\geq 3)$ follows from (4.9), (iii).

Corollary 5.7. *$(\beta_{(1)}\delta)^p\alpha=\alpha(\delta\beta_{(1)})^p=0$. If $s\geq 2$ $\beta_{(s)}((\beta_{(1)}\delta)^p+(\delta\beta_{(1)})^p)=0$, hence $\beta_{(s)}(\delta\beta_{(1)})^p=0$ further if $s\not\equiv -1$ (mod p). If $s\geq 3$ then $\beta_{(s)}\delta(\beta_{(1)}\delta)^p=\frac{y}{x}\beta_{(s)}\delta\beta_{(p-1)}\delta\alpha$.*

This follows directly from Theorems 5.6, 5.5 and 5.1.

Theorem 5.8. *If $s \geqq 2$, then $\beta_s \beta_1^p = 0$ and $\beta_s^{p+1} = 0$.*

Proof. By Corollary 5.7, $\beta_s \beta_1^p = \pi \beta_{(s)}(\delta \beta_{(1)})^p i = \pi \beta_{(s)}((\beta_{(1)}\delta)^p + (\delta \beta_{(1)})^p) i = 0$. By Theorem 5.3, (ii) β_s^{p+1} is a multiple of $\beta_{sp+s-p}\beta_1^p = 0$ or $\beta_{sp+s-p-1}\beta_2\beta_1^{p-1}$. In the latter case $sp+s-p-1 \equiv -1 \pmod{p}$, then $s \equiv 0 \pmod{p}$ and $\beta_s^2 = 0$ by Theorem 5.3, (i). q.e.d.

The following problem seems very difficult.

Problem. *Is $\gamma_1 = \pi \gamma_{(1)} i \in G_{(p^2-1)q-3}$ non-trivial?*

Proposition 5.9. *If $\gamma_1 \neq 0$, then*

$$\alpha_1 \beta_{p-1} \beta_s = 0 \qquad \textit{for } s \geqq 3,$$

hence

$$\alpha_1 \beta_1 \beta_k = 0 \qquad \textit{if } k \not\equiv -2 \pmod{p} \textit{ and } k \geqq p,$$

$$\alpha_1 \beta_2 \beta_{k-1} = 0 \qquad \textit{if } k \not\equiv -2 \pmod{p} \textit{ and } k \geqq p+1.$$

Proof. By Theorem 5.5,

$$\gamma_1 = \pi \gamma_{(1)} i = y \beta_{p-1} \alpha_1. \tag{5.12}$$

If $\gamma_1 \neq 0$, then $y \not\equiv 0 \pmod{p}$, and by Corollary 5.7, $\alpha_1 \beta_{p-1} \beta_s = \pi \beta_{(s)} \delta \beta_{(p-1)} \delta \alpha i = \frac{x}{y} \pi_1 \beta_{(s)} \delta (\beta_{(1)} \delta)^p i = 0$. The remaining part follows from Theorem 5.3, (i).

6. The Case $p=3$.

The case $p=3$ is quite different from the other cases. For $p=3$, M and $V(1)$ are not associative, $V(2)$ and β do not exist and the products $\alpha''\alpha'$, $\alpha'\alpha''$ are not trivial.

First we consider the effect of non-associativity. In this section we assume that each Z_p-space X is a finite CW-complex and $\mathscr{A}_1(X) = 0$. Then an element (associator) $\alpha_X \in \mathscr{A}_2(X)$ of Proposition 2.1 is associated for each Z_p-space X. Theorem 2.2, (iv) and Theorem 2.4,

(v) are generalized as follows.

Theorem 6.1. (i) *For* $\gamma\in\pi_k^S(X;\,Y)$,

$$\theta(\theta(\gamma))=\alpha_Y\gamma-\gamma\alpha_X.$$

(ii) *For* $\xi\in\mathscr{A}_t(M_p)$,

$$\theta(\lambda_X(\xi))=-2\lambda_X(\theta(\xi))=2(\alpha_X\lambda_X(\delta\xi)-(-1)^t\lambda_X(\xi\delta)\alpha_X).$$

Proof. By Proposision 2.1 and Theorem 2.6,

$$\begin{aligned}\theta\theta(\gamma)&=\mu_Y(1_M\wedge\mu_Y)(1_M\wedge 1_M\wedge\gamma)(1_M\wedge\varphi_X)\varphi_X\\&=(\alpha_Y(\pi\wedge\pi\wedge 1_X)-\mu_Y(\mu_M\wedge 1_Y))(1_M\wedge 1_M\wedge\gamma)\\&\qquad((i\wedge i\wedge 1_X)\alpha_X+(\varphi_M\wedge 1_X)\varphi_X)\\&=\alpha_Y(\pi i\wedge\pi i\wedge\gamma)\alpha_X+\alpha_Y\gamma(\pi\wedge 1_X)((\pi\wedge 1_M)\varphi_M\wedge 1_X)\varphi_X\\&\quad-\mu_Y(\mu_M(i\wedge 1_M)\wedge 1_Y)\gamma\alpha_X-\mu_Y(\mu_M\varphi_M\wedge\gamma)\varphi_X\\&=\alpha_Y\gamma-\gamma\alpha_X,\end{aligned}$$

$$\begin{aligned}\lambda_X(\theta(\xi))&=-\lambda_X(\lambda_M(\xi))=-\mu_X(\mu_M\wedge 1_X)(\xi\wedge 1_M\wedge 1_X)(\varphi_M\wedge 1_X)\varphi_X\\&=(\mu_X(1_M\wedge\mu_X)-\alpha_X(\pi\wedge\pi\wedge 1_X))(\xi\wedge 1_M\wedge 1_X)\\&\qquad((1_M\wedge\varphi_X)\varphi_X-(i\wedge i\wedge 1)\alpha_X)\\&=-(-1)^{t+1}\mu_X(\xi\wedge 1_X)(1_M\wedge\mu_X(i\wedge 1_M))(1_0\wedge i\wedge 1_X)\alpha_X\\&\quad-\alpha_X(\pi\wedge 1_X)(1_M\wedge(\pi\wedge 1_M)\varphi_X)(\xi\wedge 1_X)\varphi_X\\&=(-1)^t\mu_X(\xi i\wedge 1_X)(\pi\wedge 1_X)\varphi_X\alpha_X\\&\quad-\alpha_X\mu_X(i\wedge 1_X)(\pi\xi\wedge 1_X)\varphi_X\\&=(-1)^t\lambda_X(\xi\delta)\alpha_X-\alpha_X\lambda_X(\delta\xi),\end{aligned}$$

$$\begin{aligned}\theta(\lambda_X(\xi))&=\mu_X(1_M\wedge\mu_X)(1_M\wedge\xi\wedge 1_X)(1_M\wedge\varphi_X)\varphi_X\\&=(\alpha_X(\pi\wedge\pi\wedge 1_X)-\mu_X(\mu_M\wedge 1_X))(1_M\wedge\xi\wedge 1_X)\\&\qquad((i\wedge i\wedge 1_X)\alpha_X+(\varphi_M\wedge 1_X)\varphi_X)\\&=\alpha_X(1_0\wedge\pi\xi\wedge 1_X)((\pi\wedge 1_M)\varphi_M\wedge 1_X)\varphi_X-\lambda_X(\theta(\xi))\end{aligned}$$

$$-(-1)^t \mu_X(\mu_M(i \wedge 1_M) \wedge 1_X)(1_0 \wedge \xi i \wedge 1_X)\alpha_X$$

$$=\alpha_X \mu_X(i\pi\xi \wedge 1_X)\varphi_X - \lambda_X(\theta(\xi)) - (-1)^t \mu_X(\xi i\pi \wedge 1_X)\varphi_X \alpha_X$$

$$=-2\lambda_X(\theta(\xi)). \qquad \text{q.e.d.}$$

In our case $p=3$, Theorem 6.1, (ii) says

$$\theta(\lambda_X(\xi)) = \lambda_X(\theta(\xi)) = (-1)^t \lambda_X(\xi\delta)\alpha_X - \alpha_X \lambda_X(\delta\xi). \tag{6.1}$$

Some of the results in the previous sections are valid for the case $p=3$, and we shall recall them. The sections 1, 2 and the beginning of the section 3 are general theories and can be applied here. From the existence of $M=V(0)$ and $V(1)$, we can define the elements i, π, $\delta=i\pi$, i_1, π_1, $i_0=i_1 i$, $\pi_0=\pi\pi_1$, $\delta_1=i_1\pi_1$, $\delta_0=i_0\pi_0$, α, $\alpha_1=\pi\alpha i$, $a'=\lambda_{V(1)}(\delta\alpha\delta)$. The following are valid.

(3.3) $\quad \pi i = i_1\alpha = \pi_1 i_1 = \alpha\pi_1 = 0,$

(3.7) $\quad \theta(\alpha) = \theta(i_1) = \theta(\pi_1) = 0$

(3.8) $\quad \lambda_M(\delta\alpha\delta) = \alpha\delta - \delta\alpha$

(3.8)′ $\quad (\alpha\delta - \delta\alpha)\xi = (-1)^t \xi(\alpha\delta - \delta\alpha)$

(3.9) $\quad \alpha'\xi = (-1)^t \xi\alpha', \quad \alpha'\xi' = (-1)^{t'} \xi'(\alpha\delta - \delta\alpha),$

$\quad \alpha'\xi'' = (-1)^{t''} \xi''\alpha_1.$

Some relations follow from these, for example (3.8)″ and (3.10) valid, and we have the existence of $\alpha'' \in \mathscr{A}_2(V(1))$ (Lemma 3.1).

By Lemma 6.2 of [14], M_3 is not associative: $\alpha_M \neq 0$ in $\mathscr{A}_2(M_3) = \{\delta\alpha\delta\} \approx Z_3$. By changing the sign of α, if it is necessary, we have

$$\alpha_M = \delta\alpha\delta. \tag{6.2}$$

Even though β does not exist, an element

$$[\beta i_1] \in \pi_{16}^S(M;\ V(1))$$

does exist since as a mapping cone of it the existence of $V\left(1\frac{1}{2}\right)$ is assured [14] for $p=3$. Here the notation $[\xi\eta]$ indicates a single element which is not a product of elements, one of ξ and η is an imaginary element.

We define

$$\beta_{(1)}=\pi_1[\beta i_1]\in\mathscr{A}_{11}(M),\quad \beta_1=\pi\beta_{(1)}i\in G_{10}$$

and

$$\beta'=\lambda_{V(1)}(\delta\beta_{(1)}\delta)\in\mathscr{A}_{10}(V(1)).$$

Then (3.8), (3.8)′ for $s=1$ and (3.9) hold. Using $[\beta i_1]$, $[\beta i_0]=[\beta i_1]i$ and $[\beta\delta_1]=[\beta i_1]\pi_1$ in place of βi_1, βi_0 and $\beta\delta_1$, we see that Lemma 3.4 holds and

(6.3)′ *Theorem* 3.2 *holds for deg*<26, *Corollary* 3.3 *holds for deg*<25 *and Theorem* 3.6 *holds for degree*<10.

Now we change the sign of $[\beta i_1]$, if it is necessary, so that the following theorem holds with the right sign.

Theorem 6.2. $\alpha''\alpha''=\beta'\delta_0$ *and* $\alpha'\alpha''=\beta'\delta_1$.

Proof. First we prove $\alpha''\alpha''\neq 0$, then the first formula holds by suitable choice of the sign of $[\beta i_1]$ since $\mathscr{A}_4(V(1))$ is spanned by $\beta'\delta_0$. As is seen in the proof of Theorem 3.2, in the mapping cone $C_{\alpha''}$ of α'', $\mathscr{P}^1(e^n)=e^{n+q}$ and hence $\Delta\mathscr{P}^1\mathscr{P}^1(e^n)=e^{n+2q+1}$(top cell). By use of Adem relation $2\mathscr{P}^1\Delta\mathscr{P}^1=\mathscr{P}^1\mathscr{P}^1\Delta+\Delta\mathscr{P}^1\mathscr{P}^1$, we see $\mathscr{P}^1\mathscr{P}^1\Delta(e^n)=\mathscr{P}^1(e^{n+q+1})=-e^{n+2q+1}$. Then assuming $\alpha''\alpha''=0$ and considering an extension $A: M^{q-2}C_{\alpha''}\to V(1)$ of α'', we see $\mathscr{P}^1\mathscr{P}^1\mathscr{P}^1(e^n)\neq 0$ in C_A which contradicts to $\mathscr{P}^1\mathscr{P}^1\mathscr{P}^1=0$ $(p=3)$. Thus $\alpha''\alpha''\neq 0$. By use of the following (6.3) and (3.9), we have

$$\alpha'\alpha''=-2\alpha'\alpha''=-\theta(\alpha'')\alpha''-\alpha''\theta(\alpha'')=-\theta(\alpha''\alpha'')$$
$$=-\theta(\beta'\delta_0)=-\beta'\theta(\delta_0)=\beta'\delta_1.$$

(6.3) $\theta(\delta_0)=-\delta_1$, $\theta(\alpha'')=\alpha'$ *and* $\theta(\delta_1)=\theta(\alpha')=\theta(\beta')=0$.

The proof is same as one of Theorem 4.1, but use Theorem 6.1, (ii) in place of Theorem 2.4, (v).

For the convenience of discussions, we introduce the results on the additive structure of $\mathscr{A}_*(M)$ which is directly computed from the results on G_* by use of (3.5), (3.5)*.

(6.4) *For degree* <32

$$\mathscr{A}_*(M)=P(\alpha)\otimes E(\delta, \alpha\delta-\delta\alpha)$$
$$+E(\delta)\otimes\{1, \alpha\delta\}\otimes\{\beta_{(1)}, \beta_{(2)}\}\otimes P(\delta\beta)\otimes E(\delta),$$

where $\beta_{(2)}\in\mathscr{A}_{27}(M)$ *satisfies* $\pi\beta_{(2)}i(=\beta_2)\neq 0$.

Lemma 6.3. *There exists an element* $[\pi_1\beta]\in\pi_{11}(V(1); M)$ such that $[\pi_1\beta]i_1=\pi_1[\beta i_1]=\beta_{(1)}$.

Proof. By the exactness of the sequence (3.6)* for $X=M$, it is sufficient to prove $\beta_{(1)}\alpha=0$. By (6.4), $\beta_{(1)}\alpha=x\alpha^4\delta+y\alpha^3\delta\alpha$ for some $x, y\in Z_p$. Then $x\alpha^5\delta+y\alpha^4\delta\alpha=\alpha\beta_{(1)}\alpha=\alpha\pi_1[\beta i_1]=0$ by (3.3). Thus $x=y=0$ and $\beta_{(1)}\alpha=0$. q.e.d.

We put

$$\beta_{(2)}=[\pi_1\beta][\beta i_1] \quad \text{and} \quad \beta_2=\pi\beta_{(2)}i\in G_{26},$$

then the non-triviality of β_2 is proved as in the proof of Theorem 3.2, and (6.4) holds for this $\beta_{(2)}$.

The following parts of Yamamoto's formula hold for $p=3$.

(6.5) (i) $\delta\delta=\alpha\beta_{(1)}=\beta_{(1)}\alpha=0$,

(ii) $\delta\alpha^2=2\alpha\delta\alpha-\alpha^2\delta$,

(iii) $\alpha\delta\beta_{(1)}=\beta_{(1)}\delta\alpha$.

For, (ii) is (3.8)″, $\delta\delta=0$ is obvious and $\alpha\beta_{(1)}=\alpha\pi_1[\beta i_1]=0$, $\beta_{(1)}\alpha=[\pi_1\beta]i_1\alpha=0$ by (3.3). Then (iii) follows from the relation $(\alpha\delta-\delta\alpha)\beta_{(1)}=-\beta_{(1)}(\alpha\delta-\delta\alpha)$ of (3.8)′.

Theorem 6.4. $\theta(\beta_{(1)})=\delta\alpha\delta\beta_{(1)}\delta$ *and* $\theta([\beta i_1])=\alpha''[\beta i_1]\delta$, *thus* $V\left(1\frac{1}{2}\right)/V(0)$ *and* $V\left(1\frac{1}{2}\right)$ *are not Z_p-spectra.*

Proof. Since $\theta(\beta_{(1)})\in\mathscr{A}_{12}(M)=\{\alpha^3, \delta\alpha\delta\beta_{(1)}\delta\}$, we put $\theta(\beta_{(1)})=x\alpha^3+y\delta\alpha\delta\beta_{(1)}\delta$. By (6.5) and (3.10),

$$0=\theta(\alpha\beta_{(1)})=\alpha\theta(\beta_{(1)})=x\alpha^4+y\alpha\delta\alpha\delta\beta_{(1)}\delta$$
$$=x\alpha^4+y\delta\alpha\delta\alpha\beta_{(1)}\delta=x\alpha^4.$$

Thus $x=0$. By Theorem 6.1, (6.2) and (6.5)

$$\theta(\theta(\beta_{(1)}))=\delta\alpha\delta\beta_{(1)}-\beta_{(1)}\delta\alpha\delta=\delta\alpha\delta\beta_{(1)}-\alpha\delta\beta_{(1)}\delta$$

and

$$\theta(\delta\alpha\delta\beta_{(1)}\delta)=-\alpha\delta\beta_{(1)}\delta+\delta\alpha\beta_{(1)}\delta+\delta\alpha\delta\beta_{(1)}$$
$$=\delta\alpha\delta\beta_{(1)}-\alpha\delta\beta_{(1)}\delta.$$

Thus $y=1$ and the first formula is proved. Next $\theta([\beta i_1])\in\pi_{17}(M; V(1))=\{\alpha''[\beta i_1]\delta\}$ by (6.3). Put $\theta([\beta i_1])=x\alpha''[\beta i_1]\delta$, then by (5.6)

$$\delta\alpha\delta\beta_{(1)}\delta=\theta(\beta_{(1)})=\theta(\pi_1[\beta i_1])=-\pi_1\theta([\beta i_1])$$
$$=-x\pi_1\alpha''[\beta i_1]\delta=x\delta\alpha\delta\pi_1[\beta i_1]\delta=x\delta\alpha\delta\beta_{(1)}\delta.$$

Thus $x=1$ and the second formula follows. The last statement follows from Lemma 2.3. q.e.d.

Lemma 6.5. *$\alpha_V=\alpha''$ for $V=V(1)$.*

Proof. Since $\alpha_V\in\mathscr{A}_2(V(1))=\{\alpha''\}$, $\alpha_V=x\alpha''$ for some x. Since $\theta(i_1)=0$, i_1 is a Z_p-map: $i_1\mu_M=\mu_V(1_M\wedge i_1)$. Then we have

$$x\alpha''i_1(\pi\wedge\pi\wedge 1_M)=\alpha_V(\pi\wedge\pi\wedge 1_V)(1_M\wedge 1_M\wedge i_1)$$
$$=(\mu_V(1_M\wedge\mu_V)+\mu_V(\mu_M\wedge 1_V))(1_M\wedge 1_M\wedge i_1)$$
$$=\mu_V(1_M\wedge i_1\mu_M)+\mu_V(1_M\wedge i_1)(\mu_M\wedge 1_M)$$
$$=i_1(\mu_M(1_M\wedge\mu_M)+\mu_M(\mu_M\wedge 1_M))$$
$$=i_1\delta\alpha\delta(\pi\wedge\pi\wedge 1_M)=\alpha''i_1(\pi\wedge\pi\wedge 1_M).$$

Since $(\pi\wedge\pi\wedge 1_M)(\varphi_M\wedge 1_M)\varphi_M=1_M$, we have $x\alpha''i_1=\alpha''i_1$, $x=1$ and $\alpha_V=\alpha''$.

Lemma 6.6. $\lambda_V(\alpha\delta)=\beta'\delta_0$ *for* $V=V(1)$.

Proof. $\lambda_V(\alpha\delta)\in\mathscr{A}_4(V)=\{\beta'\delta_0\}$. Put $\lambda_V(\alpha\delta)=x\beta'\delta_0$, then

$$\begin{aligned} x\beta'\delta_1 &= -x\beta'\theta(\delta_0)=-x\theta(\beta'\delta_0)=-\theta(\lambda_V(\alpha\delta)) \\ &= -\lambda_V(\alpha\delta\delta)\alpha_V+\alpha_V\lambda_V(\delta\alpha\delta) && \text{by } (6.1) \\ &= \alpha''\alpha' && \text{by Lemma 6.5} \\ &= \beta'\delta_1 && \text{by Theorem 6.2.} \end{aligned}$$

Thus $x=1$, and $\lambda_V(\alpha\delta)=\beta'\delta_0$.

The following values are the obstructions to the existence of $V(2)$ and $V\left(1\frac{3}{4}\right)$, (see Lemma 6.4 of [14]).

Theorem 6.7. $[\beta i_1]\alpha=(\beta')^2 i_1+\beta'\delta_1[\beta i_1]\delta$, *thus* $[\beta i_1]\alpha i=(\beta')^2 i_0=i_0(\beta_1)^2$.

Proof. By Theorem 2.6, $\lambda_M(\alpha\delta)=-\theta(\alpha\delta)=-\alpha\theta(\delta)=\alpha$. Then by Theorem 2.4, (iii), Theorems 6.4, 6.2,

$$\begin{aligned} \lambda_V(\alpha\delta)[\beta i_1] &= [\beta i_1]\lambda_M(\alpha\delta)+\theta([\beta i_1])\lambda_M(\delta\alpha\delta) \\ &= [\beta i_1]\alpha+\alpha''[\beta i_1]\delta(\alpha\delta-\delta\alpha) \\ &= [\beta i_1]\alpha+\alpha'\alpha''[\beta i_1]\delta \\ &= [\beta i_1]\alpha+\beta'\delta_1[\beta i_1]\delta \end{aligned}$$

and by Lemma 6.6 and Lemma 3.4

$$\begin{aligned} \lambda_V(\alpha\delta)[\beta i_1] &= \beta'\delta_0[\beta i_1] \\ &= (\beta')^2 i_1-\beta'\delta_1[\beta i_1]\delta, \end{aligned}$$

and the theorem follows.

The following theorem corrects the parts of Yamamoto's relations which do not hold for $p=3$, (see Theorem 5.1).

Theorem 6.8. ($p=3$).

(i) $\beta_{(1)}\beta_{(1)}=\delta\alpha\delta\beta_{(1)}\delta\beta_{(1)}\delta$,

(ii) $\alpha\beta_{(2)}=\beta_{(2)}\alpha=\beta_{(1)}\delta\beta_{(1)}\delta\beta_{(1)}$,

(iii) $\beta_{(2)}\delta\alpha=\alpha\delta\beta_{(2)}+(\beta_{(1)}\delta)^3-(\delta\beta_{(1)})^3$,

(iv) $\theta(\beta_{(2)})=\delta\alpha\delta\beta_{(2)}\delta$.

Proof. By Theorem 6.4, $\theta(\beta_{(1)})\delta=\delta\theta(\beta_{(1)})=0$. Then by (2.10) and (6.5), (iii)

$$\beta_{(1)}\beta_{(1)}=-(\beta_{(1)}\beta_{(1)}+\beta_{(1)}\beta_{(1)})=\delta\beta_{(1)}\theta(\beta_{(1)})$$
$$=\delta\beta_{(1)}\delta\alpha\delta\beta_{(1)}\delta=\delta\alpha\delta\beta_{(1)}\delta\beta_{(1)}\delta.$$

Remark that

(6.6) $$\beta_{(1)}\beta_{(1)}\delta=\delta\beta_{(1)}\beta_{(1)}=0 \quad \text{and}$$
$$(\beta_{(1)}\delta+\delta\beta_{(1)})^k=(\beta_{(1)}\delta)^k+(\delta\beta_{(1)})^k.$$

Next Theorem 6.7 and (3.9) imply

$$\beta_{(2)}\alpha=[\pi_1\beta][\beta i_1]\alpha=[\pi_1\beta]((\beta')^2 i_1+\beta'\delta_1[\beta i_1]\delta)$$
$$=[\pi_1\beta]i_1(\beta_{(1)}\delta+\delta\beta_{(1)})^2+[\pi_1\beta]\delta_1[\beta i_1]\delta(\beta_{(1)}\delta+\delta\beta_{(1)})$$
$$=\beta_{(1)}(\beta_{(1)}\delta+\delta\beta_{(1)})^2+\beta_{(1)}\beta_{(1)}\delta\beta_{(1)}\delta$$
$$=\beta_{(1)}\delta\beta_{(1)}\delta\beta_{(1)}.$$

By (2.10)

$$\beta_{(2)}\alpha-\alpha\beta_{(2)}=\alpha\delta\theta(\beta_{(2)})-\delta\theta(\beta_{(2)})\alpha,$$

where $\theta(\beta_{(2)})\in\mathscr{A}_{28}=\{\alpha^7,\ \delta\alpha\delta\beta_{(2)}\delta\}$. Then $\beta_{(2)}\alpha-\alpha\beta_{(2)}$ is a multiple of $\alpha\delta\alpha^7-\alpha^7\delta\alpha=0$. Thus (ii) is proved.

By (3.8)′, $(\alpha\delta-\delta\alpha)\beta_{(2)}=-\beta_{(2)}(\alpha\delta-\delta\alpha)$. Thus

$$\beta_{(2)}\delta\alpha=\alpha\delta\beta_{(2)}-\delta\alpha\beta_{(2)}+\beta_{(2)}\alpha\delta$$
$$=\alpha\delta\beta_{(2)}-(\delta\beta_{(1)})^3+(\beta_{(1)}\delta)^3.$$

Finally, put $\theta(\beta_{(2)})=x\alpha^7+y\delta\alpha\delta\beta_{(2)}\delta$, then as in the proof Theorem 6.4.,

$$0=\theta(\beta_{(1)}\delta\beta_{(1)}\delta\beta_{(1)})\delta \qquad \text{by (6.6)}$$

$$=\theta(\alpha\beta_{(2)})\delta=\alpha\theta(\beta_{(2)})\delta=x\alpha^7\delta,$$

and

$$\theta\theta(\beta_{(2)})=\delta\alpha\delta\beta_{(2)}-\beta_{(2)}\delta\alpha\delta$$

$$=\delta\alpha\delta\beta_{(2)}-\alpha\delta\beta_{(2)}\delta+(\delta\beta_{(1)})^3\delta,$$

$$\theta(\delta\alpha\delta\beta_{(2)}\delta)=-\alpha\delta\beta_{(2)}\delta+\delta\alpha\beta_{(2)}\delta+\delta\alpha\delta\beta_{(2)}$$

$$=\delta\alpha\delta\beta_{(2)}-\alpha\delta\beta_{(2)}\delta+(\delta\beta_{(1)})^3\delta.$$

Thus we have $x=0$, $y=1$ and (iv) follows. q.e.d.

By this theorem and (6.4), (6.5) the algebra structure of $\mathscr{A}_*(M)$ is determined for degree <32.

The module $\pi_*(M; V(1))$ has the following basis.

Proposition 6.9. For degree <32

$$\pi_*(M; V(1))=[P(\beta')\otimes B+\{1, \beta'\}\otimes\{[\delta_1\beta]i_1\}]\otimes E(\delta)$$

where $B=\{i_1, \alpha' i_1, [\beta i_1], \alpha''[\beta i_1], \alpha'[\beta i_1], [\delta_0\beta][\beta i_1], [\delta_1\beta][\beta i_1], \alpha'[\delta_0\beta][\beta i_1]\}$. *Comparing with Corollary* 3.3 *the differences are the relations* $(\beta')^2\delta_1[\beta i_1]=(\beta')^2\delta_1[\beta i_1]\delta=0$ *and the lack of* $\beta^2 i_1$ *and* $\beta^2 i_1\delta$.

This is computed from (6.4) as a converse of the proof of Theorem 5.2. The only difference is

$$\alpha_*(\beta_{(2)})\delta^\varepsilon)=\alpha\beta_{(2)}\delta^\varepsilon=\beta_{(1)}(\delta\beta_{(1)})^2\delta^\varepsilon \qquad (\varepsilon=0, 1)$$

and thus $i_{1*}(\beta_{(1)}(\delta\beta_{(1)})^2\delta^\varepsilon)=i_1\beta_{(1)}(\beta_{(1)}\delta+\delta\beta_{(1)})^2\delta^\varepsilon=(\beta')^2 i_1\beta_{(1)}\delta^\varepsilon=(\beta')^2\delta_1[\beta i_1]\delta^\varepsilon=0$.

Next consider the exact sequence (3.6)*:

$$\cdots\rightarrow\mathscr{A}_k(V(1))\xrightarrow{i_{1*}}\pi_k(M; V(1))\xrightarrow{\alpha^*}\pi_{k+4}(M; V(1))\xrightarrow{\pi_{1*}}\cdots,$$

and check the proof of Theorem 3.6. First consider α^* for $k+4<32$. For $\xi=1, \alpha', [\delta_1\beta]$,

$$\alpha^*((\beta')^r\xi i_1)=(\beta')^r\xi i_1\alpha=0$$

and

$$\alpha_*((\beta')^r\xi i_1\delta)=(\beta')^r\xi i_1\delta\alpha=(\beta')^r\xi i_1(\delta\alpha-\alpha\delta)$$
$$=\pm(\beta')^r\alpha'\xi i_1 \qquad \text{by (3.9).}$$

Thus $\alpha^*((\beta')^r\xi i_1\delta)=(\beta')^r\alpha' i_1$, 0, 0 for each value of ξ. For $\eta=1, \alpha''$, α', $[\delta_0\beta]$, $[\delta_1\beta]$, we have by Theorem 6.7 (omitting $(\beta')^r$)

$$\alpha^*(\eta[\beta i_1])=(\beta')^2\eta i_1+\beta'\eta\delta_1[\beta i_1]\delta$$

and

$$\alpha^*(\eta[\beta i_1]\delta)=\eta[\beta i_1]\alpha\delta-\eta[\beta i_1](\alpha\delta-\delta\alpha)$$
$$=(\beta')^2\eta i_1\delta-\alpha'\eta[\beta i_1].$$

The case $\eta=1$ is obvious. For $\eta=\alpha''$, $\alpha'' i_1=\alpha' i_1\delta$, $\alpha'\alpha''=\beta'\delta_1$ and $\alpha''\delta_1[\beta i_1]=\alpha'\delta_0[\beta i_1]\delta=\alpha' i_0\beta_1\pi=\beta'\alpha' i_1\delta$. Thus

$$\alpha^*(\alpha''[\beta i_1])=-(\beta')^2\alpha' i_1\delta \quad \text{and} \quad \alpha^*(\alpha''[\beta i_1]\delta)=\beta'[\delta_1\beta]i_1.$$

For $\eta=\alpha'$, since $\alpha'\delta_1=\alpha'\alpha'=0$,

$$\alpha^*(\alpha'[\beta i_1])=(\beta')^2\alpha' i_1 \quad \text{and} \quad \alpha^*(\alpha'[\beta i_1]\delta)=(\beta')^2\alpha' i_1\delta.$$

For $\eta=[\delta_0\beta]$, $[\delta_0\beta]i_1=\beta' i_1-[\delta_1\beta]i_1\delta$ by Lemma 3.4, $[\delta_0\beta]\delta_1[\beta i_1]\delta$ $=\beta'\delta_1[\beta i_1]\delta-[\delta_1\beta]\delta_0[\beta i_1]\delta=\beta'[\delta_1\beta]i_1\delta-[\delta_1\beta]\beta' i_1\delta=0$, $[\delta_0\beta]i_1\delta$ $=\beta' i_1\delta$, thus

$$\alpha^*([\delta_0\beta][\beta i_1])=(\beta')^3 i_1 \qquad \text{by Proposition 6.9}$$

and

$$\alpha^*([\delta_0\beta][\beta i_1]\delta)=(\beta')^3 i_1\delta-\alpha'[\delta_0\beta][\beta i_1].$$

For $\eta=[\delta_1\beta]$, $\alpha^*([\delta_1\beta][\beta i_1])=0$ since $\pi_{31}(M; V(1))=0$, $(\beta')^2[\delta_1\beta]i_1\delta=0$ by Proposition 6.9, and

$$\alpha'[\delta_1\beta][\beta i_1]=\alpha' i_1\beta_{(2)}=i_1\delta\alpha\beta_{(2)}=i_1(\delta\beta_{(1)})^3$$
$$=i_1(\delta\beta_{(1)}+\beta_{(1)}\delta)^3-i_1(\delta\beta_{(1)}+\beta_{(1)}\delta)^2\beta_{(1)}\delta$$
$$=(\beta')^3 i_1-(\beta')^2\delta_1[\beta i_1]\delta=(\beta')^3 i_1.$$

Thus

$$\alpha^*([\delta_1\beta][\beta i_1])=0 \quad \text{and} \quad \alpha^*([\delta_1\beta][\beta i_1]\delta)=-(\beta')^3 i_1.$$

These formulas determine α^*. As relations of the form $0=\pi_1^*\alpha^*(\ )=[\alpha^*(\ )]\pi_1$, we have

Theorem 6.10. *The following relations hold* ($p=3$).

(i) $(\beta')^2\delta_0=\alpha'[\beta\delta_1]$, $\qquad (\beta')^2\delta_1=-\beta'[\delta_1\beta]\delta_0$,

(ii) $(\beta')^2\alpha'\delta_0=\beta'[\delta_1\beta]\delta_1=(\beta')^3\delta_1=(\beta')^2[\delta_1\beta]\delta_0=0$.

The kernel of α^* is spanned by $(0\leq r\leq 2,\ 0\leq \varepsilon\leq 1)$

$$(\beta')^r\xi i_1=i_1^*((\beta')^r\xi) \qquad \text{for } \xi=1, \alpha',$$

$$(\beta')^r\alpha' i_1\delta=i_1^*((\beta')^r\alpha''),$$

$$(\beta')^\varepsilon[\delta_1\beta]i_1=i_1^*((\beta')^\varepsilon[\delta_1\beta]),$$

$$(\beta')^\varepsilon[\delta_1\beta]i_1\delta=i_1^*((\beta')^{\varepsilon+1}-(\beta')^\varepsilon[\delta_0\beta]),$$

$$\alpha''[\beta i_1]+\alpha'[\beta i_1]\delta=i_1^*[\alpha''\beta+\beta\alpha''],$$

$$\alpha'[\beta i_1]-(\beta')^2 i_1\delta=i_1^*[\alpha'\beta],$$

$$[\delta_1\beta][\beta i_1]\delta+\beta'[\beta i_1]=i_1^*[\beta\delta_0\beta]$$

and

$$[\delta_0\beta][\beta i_1]-\beta'[\beta i_1]=i_1^*[\delta_0\beta^2]$$

where the last four equations define new elements in the right hand sides.

The image of π_1^* is spanned by (degree<27)

$$(\beta')^r\delta_0(0\leq r\leq 3),\ (\beta')^r(0\leq r\leq 2),$$

$$(\beta')^\varepsilon\alpha'\delta_0,\ [\delta_1\beta]\delta_{\varepsilon'},\ (\beta')^\varepsilon[\beta\delta_{\varepsilon'}],\ (\beta')^\varepsilon\alpha''[\beta\delta_{\varepsilon'}],$$

$$(\beta')^\varepsilon\alpha'[\beta\delta_0],\ [\delta_\varepsilon\beta][\beta\delta_{\varepsilon'}],\ \alpha'[\delta_0\beta][\beta\delta_0],$$

where $\varepsilon, \varepsilon'=0, 1$.

Consequently we have

Theorem 6.11. ($p=3$). *For degree* $<27=(p^2-1)q-5$, $\mathscr{A}_*(V(1))$ *have a basis which consists of the above* π_1^**-images and* i_1^**-anti-images. More precisely,* $\mathscr{A}_k(V(1))$ *has a basis as in Theorem* 3.6 *for* $k\leq 14$ *and for* $20\leq k\leq 24$, $\mathscr{A}_{15}=\{(\beta')^2\delta_1\}$, $\mathscr{A}_{16}=\mathscr{A}_{17}=0$, $\mathscr{A}_{18}=\{[\alpha''\beta+\beta\alpha'']\}$, $\mathscr{A}_{19}=\{[\alpha'\beta]\}$, $\mathscr{A}_{25}=0$, $\mathscr{A}_{26}=\{[\beta\delta_0\beta], [\delta_0\beta^2]\}$.

Finally we remark on some of the easy relations:

$$[\delta_\varepsilon\beta]\delta_{\varepsilon'}=\delta_\varepsilon[\beta\delta_{\varepsilon'}],$$

$$[\delta_0\beta]\delta_0=\beta'\delta_0,$$

$$[\delta_0\beta]\delta_1+[\delta_1\beta]\delta_0=\beta'\delta_1,$$

$$\alpha''\delta_1=\delta_1\alpha''=\alpha'\delta_0, \quad \delta_\varepsilon\delta_{\varepsilon'}=0, \quad \alpha'\alpha'=0,$$

$$[\beta\delta_\varepsilon][\delta_{\varepsilon'}\beta]=0,$$

$$[\beta\delta_0]\alpha''=\alpha''[\delta_0\beta]=0,$$

$$[\beta\delta_1]\alpha''=\alpha'[\beta\delta_0].$$

7. The Case $p=5$.

For the case $p=5$, the only difference from the general p is Theorem 4.5:

$$\beta''\beta''=(\beta')^2\delta_1\beta\delta_1 \qquad (p=5)$$

up to non-zero coefficient. Here $\beta''\beta''\in\mathscr{A}_{(2p+4)q-6}(V(1))=\mathscr{A}_{(3p-1)q-6}(V(1))=\{(\beta')^2\delta_1\beta\delta_1\}$.

Let n be sufficiently large, $C_{\alpha_1}=S^n\cup_{\alpha_1}e^{n+q}$ a mapping cone of α_1 and let

$$S^n\xrightarrow{i_C}C_{\alpha_1}\xrightarrow{\pi_C}S^{n+q}$$

be the cofibering.

Lemma 7.1. (i) *There exists an element* α_0 *of the p-component of* $\mathscr{A}_{2q-1}(C_{\alpha_1})$ *such that* $\pi_C\alpha_0 i_C=\alpha_1$.

(ii) *If* $p=5$, $\alpha_0\alpha_0=i_C\beta_1\pi_C$ *up to non-zero coefficient.*

Proof. By the exactness of the sequences (1.11), (1.11)* for $\alpha=\alpha_1$ and by the triviality of the p-components of G_{2q-2} and G_{3q-2}, $\pi_{C^*}: \pi^S_{n+2q-1}(C_{\alpha_1})\to G_{q-1}$ and $i^*_C: \mathscr{A}_{2q-1}(C_{\alpha_1})\to\pi^S_{n+2q-1}(C_{\alpha_1})$ are epimorphisms of the p-components. Then (i) follows. Similarly from the mod p triviality of G_{3q-2} and G_{4q-2}, $\pi^*_C: G_{5q-2}\to\pi^S_{n+4q-2}(C_{\alpha_1}; S^n)$ and $i_{C^*}: \pi^S_{n+4q-2}(C_{\alpha_1}; S^n)\to\mathscr{A}_{4q-2}(C_{\alpha_1})$ are mod p epimorphisms. For $p=5$, the p-component of G_{5q-2} is generated by β_1. Thus $\alpha_0\alpha_0=xi_C\beta_1\pi_C$ for some $x\in Z_p$. Now assume that $x=0$, then $\alpha_0\alpha_0=0$ and there exists an extension $A: \sum^{2q-1}C_{\alpha_0}=\sum^{2q-1}(C_{\alpha_1}\cup C\sum^{2q-1}C_{\alpha_1})\to C_{\alpha_1}$ of α_0. Since $\mathscr{P}^1\neq 0$ in C_{α_1} and since α_1 is detected by $\mathscr{P}^1$, $\mathscr{P}^1\mathscr{P}^1\mathscr{P}^1=\mathscr{P}^3\neq 0$ in C_{α_0}. Thus $\mathscr{P}^3\mathscr{P}^1\mathscr{P}^1\neq 0$ in C_A but this contradicts to the Adam relation $\mathscr{P}^3\mathscr{P}^1\mathscr{P}^1=10\mathscr{P}^5=0$ $(p=5)$. Thus $x\neq 0$ and the lemma is proved. q.e.d.

$$\alpha_1=\pi\alpha i: S^{n+q-1}\to S^n \qquad \text{for } \alpha: \textstyle\sum^{n+q-2}M_p\to\sum^{n-2}M_p.$$

Then $C_\alpha=V(1)_{n-1}$, $C_{\alpha i}=V\left(\frac{1}{2}\right)_{n-1}$, $C_{\pi\alpha}=V(1)_{n-1}/S^{n-1}$ and we have the following commutative diagram of cofiberings.

$$\begin{array}{ccccc}
S^{n-1} & & S^{n-1} & & \\
\downarrow & & \downarrow & & \\
C_{\alpha i} & \xrightarrow{i'} & C_\alpha & \longrightarrow & S^{n+q+1} \\
\downarrow{\scriptstyle \pi''} & & \downarrow{\scriptstyle \pi'} & & \\
C_{\alpha_1} & \xrightarrow{i''} & C_{\pi\alpha} & \longrightarrow & S^{n+q+1}
\end{array}$$

Lemma 7.2 *There exists an element β''_0 of $\pi^S_{(p+2)q-3}((C_{\pi\alpha}; C_{\alpha i})$ such that $\pi''\beta''_0 i''=\beta_1\wedge\alpha_0$ and $i'\beta''_0\pi'=\beta''$.*

Proof. From the cofibering $S^{n-1}\to C_{\alpha i}\xrightarrow{\pi''}C_{\alpha_1}$ we have and exact sequence $\pi^S_{(p+2)q-3}(C_{\alpha_1}; C_{\alpha i})\xrightarrow{\pi''_*}\mathscr{A}_{(p+2)q-3}(C_{\alpha_1})\to\pi^S_{(p+2)q-4}(C_{\alpha_1}; S^{n-1})$. The mod p triviality of the last group follows from that of $G_{(p+2)q-3}$ and $G_{(p+3)q-3}$. Then π''_* is an epimorphism of the p-components. Similarly, we have the mod p triviality of $\pi_{(p+2)q-4}(S^{n+q+1}; C_{\alpha i})$ from that of $G_{(p+2)q-3}$, $G_{(p+3)q-2}$, $G_{(p+3)q-3}$, and thus $i''^*: \pi^S_{(p+2)q-3}(C_{\pi\alpha}; C_{\alpha i})\to\pi^S_{(p+2)q-3}(C_{\alpha_1}; C_{\alpha i})$ is a mod p epimorphism. This shows the existence of

β_0'' satisfying $\pi''\beta_0''i''=\beta_1\wedge\alpha_0\in\mathscr{A}_{(p+2)q-3}(C_{\alpha_1})$. Next $i'\beta_0''\pi'\in\mathscr{A}_{(p+2)q-3}(C_\alpha=V(1)_{n-1})=\{\beta''\}$, so $i'\beta_0''\pi'=x\beta''$. By Lemma 3.5, (4.4) and (4.1)

$$\delta_1\beta''i_1=\delta_1\alpha''\beta i_1\delta=\alpha'\delta_0\beta i_0\pi=\alpha' i_0\beta_1\pi$$
$$=\beta'\alpha' i_0\pi=\beta'\alpha' i_1\delta.$$

Apparently $\pi_1 i'=i\pi_C\pi''$ and $\pi' i_1=i'' i_C\pi$, then

$$\delta_1(i'\beta_0''\pi')i_1=i_1\pi_1 i'\beta_0''\pi' i_1=i_0\pi_C\pi''\beta_0'' i'' i_C\pi$$
$$=i_0(1_0\wedge\pi_C)(\beta_1\wedge\alpha_0)(1_0\wedge i_C)\pi=i_0(\beta_1\wedge\alpha_1)\pi$$
$$=i_0\beta_1\alpha_1\pi=\beta'\alpha' i_1\delta.$$

By Corollary 3.3, $\beta'\alpha' i_1\delta\neq 0$. Thus $x=1$ and $i'\beta_0''\pi'=\beta''$. q.e.d.

Proof of Theorem 4.5. By Lemmas 7.1 and 7.2 we have

$$\pi'(\beta''\beta'')i'=\pi' i'\beta_0''\pi' i'\beta_0''\pi' i'$$
$$=i''\pi''\beta_0'' i''\pi''\beta_0'' i''\pi''=i''(\beta_1\wedge\alpha_0)(\beta_1\wedge\alpha_0)\pi''$$
$$=i''(\beta_1^2\wedge i_C\beta_1\pi_C\pi''=i'' i_C\beta_1^3\pi_C\pi''$$
$$=i'' i_C\beta_1^2\pi\beta_{(1)} i\pi_C\pi''=i'' i_C\pi(\beta_1^2\wedge 1_M)\beta_{(1)} i\pi_C\pi''$$
$$=\pi' i_1(\beta_1^2\wedge 1_M)\beta_{(1)}\pi_1 i'=\pi'(\beta')^2 i_1\beta_{(1)}\pi_1 i'$$
$$=\pi'((\beta')^2\delta_1\beta\delta_1)i'.$$

Thus it is sufficient to prove that $i'^*\colon\mathscr{A}_{(3p-1)q-6}(C_\alpha)\to\pi^S_{(3p-1)q-6}(C_{\alpha i};C_\alpha)$ and $\pi'_*\colon\pi^S_{(3p-1)q-6}(C_{\alpha i};\ C_\alpha)\to\pi^S_{(3p-1)q-6}(C_{\alpha i};C_{\pi\alpha})$ are mod p monomorphism. The kernel of these homomorphisms are images of $\pi^S_{(3p-1)q-6}(S^{n+q+1};C_\alpha)$ and $\pi^S_{(3p-1)q-6}(C_{\alpha i};\ S^{n-1})$. By Theorem 3.2, $\pi^S_{3(p-1)q-6}(S^{n+q+1};C_\alpha)=\pi^S_{(3p-1)q-6}(S^{n+q+1};\ V(1)_{n-1})=\pi_{3pq-4}(V(1))=0$. Also the mod p triviality of $\pi^S_{(3p-1)q-6}(C_{\alpha i};S^{n-1})$ follows from that of $G_{(3p-1)q-6}$, $G_{(3p-1)q-5}$ and G_{3pq-5}. Thus i'^* and π'_* are mod p monomorphisms and we have obtained the equality $\beta''\beta''=(\beta')^2\delta_1\beta\delta_1$.

KYOTO UNIVERSITY

Bibliography

[1] J.F. Adams, *On the group $J(X)$-IV*, Topology, **5** (1966), 21-71.

[2] S. Araki and H. Toda, *Multiplicative structures in* mod q *cohomology theories I, II*, Osaka J. Math., **2** (1965), 71-115, 3 (1966), 81-120.

[3] P. Hoffman, *Relations in the stable homotopy rings of Moore spaces*, Proc. London Math. Soc., (3) **18** (1968), 621-634.

[4] A. Liulevicius, *The factorization of cyclic reduced powers by secondary cohomology operations*, Mem. Amer. Math. Soc., **42** (1962).

[5] N. Shimada and T. Yamanoshita, *On triviality of the* mod p *Hopf invariant*, Jap. Jour. of Math., **31** (1961), 1-24.

[6] L. Smith, *On realizing complex bordism modules. Applications to the homotopy of spheres*, Amer. J. Math., **92** (1970), 793-856.

[7] N.E. Steenrod and D.B.A. Epstein, *Cohomology operations*, Princeton, 1962.

[8] H. Toda, *p-primary components of homotopy groups III*, Mem. Coll. Sci., Univ. of Kyoto, **31** (1958), 191-210.

[9] H. Toda, *Order of the identity class of a suspension space*, Ann. of Math., **78** (1963), 300-325.

[10] H. Toda, *Composition methods in homotopy groups of spheres*, Princeton, 1962.

[11] H. Toda, *An important relations in homotopy groups of spheres*, Proc. Japan Acad. Sci., **43** (1967). 839-842.

[12] H. Toda, *On iterated suspensions III*, J. of Math., Kyoto Univ., **8** (1969), 101-130.

[13] H. Toda, *Extended powers of complexes and applications to homotopy theory*, Proc. Japan Acad. Sci., **44** (1968), 108-203.

[14] H. Toda, *On spectra realizing exterior parts of the Steenrod algebra*, Topology, **10** (1971), 53-65.

[15] N. Yamamoto, *Algebra of stable homotopy of Moore spaces*, J. Math., Osaka City Univ., **14** (1963), 45-67.

[16] N. Yamamoto, *An application of functional higher operation*, Osaka J. Math., **2** (1965), 37-62.