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On stochastic differential equations for multi-dimensional diffusion processes with boundary conditions II

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The present paper consists of complementary remarks on our previous paper [3]. We will discuss here the case of general sticky boundaries. Thus, our probabilistic construction can cover the diffusion processes with Wentzell's boundary conditions without jump terms.

Let $\bar{D}=R_n^+=\{x=(x^1, x^2, \cdots, x^n)\in R^n;\ x^1\geqq 0\}$,

$$D=\{x\in\bar{D};\ x^1>0\} \text{ and } \partial D=\{x\in\bar{D};\ x^1=0\}.$$

Let σ, b, τ, β and ρ be given as follows:

$$\sigma=(\sigma_j^i)_{i,j=1}^n;\qquad \bar{D}\to R^n\otimes R^n,$$

$$b=(b^i)_{i=1}^n\ ;\qquad \bar{D}\to R^n,$$

$$\tau=(\tau_j^i)_{i,j=2}^n;\qquad \partial D\to R^{n-1}\otimes R^{n-1},$$

$$\beta=(\beta^i)_{i=2}^n\ ;\qquad \partial D\to R^{n-1},$$

$$\rho;\qquad \partial D\to[0, \infty).$$

We assume that they are all bounded and Borel measurable. We consider the following stochastic differential equation of the process $x_t=(x_t^1, x_t^2, \cdots, x_t^n)$;

$$(1)\qquad \begin{cases} dx_t^1=\sigma^1(x_t)\,I_D(x_t)\,dB_t+b^1(x_t)\,I_D(x_t)\,dt+d\varphi_t, \\ dx_t^i=\sigma^i(x_t)\,I_D(x_t)\,dB_t+b^i(x_t)\,I_D(x_t)\,dt+\tau^i(x_t)\,I_{\partial D}(x_t)\,dM_t \end{cases}$$

$$\left|\begin{array}{l} \qquad +\beta^i(x_t)\, I_{\partial D}(x_t)\, d\varphi_t, \qquad i=2, 3, \cdots, n, \\ I_{\partial D}(x_s)\, ds=\rho(x_s)\, d\varphi_s, \end{array}\right.$$

where $B_t=(B_t^1, B_t^2, \cdots, B_t^n)$, $M_t=(M_t^2, \cdots, M_t^n)$,

$$\sigma^i(x_t)\, I_D(x_t)\, dB_t=\sum_{j=1}^n \sigma_j^i(x_t)\, I_D(x_t)\, dB_t^j, \qquad i=1, 2, \cdots, n,$$

and

$$\tau^i(x_t)\, I_{\partial D}(x_t)\, dM_t=\sum_{j=2}^n \tau_j^i(x_t)\, I_{\partial D}(x_t)\, dM_t^j, \qquad i=2, 3, \cdots, n.$$

A precise formulation is as follows; by a probability space with an increasing family of Borel fields $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$, we mean a probability space $(\Omega, \mathscr{F}, P)$ with a system $\{\mathscr{F}_t\}_{t\in[0,\infty)}$ of sub-Borel fields of $\mathscr{F}$ such that it is increasing and right-continuous, i.e., $\mathscr{F}_t\subset\mathscr{F}_s$ if $t<s$ and $\mathscr{F}_{t+0}\equiv\bigcap_{\varepsilon>0}\mathscr{F}_{t+\varepsilon}=\mathscr{F}_t$. We assume always that $(\Omega, \mathscr{F}, P)$ is a standard probability space in the sense of K. Ito [1].

Definition 1. By a solution of equation (1), we mean a family of stochastic processes $\mathfrak{X}=\{x_t=(x_t^1, x_t^2, \cdots, x_t^n), B_t=(B_t^1, B_t^2, \cdots, B_t^n), M_t=(M_t^2, \cdots, M_t^n), \varphi_t\}$ defined on a probability space with an increasing family of Borel fields $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$ such that,

(i) with probability one, they are all continuous in t such that $B_0=0$, $M_0=0$ and $\varphi_0=0$,

(ii) they are all adapted to $\mathscr{F}_t$, i.e., for fixed t, they are $\mathscr{F}_t$-measurable,

(iii) with probability one, $x_t\in\bar{D}$ for all t and φ_t is non-decreasing; further, φ_t increases only when $x_t^1=0$, i.e.,

$$\varphi_t=\int_0^t I_{\partial D}(x_s)\, d\varphi_s,$$

(iv) (B_t, M_t) is a system of $\mathscr{F}_t$-martingales such that

$$\langle B^i, B^j\rangle_t=\delta_{ij}t, \ \langle B^i, M^j\rangle_t=0 \quad \text{and} \quad \langle M^i, M^j\rangle_t=\delta_{ij}\varphi_t,$$

(v) $\mathfrak{X}=\{x_t, B_t, M_t, \varphi_t\}$ satisfies

$$(1') \quad \begin{cases} x_t^1 - x_0^1 = \int_0^t \sigma^1(x_s)\, I_D(x_s)\, dB_s + \int_0^t b^1(x_s)\, I_D(x_s)\, ds + \varphi_t, \\ x_t^i - x_0^i = \int_0^t \sigma^i(x_s)\, I_D(x_s)\, dB_s + \int_0^t b^i(x_s)\, I_D(x_s)\, ds \\ \qquad + \int_0^t \tau^i(x_s)\, I_{\partial D}(x_s)\, dM_s + \int_0^t \beta^i(x_s)\, I_{\partial D}(x_s)\, d\varphi_s, \\ \qquad\qquad i = 2, 3, \cdots, n, \\ \int_0^t I_{\partial D}(x_s)\, ds = \int_0^t \rho(x_s)\, d\varphi_s, \end{cases}$$

where $\int dB$ and $\int dM$ are stochastic integrals.

Definition 2. We shall say that the uniqueness holds for (1) if, for any two solutions $\mathfrak{X}=(x_t, B_t, M_t, \varphi_t)$ and $\mathfrak{X}'=(x_t', B_t', M_t', \varphi_t')$ (which may be defined on different probability spaces) such that $x_0=x$ and $x_0'=x$ a.s. for some $x \in \bar{D}$, the probability law of the processes x_t and x_t' on the space $\{W^+, \mathscr{B}(W^+)\}$ coinsides, where W^+ is the Fréchet space of all $\bar{D}$-valued continuous functions on $[0, \infty)$ with the compact uniform topology and $\mathscr{B}(W^+)$ is the topological Borel field on W^+.

Now Proposition 1 of [3] holds and hence, if we can show the existence and the uniqueness of solutions $\mathfrak{X}=(x_t, B_t, M_t, \varphi_t)$, then x_t is a diffusion process on $\bar{D}$.

Theorem 1. *Suppose σ, b, τ, β and ρ are bounded and Borel measurable. Suppose, further, that σ, b, τ, β are Lipschitz continuous and that a constant $c>0$ exists such that*

$$|\sigma^1(x)| = \left(\sum_{j=1}^n \sigma_j^1(x)^2\right)^{1/2} \geqq c.$$

Then, for any probability law μ on $\bar{D}$, a solution $\mathfrak{X}=(x_t, B_t, M_t, \varphi_t)$ of (1) *exists such that $P(x_0 \in dx) = \mu(dx)$. Furthermore, the uniqueness of solutions holds.*

Proof. (1°) *Existence.* The existence is proved in [3] if $\rho\equiv 0$. $\left(\text{It is implicitly shown in [3] that the solution constructed satisfies } \int_0^t I_{\partial D}(x_s)\,ds=0.\right)$

Let $\mathfrak{X}=(x_t, B_t, M_t, \varphi_t)$ be a solution corresponding to $[\sigma, b, \tau, \beta, \rho\equiv 0]$ on $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$. We take $(\Omega, \mathscr{F})$ sufficiently large so that there exists an n-dimensional Brownian motion $\hat{B}_t$ on $(\Omega, \mathscr{F}, P)$ such that $\hat{B}$ and $\mathfrak{X}$ are independent. Let $A_t=t+\int_0^t \rho(x_s)\,d\varphi_s$ and A_t^{-1} be the inverse function of $t\to A_t$. Set

$$\tilde{x}_t=x_{A_t^{-1}},\quad \tilde{M}_t=M_{A_t^{-1}},\quad \tilde{\varphi}_t=\varphi_{A_t^{-1}},$$

$$\tilde{\mathscr{F}}_t=\text{the } \sigma\text{-field generated by } \mathscr{F}_{A_t^{-1}} \text{ and } \{\hat{B}_s, s\leq t\}$$

and
$$\tilde{B}_t=B_{A_t^{-1}}+\int_0^t I_{\partial D}(\tilde{x}_s)\,d\hat{B}_s.$$

Then $\tilde{\mathfrak{X}}=(\tilde{x}_t, \tilde{B}_t, \tilde{M}_t, \tilde{\varphi}_t)$ is a solution on $(\Omega, \mathscr{F}, P; \tilde{\mathscr{F}}_t)$ corresponding to $[\sigma, b, \tau, \beta, \rho]$. This can be proved easily by Doob's optional sampling theorem and the following easily verified relations;

$$A_t^{-1}=\int_0^t I_D(\tilde{x}_s)ds,\quad \int_0^t I_{\partial D}(\tilde{x}_s)ds=\int_0^t \rho(\tilde{x}_s)d\tilde{\varphi}_s.$$

(2°) *Uniqueness.* Let $\tilde{\mathfrak{X}}=(\tilde{x}_t, \tilde{B}_t, \tilde{M}_t, \tilde{\varphi}_t)$ be a solution corresponding to $[\sigma, b, \tau, \beta, \rho]$. Set $\tilde{A}_t=\int_0^t I_D(\tilde{x}_s)\,ds$. Then, with probability one, $\tilde{A}_t$ is strictly increasing in t. In fact, if, for rationals $0\leqq r_1<r_2$, $\tilde{A}_{r_2}-\tilde{A}_{r_1}=0$, this would imply $\tilde{x}_s\in\partial D$, ${}^\forall s\in[r_1, r_2]$ and hence,

$$\int_{r_1}^{r_2} I_{\partial D}(\tilde{x}_s)\,ds=\int_{r_1}^{r_2}\rho(\tilde{x}_s)d\tilde{\varphi}_s=r_2-r_1>0,$$

impling $\tilde{\varphi}_{r_2}-\tilde{\varphi}_{r_1}>0$. On the other hand,

$$\int_{r_1}^{r_2}\sigma^1(\tilde{x}_s)\,I_D(\tilde{x}_s)\,d\tilde{B}_s=0,\quad \int_{r_1}^{r_2} b^1(\tilde{x}_s)\,I_D(\tilde{x}_s)\,ds=0$$

and hence, we would have

$$\tilde{x}^1_{r_2} = \tilde{x}^1_{r_1} + \tilde{\varphi}_{r_2} - \tilde{\varphi}_{r_1} > \tilde{x}^1_{r_1}$$

and this contradicts $\tilde{x}^1_{r_2} = 0$.

Thus, the inverse function $\tilde{A}_t^{-1}$ is continuous. Define $\mathfrak{X} = (x_t, B_t, M_t, \varphi_t)$ by

$$x_t = \tilde{x}_{\tilde{A}_t^{-1}}, \quad B_t = \int_0^{\tilde{A}_t^{-1}} I_D(\tilde{x}_s)\, d\tilde{B}_s, \quad M_t = \tilde{M}_{\tilde{A}_t^{-1}} \quad \text{and} \quad \varphi_t = \tilde{\varphi}_{\tilde{A}_t^{-1}}.$$

Then $\mathfrak{X} = (x_t, B_t, M_t, \varphi_t)$ is a solution corresponding to $[\sigma, b, \tau, \beta, \rho \equiv 0]$. Furthermore, since

$$t = \int_0^t I_D(\tilde{x}_s)\, ds + \int_0^t I_{\partial D}(\tilde{x}_s)\, ds = A_t + \int_0^t \rho(\tilde{x}_s)\, d\tilde{\varphi}_s,$$

we have

$$\tilde{A}_t^{-1} = t + \int_0^t \rho(x_s)\, d\varphi_s.$$

This shows that $\tilde{\mathfrak{X}}$ is obtained from $\mathfrak{X}$ as in (1°). Since, as we have shown in [3], the joint distribution of the solution $\mathfrak{X}$ is unique, we have the uniqueness of $\tilde{\mathfrak{X}}$. Q.E.D.

Remark 1. By a generalized Ito's formula on stochastic integrals (cf. [2]), we have, for $f \in \mathbf{C}_0^2(\bar{D})$ (=the space of all twice continuously differentiable functions on $\bar{D}$ with compact support),

$$\begin{aligned} f(x_t) - f(x_0) &= \text{a martingale} + \int_0^t Af(x_s)\, I_D(x_s)\, ds \\ &\quad + \int_0^t Lf(x_s)\, I_{\partial D}(x_s)\, d\varphi_s \\ &= \text{a martingale} + \int_0^t Af(x_s)\, ds + \int_0^t Lf(x_s)\, I_{\partial D}(x_s)\, d\varphi_s \\ &\quad - \int_0^t Af(x_s)\, I_{\partial D}(x_s)\, ds \\ &= \text{a martingale} + \int_0^t Af(x_s)\, ds + \int_0^t (Lf - \rho \cdot Af)(x_s)\, I_{\partial D}(x_s)\, d\varphi_s, \end{aligned}$$

where

$$Af(x)=\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 f}{\partial x^i \partial x^j}+\sum_{i=1}^{n} b^i(x)\frac{\partial f}{\partial x^i}, \qquad x\in \bar{D}$$

and

$$Lf(x)=\sum_{i,j=2}^{n} \alpha^{ij}(x)\frac{\partial^2 f}{\partial x^i \partial x^j}+\sum_{i=2}^{n} \beta^i(x)\frac{\partial f}{\partial x^i}+\frac{\partial f}{\partial x^1}, \qquad x\in \partial D$$

with

$$2a^{ij}=\sum_{k=1}^{n}\sigma_k^i\sigma_k^j \quad\text{and}\quad 2\alpha^{ij}=\sum_{k=2}^{n}\tau_k^i\tau_k^j.$$

Thus, we see that the infinitesimal generator of the semigroup of the diffusion process constructed in Theorem 1 (which turns out to be Fellerian if ρ is continuous) is an extention of the differential operator A with the domain $\mathscr{D}(A)=\{f\in \mathbf{C}_0^2(\bar{D});\ Lu=\rho\cdot Au \text{ on } \partial D\}$.

Remark. 2. The time-dependent case, i.e., the case when the coefficients $[\sigma, b, \tau, \beta, \rho]$ are functions of (t, x), $t_0\leqq t<\infty$, $x\in\bar{D}$, can be discussed in our framework: for simplicity, we assume $t_0=0$. Let $\bar{D}'=\{\tilde{x}=(x, x^{n+1});\ x\in\bar{D},\ x^{n+1}\in R^1\}\equiv R_+^{n+1}$ and set

$$\tilde{\sigma}^{ij}(\tilde{x})=\begin{cases}\sigma^{ij}(x^{n+1}, x), & i, j=1, 2, \cdots, n, \quad x\in\bar{D}\\ 0, & \text{if } i=n+1 \text{ or } j=n+1\end{cases}$$

$$\tilde{b}^i(\tilde{x})=\begin{cases}b^i(x^{n+1}, x), & i=1, 2, \cdots, n, \quad x\in\bar{D}\\ 1, & i=n+1\end{cases}$$

$$\tilde{\tau}^{ij}(\tilde{x})=\begin{cases}\tau^{ij}(x^{n+1}, x), & i, j=2, 3, \cdots, n, \quad x\in\partial D\\ 0, & \text{if } i=n+1 \text{ or } j=n+1\end{cases}$$

$$\tilde{\beta}^i(\tilde{x})=\begin{cases}\beta^i(x^{n+1}, x), & i=2, 3, \cdots, n, \quad x\in\partial D\\ \tilde{\rho}(\tilde{x}), & i=n+1\end{cases}$$

$$\tilde{\rho}(\tilde{x})=\rho(x^{n+1}, x), \qquad x\in\partial D.$$

Consider the stochastic differential equation of $\tilde{x}_t=(x_t, x_t^{n+1})$ for $[\tilde{\sigma}, \tilde{b},$

$\tilde{\tau}$, $\tilde{\beta}$, $\tilde{\rho}$] with an initial condition $x_0^{n+1}=0$. Then, the last equation reduces to

$$dx_t^{n+1}=I_{D'}(\tilde{x}_t)dt+I_{\partial D'}(\tilde{x}_t)\,dt=dt$$

and hence $x_t^{n+1}\equiv t$. Thus, we see from the theorem 1 that, if $[\sigma, b, \tau, \beta, \rho]$ are bounded and Lipschitz continuous in $(t, x)\in[0, \infty)\times\bar{D}'$ and if a constant $c>0$ exists such that $|\sigma^1(t, x)|\geqq c$, $(t, x)\in[0, \infty)\times\bar{D}$, then the solution exists and is unique.

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References

[1] K. Ito, Canonical measurable random functions, Proc. International Conference on Functional Analysis and Related Topics, Univ. Tokyo Press 1970, 369-377.

[2] H. Kunita and S. Watanabe, On square integrable martingales, Nagoya Math. J.**30** (1967), 209-245.

[3] S. Watanabe, On stochastic differential equations for multi-dimensional diffusion processes with boundary conditions, J. Math. Kyoto Univ. **11**-1 (1971) 169-180.