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On the Riemann-Roch theorem on open Riemann Surfaces

By

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Introduction

To generalize the classical theory of algebraic functions to open Riemann surfaces, much effort has been made in the last three decades. As for Riemann-Roch theorem and Abel's theorem, similar formulations as classical were obtained by L. Ahlfors [2] [3] [4], Y. Kusunoki [6], B. Rodin [15] and H. L. Royden [16] for some class of open surfaces. The results but for [6] are described in terms of *distinguished* harmonic differentials introduced by Ahlfors. Although restrictions for surfaces are not explicitly mentioned, they seem to be meaningful only for surfaces with small boundaries, say, those of class O_{KD}. Otherwise, a single-valued meromorphic function whose differential is distinguished would reduce to a constant. As was pointed out by R.D.M. Accola [1], the same situation occurs if the surface belongs to the class $O_{HD}-O_{G}$. For surfaces of class $O_{KD}-O_{HD}$, it seems yet unknown whether or not non-constant meromorphic function f exists such that df is distinguished. While, the results by Y. Kusunoki [7] [8] [9] are meaningful for general surfaces. His results are given in terms of *canonical semiexact* differentials and functions introduced by himself, which have some restrictions only in their real parts. M. Mori [13] pointed out that canonical semiexact differentials are identical with meromorphic differentials whose real parts are distinguished (in the real sense). Recently

H. Mizumoto [11] and M. Yoshida [18] obtained further generalizations along Kusunoki's program. Some interesting applications of these theories can be seen in [7] [8], [11] and M. Mori [12], M. Ota [14].

In the present paper, by modifying those methods we shall show a slightly more extended formulation for Riemann-Roch theorem in general surfaces, that is, it will be given in terms of the single-valued meromorphic functions (multiples of a divisor δ) with a certain boundary behavior and the meromorphic differentials (multiples of $1/\delta$) with another behavior which is dual in some sense to the behavior above. Our result generalizes the corresponding theorems in [7] [8] [9], [11] and [18]. Even in the case of finite genus, our formulation yields somewhat new canonical conformal mappings. Our treatment seems to be analogous to Yoshida's one, but different in some respects. Actually, our starting point is to consider the totality of square integrable *complex* differentials as a *real* Hilbert space, which differs from the customary ones. Moreover, we shall not necessarily require that the real parts of differentials under consideration are exact near the ideal boundary. We impose restrictions onto differentials rather than functions and by doing so, we shall be able to take into account a wider class of differentials with an infinite number of non-vanishing periods.

Now we shall sketch our program. In the first section we consider the space Λ of square integrable complex differentials on a Riemann surface as a real Hilbert space, and show some fundamental lemmas including de Rham's decomposition, Dirichlet principle and Weyl's lemma etc.. The definition of Λ_0-behavior is given in section 2. In section 3, we shall show the uniqueness and existence theorems of elementary differentials with Λ_0-behaviors (Theorems 1, 2 and 3) by means of orthogonal projection method.

In section 4, the notion of *dual boundary behaviors* is introduced and some lemmas will be prepared for the following sections. In section 5, we shall formulate the Riemann-Roch theorem in terms of differentials with dual boundary behaviors (Theorem 4 and its Corollary).

The proof is analogous to Kusunoki's one [6] [7] [9] (see also Yoshida [18]). Our results generalize the known cases, e.g. canonical semiexact differentials ([7] [8] [9] cf. also [10]), Mizumoto's results [11] and Yoshida's Γ_χ-behaviors [18]. We refer to these cases in section 6.

The last two sections, 7 and 8 are devoted to show examples which exhibit the merits of our standpoint. The first example (sec. 7) gives a canonical conformal mapping of a compact bordered Riemann surface onto a region with slits whose directions are arbitrarily prescribed. This can be regarded as a generalization of P. Koebe's classical results [5]. The second example (sec. 8) shows that differentials with an infinite number of non-vanishing periods may appear in our theory.

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§I. Preliminaries and existence theorems.

1. Let W be an arbitrary Riemann surface. A Lebesgue measurable complex differential $\lambda=a(z)\,dx+b(z)\,dy$ on W is said to be square integrable, if the integral $\iint_W(|a|^2+|b|^2)dxdy$ is finite. The totality of square integrable *complex* differentials on W forms a Hilbert space $\tilde{\Lambda}=\tilde{\Lambda}(W)$ over the complex number field $\mathbf{C}$ with the usual inner product defined by

$$(\lambda_1, \lambda_2)=\iint_W \lambda_1 \wedge \bar{\lambda}_2^* = \iint_W (a_1\bar{a}_2+b_1\bar{b}_2)\,dx\,dy$$

where $\lambda_j=a_j(z)\,dx+b_j(z)\,dy$ for a local parameter $z=x+iy$. We denote by $\bar{\lambda}$ the complex conjugate of λ, and by λ^* the conjugate differential of λ. The norm in $\tilde{\Lambda}$ is denoted by $\|\cdot\|$. Similarly, the totality of square integrable *real* differentials on W forms a Hilbert space $\Gamma=\Gamma(W)$ over the real number field $\mathbf{R}$ with the same inner product as above. $\tilde{\Lambda}$ can also be considered as a linear space over $\mathbf{R}$, and $\tilde{\Lambda}$ so understood is denoted by $\Lambda=\Lambda(W)$. We introduce another inner product

$$<\lambda_1, \lambda_2> = \mathrm{Re}(\lambda_1, \lambda_2).$$

Then it is easily verified that Λ forms a real Hilbert space with respect to this new inner product. We shall use symbol $|||\cdot|||$ to indicate the norm in Λ. It is evident that $|||\lambda||| = ||\lambda||$ for any square integrable complex differential λ, and therefore we could do without the symbol $|||\cdot|||$. But we prefer to use it to make clear the structure of the space under consideration. As has been pointed out, Λ has the same topology as $\tilde{\Lambda}$. However, the orthogonality in Λ does not always imply the orthogonality in $\tilde{\Lambda}$, while its converse is true.

For any complex differential $\lambda = a(z)dx + b(z)dy$ we set

$$\sigma = \mathrm{Re}\,\lambda = (\mathrm{Re}\,a)\,dx + (\mathrm{Re}\,b)\,dy,$$

$$\tau = \mathrm{Im}\,\lambda = (\mathrm{Im}\,a)\,dx + (\mathrm{Im}\,b)\,dy.$$

Then $\lambda = \sigma + i\tau$ and σ, τ are real differentials. Indeed, their coefficients are covariant because of the fact that the transformation matrix is real. (Ahlfors-Sario [4] cf. also Weyl [17] p. 56). Conversely if σ and τ are real differentials, then $\sigma + i\tau$ evidently defines a complex differential. Further we can easily see that $|||\lambda|||^2 = |||\sigma|||^2 + |||\tau|||^2$. More generally, if $\lambda_j = \sigma_j + i\tau_j (j=1, 2)$ are two complex differentials, then we have

$$\begin{aligned} <\lambda_1, \lambda_2> &= <\sigma_1, \sigma_2> + <\tau_1, \tau_2> \\ &= (\sigma_1, \sigma_2) + (\tau_1, \tau_2). \end{aligned}$$

The space Γ can be considered as a closed linear subspace of Λ. If we write $i\Gamma = \{i\omega; \omega \in \Gamma\}$, $i\Gamma$ is also a closed linear subspace of Λ and, by just obtained identity, it is evident that the orthogonal complement of Γ in Λ is exactly $i\Gamma$, i.e.

$$\Lambda = \Gamma \dot{+} i\Gamma \qquad \text{(direct sum)}.$$

It should be noted that the meanings of the letters Λ and Γ are different from those in Ahlfors-Sario [4], Mizumoto [11], Rodin [15] and Yoshida [18] etc.. With only these exceptions, we inherit the terminologies and notations of Ahlfors-Sario [4], if not mentioned further.

For example, Γ_e, Γ_{e0}, Γ_c, Γ_{c0}, Γ_h, $\cdots$ stand for the real Hilbert spaces of *real* square integrable differentials (on W) with some restricted properties; Γ_e (resp. Γ_{e0}) is defined by $\mathrm{Cl}\,\Gamma_e^1$ (resp. $\mathrm{Cl}\,\Gamma_{e0}^1$), where Γ_e^1 (resp. Γ_{e0}^1) is the linear space formed by all exact real C^1-differentials (resp. exact real C^1-differentials with compact supports) on W and Cl denotes the closure in Γ. If we consider the corresponding linear spaces (over $\mathbf{R}$) formed by complex differentials, Λ_e and Λ_{e0} are defined analogously. That is, we define Λ_e (resp. Λ_{e0}) to be the closure of Λ_e^1 (resp. Λ_{e0}^1) in Λ. Γ_c (resp. Γ_{c0}) was defined to be the orthogonal complement of Γ_{e0}^* (resp. Γ_e^*) in Γ. Now it should be noted that Λ_c and Λ_{c0} are defined to be the orthogonal complements of Λ_{e0}^* and Λ_e^* respectively with respect to the inner product $<\ ,\ >$ (not $(\ ,\ )$!). With these notations it is trivial that the following relations hold.

$$\Lambda_e=\tilde{\Lambda}_e=\Gamma_e \dotplus i\Gamma_e$$

$$\Lambda_{e0}=\tilde{\Lambda}_{e0}=\Gamma_{e0} \dotplus i\Gamma_{e0}.$$

In order to obtain other important decompositions we need the following

LEMMA 1. *Let Γ_1 and Γ_2 be two closed linear subspaces of Γ and $\Lambda_1=\Gamma_1 \dotplus i\Gamma_2$. Then $\Lambda_1^{\perp}=\Gamma_1^{\perp} \dotplus i\Gamma_2^{\perp}$, where $\Lambda_1^{\perp}$ means the orthogonal complement of Λ_1 in Λ and $\Gamma_j^{\perp}$ mean the orthogonal complements of Γ_j in $\Gamma(j=1, 2)$.*

PROOF. It is evident that $\Lambda_1^{\perp}\supset\Gamma_1^{\perp} \dotplus i\Gamma_2^{\perp}$. To show the converse, suppose that $\lambda\in\Lambda_1^{\perp}$. By assumption $\lambda_1=\sigma_1+i\sigma_2\in\Lambda_1$ for any $\sigma_j\in\Gamma_j$. If we set $\lambda=\tau_1+i\tau_2$, $\tau_j\in\Gamma$,

$$0=<\lambda_1, \lambda>=<\sigma_1, \tau_1>+<\sigma_2, \tau_2>.$$

We can take σ_2 to be the zero element of Γ_2, and obtain that $<\sigma_1, \tau_1>=0$ for any $\sigma_1\in\Gamma_1$. Hence $\tau_1\in\Gamma_1^{\perp}$. Similarly $\tau_2\in\Gamma_2^{\perp}$. Therefore $\lambda=\tau_1+i\tau_2\in\Gamma_1 \dotplus i\Gamma_2^{\perp}$, q.e.d.

The following lemma will justify the definitions of Λ_c and Λ_{c0}.

LEMMA 2.

$$\Lambda_c=\Gamma_c\dot{+}i\Gamma_c$$

$$\Lambda_{c0}=\Gamma_{c0}\dot{+}i\Gamma_{c0}.$$

PROOF. We give the proof only for the first decomposition, since the second case can be proved analogously. By Lemma 1, $\Lambda_c=\Lambda_{e0}^{*\perp}=(\Gamma_{e0}^{*}\dot{+}i\Gamma_{e0}^{*})^{\perp}=\Gamma_{e0}^{*\perp}\dot{+}i\Gamma_{e0}^{*\perp}=\Gamma_c\dot{+}i\Gamma_c$ and this is the desired relation.

We define the space of square integrable complex harmonic differentials on W, Λ_h, to be the class $\Lambda_c^1\cap\Lambda_c^{1*}$. Then we have

LEMMA 3. (*Weyl's lemma*).

$$\Lambda_h=\Lambda_c\cap\Lambda_c^*.$$

PROOF. It is evident by Weyl's lemma for Γ_h (cf. Ahlfors-Sario [4] p. 281) and Lemma 2.

LEMMA 4. (*de Rham's decomposition*).

$$\Lambda=\Lambda_h\dot{+}\Lambda_{e0}\dot{+}\Lambda_{e0}^*.$$

PROOF. On account of the preceding lemma, $(\Lambda_{e0}\dot{+}\Lambda_{e0}^*)^{\perp}=\Lambda_{e0}^{\perp}\cap\Lambda_{e0}^{*\perp}=\Lambda_c\cap\Lambda_c^*=\Lambda_h$, which is to be proved.

As an immediate consequence of this lemma we have

LEMMA 5. (*Dirichlet principle*).

$$\Lambda_c=\Lambda_h\dot{+}\Lambda_{e0}.$$

The following lemma can be regarded as a generalization of Green's formula:

LEMMA 6. *Let φ_1 and φ_2 be closed C^1-differentials on $\bar{\Omega}$ where Ω is a canonical regular region of W. Let $\Xi(W)=\{A_j, B_j\}$ be a* *canonical homology basis of W modulo dividing cycles such that $\Xi(W)\cap\bar{\Omega}$ forms a canonical homology basis of $\bar{\Omega}$ modulo $\partial\Omega$. If φ_1 is semiexact, then*

$$(\varphi_1, \varphi_2^*)_\Omega = -\int_{\partial\Omega}\Big(\int\varphi_1\Big)\bar{\varphi}_2 + \sum_{\Omega}\Big(\int_{A_j}\varphi_1\int_{B_j}\bar{\varphi}_2 - \int_{B_j}\varphi_1\int_{A_j}\bar{\varphi}_2\Big).$$

Here $\sum_{\Omega}$ stands for the sum only for A_j and B_j which are contained in $\bar{\Omega}$. The precise meaning of the integral $\int\varphi_1$ is given in the proof.

PROOF. We cut Ω along A_j and B_j, and obtain a planar surface Ω_0. Since φ_1 is semiexact, there exists a C^2-function f on Ω_0 such that $df=\varphi_1$. We apply the well-known Green's formula on Ω_0. Then

$$\begin{aligned}(\varphi_1, \varphi_2^*)_\Omega &= (\varphi_1, \varphi_2^*)_{\Omega_0} = -\iint_{\Omega_0}\varphi_1\wedge\bar{\varphi}_2 = -\iint_{\Omega_0} df\wedge\bar{\varphi}_2 \\ &= -\int_{\partial\Omega_0} f\bar{\varphi}_2 = -\int_{-\Sigma A_jB_jA_j^{-1}B_j^{-1}+\partial\Omega} f\varphi_2 \\ &= -\int_{\partial\Omega} f\bar{\varphi}_2 + \Sigma\Big(\int_{A_j}\varphi_1\int_{B_j}\bar{\varphi}_2 - \int_{B_j}\varphi_1\int_{A_j}\bar{\varphi}_2\Big),\end{aligned}$$ q.e.d.

REMARK. Note that f is determined up to an additive constant. But the choice of f has no effect on the integral $\int_{\partial\Omega} f\bar{\varphi}_2$, because of the closedness of φ_2.

For later use we shall prove the following

LEMMA 7. *Let ζ be an arbitrary non-zero complex number. If Λ_1 is a closed linear subspace of Λ, then $\zeta\Lambda_1=\{\zeta\lambda;\lambda\in\Lambda_1\}$ is also a closed linear subspace of Λ and $(\zeta\Lambda_1)^\perp=\zeta\Lambda_1^\perp$.*

PROOF. The first assertion is trivial. To show the remaining part, we need only note that $<\zeta\lambda', \zeta\lambda''>=|\zeta|^2<\lambda', \lambda''>$. From this relation we know that if $\lambda\in(\zeta\Lambda_1)^\perp$ then $\zeta^{-1}\lambda\in\Lambda_1^\perp$ and vice versa, q.e.d.

2. From now on, we regard the complex plane **C** as a two-dimensional linear space over $\mathbf{R}$, and consider a family $\mathscr{L}=\{L_j\}_{j=1}^{g}$ of (at most a countable number of) one-dimensional subspaces L_j of $\mathbf{C}$. Here g denotes the genus of W which may be infinity. Once for all, we fix a canonical homology basis $\Xi=\Xi(W)=\{A_j, B_j\}_{j=1}^{g}$ modulo dividing cycles and consider a space $\Lambda_0=\Lambda_0(\Lambda_1; \mathscr{L})$ such that

(1) Λ_0 is a linear subspace (not necessarily closed) of Λ_{hse},

(2) there exists a closed linear subspace Λ_1 of Λ_h such that

$$\Lambda_0 \supset \Lambda_1 + i\Lambda_1^{\perp *}$$

where $\Lambda_1^{\perp}$ is the orthogonal complement of Λ_1 in Λ_h,

(3) $<\lambda_0, i\lambda_0^*>=0$ for any $\lambda_0 \in \Lambda_0$,

(4) $\int_{A_j, B_j} \lambda_0 \in L_j$ for every $\lambda_0 \in \Lambda_0$ and $j=1, 2, \cdots, g$.

Such a space $\Lambda_0=\Lambda_0(\Lambda_1; \mathscr{L})$ will be called a *behavior space* associated with Λ_1 and $\mathscr{L}$.

If $\Lambda_0=\Lambda_0(\Lambda_1; \mathscr{L})$ is a behavior space, so is $\bar{\Lambda}_0=\{\lambda \in \Lambda_h;\ \bar{\lambda} \in \Lambda_0\}$. Indeed, (1) and (3) are easily verified, because $\bar{\Lambda}_0$ is evidently a linear subspace of Λ_{hse} and $<\bar{\lambda}_0, i\bar{\lambda}_0^*>=<\lambda_0, -i\lambda_0^*>=-<\lambda_0, i\lambda_0^*>=0$ for any $\lambda_0 \in \Lambda_0$. Next, $\bar{\Lambda}_0 \supset \overline{\Lambda_1 + i\Lambda_1^{\perp *}}=\bar{\Lambda}_1 + i\bar{\Lambda}_1^{\perp *}$ and this proves (2). Finally for every $\lambda_0 \in \Lambda_0$ and $j=1, 2, \cdots, g$

$$\int_{A_j, B_j} \bar{\lambda}_0 = \overline{\int_{A_j, B_j} \lambda_0} \in \bar{L}_j$$

where $\bar{L}_j=\{z \in \mathbf{C};\ \bar{z} \in L_j\}$. $\bar{L}_j$ is obviously a one-dimensional linear subspace of $\mathbf{C}$. Set $\bar{\mathscr{L}}=\{L;\ \bar{L} \in \mathscr{L}\}$. Then we can write $\bar{\Lambda}_0=\Lambda_0(\bar{\Lambda}_1; \bar{\mathscr{L}})$.

Let $\mathscr{E}(W)$ be the collection of all $V \subset W$ for which there exists a canonical regular region Ω such that $V=W-\bar{\Omega}$. Each element of $\mathscr{E}(W)$ is a neighborhood of the ideal boundary β of W.

DEFINITION. Let Λ_0 be a behavior space. A meromorphic differential φ, defined on a neighborhood of β, is called to have Λ_0-behavior if there exist $U \in \mathscr{E}(W)$, $\lambda_0 \in \Lambda_0$ and $\lambda_{e0} \in \Lambda_{e0} \cap \Lambda^1$ such that

$$\varphi=\lambda_0+\lambda_{e0} \qquad \text{on } U.$$

A meromorphic function f (not necessarily single-valued), defined near β, is called to have Λ_0-behavior if differential df has Λ_0-behavior in the above sense.

3. Existence and uniqueness theorems of the elementary differentials with Λ_0-behaviors.

THEOREM 1. (*uniqueness*). *A regular analytic differential φ which has Λ_0-behavior ($\Lambda_0=\Lambda_0(\Lambda_1;\mathscr{L})$, $\mathscr{L}=\{L_j\}_{j=1}^{g}$) is identically zero provided that*

$$\int_{\substack{A_j\\B_j}}\varphi\in L_j \qquad (j=1,2,\cdots,g).$$

REMARK. Let $\mathscr{L}'=\{L_j'\}_{j=1}^{g}$ be another family of one-dimensional subspaces of $\mathbf{C}$ such that $L_j'=L_j$ for all but a finite number of j. Suppose that $\int_{\substack{A_j\\B_j}}\varphi\in L_j'$ $(j=1,2,\cdots,g)$. Then we have the same conclusion.

PROOF. Since φ has Λ_0-behavior, there exist $U\in\mathscr{E}(W)$, $\lambda_0\in\Lambda_0$ and $\lambda_{e0}\in\Lambda_{e0}\cap\Lambda^1$ such that

$$\varphi=\lambda_0+\lambda_{e0} \qquad \text{on } U.$$

We take a sufficiently large canonical regular region Ω whose relative boundary $\partial\Omega$ is contained in U. We may assume that $\Xi\cap\bar{\Omega}$ forms a canonical homology basis of $\bar{\Omega}$ modulo the border. Then, using Lemma 6 twice, because of the analyticity of φ,

$$\begin{aligned}|||\varphi|||_{\Omega}^2&=\|\varphi\|_{\Omega}^2=(\varphi,\varphi)_{\Omega}=-i(\varphi,\varphi^*)_{\Omega}\\&=i\int_{\partial\Omega}\left(\int\varphi\right)\bar{\varphi}-i\sum_{\Omega}\left(\int_{A_j}\varphi\int_{B_j}\bar{\varphi}-\int_{B_j}\varphi\int_{A_j}\bar{\varphi}\right)\\&=i\int_{\partial\Omega}\left(\int(\lambda_0+\lambda_{e0})\right)\overline{(\lambda_0+\lambda_{e0})}-i\sum_{\Omega}\left(\int_{A_j}\varphi\int_{B_j}\bar{\varphi}-\int_{B_j}\varphi\int_{A_j}\bar{\varphi}\right)\end{aligned}$$

$$= -i(\lambda_0+\lambda_{e0},\ \lambda_0^*+\lambda_{e0}^*)_\Omega + i\sum_\Omega\left(\int_{A_j}\lambda_0\int_{B_j}\bar{\lambda}_0 - \int_{B_j}\lambda_0\int_{A_j}\bar{\lambda}_0\right)$$

$$-i\sum_\Omega\left(\int_{A_j}\varphi\int_{B_j}\bar{\varphi} - \int_{B_j}\varphi\int_{A_j}\bar{\varphi}\right)$$

$$= -i(\lambda_0+\lambda_{e0},\ \lambda_0^*+\lambda_{e0}^*)_\Omega - 2\ \mathrm{Im}\sum_\Omega\left(\int_{A_j}\lambda_0\int_{B_j}\bar{\lambda}_0 - \int_{A_j}\varphi\int_{B_j}\bar{\varphi}\right).$$

Hypothesis in the theorem and the condition (4) for Λ_0 imply that

$$\int_{A_j}\varphi\int_{B_j}\bar{\varphi} \quad \text{and} \quad \int_{A_j}\lambda_0\int_{B_j}\bar{\lambda}_0$$

are both real and consequently,

$$|||\varphi|||_\Omega^2 = -i(\lambda_0+\lambda_{e0},\ \lambda_0^*+\lambda_{e0}^*)_\Omega = (\lambda_0,\ i\lambda_0^*)_\Omega - i\varepsilon_\Omega$$

where

$$\varepsilon_\Omega = (\lambda_{e0},\ \lambda_0^*)_\Omega + (\lambda_0,\ \lambda_{e0}^*)_\Omega + (\lambda_{e0},\ \lambda_{e0}^*)_\Omega.$$

Let Ω tend to W, then since $\lim_{\Omega\to W}\varepsilon_\Omega = 0$ we obtain the equality

$$|||\varphi|||^2 = (\lambda_0,\ i\lambda_0^*) = \langle\lambda_0,\ i\lambda_0{}^*\rangle.$$

The final term is zero because of the condition (3) for Λ_0 and we conclude that $\varphi\equiv 0$. q.e.d.

Next we prove the existence theorems of certain elementary differentials with preassigned periods and singularities. Let L be a one-dimensional subspace of **C**. For two complex numbers z_1 and z_2 we shall write $z_1\equiv z_2 \bmod L$ if $z_1 - z_2 \in L$.

THEOREM 2. *Let* α_j *and* β_j *be given complex numbers, such that* $\alpha_j \not\equiv 0$, $\beta_j \not\equiv 0 \bmod L_j$. *Then there are square integrable holomorphic differentials* $\phi_{\alpha_j}(A_j)$, $\phi_{\beta_j}(B_j)$ *which have the following properties*:

(i) $\phi_{\alpha_j}(A_j)$, $\phi_{\beta_j}(B_j)$ *have* Λ_0*-behaviors*,

(ii) $\displaystyle\int_{B_k}\phi_{\alpha_j}(A_j) \equiv \alpha_j(A_j\times B_k) = \begin{cases}\alpha_j & (k=j)\\ 0 & (k\neq j)\end{cases} \bmod L_k$

$$\int_{A_k} \phi_{\alpha_j}(A_j) \equiv \alpha_j(A_j \times A_k) = 0 \qquad \mathrm{mod}\ L_k.$$

(ii)′ *Similar relations hold for* $\phi_{\beta_j}(B_j)$.

The $\phi_{\alpha_j}(A_j)$ *and* $\phi_{\beta_j}(B_j)$ *are uniquely determined for each* j.

PROOF. We give the proof only for the case of $\phi_{\alpha_j}(A_j)$, since the case of $\phi_{\beta_j}(B_j)$ will be analogously proved. We may assume that the given cycle A_j is an oriented analytic Jordan curve. Let R be a relatively compact ring domain containing A_j. We consider a C^2-function v on $R-A_j$ such that

$$v=\begin{cases}\alpha_j & \text{on the left side of } A_j\\ 0 & \text{on the right side of } A_j.\end{cases}$$

We can extend v as $\hat{v} \in C_0^2(W-A_j)$. Then $d\hat{v}$ is a closed C^1-differential with finite norm, that is, $d\hat{v} \in \Lambda_c^1(W)$. So, by use of Dirichlet principle (Lemma 5) $\Lambda_c=\Lambda_h \dotplus \Lambda_{e0}$ and the orthogonal decomposition $\Lambda_h=\Lambda_1 \dotplus \Lambda_1^{\perp}$, we know the existence of differentials $\lambda_1 \in \Lambda_1$, $\lambda_1^{\perp} \in \Lambda_1^{\perp}$ and $\lambda_{e0} \in \Lambda_{e0} \cap \Lambda^1$ such that

$$d\hat{v}=\lambda_1+\lambda_1^{\perp}+\lambda_{e0}.$$

Since $\alpha_j \not\equiv 0 \ \mathrm{mod}\, L_j$, $\lambda_1^{\perp}$ is not identically zero.

We set

$$\begin{aligned}\phi_{\alpha_j}(A_j) &= \lambda_1^{\perp}+i(\lambda_1^{\perp})^*\\ &= d\hat{v}-(\lambda_1-i\,\lambda_1^{\perp *})-\lambda_{e0}.\end{aligned}$$

Then $\phi_{\alpha_j}(A_j)$ is a regular analytic differential. Since $d\hat{v}$ has a compact support, and further Λ_0 contains $\Lambda_1+i\Lambda_1^{\perp *}$, we can conclude that $\phi_{\alpha_j}(A_j)$ has Λ_0-behavior. Moreover, for any cycle γ, we have

$$\begin{aligned}\int_\gamma \phi_{\alpha_j}(A_j) &= \int_\gamma d\hat{v}-\int_\gamma(\lambda_1-i\,\lambda_1^{\perp *})\\ &= \alpha_j\cdot(A_j\times\gamma)-\int_\gamma \lambda_0\end{aligned}$$

provided that $\lambda_0=\lambda_1-i\lambda_1^{\perp *}(\in \Lambda_0)$. If we choose A_k resp. B_k as γ, condition (4) for Λ_0 implies (ii).

Next we show the uniqueness of $\phi_{\alpha_j}(A_j)$. Suppose that ϕ_1 and ϕ_2 are admissible differentials. The difference $\phi_1-\phi_2$ is then a regular analytic differential with Λ_0-behavior and satisfies

$$\int_{\substack{A_k\\B_k}}(\phi_1-\phi_2)\in L_k \qquad (k=1, 2, \cdots, g).$$

Therefore by Theorem 1, $\phi_1-\phi_2\equiv 0$. This completes the proof.

REMARK. More generally we can prove that $\phi_{\alpha_j}(A_j)=\phi_{\alpha_j'}(A_j)$ if $\alpha_j\equiv\alpha_j'$ mod L_j. Indeed, Theorem 1 is sufficient to draw this conclusion.

Let p_0 be a point of W and $z=z(p)$ be a local parameter near p_0 for which $z(p_0)=0$. Conventionally, by an analytic singularity at p_0 we understand a differential Θ_0 which is defined in a punctured neighborhood U_0 of p_0 and is analytic on U_0. It may be assumed that Θ_0 is represented as follows:

$$\Theta_0=\sum_{n=1}^{\infty} b_n z^{-n}dz \qquad (b_n\in \mathbf{C}).$$

For sufficiently small $r>0$ the quantity $\frac{1}{2\pi i}\int_{|z|=r}\Theta_0=b_1$ is known as the residue of Θ_0 at p_0, which is independent of the choice of local parameters. For further details, refer to Ahlfors-Sario [4] p. 299, p. 305.

THEOREM 3. *Let $p_1, p_2, \cdots, p_N$ be a finite number of points on W, and Θ_j an analytic singularity given at each $p_j (j=1, 2, \cdots, N)$. Consider a differential Θ which is equal to Θ_j near p_j. Suppose that the sum of residues of Θ* is *zero. Then there exists a differential $\varphi=\varphi_\Theta$ such that*

(i) *φ has Λ_0-behavior,*

(ii) *φ is regular analytic except at p_j $(j=1, 2, \cdots, N)$,*

(iii) *φ has the singularity Θ, that is, $|||\varphi-\Theta_j|||_{U_j}<\infty$ for some punctured neighborhood U_j of $p_j (j=1, 2, \cdots, N)$.*

To prove this theorem we need the following extension lemma for differentials which is due to H. Yamaguchi (Lemma 1, in Yoshida [18]).

LEMMA 8. *Let G be a regularly imbedded connected subregion of W whose relative boundary ∂G is compact, and let V be the complement of $\bar{G}$. For any closed C^1-differential, σ, defined on a neighborhood of $\bar{V}$, the following two statements are equivalent*:

(i) *$\sigma|_V$, the restriction of σ onto V, can be extended as a closed C^1-differential $\hat{\sigma}$ on W so that the support of $\hat{\sigma}$ has a compact intersection with $\bar{G}$.*

(ii) $\int_{\partial G}\sigma=0.$

Proof of Theorem 3. Take sufficiently small parametric disks Δ_j about p_j whose closures are mutually disjoint. We set $W'=W-\{p_j\}_{j=1}^N$, $\Delta'_j=\Delta_j-\{p_j\}$ and $V=\bigcup_{j=1}^N \Delta'_j$. For a while we focus our attention on the new Riemann surface W'. Then $\Omega=W'-\bar{V}$ is a regularly imbedded connected subregion of W', and its relative boundary $\partial\Omega=\bigcup_{j=1}^N \partial\Delta_j$ is compact. It is evident that $V=W'-\bar{\Omega}$. By our assumption, Θ is a closed C^1-differential on a neighborhood of $\bar{V}$ and satisfies that

$$\int_{\partial\Omega}\Theta=\sum_{j=1}^N\int_{\partial\Delta_j}\Theta_j=0.$$

For the sum of residues of Θ vanishes. Therefore, by Lemma 8, we can extend $\Theta|_V$ as a closed C^1-differential on W' with compact support, which we denote by $\hat{\Theta}$. (As for the constructive method to obtain $\hat{\Theta}$, refer to Ahlfors-Sario [4] p. 301, p. 306. See also Ahlfors [2].)

On the other hand, since Θ gives an analytic singularity at each p_j, $\Theta-i\Theta^*=0$ near p_j and so $\hat{\Theta}$ satisfies the relation

$$\hat{\Theta}-i\hat{\Theta}^*=0 \qquad \text{near } p_j \text{ and near } \beta.$$

Hence $\hat{\Theta}-i\hat{\Theta}^*\in\Lambda^1(W)\subset\Lambda(W)$.

Now the de Rham's decomposition (Lemma 4) $\Lambda=\Lambda_h\dotplus\Lambda_{e0}\dotplus\Lambda_{e0}^*$ and the decomposition $\Lambda_h=\Lambda_1\dotplus\Lambda_1^\perp$ show that there are differentials $\lambda_1\in\Lambda_1$; $\lambda_1^\perp\in\Lambda_1^\perp$; λ'_{e0}, $\lambda''_{e0}\in\Lambda_{e0}$ satisfying

$$\hat{\Theta}-i\hat{\Theta}^*=\lambda_1+\lambda_1^\perp+\lambda'_{e0}+\lambda''^*_{e0}.$$

Define

$$\tau=\hat{\Theta}-\lambda_1-\lambda'_{e0}=\lambda_1^\perp+\lambda''^*_{e0}+i\hat{\Theta}^*,$$

then we know that τ is a complex harmonic differential with singularity Θ. Consequently, λ'_{e0}, $\lambda''_{e0}\in\Lambda_{e0}\cap\Lambda^1$. Now it is obviously seen that $\varphi=\frac{1}{2}(\tau+i\tau^*)$ has the desired properties.

REMARK. Up to this point, we can not insist that the so constructed differentials are uniquely determined. But under certain normalization those are unique. We require that φ should satisfy

$$\int_{\substack{A_j\\B_j}}\varphi\in L_j \qquad (j=1, 2, \cdots, g).$$

It is easy to show that this normalization is always possible. Indeed, if x_j(resp. y_j) are A_j- (resp. B_j-) periods of φ, only a finite number of x_j and y_j are $\not\equiv 0 \bmod L_j$. Set

$$\tilde{\varphi}=\varphi-\sum_j(-\phi_{x_j}(B_j)+\phi_{y_j}(A_j)).$$

The sum in the right hand runs over j for which $x_j\not\equiv 0$ or $y_j\not\equiv 0 \bmod L_j$, and is therefore a finite sum. $\tilde{\varphi}$ preserves the singularity and satisfies the required normalization condition. As for uniqueness, we need only Theorem 1.

Thus, if we take a local parameter z_j near p_j such that $z_j(p_j)=0$, the following normalized differentials always exist and are unique:

(I) $\varphi_{p_j,n}$: differential with Λ_0-behavior, regular analytic except at p_j, where $\varphi_{p_j,n}$ has the singlarity dz_j/z_j^n $(n=2, 3, \cdots)$.

(II) $\tilde{\varphi}_{p_j,n}$: differential with Λ_0-behavior, regular analytic except at p_j, where $\tilde{\varphi}_{p_j,n}$ has the singuularity $i dz_j/z_j^n$ $(n=2, 3, \cdots)$.

(III) $\psi_{p,q}$: meromorphic differential with Λ_0-behavior, which has residues 1 at p, -1 at $q(p, q\in W)$ and is regular analytic elsewhere.

(IV) $\tilde{\psi}_{p,q}$: meromorphic differential with Λ_0-behavior, which has residues i at p, $-i$ at q $(p, q\in W)$ and is regular analytic elsewhere.

These normalized differentials together with holomorphic differentials $\phi_{\alpha_j}(A_j)$, $\phi_{\beta_j}(B_j)$ whose existence and uniqueness are guaranteed by Theorem 2 will play a fundamental role later.

§II Dual boundary behaviors and Riemann-Roch theorem.

4. For our purposes we consider here two boundary behaviors. Let $\Lambda_0^{(k)}=\Lambda_0(\Lambda_1^{(k)}; \mathscr{L}_k)$ $(k=1, 2)$ be two behavior spaces corresponding to the spaces $\Lambda_1^{(1)}$, $\Lambda_1^{(2)}(\subset\Lambda_h)$ and the families $\mathscr{L}_1$, $\mathscr{L}_2$. Let L_0 be a one-dimensional subspace of $\mathbf{C}$. Suppose that $\mathscr{L}_k=\{L_j^{(k)}\}_{j=1}^{g}$ $(k=1, 2)$. We say that $\Lambda_0^{(1)}$-behavior and $\Lambda_0^{(2)}$-behavior are *dual* to one another with respect to L_0 if the following two conditions are fulfilled:

1°) $(\lambda_0^{(1)}, \overline{\lambda_0^{(2)*}})\equiv 0$ *mod* L_0 *i.e.*, $<\lambda_0^{(1)}, \overline{\lambda_0^{(2)*}}>+i<\lambda_0^{(1)}, i\overline{\lambda_0^{(2)*}}>\in L_0$ *for all* $\lambda_0^{(1)}\in\Lambda_0^{(1)}$ *and* $\lambda_0^{(2)}\in\Lambda_0^{(2)}$.

2°) *For each* j,

$$L_j^{(1)}\times L_j^{(2)}=L_0$$

where the left term is defined by the set $\{z\in\mathbf{C};\ z=\zeta_j^{(1)}\zeta_j^{(2)}$ *for some* $\zeta_j^{(k)}\in L_j^{(k)}$ $(k=1, 2)\}$.

If a behavior space $\Lambda_0=\Lambda_0(\Lambda_1; \mathscr{L})$ satisfies a stronger condition

3′) $<\lambda_0', i\lambda_0''^*>=0$ for any $\lambda_0', \lambda_0''\in\Lambda_0$

then, Λ_0- and $\bar{\Lambda}_0$-behaviors are dual to one another. In fact, we already know that $\bar{\Lambda}_0$ always defines a boundary behavior if Λ_0 does (see sec. 2). Hence, we need only check the conditions 1^0) and 2^0). To do this, suppose that $\lambda_0', \lambda_0'' \in \Lambda_0$. Then, on account of (3′) we have

$$(\lambda_0', \overline{(\overline{\lambda_0''})^*}) = <\lambda_0', \lambda_0''^*> + i<\lambda_0', i\lambda_0''^*> = <\lambda_0', \lambda_0''^*> \in \mathbf{R}$$

that is, $(\lambda_0', \overline{(\overline{\lambda_0''})^*}) \equiv 0 \mod \mathbf{R}$. What is more, $L_j \times \bar{L}_j = \mathbf{R}$ for each $j=1, 2, \cdots, g$. Thus we have shown

LEMMA 9. *Let $\Lambda_0 = \Lambda_0(\Lambda_1; \mathscr{L})$ be a behavior space which satisfies the condition*

(3′) $<\lambda_0', i\lambda_0''^*> = 0$ *for all* $\lambda_0', \lambda_0'' \in \Lambda_0$.

Then Λ_0- and $\bar{\Lambda}_0$-behaviors are dual to one another with respect to $\mathbf{R}$.

We shall make use of the following lemma which is essentially due to Y. Kusunoki [6] [7] [9] (see also Yoshida [18]).

LEMMA 10. *Let $\Lambda_0^{(1)}$ and $\Lambda_0^{(2)}$ define dual boundary behaviors to each other with respect to L_0. Let φ be an Abelian differential (of 1st or 2nd kind) with $\Lambda_0^{(1)}$-behavior and ψ any Abelian differential with $\Lambda_0^{(2)}$-behavior. We cut W along A_j and B_j to make it a planar Riemann surface W_0. Then*

(i) *there exists a single-valued meromorphic function f on W_0 such that $df = \varphi$,*

(ii) $2\pi i \sum \mathrm{Res}.\, f\psi \equiv -\sum_{j=1}^{g}\left(\int_{A_j}\varphi\int_{B_j}\psi - \int_{B_j}\varphi\int_{A_j}\psi\right) \mod L_0$.

PROOF. (i) is evident by assumptions. In order to prove (ii), we apply Lemma 6 on Ω_0, the region obtained from a sufficiently large canonical regular region Ω by taking off mutually disjoint parametric disks about the singularities of φ and ψ. As before (the proof of Theorem 1), we may suppose that $\Xi \cap \bar{\Omega}$ forms a canonical homology basis of $\bar{\Omega}$ modulo $\partial\Omega$. Then

$$2\pi i \sum \mathrm{Res.}\, f\psi = -\sum_{\Omega}\left(\int_{A_j}\varphi\int_{B_j}\psi - \int_{B_j}\varphi\int_{A_j}\psi\right) + \int_{\partial\Omega} f\psi.$$

We can assume that for some $\lambda_0^{(k)} \in \Lambda_0^{(k)}$ ($k=1, 2$) and some λ'_{e0}, $\lambda''_{e0} \in \Lambda_{e0} \cap \Lambda^1$

$$\varphi = \lambda_0^{(1)} + \lambda'_{e0}, \quad \psi = \lambda_0^{(2)} + \lambda''_{e0}$$

on a $U \in \mathscr{E}(W)$, in particular near $\partial\Omega$. And by the usual techniques including the use of Lemma 6, we have

$$\int_{\partial\Omega} f\psi = -(\lambda_0^{(1)}, \overline{\lambda_0^{(2)*}})_\Omega + \sum_{\Omega}\left(\int_{A_j}\lambda_0^{(1)}\int_{B_j}\lambda_0^{(2)} - \int_{B_j}\lambda_0^{(1)}\int_{A_j}\lambda_0^{(2)}\right) + \varepsilon_\Omega$$

where

$$\varepsilon_\Omega = -[(\lambda_0^{(1)}, \overline{\lambda''^*_{e0}})_\Omega + (\lambda'_{e0}, \overline{\lambda_0^{(2)}}{}^*)_\Omega + (\lambda'_{e0}, \overline{\lambda''^*_{e0}})_\Omega].$$

Now our assumptions 1°) and 2°) yield that

$$\left.\begin{aligned} &(\lambda_0^{(1)}, \overline{\lambda_0^{(2)*}}) \equiv 0 \\ &\sum_{j=1}^{g}\left(\int_{A_j}\lambda_0^{(1)}\int_{B_j}\lambda_0^{(2)} - \int_{B_j}\lambda_0^{(1)}\int_{A_j}\lambda_0^{(2)}\right) \equiv 0 \end{aligned}\right\} \mod L_0.$$

On the other hand, since $\lim_{\Omega\to W}\varepsilon_\Omega = 0$, it follows that

$$2\pi i \sum \mathrm{Res.}\, f\psi \equiv -\sum_{j=1}^{g}\left(\int_{A_j}\varphi\int_{B_j}\psi - \int_{B_j}\varphi\int_{A_j}\psi\right) \mod L_0,$$

which is to be proved.

REMARK. Note that we have

$$\int_{A_j}\varphi\int_{B_j}\psi - \int_{B_j}\varphi\int_{A_j}\psi \equiv 0 \mod L_0$$

except for a finite number of j. Indeed, φ and ψ have $\Lambda_0^{(1)}$- and $\Lambda_0^{(2)}$-behaviors respectively and these behaviors are dual to one another with respect to L_0. So, by the condition 2°) of dual boundary behaviors, we can conclude the desired congruence relation.

As in the traditional cases (Ahlfors-Sario [4] pp. 325–329, Kusunoki

[6] [7] [9], Mizumoto [11], Rodin [15], Yoshida [18] etc.), the following well-known algebraic lemma will be needed. For its proof, refer to [18] (Lemma 4), for instance.

LEMMA 11. *Let K be a field and X, Y two linear spaces over K. Suppose that h is a bilinear form defined on the product space $X\times Y$, and that X_0 (resp. Y_0) is the left-(resp. right-) kernel of h, that is, $X_0=\{x\in X; h(x, y)=0$ for all $y\in Y\}$ and $Y_0=\{y\in Y; h(x, y)=0$ for all $x\in X\}$. Then we have an isomorphism*

$$X/X_0\cong Y/Y_0$$

provided that at least one of the quotient spaces X/X_0, Y/Y_0 is finite dimensional.

5. Let $\delta=\delta_p/\delta_q$ be a finite divisor on W, where $\delta_p=p_1^{m_1}p_2^{m_2}\cdots p_r^{m_r}$ and $\delta_q=q_1^{n_1}q_2^{n_2}\cdots q_s^{n_s}$ are disjoint integral divisors. Let L_0 be a one-dimensional subspace of $\mathbf{C}$. Let spaces $\Lambda_0^{(1)}=\Lambda_0(\Lambda_1^{(1)}; \mathcal{L}_1)$ and $\Lambda_0^{(2)}=\Lambda_0(\Lambda_1^{(2)}; \mathcal{L}_2)(\mathcal{L}_k=\{L_j^{(k)}\}_{j=1}^{g}(k=1, 2))$ define dual boundary behaviors with respect to L_0. For each $L_j^{(k)}$ (resp. L_0) we take a complex number $\zeta_j^{(k)}$(resp. ζ_0) of modulus 1 which determines $L_j^{(k)}$ (resp. L_0). We consider the following sets which evidently form linear spaces over $\mathbf{R}$:

$S(\Lambda_0^{(1)}; 1/\delta)=\{f$; (i) f is a single-valued meromorphic function on W, (ii) f has $\Lambda_0^{(1)}$-behavior, (iii) f is a multiple of $1/\delta$.$\}$,

$M(\Lambda_0^{(1)}; 1/\delta_p)=\{f$; (i) f is a (multi-valued) meromorphic function on W, (ii) f has $\Lambda_0^{(1)}$-behavior, (iii) f is a multiple of $1/\delta_p$, (iv) periods of df are normalized.$\}$,

$D(\Lambda_0^{(2)}; \delta)=\{\alpha$; (i) α is a meromorphic differential on W, (ii) α has $\Lambda_0^{(2)}$-behavior, (iii) α is a multiple of δ.$\}$,

$D(\Lambda_0^{(2)}; 1/\delta_q)=\{\alpha$; (i) α is a meromorphic differential on W, (ii) α has $\Lambda_0^{(2)}$-behavior, (iii) α is a multiple of $1/\delta_q$.$\}$.

Here, in the case that $\delta_q\neq 1$ we identify two elements f_1, f_2 of

$M(\Lambda_0^{(1)}; 1/\delta_p)$ if and only if $f_1 - f_2 = \text{const.}$ $(\in \mathbf{C})$.

THEOREM 4. (*Riemann-Roch theorem*). *Suppose that $\Lambda_0^{(1)}$- and $\Lambda_0^{(2)}$-behaviors are dual to each other. Let $\delta = \delta_p/\delta_q$ be a finite divisor on W, where δ_p and δ_q are disjoint integral divisors. Then*

$$\dim S(\Lambda_0^{(1)}; 1/\delta) = 2[\operatorname{ord}\delta_p + 1 - \min(\operatorname{ord}\delta_q, 1)] - \\ - \dim[D(\Lambda_0^{(2)}; 1/\delta_q)/D(\Lambda_0^{(2)}; \delta)].$$

PROOF. First of all, we shall find the dimension of $M(\Lambda_0^{(1)}; 1/\delta_p)$. To do so, we need the integrals of the elementary differentials with $\Lambda_0^{(1)}$-behaviors obtained in sec. 3

$$\int \varphi_{p_j,\mu}^{(1)}, \quad \int \tilde{\varphi}_{p_j,\mu}^{(1)} \qquad \begin{matrix} 1 \leq j \leq r \\ 2 \leq \mu \leq m_j + 1 \end{matrix}$$

where the superscript denotes that they have $\Lambda_0^{(1)}$-behaviors. It is easily seen that if $\delta_q \neq 1$ these integrals span $M(\Lambda_0^{(1)}; 1/\delta_p)$, and if $\delta_q = 1$, those integrals and constants 1, i make a basis of $M(\Lambda_0^{(1)}; 1/\delta_p)$. So we find that

$$\dim M(\Lambda_0^{(1)}; 1/\delta_p) = \begin{cases} 2\sum_{j=1}^{r} m_j + 2 = 2\operatorname{ord}\delta_p + 2 & (\delta_q = 1) \\ 2\sum_{j=1}^{r} m_j \quad = 2\operatorname{ord}\delta_p & (\delta_q \neq 1) \end{cases}$$

$$= 2(\operatorname{ord}\delta_p + 1 - \min(\operatorname{ord}\delta_q, 1)).$$

Now we consider a (real-valued) bilinear form defined on the product space $M(\Lambda_0^{(1)}; 1/\delta_p) \times D(\Lambda_0^{(2)}; 1/\delta_q)$

$$h_{L_0}(f, \alpha) = \operatorname{Re}[\zeta_0 \sum_j \operatorname{Res.}_{p_j} f\alpha] \qquad \begin{matrix} f \in M(\Lambda_0^{(1)}; 1/\delta_p) \\ \alpha \in D(\Lambda_0^{(2)}; 1/\delta_q). \end{matrix}$$

Since α is regular at each p_j, additive constants (including periods) of f have no effect on the residue of $f\alpha$ at p_j, and hence h_{L_0} is well-defined. By Lemma 10 we have

$$h_{L_0}(f,\alpha)=-\frac{1}{2\pi}\,\mathrm{Im}\Big[\zeta_0\sum_{j=1}^{g}\Big(\int_{A_j}df\int_{B_j}\alpha-\int_{B_j}df\int_{A_j}\alpha\Big)\Big]$$
$$-\mathrm{Re}\Big[\zeta_0\sum_k \underset{q_k}{\mathrm{Res.}} f\alpha\Big].$$

Then we can determine the left- and right-kernels of h_{L_0}. In fact, if f is an element of the left-kernel of h_{L_0}, we choose $\phi^{(2)}_{i\zeta^{(2)}_k}(A_k)$ as α and know that $\int_{A_k} df=0$. Similarly we find that $\int_{B_k} df=0$ by choosing $\phi^{(2)}_{i\zeta^{(2)}_k}(B_k)$ as α. Hence f is single-valued on the whole of W.

If δ is integral, then $\delta_p=\delta$ and therefore $f\in S(\Lambda_0^{(1)};1/\delta)$. Next, in the case that δ is non-integral, we set $\alpha=\psi^{(2)}_{q_1,q_k}$. Then we know that $\mathrm{Im}[\zeta_0 f(q_1)]=\mathrm{Im}[\zeta_0 f(q_k)]$ for $k=2,3,\cdots,s$. It is entirely similar for $\mathrm{Re}[\zeta_0 f]$, and so we can conclude that the function $f-f(q_1)$ has zeros at $q_k(k=2,\cdots,s)$. Moreover, if we take $\varphi^{(2)}_{q_k,\nu}$ and $\tilde{\varphi}^{(2)}_{q_k,\nu}$ as α $(1\leq k\leq s;\,2\leq\nu\leq n_k)$, it follows immediately that the function $f-f(q_1)$ has at least n_k-ple zeros at $q_k(1\leq k\leq s)$. By the equivalence relation in $M(\Lambda_0^{(1)};1/\delta_q)$ we know that $f\in S(\Lambda_0^{(1)};1/\delta)$. Conversely, it is obvious that the left-kernel of h_{L_0} contains $S(\Lambda_0^{(1)};1/\delta)$. Therefore the left-kernel of h_{L_0} is exactly equal to $S(\Lambda_0^{(1)};1/\delta)$. Concerning the right-kernel, we proceed analogously. In this case also, it is easily verified that $D(\Lambda_0^{(2)};\delta)$ is contained in the right-kernel, for $f\alpha$ is regular analytic at p_j if $f\in M(\Lambda_0^{(1)};1/\delta_p)$ and $\alpha\in D(\Lambda_0^{(2)};\delta)$. The converse implication is proved by taking the integrals

$$\int\varphi^{(1)}_{p_j,\mu}\quad\text{and}\quad\int\tilde{\varphi}^{(1)}_{p_j,\mu}$$

as $f(1\leq j\leq r;\,2\leq\mu\leq m_j+1)$. Therefore the right-kernel is $D(\Lambda_0^{(2)};\delta)$.

Now Lemma 11 is applicable and it follows that $M(\Lambda_0^{(1)};1/\delta_p)/S(\Lambda_0^{(1)};1/\delta)\cong D(\Lambda_0^{(2)};1/\delta_q)/D(\Lambda_0^{(2)};\delta)$, for we already know that $M(\Lambda_0^{(1)};1/\delta_p)$ is finite-dimensional. This isomorphism yields the desired dimension relation. q.e.d.

If g, the genus of W, is finite, we can easily find a basis for $D(\Lambda_0^{(2)};1/\delta_q)$ as usual:

(a) if $\delta_q = 1$,

$\{\phi^{(2)}_{\alpha_j}(A_j), \phi^{(2)}_{\beta_j}(B_j)\}_{1 \leq j \leq g}$ span $D(\Lambda_0^{(2)}; 1/\delta_q)$, and

(b) if $\delta_q \neq 1$,

$$\{\phi^{(2)}_{\alpha_j}(A_j), \phi^{(2)}_{\beta_j}(B_j); \varphi^{(2)}_{q_k,\nu}; \tilde{\varphi}^{(2)}_{q_k,\nu}; \psi^{(2)}_{q_1,q_l}, \tilde{\psi}^{(2)}_{q_1,q_l}\}_{\substack{1 \leq j \leq g, 1 \leq k \leq s \\ 2 \leq l \leq s, 2 \leq \nu \leq n_k}}$$

span $D(\Lambda_0^{(2)}; 1/\delta_q)$,

provided that in both cases we choose α_j and β_j appropriately, say, $\alpha_j = \beta_j = i\zeta_j^{(2)}$. Hence

$$\dim D(\Lambda_0^{(2)}; 1/\delta_q) = \begin{cases} 2g & (\delta_q = 1) \\ 2[g + \sum_{k=1}^{s}(n_k - 1) + s - 1] & (\delta_q \neq 1) \end{cases}$$

$$= 2[g - \min(\operatorname{ord}\delta_q, 1) + \operatorname{ord}\delta_q].$$

And therefore Theorem 4 reduces to the following rather classical form:

COROLLARY. *If $\Lambda_0^{(1)}$- and $\Lambda_0^{(2)}$-behaviors are dual to each other, then for any finite divisor δ on W*

$$\dim S(\Lambda_0^{(1)}; 1/\delta) - \dim D(\Lambda_0^{(2)}; \delta) = 2(\operatorname{ord}\delta - g + 1).$$

6. In this section we mention about the important particular cases.

(a) Let Γ_χ be a closed subspace of Γ_{he}, containing Γ_{hm}. Set $\Lambda_\chi = \Gamma_\chi \dotplus i\Gamma_\chi^{\perp *}$, where $\Gamma_\chi^{\perp}$ is the orthogonal complement of Γ_χ in Γ_h. It is easily seen that conditions (1)—(4) for behavior spaces in sec. 2 are all satisfied. Note that $\Lambda_\chi + i\Lambda_\chi^{\perp *} = \Lambda_\chi$ where $\Lambda_\chi^{\perp}$ is the orthogonal complement of Λ_χ in Λ_h. Further, it should be noted that $\mathscr{L}$ consists of only one element $L = i\mathbf{R}$. It is easy to verify that Λ_χ satisfies the stronger condition (3′). Hence, by Lemma 9, we know that Λ_χ- and $\bar{\Lambda}_\chi$-behaviors are dual to each other (with respect to $\mathbf{R}$). What is more, $\bar{\Lambda}_\chi=\Lambda_\chi$ and therefore Λ_χ-behavior is self-dual. Thus we get Riemann-Roch theorem for differentials with Λ_χ-behaviors. Λ_χ-behavior is nothing other than *Γ_χ-behavior* in Yoshida [18]. In particular, the case of $\Gamma_\chi=\Gamma_{hm}$ tells us the results for *canonical semiexact differentials* obtained in Kusunoki [7], [8] and [9]. It is obvious that Γ_χ may be any intermediate space between Γ_{h0} and Γ_{hse} (cf. (b)).

(b) If ζ is a non-zero complex number and Λ_χ is such a one as in (a), then on account of Lemma 7 we can speak of $\zeta\Gamma_\chi$-behavior and obtain Riemann-Roch theorem for differentials with this behavior. The extreme case $\Gamma_\chi=\Gamma_{hm}$ gives an at most $(g+1)$-valent conformal mapping of W onto a region with parallel slits, provided that W is of finite genus g (cf. Mori [12]).

(c) We consider two (distinct) boundary behaviors. Let $\Lambda_0^{(1)}=\Gamma_\chi+i\Gamma_\chi^{\perp *}$, $\Lambda_0^{(2)}=\Gamma_\chi^{\perp *}+i\Gamma_\chi$($\Gamma_\chi$ and $\Gamma_\chi^{\perp}$ are the same ones as in (a)), then they have all our required properties and therefore we obtain a dimension relation between $S(\Lambda_0^{(1)};1/\delta)$ and $D(\Lambda_0^{(2)};1/\delta_q)/D(\Lambda_0^{(2)};\delta)$. Note that $\mathscr{L}_1=\{i\mathbf{R}\}$, $\mathscr{L}_2=\{\mathbf{R}\}$ and $L_0=i\mathbf{R}$. The extreme case that $\Gamma_\chi=\Gamma_{hm}$ now has a connection with the results in Royden [16]. Further, as in (b), a pair of behavior spaces $\zeta_1\Lambda_0^{(1)}$, $\zeta_2\Lambda_0^{(2)}$ gives a similar example ($\zeta_1, \zeta_2\in\mathbf{C}-\{0\}$).

(d) We can also construct somewhat more general examples (sec. 7) which exhibit that our result strictly contains the already known ones.

§III Applications and examples.

7. Each element L_j of $\mathscr{L}$ is representable as a straight line which passes through the origin. In the sequel we use the term "straight lines passing through the origin" or simply "lines" instead of "one-dimensional linear subspaces of $\mathbf{C}$".

Let γ be a differentiable curve on W and γ: $z=z(t)$ $t\in I=[0, 1]$ be one of its representations. A complex differential $\lambda=a(z)dx+b(z)dy$ is said to be zero along γ if

$$a(z(t))x'(t)+b(z(t))y'(t)=0$$

for all $t\in I$, where $x'(t)$, $y'(t)$ are the derivatives of x, y with respect to t. This notion does not depend on the choice of representations of γ. Similarly we say that λ is real along γ if $\tau=\operatorname{Im}\lambda$ is zero along γ. We generalize this notion and say λ to be *l-valued along γ*, l being a line, if and only if

$$a(z(t))x'(t)+b(z(t))y'(t)\in l$$

for all $t\in I$. In the case that $l=\mathbf{R}$ (the real axis), to say that λ is l-valued along γ is nothing other than saying that λ is real along γ.

Now we shall construct an example announced in sec. 6 (d). We shall also obtain some canonical conformal mappings, Theorem 5 below.

Let $\bar{W}$ be a compact bordered Riemann surface of genus g, and W its interior. Let $\Xi=\{A_j, B_j\}_{j=1}^{g}$ be a canonical homology basis of $\bar{W}$ modulo the border. Suppose that $\beta=\beta(\bar{W})$, the border of $\bar{W}$, consists of h boundary components $\beta_1, \beta_2, \cdots, \beta_h$. With each $j\ (1\leq j\leq g)$ and each $k\ (1\leq k\leq h)$, associate lines L_j and l_k. Let $\tilde{L}_j$ (resp. $\tilde{l}_k$) denote the line which is determined by $\tilde{\zeta}_j=i\zeta_j$ (resp. $\tilde{z}_k=iz_k$), where ζ_j (resp. z_k) is a complex number on L_j (resp. l_k) with $|\zeta_j|=|z_k|=1$. We set $\mathscr{L}=\{L_j\}$, $\ell=\{l_k\}$ and $\tilde{\mathscr{L}}=\{\tilde{L}_j\}$, $\tilde{\ell}=\{\tilde{l}_k\}$.

Define

$$\Lambda_q^1(\bar{W})=\Lambda_q^1(\bar{W};\mathscr{L},\ell)=\Big\{\lambda\in\Lambda_c^1(\bar{W});\ \text{(i) } \lambda \text{ is semiexact, i.e.,}$$

$$\int_{\beta_k}\lambda=0 \quad \text{for all } \beta_k(1\leq k\leq h),\ \text{(ii)} \int_{\substack{A_j\\B_j}}\lambda\equiv 0 \bmod L_j\ (1\leq j\leq g),$$

$$\text{(iii) } \lambda \text{ is } l_k\text{-valued along } \beta_k\ (1\leq k\leq h).\Big\}.$$

If we denote by $\tilde{\Lambda}_q^1(\bar{W})$ the class $\Lambda_q^1(\bar{W};\tilde{\mathscr{L}},\tilde{\ell})$, it is evident that the following lemma holds.

LEMMA 12.

$$\tilde{\Lambda}_q^1(\bar{W})=i\Lambda_q^1(\bar{W}).$$

What is more, we can show the following

LEMMA 13. *$\Lambda_q^1(\bar{W})$ is the orthogonal complement of $\tilde{\Lambda}_q^1(\bar{W})^*$ in $\Lambda_c^1(\bar{W})$.*

PROOF. First we shall show that $\tilde{\Lambda}_q^1(\bar{W})^* \perp \Lambda_q^1(\bar{W})$. Take $\lambda_q \in \Lambda_q^1(\bar{W})$ and $\lambda \in \tilde{\Lambda}_q^1(\bar{W})$. Then, by Lemma 6, we have

$$\begin{aligned} <\lambda_q, \lambda^*> &= \mathrm{Re}\,(\lambda_q, \lambda^*) \\ &= -\sum_{k=1}^{h} \mathrm{Re}\int_{\beta_k} f_k \bar{\lambda} + \sum_{j=1}^{g} \mathrm{Re}\left(\int_{A_j} \lambda_q \int_{B_j} \bar{\lambda} - \int_{B_j} \lambda_q \int_{A_j} \bar{\lambda}\right) \end{aligned}$$

provided that $df_k = \lambda_q$ near $\beta_k\,(k=1, 2, \cdots, h)$. Because of the semiexactness of λ we can take functions f_k separately on each boundary component.

The condition that λ_q is l_k-valued along β_k implies that $\bar{z}_k \lambda_q$ is real along β_k i.e. $\mathrm{Im}\,(\bar{z}_k f_k) = \text{const.}$ on β_k. Similarly, we know that $\bar{z}_k \lambda$ is imaginary along β_k, that is, $\mathrm{Re}(\bar{z}_k \lambda)$ is zero along β_k. Therefore

$$\begin{aligned} \mathrm{Re}\int_{\beta_k} f_k \bar{\lambda} &= \mathrm{Re}\int_{\beta_k} (\bar{z}_k f_k)\,(\overline{\bar{z}_k \lambda}) \\ &= \int_{\beta_k} \mathrm{Re}\,(\bar{z}_k f_k)\,\mathrm{Re}\,(\bar{z}_k \lambda) + \int_{\beta_k} \mathrm{Im}(\bar{z}_k f_k)\,\mathrm{Im}(\bar{z}_k \lambda) \end{aligned}$$

vanishes because of the semiexactness of $\bar{z}_k \lambda$.

On the other hand, the period conditions for λ_q and λ yield that

$$\int_{\substack{A_j \\ B_j}} \lambda_q \int_{\substack{B_j \\ A_j}} \bar{\lambda} \in L_j \times \overline{\bar{L}_j} = L_j \times i\bar{L}_j = i\mathbf{R}.$$

Hence $\mathrm{Re}\left(\int_{A_j} \lambda_q \int_{B_j} \bar{\lambda} - \int_{B_j} \lambda_q \int_{A_j} \bar{\lambda}\right) = 0$. Since these reasonings are valid for all $k, j\,(1 \leq k \leq h;\ 1 \leq j \leq g)$, it follows that $<\lambda_q, \lambda^*> = 0$.

Next we shall show the converse. Before carrying out the proof we note that for each k_0 and j_0 $(1 \leq k_0 \leq h;\ 1 \leq j_0 \leq g)$ we can readily construct a semiexact C^1-differential $\lambda_{k_0 j_0} = \lambda_{k_0 j_0}(u_{k_0}, c_{k_0}, C_{j_0}) \in \Lambda_q^1(\bar{W})$ such that

$$\text{(i)}\qquad \int \lambda_{k_0 j_0} = \begin{cases} z_{k_0}(u_{k_0}+ic_{k_0}) & \text{on } \beta_{k_0} \\ 0 & \text{on } \beta_k \quad (k \neq k_0), \end{cases}$$

$$\text{(ii)}\qquad \int_{A_{j_0}} \lambda_{k_0 j_0} = C_{j_0}\cdot\zeta_{j_0}, \quad \int_{B_{j_0}} \lambda_{k_0 j_0} = 0$$

$$\int_{A_j} \lambda_{k_0 j_0} = \int_{B_j} \lambda_{k_0 j_0} = 0 \qquad (j \neq j_0),$$

where $u_{k_0} \in C^2_{\mathbf{R}}(\beta_{k_0}) = \{$all the real-valued twice continuously differentiable functions defined on $\beta_{k_0}\}$ and c_{k_0}, $C_{j_0} \in \mathbf{R}$. (In (i) the integral $\int \lambda_{k_0 j_0}$ is understood in the sense of Lemma 6. That is, we cut $\bar{W}$ along A_j, B_j to make it a planar surface $\bar{W}_0$ and consider the integral $\int \lambda_{k_0 j_0}$ on $\bar{W}_0$). Indeed, such a differential is obtained by a standard method as follows: Let R be a relatively compact ring domain containing B_{j_0} which may be assumed to be an orientable analytic Jordan curve. For any $u_{k_0} \in C^2_{\mathbf{R}}(\beta_{k_0})$ and c_{k_0}, $C_{j_0} \in \mathbf{R}$ we take a function F defined on $R \cup \beta$ such that $F = z_{k_0}(u_{k_0}+ic_{k_0})$ on β_{k_0}, $F = C_{j_0}\zeta_{j_0}$ on the right part of R and $F=0$ elsewhere. We can extend F so as $\hat{F}$ blongs to $C^2(\bar{W}-B_{j_0})$. If we set $\lambda_{k_0 j_0} = d\hat{F}$, $\lambda_{k_0 j_0}$ is the desired differential.

Now suppose that $<\lambda_q, \lambda^*> = 0$ for all $\lambda_q \in \Lambda^1_q(\bar{W})$. By Lemma 6, we have

$$\sum_k \mathrm{Re}\int_{\beta_k}\Big(\bar{z}_k \int \lambda_q\Big)\overline{\bar{z}_k \lambda} + \sum_j \mathrm{Re}\Big(\int_{A_j}\lambda_q \int_{B_j}\bar{\lambda} - \int_{B_j}\lambda_q \int_{A_j}\bar{\lambda}\Big) = 0.$$

We can take $\lambda_{k_0 j_0}(0, 1, 0)$ and $\lambda_{k_0 j_0}(1, 0, 0)$ as λ_q and obtain that $\mathrm{Re}\int_{\beta_{k_0}} \overline{\bar{z}_{k_0}\lambda} = \mathrm{Re}\int_{\beta_{k_0}} i\overline{\bar{z}_{k_0}\lambda} = 0$, which proves the semiexactness of λ. Setting $\lambda_q = \lambda_{k_0 j_0}(u_{k_0}, c_{k_0}, 0)$, we have

$$\int_{\beta_{k_0}} u_{k_0} \mathrm{Re}(\bar{z}_{k_0}\lambda) = 0.$$

This holds for all $u_{k_0} \in C^2_{\mathbf{R}}(\beta_{k_0})$, and therefore we can conclude that $\mathrm{Re}(\bar{z}_{k_0}\lambda) = 0$ along β_{k_0}, that is, λ is $\tilde{l}_{k_0}$-valued along β_{k_0}.

Finally we set $\lambda_q = \lambda_{k_0 j_0}(u_{k_0}, c_{k_0}, 1)$. Then it follows that

$\mathrm{Re}\left[\zeta_{j_0}\int_{Bj_0}\bar{\lambda}\right]=0$, that is, the B_{j_0}-period of λ lies on the line $\tilde{L}_{j_0}$. We can discuss about A_{j_0} analogously.

Since these reasonings are valid for all k_0 and j_0 $(1\leq k_0\leq h;\ 1\leq j_0\leq g)$, we can conclude that $\lambda\in\tilde{\Lambda}_q^1(\bar{W})$, q.e.d.

If we restrict ourselves to harmonic differentials, we have the following

LEMMA 14.

$$\tilde{\Lambda}_0(\bar{W})=i\Lambda_0(\bar{W})$$

$$\Lambda_h(\bar{W})=\Lambda_0(\bar{W})\dotplus\tilde{\Lambda}_0(\bar{W})^*.$$

Here, by definition,

$$\Lambda_0(\bar{W})=\Lambda_q^1(\bar{W})\cap\Lambda_h(\bar{W}) \quad \textit{and} \quad \tilde{\Lambda}_0(\bar{W})=\tilde{\Lambda}_q^1(\bar{W})\cap\Lambda_h(\bar{W}),$$

which are evidently closed linear subspaces of Λ_h.

The class $\Lambda_0(\bar{W})$ satisfies all the conditions (1)–(4) in sec. 2. In fact, first, $\Lambda_0(\bar{W})\subset\Lambda_{hse}(\bar{W})$ is obvious. Second, $\Lambda_0(\bar{W})+i\Lambda_0(\bar{W})^{\perp *}=\Lambda_0(\bar{W})+i\tilde{\Lambda}_0(\bar{W})=\Lambda_0(\bar{W})$ by Lemma 14. Third, $<\lambda_0', i\lambda_0''^*>=0$ for any λ_0' and λ_0'' belonging to $\Lambda_0(\bar{W})$, since by Lemmas 7 and 14 $i\lambda_0''^*\in i\Lambda_0(\bar{W})^*=i\tilde{\Lambda}_0(\bar{W})^{\perp}=\Lambda_0(\bar{W})^{\perp}$. And finally by the definition of $\Lambda_0(\bar{W})$, the values $\int_{A_j}\lambda_0$ and $\int_{B_j}\lambda_0$ evidently belong to L_j for each $\lambda_0\in\Lambda_0(\bar{W})$ and $j=1, 2, \cdots, g$. Therefore "$\Lambda_0(\bar{W})$-behavior" is well-defined. Moreover, as has been verified, $\Lambda_0(\bar{W})$ satisfies the stronger condition (3′). Hence by Lemma 9 we know that $\Lambda_0(\bar{W})$ and $\overline{\Lambda_0(\bar{W})}=\tilde{\Lambda}_0(\bar{W})$ define dual boundary behaviors (with respect to $\mathbf{R}$). Riemann-Roch theorem is now applicable for these boundary behaviors, and we know that there exists a non-constant meromorphic function f with $\Lambda_0(\bar{W})$-behavior whose possible poles are arbitrarily prescribed $(g+1)$ points $p_r(0\leq r\leq g)$. Indeed, Corollary to Theorem 4 yields that

$\dim S(\Lambda_0(\bar{W}); 1/\delta) = \dim D(\bar{\Lambda}_0(\bar{W}); \delta) + 2(\text{ord}\,\delta - g + 1) \geq 2(\text{ord}\,\delta - g + 1)$.

If we set $\delta = p_0 p_1 \cdots p_g$, then $\dim S(\Lambda_0(\bar{W}); 1/\delta) \geq 4 > 2$. The function f has $\Lambda_0(\bar{W})$-behavior and so df is l_k-valued along β_k, that is, $\text{Re}(\bar{z}_k f)$ is constant on β_k. It follows that f maps the border β_k to a slit which is parallel to l_k.

Thus, by use of an argument which is similar to Kusunoki [7] (pp. 256–257) we have

THEOREM 5. *Let W be the interior of a compact bordered Riemann surface. Then there is a meromorphic function f on W such that*

(i) *f maps W onto a region with slits whose directions are arbitrarily prescribed,*

(ii) *f maps some of $(g+1)$ preassigned points on W to the point at infinity,*

(iii) *$f(W)$, the image of W under f, is at most $(g+1)$-sheeted over the Riemann sphere.*

REMARKS. (1) In connection with Theorem 5, cf. Koebe's classical work (Koebe [5], especially pp. 198–215) which deals with planar case $(g=0)$.

(2) Our starting point (to consider the totality of square integrable complex differentials as a real Hilbert space) and the notion of dual boundary behaviors essentially contribute to ensuring the existence of such a conformal mapping as in Theorem 5.

(3) If all l_k and L_j coincide with the imaginary axis $i\mathbf{R}$, then meromorphic differentials with $\Lambda_0(\bar{W}; \mathscr{L}, \ell)$-behaviors are nothing other than canonical semiexact differentials. See the first example of sec. 6. Cf. also [7] Theorems 12–14 and [12] Theorem 4.

(3′) Even if $L_j \neq i\mathbf{R}$ for some j, a meromorphic differential with $\Lambda_0(\bar{W}; \mathscr{L}, \ell)$-behavior is canonical semiexact and vice versa, provided that $l_k = i\mathbf{R}$ for all $k = 1, 2, \cdots, h$. See [10] Theorem 1. Compare with the example in the following section.

8. Finally we shall construct another example, which shows our Riemann-Roch theorem is valid for some functions and differentials with an infinite number of non-vanishing periods.

Let W be an open Riemann surface of infinite genus whose ideal boundary β consists of only two Stoïlow components ${}_{\#}\beta$ and $\beta_{\#}$. Let Q be the canonical partition of β

$$Q:\ \beta={}_{\#}\beta\cup\beta_{\#}.$$

Let $\{\Omega_{\nu}\}_{\nu=1}^{\infty}$ be a canonical regular exhaustion of W and $\beta(\bar{\Omega}_{\nu})$ be the border of the compact bordered surface $\bar{\Omega}_{\nu}$. By Q_{ν} we denote the partition of $\beta(\bar{\Omega}_{\nu})$ induced by Q;

$$Q_{\nu}:\ \beta(\bar{\Omega}_{\nu})={}_{\nu}\beta\cup\beta_{\nu},$$

where ${}_{\nu}\beta$ (resp. β_{ν}) is the common relative boundary of Ω_{ν} and the regularly imbedded open neighborhood ${}_{\nu}V$ (resp. V_{ν}) of ${}_{\#}\beta$ (resp. $\beta_{\#}$) ($\nu=1, 2, \ldots$). For each m and n we put ${}_{1}^{n}V={}_{1}V-{}_{n}V$, $V_{1}^{m}=V_{1}-V_{m}$, ${}_{n}W_{m}={}_{1}^{n}V\cup\Omega_{1}\cup V_{1}^{m}$, ${}_{n}W=\bigcup_{m=1}^{\infty}{}_{n}W_{m}$ and $W_{m}=\bigcup_{n=1}^{\infty}{}_{n}W_{m}$. Then clearly ${}_{n}W_{n}=\Omega_{n}$, ${}_{n}W\cap W_{m}={}_{n}W_{m}$ and $W=\bigcup_{n=1}^{\infty}{}_{n}W=\bigcup_{m=1}^{\infty}W_{m}=\bigcup_{n,m=1}^{\infty}{}_{n}W_{m}$.

We take a canonical homology basis $\Xi=\{A_{j}, B_{j}\}_{j\in J}$ of W modulo dividing cycles such that $\Xi\cap{}_{n}\bar{W}_{m}$ forms a canonical homology basis of ${}_{n}\bar{W}_{m}$ modulo the border ($m, n=1, 2, \ldots$). We divide J into two disjoint classes J_{1} and J_{2} which are both infinite sets. Set $\Xi_{k}=\{A_{j}, B_{j}\}_{j\in J_{k}}$ ($k=1, 2$).

We begin with compact bordered surfaces ${}_{n}\bar{W}_{m}$ ($m, n=1, 2, \ldots$). For the time being, we confine differentials in real ones. Define

$${}_{\#}\Gamma({}_{n}\bar{W}_{m})=\Big\{\sigma\in\Gamma_{hse}({}_{n}\bar{W}_{m});\quad \text{(i)}\ \int_{\substack{A_j\\B_j}}\sigma=0\quad\text{if } A_{j}, B_{j}\in\Xi_{1}\cap{}_{n}\bar{W}_{m},$$

$$\text{(ii)}\quad \sigma=0\quad\text{along } {}_{n}\beta\Big\}$$

and

$$\Gamma_{\sharp}({}_n\bar{W}_m)=\left\{\tau\in\Gamma_{hse}({}_n\bar{W}_m);\ \text{(i)}\ \int_{A_j,B_j}\tau=0\quad\text{if } A_j,\ B_j\in\Xi_2\cap{}_n\bar{W}_m,\right.$$

$$\left.\text{(ii)}\quad\tau=0\quad\text{along } \beta_m\right\}.$$

Then, repeating the discussions in the proofs of Lemmas 13 and 14, we have

LEMMA 14′. *For* $m, n=1, 2, \cdots,$

$$\Gamma_h({}_n\bar{W}_m)={}_{\sharp}\Gamma({}_n\bar{W}_m)\dotplus\Gamma_{\sharp}({}_n\bar{W}_m)^*={}_{\sharp}\Gamma({}_n\bar{W}_m)^*\dotplus\Gamma_{\sharp}({}_n\bar{W}_m).$$

For each m we define the space ${}_{\sharp}\Gamma(\bar{W}_m)$ as the set of $\sigma_m\in\Gamma_{hse}(\bar{W}_m)$ which is approximated by ${}_n\sigma_m\in{}_{\sharp}\Gamma({}_n\bar{W}_m)$ in the sense of norm. That is, σ_m belongs to ${}_{\sharp}\Gamma(\bar{W}_m)$ if and only if for any $\varepsilon>0$ and any compact set $E\subset\bar{W}_m$ there exist ${}_n\bar{W}_m\supset E$ and ${}_n\sigma_m\in{}_{\sharp}\Gamma({}_n\bar{W}_m)$ such that $\|\sigma_m-{}_n\sigma_m\|_{{}_nW_m}<\varepsilon$. On the other hand, we define $\Gamma_{\sharp}(\bar{W}_m)$ directly:

$$\Gamma_{\sharp}(\bar{W}_m)=\left\{\tau\in\Gamma_{hse}(\bar{W}_m);\ \text{(i)}\ \int_{A_j,B_j}\tau=0\quad\text{if } A_j,\ B_j\in\Xi_2\cap\bar{W}_m,\right.$$

$$\left.\text{(ii)}\quad\tau=0\quad\text{along } \beta_m\right\}.$$

With these definitions we can prove the following

LEMMA 15. *For* $m=1, 2, \cdots,$ *we have the orthogonal decompositions*

$$\Gamma_h(\bar{W}_m)={}_{\sharp}\Gamma(\bar{W}_m)\dotplus\Gamma_{\sharp}(\bar{W}_m)^*={}_{\sharp}\Gamma(\bar{W}_m)^*\dotplus\Gamma_{\sharp}(\bar{W}_m).$$

Since the proof is substantially the same as in Ahlfors-Sario [4] (see pp. 292–295), we omit it. (In addition to the ordinary discussions, we need to verify that $(\sigma-\sigma_m)^*$ vanishes along β_m, σ_m being the limit differential of ${}_n\sigma_m$. However, this can be done without difficulty if we use the Schwarz' reflection principle on β_m.)

Now we pass to the open surface W. We define ${}_{\sharp}\Gamma={}_{\sharp}\Gamma(W)$ to be the set of all the elements $\sigma\in\Gamma_{hse}(W)$ which is approximated by ${}_n\sigma\in{}_{\#}\Gamma({}_n\bar{W})$ in the sense of norm. $\Gamma_{\#}=\Gamma_{\#}(W)$ is similarly defined.

Suppose that $\sigma\in{}_{\#}\Gamma$ and $\tau\in\Gamma_{\#}$. Then, for any $\varepsilon>0$ and any compact set E, there are ${}_n\bar{W}$, $\bar{W}_m({}_nW_m\supset E)$, ${}_n\sigma\in{}_{\#}\Gamma({}_n\bar{W})$ and $\tau_m\in\Gamma_{\#}(\bar{W}_m)$ such that

$$|||\sigma-{}_n\sigma|||_{{}_nW}<\varepsilon,\ |||\tau-\tau_m|||_{W_m}<\varepsilon.$$

Note that ${}_n\sigma|{}_n\bar{W}_m\in{}_{\#}\Gamma({}_n\bar{W}_m)$ and $\tau_m|{}_n\bar{W}_m\in\Gamma_{\#}({}_n\bar{W}_m)$. Hence

$$\begin{aligned}|<\sigma^*,\tau>_{{}_nW_m}|&=|<\sigma^*-{}_n\sigma^*,\ \tau>_{{}_nW_m}+<{}_n\sigma^*,\ \tau-\tau_m>_{{}_nW_m}|\\&\leq|||\sigma-{}_n\sigma|||_{{}_nW}\cdot|||\tau|||_W+|||{}_n\sigma|||_{{}_nW}\cdot|||\tau-\tau_m|||_{W_m}\\&\leq\varepsilon\cdot|||\tau|||_W+(|||\sigma|||_W+\varepsilon)\cdot\varepsilon\end{aligned}$$

for $<{}_n\sigma^*,\ \tau_m>_{{}_nW_m}$ vanishes by Lemma 14$'$. It follows that ${}_{\#}\Gamma^*\perp\Gamma_{\#}$.

Suppose that $\tau\perp{}_{\#}\Gamma^*$. We have decomposition $\tau=\sigma_m^*+\tau_m$ on $\bar{W}_m$ where $\sigma_m\in{}_{\#}\Gamma(\bar{W}_m)$ and $\tau_m\in\Gamma_{\#}(\bar{W}_m)$. In fact, as for $\bar{W}_m$ we have Lemma 15. The argument in the proof of Lemma 15 may be repeated to conclude that τ_m converges to a harmonic differential $\tau_\infty\in\Gamma_{\#}$ (uniformly on compacta and then in the sense of norm) and that $\tau=\tau_\infty$. Therefore $\tau\in\Gamma_{\#}(W)$. These reasonings imply the following

LEMMA 16. *For the open surface W*

$$\Gamma_h(W)=\Gamma_{\#}(W)\dotplus{}_{\#}\Gamma(W)^*=\Gamma_{\#}(W)^*\dotplus{}_{\#}\Gamma(W).$$

Now we set

$$\Lambda_{\#}=\Lambda_{\#}(W)={}_{\#}\Gamma(W)\dotplus i\Gamma_{\#}(W).$$

Then $\Lambda_{\#}$ is a closed linear subspace of Λ_{hse}. Due to the last lemma we know that $\Lambda_{\#}$ is a behavior space. What is more, $\Lambda_{\#}$-behavior is self-dual since $\Lambda_{\#}=\bar{\Lambda}_{\#}$. For the details of the proof, refer to the preceding section. Corresponding family of lines consists of only two lines $\mathbf{R}$ and $i\mathbf{R}$. For any $\zeta\in\mathbf{C}-\{0\}$ and any $\lambda\in\Lambda_{\#}$ both $\mathrm{Re}(\zeta\lambda)$ and $\mathrm{Im}(\zeta\lambda)$ have an infinite number of non-vanishing periods. Nevertheless, *our Riemann-Roch theorem is valid for functions and differentials with*

$\Lambda_{\#}$*-behaviors.*

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References

[1] Accola, R. D. M.: Some classical theorems on open Riemann surfaces, Bull. Amer. Math. Soc., **73** (1967), 13-26.
[2] Ahlfors, L. V.: Abel's theorem for open Riemann surfaces, Seminars on Analytic Functions, II, 7-19, Institute for Advanced Study, Princeton 1958.
[3] ————: The method of orthogonal decomposition for differentials on open Riemann surfaces, Ann. Acad. Sci. Fenn. Ser. A.I. no. 249/7 (1958), 15 pp.
[4] Ahlfors, L. V. & Sario, L.: Riemann Surfaces, Princeton Univ. Press 1960.
[5] Koebe, P.: Abhandlungen zur Theorie der konformen Abbildung V. Abbildung mehrfach zusammenhängender schlichter Bereiche auf Schlitzbereiche (Erste Fortsetzung), Math. Z., **2** (1918), 198-236.
[6] Kusunoki, Y.: Contributions to Riemann-Roch's theorem, Mem. Col. Sci. Univ. Kyoto Ser. A. Math., **31** (1958), 161-180.
[7] ————: Theory of Abelian integrals and its applications to conformal mappings, Ibid., **32** (1959), 235-258.
[8] ————: Supplements and corrections to my former papers, Ibid., **33** (1961), 429-433.
[9] ————: Abelian differentials on open Riemann surfaces (in Japanese), mimeographed. 24 pp. 1961.
[10] ————: Characterizations of canonical differentials, J. Math. Kyoto Univ., **5** (1966), 197-207.
[11] Mizumoto, H.: Theory of Abelian differentials and relative extremal length with applications to extremal slit mappings, Jap. J. Math., **37** (1968), 1-58.
[12] Mori, M.: Canonical conformal mappings of open Riemann surfaces, J. Math. Kyoto Univ., **3** (1964), 169-192.
[13] ————: Contributions to the theory of differentials on open Riemann surfaces, Ibid., **4** (1964) 77-97.
[14] Ota, M.: A duality theorem for open Riemann surfaces, Mem. of Konan Univ. Sci. Ser., **11** (1968), 63-72.
[15] Rodin, B.: Reproducing kernels and principal functions, Proc. Amer. Math. Soc., **13** (1962), 982-992.
[16] Royden, H. L.: The Riemann-Roch theorem, Comm. Math. Helv., **34** (1960), 37-51.
[17] Weyl, H.: Die Idee der Riemannschen Flächen, Dritte Auflage. Stuttgart 1955.
[18] Yoshida, M.: The method of orthogonal decomposition for differentials on open Riemann surfaces, J. Sci. Hiroshima Univ., Ser. A-I, **32** (1968), 181-210.