J. Math. Kyoto Univ. (JMKYAZ)
11-3 (1971) 449-484

# First order hyperbolic mixed problems

By

Kunihiko KAJITANI

(Received April 5, 1971)

## §1. Introduction

We consider the mixed problems for the first order hyperbolic systems in a quarter space, $t>0$, $x>0$, $y\in\mathbf{R}^{n-1}$;

$$(1.1)\qquad \begin{cases} \dfrac{\partial}{\partial t}u(t)=Lu(t)+f(t) \\ u(0)=g \\ Bu(t)|_{x=0}=h, \end{cases}$$

where $L=A\dfrac{\partial}{\partial x}+\sum_{j=1}^{n-1}B_j\dfrac{\partial}{\partial y_j}+K$, $A$, $B_j$ and $K$ are $N\times N$ matrices and $B$ is a $l\times N$ matrix.

The aim of this article is to derive energy inequalities of the solutions for the mixed problems (1.1).

We assume as follows;

A.I) The coefficients of $(L, B)$ are independent of $t$, sufficiently smooth with respect to $(x, y)$ in $\mathbf{R}^n$ and constant outside a compact set in $\mathbf{R}^n$. The coefficients of $L$ are real valued and $A$ is non singular.

A.II) is strictly hyperbolic, that is, $A\xi+\Sigma B_j\eta_j$ has only real distinct eigen values for $(x, y)\in\mathbf{R}^n$, $(\xi, \eta)\in\mathbf{R}^n$, $(\xi, \eta)\neq 0$. Hence

$$M(x, y; \lambda, \eta)=A^{-1}(\lambda-i\Sigma B_j\eta_j),\ \mathrm{Re}\,\lambda>0,\ \eta\in\mathbf{R}^{n-1}$$

has not real eigen values. Let $k$ of eigen values have negative real parts. Then we can find a smooth $N\times N$ matrix $U(x, y: \lambda, \eta)$ homogeneous of degree zero with respect to $(\lambda, \eta)$ such that for $(x, y) \in \mathbf{R}^n$, $\eta \in \mathbf{R}^{n-1}$, $\operatorname{Re}\lambda \geqq 0$, $(\lambda, \eta) \neq 0$,

$$|U(x, y: \lambda, \eta)|\,|U^{-1}(x, y: \lambda, \eta)| \leqq \text{const.},$$

and

$$U^{-1}(x, y: \lambda, \eta) M(x, y: \lambda, \eta) U(x, y: \lambda, \eta) = \begin{pmatrix} M_1(x, y: \lambda, \eta) & * \\ 0 & M_2(x, y: \lambda, \eta) \end{pmatrix},$$

where $M_1$(resp. $M_2$) is a $k \times k$(resp. $(N-k) \times (N-k)$) matrix which has only eigen values with negative (resp. positive) real parts. Let decompose $U = (U_1, U_2)$, where $U_1$(resp. $U_2$) is a $N \times k$(resp. $N \times (N-k)$) matrix.

A.III) $(L, B)$ satisfies the uniform Lopatinski's condition, that is, $l = k$ and the $l \times l$ matrix $(B(y) \cdot U_1(0, y; \lambda, \eta))$ is non singular for $y \in \mathbf{R}^{n-1}$, $\eta \in \mathbf{R}^{n-1}$, $\operatorname{Re}\lambda \geqq 0$, $|\lambda|^2 + |\eta|^2 = 1$.

Moreover we assume

A.IV) $A = A(x)$ is independent of $y$ in $\mathbf{R}^{n-1}$.

Then we have

**Theorem.** Assume A.I, A.II, A.III and A.IV. Let $u(t)$ belong to $\mathscr{E}_t^0(H^1(\mathbf{R}_+^n))$ and $\mathscr{E}_t^1(L^2(\mathbf{R}_+^n))$. Then there exist positive constants $C$ and $\mu_0$ such that

$$\|e^{-\mu t} u(t)\|^2 + \int_0^t \mu \|e^{-\mu s} u(s)\|^2 + \langle e^{-\mu s} u(s, 0)\rangle^2 ds$$

$$\leqq C \cdot \{\|g\|^2 + \int_0^t \frac{1}{\mu} \|e^{-\mu s} f(s)\|^2 + \langle e^{-\mu s} h(s)\rangle^2 ds\},$$

for $t \geqq 0$, $\mu \geqq \mu_0$. (Notation is explained in the next section.)

## §2. Notation

$\mathbf{R}^n(\mathbf{C}^n)$: $n$-dimensional real (complex) Euclidian space.

$\mathbf{R}_+^n$: the set $\{(x, y); x > 0, y \in \mathbf{R}^{n-1}\}$.

$H^s(\mathbf{R}^n)(H^s(\mathbf{R}_+^n))$: the space of functions of which $s$ times derivatives are square integrable in $\mathbf{R}^n(\mathbf{R}_+^n)$.

$\mathscr{E}_t^p(E)$: the space of functions of which $p$ times derivatives are continuous in $E$ with respect to $t$.

$(\ ,\ )$: the inner product in $L^2(\mathbf{R}_+^n)$.

$<\ ,\ >$: the inner product in $L^2(\mathbf{R}^{n-1})$.

$\|\cdot\|(<\cdot>)$: the norm of $L^2(\mathbf{R}_+^n)(L^2(\mathbf{R}^{n-1}))$.

$[A, B]$: the commutator of two operators $A$ and $B$.

$u_0$: the extension of $u$ in $L^2(\mathbf{R}_+^n)$ such that

$$u_0(x, y)=\begin{cases} u(x, y), & \text{for } x>0 \\ 0 & \text{for } x<0. \end{cases}$$

$\hat{u}(\xi, \eta)$: the Fourier transform of $u(x, y)$, $x\in R^1$, $y\in\mathbf{R}^{n-1}$

$$\hat{u}(\xi, \eta)=\int e^{-ix\xi-iy\cdot\eta}u(x, y)\,dx\,dy.$$

$\tilde{u}(\eta)$: the Fourier transform of $u(y)$, $y\in\mathbf{R}^{n-1}$

$$\tilde{u}(\eta)=\int e^{-iy\cdot\eta}u(y)\,dy.$$

$L_\mu^2(I; E)$: the space of functions $u(t)$ which satisfy

$$\int_I\|e^{-\mu t}u(t)\|^2dt<\infty,$$

where $I$ is $(0, \infty)$ or $(-\infty, \infty)$ and $\mu$ is real number.

## §3. Stationary problems and Adjoint problems

Let us consider the following boundary value problem with a parameter $\lambda$, $\mathrm{Re}\,\lambda>0$, in the half space $\mathbf{R}_+^n$:

$$\text{(3.1)}\qquad \begin{cases} (\lambda-L)v=f & \text{in } \mathbf{R}_+^n \\ Bv|_{x=0}=h & \text{in } \mathbf{R}^{n-1}. \end{cases}$$

The basic apriori estimate of the solution of (2.1) has been obtained by O. K. Kreiss [4] as follows;

**Lemma 3.1.** *Assume that* A.I, A.II *and* A.III *are valid. Let $v$ be in $H^1(\mathbf{R}_+^n)$. Then there exists a positive constant $\mu_0$ such that it* *holds for any* $\lambda$ *with* $\operatorname{Re}\lambda=\mu\geqq\mu_0$

$$\mu\|v\|^2+<v(0)>^2\leqq\text{const.}\left(\frac{1}{\mu}\|f\|^2+<h>^2\right). \tag{3.2}$$

We shall obtain the existence of the solution to the problem (2.1) by use of the estimate (3.2). For this purpose we consider the adjoint problem of (3.1);

$$\begin{cases}(\lambda-L^{(*)})w=\varphi & \text{in } \mathbf{R}^n_+\\ B'w|_{x=0}=\psi & \text{in } \mathbf{R}^{n-1},\end{cases} \tag{3.1*}$$

where $L^{(*)}$ is the formal adjoint of $L$ and $B'$ is a $(N-l)\times N$ matrix of which kernel is the complement of $(A(0, y)\operatorname{Ker}B(y))$ in $\mathbf{C}^N$.

**Lamma 3.2.** *If* $(L, B)$ *satisfies the assumptions* A.I, A.II *and* A.III, *so does* $(L^{(*)}, B')$.

Moreover the following Green's formula holds;

**Lemma 3.3.** *Let* $v$ *and* $w$ *be in* $H^1(\mathbf{R}^n_+)$. *There exist a* $l\times N$ *matrix* $C$ *and* $(N-l)N$ *matrix* $C'$ *such that*

$$\begin{aligned}((\lambda-L)v, w)-(v, (\bar{\lambda}-L^{(*)})w)\\ =<Bv(0), C'w(0)>+<Cv(0), B'w(0)>,\end{aligned} \tag{3.3}$$

and

$$\begin{cases}<v(0)>\leqq\text{const. }(<Bv(0)>+<Cv(0)>)\\ <w(0)>\leqq\text{const. }(<B'w(0)>+<C'w(0)>).\end{cases} \tag{3.4}$$

The above two lemmas shall be proved in the appendix. From Lemma 3.2 and 3.3 we have as the corollary of Lemma 3.1;

**Lemma 3.1.**$^{(*)}$ *Let* $w$ *be in* $H^1(\mathbf{R}^n_+)$. *There exists a positive constant* $\mu_0^*$ *such that it holds for any complex number* $\bar{\lambda}$ *with*

$\mathrm{Re}\,\bar{\lambda}=\mu\geqq\mu_0^*$,

$$\mu\|w\|^2+<w(0)>^2\leqq\text{const.}\left(\frac{1}{\mu}\|\varphi\|^2+<\psi>^2\right). \tag{3.1)*$$

Let $f$ and $h$ be in $L^2(\mathbf{R}_+^n)$ and $L^2(\mathbf{R}^{n-1})$ respectively. Then we say that $v$ in $L^2(\mathbf{R}_+^n)$ is a weak solution of (3.1), if it holds

$$(f, w)-(v, (\bar{\lambda}-L^{(*)})w)=<h, C'w(0)>, \tag{3.5}$$

for all $w$ in $H^1(\mathbf{R}_+^n)$ with $B'w(0)=0$.

In order to prove the existence of weak solution of (3.1), we need an essential lemma given by Lax and Phillips [5]. We denote by $H^{-1}(\mathbf{R}^{n-1})$ the dual space of $H^1(\mathbf{R}^{n-1})$.

**Lemma 3.5.** *Let $v$ be a weak solution of* (2.1). *Then $v(x)$ belongs to $\mathcal{E}_x^0(H^{-1}(\mathbf{R}^{n-1}))$ and*

$$\lim_{x\to+0}Bv(x)=h \ \text{in}\ H^{-1}(\mathbf{R}^{n-1}).$$

**Proof.** Denote by $\rho_\varepsilon(y)$ a mollifier in $\mathbf{R}^{n-1}$. We write $v_\varepsilon=\rho_\varepsilon*v$. Since $v$ satisfies (3.5), We have

$$(\lambda-L)v=f$$

in distribution sense in $\mathbf{R}_+^n$. Hence $v_\varepsilon$ belongs to $H^1(\mathbf{R}_+^n)$ and satisfies

$$A\frac{\partial}{\partial x}v_\varepsilon=\left(\lambda-\Sigma B_j\frac{\partial}{\partial y_j}\right)v_\varepsilon+\Sigma\left[B_j\frac{\partial}{\partial y_j}, \rho_\varepsilon\right]v-f_\varepsilon,$$

where $f_\varepsilon=\rho_\varepsilon*f$. Since $A$ is non singular, we have for almost every $y$,

$$v_\varepsilon(x)-v_\varepsilon(x')=\int_{x'}^{x}A^{-1}\left\{\left(\lambda-\Sigma B_j\frac{\partial}{\partial y_j}\right)v_\varepsilon+\Sigma\left[B_j\frac{\partial}{\partial y_j}, \rho_\varepsilon\right]v-f_\varepsilon\right\}dx.$$

Hence we have

$$\begin{aligned}&<v_\varepsilon(x)-v_\varepsilon(x')>^2_{H^{-1}(\mathbf{R}^{n-1})}\\&\quad\leqq\text{const.}\,|x-x'|(\|v_\varepsilon\|^2+\|v\|^2+\|f_\varepsilon\|^2).\end{aligned}$$

This shows that $v_\varepsilon(x)$ belongs to $\mathscr{E}^0_x(H^{-1}(\mathbf{R}^{n-1}))$ for any $\varepsilon>0$. And for any $\varepsilon, \varepsilon'>0$ and every $x>0$

$$<v_\varepsilon(x)-v_{\varepsilon'}(x)>^2_{H^{-1}(\mathbf{R}^{n-1})}$$

$$\leqq \text{const.}\,(\|v_\varepsilon-v_{\varepsilon'}\|^2+\|f_\varepsilon-f_{\varepsilon'}\|^2+\|g_\varepsilon-g_{\varepsilon'}\|^2)\,,$$

where $g_\varepsilon=\Sigma\left[B_j\dfrac{\partial}{\partial y_j},\ \rho_\varepsilon\right]v$. Since it holds $g_\varepsilon$ converges to zero in $L^2(\mathbf{R}^n_+)$ for $\varepsilon\to 0$, $v_\varepsilon(x)$ converges to $v(x)$ uniformly in $\mathscr{E}^0_x(H^{-1}(\mathbf{R}^{n-1}))$. Moreover by Green's formula, we obtain

$$\begin{aligned}&(f_\varepsilon,\ w)-(v_\varepsilon,\ \bar{\lambda}-L^{(*)})\,w)\\&=(g_\varepsilon,\ w)+<Av_\varepsilon(0),\ w(0)>\\&=(g_\varepsilon,\ w)+<Bv_\varepsilon(0),\ C'w(0)>.\end{aligned}$$

Hence taking $\varepsilon\to 0$, we obtain from (3.5)

$$<Bv(0)-h,\ C'w(0)>=0$$

for all $w$ in $H^1(\mathbf{R}^n_+)$ with $B'w(0)=0$. In view of (3.4), we obtain $Bv(0)=h$ (c.f. Appendix). q.e.d.

As the corollary of the above lemma we have

**Lemma 3.6.** *If $v$ is a weak solution of* (3.1), *then $v$ satisfies the inequality* (3.2).

**Proof.** Let put $v_\varepsilon=\rho_\varepsilon * v$. Then $v_\varepsilon$ satisfies

$$\begin{cases}(\lambda-L)\,v_\varepsilon=f_\varepsilon+C_\varepsilon v\\ Bv_\varepsilon(0)=h_\varepsilon+H_\varepsilon v(0),\end{cases}$$

where $C_\varepsilon=A\left[A^{-1}\left(\lambda-\Sigma B_j\dfrac{\partial}{\partial y_j}\right),\ \rho_\varepsilon\right]$, $H_\varepsilon=[\rho_\varepsilon,\ B]$. $f_\varepsilon=A\rho_\varepsilon*(A^{-1}f)$ and $h_\varepsilon=\rho_\varepsilon*h$. Since $v_\varepsilon$ is in $H^1(\mathbf{R}^n_+)$, we have

(3.6) $\quad \mu\|v_\varepsilon\|^2+<v_\varepsilon(0)>^2$

$$\leqq \text{const.}\left\{\frac{1}{\mu}(\|f_\varepsilon\|^2+\|C_\varepsilon v\|^2)+\langle h_\varepsilon\rangle^2+\langle H_\varepsilon v(0)\rangle^2\right\}.$$

From the fact that $v$ is in $L^2(R_+^n)$ and $v(0)$ in $H^{-1}(R^{n-1})$ it follows that $\|C_\varepsilon v\|^2 \to 0$ and $\langle H_\varepsilon v(0)\rangle^2 \to 0$ for $\varepsilon \to 0$ respectively. Taking $\varepsilon \to 0$ in (3.6), our assertion is proved. q.e.d.

**Theorem 3.1.** *Assume that* A.I, A.II *and* A.III *are valid. Then for any* $f$ *in* $L^2(R_+^n)$ *and* $h$ *in* $L^2(R^{n-1})$ *there exists uniquely the weak solution* $v$ *of* (3.1) *which satisfies the inequality* (3.2).

**Proof.** We first consider the homogeneous boundary condition, that is, $h=0$. We denote by $\mathscr{D}(L^{(*)})$ the graph norm closure of the set $\{w \in H^1(R_+^n);\ B'w(0)=0\}$. Then $\mathscr{D}(L^{(*)})$ is a Hilbert space with the inner product

$$\|w\|^2_{\mathscr{D}(L^{(*)})}=((\bar{\lambda}-L^{(*)})w,\ (\bar{\lambda}-L^{(*)})w).$$

Since $|(f,\ (\bar{\lambda}-L^{(*)})w)\leqq\|f\|\|(\bar{\lambda}-L^{(*)})w\|$ for any $w\in\mathscr{D}(L^{(*)})$, by Riez theorem we can find $g$ in $\mathscr{D}(L^{(*)})$ such that

$$(f,(\lambda-L^{(*)})w)=(g,w)_{\mathscr{D}(L^{(*)})}$$

Hence if we put $v=(\bar{\lambda}-L^{(*)})g$, $v$ is a weak solution of (3.1) with $h=0$. We next consider the case $h\neq 0$. Let $\{h_n\}$ in $C_0^\infty(R^{n-1})$ be the sequence such that

$$\langle h_n-h\rangle \to 0\,(n\to\infty).$$

Then we can find $\varphi_n$ in $H^1(R_+^n)$ such that

$$B\varphi_n(0)=h_n$$

Let $v_n'$ be a weak solution of

$$(\lambda-L)v_n'=f-(\lambda-L)\varphi_n$$

$$Bv_n'(0)=0.$$

Then $v_n=\varphi_n+v_n'$ is a weak solution of (3.1) with $h_n$ and satisfies

(3.2). Hence taking $n\to\infty$, $v_n$ converges and it's limit $v$ is a weak solution of (3.1). The uniqueness of the weak solution is assured by Lemma 3.6.

We next investigate the regularity of the weak solution of (3.1).

**Theorem 3.2.** *Assume that* A.I, A.II *and* A.III *are valid. Then for any* $f$ *in* $H^m(R_+^n)$ *and* $h$ *in* $H^m(R^{n-1})$ *there exists a solution* $v$ *in* $H^m(R_+^n)$ *of* (3.1) *which satisfies*

$$\mu\|v\|_m^2+\sum_{i+j=0}^{m}|\lambda|^{2j}<D_x^i v(0)>_{m-i-j}^2 \tag{3.7}$$

$$\leqq \text{const.}\left\{\sum_{j=0}^{m}|\lambda|^{2j}\left(\frac{1}{\mu}\|f\|_{m-j}^2+<h>_{m-j}^2\right)\right\},$$

*for* $\mu\geqq\mu_m>0$.

**Proof.** It suffices to prove that our statement in case $m=1$ holds. We shall show that $D_{y_j}v$ belongs to $L^2(R_+^n)(j=1,\cdots,n-1)$. From the fact that $A$ is non singular it follows that $D_x v$ is in $L^2(R_+^n)$. Let us put $V_0={}^t(D_{y_1}v,\cdots,D_{y_{n-1}}v)$. Then $V_0(x)$ is in $\mathscr{E}_x^0(H^{-2}(R^{n-1}))$, because $v(x)$ is in $\mathscr{E}_x^0(H^{-1}(R^{n-1}))$ by Lemma 3.5, and $V_0(x)$ satisfies

$$\begin{cases}(\lambda-\tilde{L})V_0=F & \text{in distribution sense}\\ \tilde{B}V_0(0)=H & \text{in } H^{-2}(R^{n-1}),\end{cases}$$

where

$$\tilde{L}=\begin{pmatrix}A & & \\ & A & 0\\ & \ddots & \\ 0 & & A\end{pmatrix}\frac{\partial}{\partial x}+\sum_{j=1}^{n-1}\begin{pmatrix}B_j & & \\ & \ddots & 0\\ 0 & & B_j\end{pmatrix}\frac{\partial}{\partial y_j}+\left(A\frac{\partial}{\partial y_i}(A^{-1}B_j)\right)$$

$$+\begin{pmatrix}K & & \\ & \ddots & 0\\ 0 & & K\end{pmatrix},\quad F=\begin{pmatrix}A\dfrac{\partial}{\partial y_1}(A^{-1}K)v+A\dfrac{\partial}{\partial y_1}(A^{-1}f)\\ \vdots\\ A\dfrac{\partial}{\partial y_{n-1}}(A^{-1}K)v+A\dfrac{\partial}{\partial y_{n-1}}(A^{-1}f)\end{pmatrix},$$

$$H=\begin{pmatrix} \frac{\partial}{\partial y_1} h-\left(\frac{\partial}{\partial y_1}\right) Bv(0) \\ \vdots \\ \frac{\partial}{\partial y_{n-1}} h-\left(\frac{\partial}{\partial y_{n-1}} B\right) v(0) \end{pmatrix}, \quad \tilde{B}=\begin{pmatrix} B & & 0 \\ & \ddots & \\ 0 & & B \end{pmatrix}.$$

In the other hand, we can show analogously to the proof of Theorem 3.1 that there exists a weak solution $V$ in $L^2(R^n_+)$ such that

$$\begin{cases} (\lambda-\tilde{L})V=F \\ \tilde{B}V(0)=H, \end{cases}$$

and satisfies

$$\mu\|V\|^2+<V(0)>^2 \leqq \text{const.}\left(\frac{1}{\mu}\|F\|^2+<H>^2\right). \tag{3.8}$$

Our assertion is that $V_0$ is equal to $V$. We put $W=V-V_0$. We define $\Lambda_\lambda$ such that

$$\Lambda_\lambda=(|\lambda|^2-\Delta_y)^{-\frac{1}{2}},$$

where $\Delta_y$ is a Laplacian in $R^{n-1}$. Then we can see easily that $\Lambda_\lambda W$ is in $L^2(R^n_+)$ and satisfies in a weak sense

$$\begin{cases} (\lambda-\tilde{L})\Lambda_\lambda W=F_\lambda \\ \tilde{B}\Lambda_\lambda W(0)=H_\lambda, \end{cases}$$

where $F_\lambda=A\left[\Lambda_\lambda, A^{-1}\left(\lambda-\Sigma B_j\frac{\partial}{\partial y_j}\right)\right]W$ and $H_\lambda=[\Lambda_\lambda, B]W(0)$. Hence we have by (3.8)

$$\mu\|\Lambda_\lambda W\|^2+<\Lambda_\lambda W(0)>^2 \leqq \text{const.}\left(\frac{1}{\mu}\|F_\lambda\|^2+<H_\lambda>^2\right). \tag{3.9}$$

On the other hand we can show easily that

$$\|F_\lambda\| \leqq \text{const.}\|\Lambda_\lambda W\|,$$

and

$$<H_\lambda> \leqq \text{const.}\frac{1}{|\lambda|}<\Lambda_\lambda W(0)>.$$

Hence there exists a constant $\mu_1$ such that for $\mu \geqq \mu_1$ we have $\Lambda_\lambda W \equiv 0$. This completes our assertion. And (3.8) implies (3.7) with $m=1$.

q.e.d.

**Remark.** Theorem 3.1 and 3.2 are valid for the adjoint problem (3.1)*.

## §4. Energy Inequalities.

To derive the energy inequality of a solution $u(t)$ of the mixed problem (1.1), we shall use the solution of the adjoint problem of (1.1). This idea is introduced by R. Sakamoto who treated the single higher order hyperbolic equations [7]. In this section we impose the assumptions A.I, A.II, A.III and A.IV stated in the introduction.

Let $u(t)$ be in $L^2_\mu((0, \infty); H^1(\mathbf{R}^n_+))$ and $\frac{\partial}{\partial t} u(t)$ in $L^2_\mu((0, \infty); L^2(\mathbf{R}^n_+))$ for $\mu > 0$. Let $\varphi(t)$ be equal to $u(t)$ for $t \geqslant 0$ and to zero for $t < 0$ and $\psi(t)$ be equal to $Cu(t, 0)$ for $t \geqslant 0$ and to zero for $t < 0$. Here $C$ is composed in Lemma 3.2. Then we can find $\varphi_n(t)$ in $C_0^\infty(R^{n+1}_+)$ and $\psi_n(t) = \psi_n(t, y)$ in $C_0^\infty(R^n)$ such that

$$\varphi_n(t) \to \varphi(t) \qquad \text{in } L^2_\mu((-\infty, \infty); L^2(R^n_+)),$$

and

$$\psi_n(t) \to \psi(t) \qquad \text{in } L^2_\mu((-\infty, \infty); L^2(R^{n-1})).$$

Then we have

**Proposition 4.1.** *For $\varphi_n(t)$ and $\psi_n(t)$ there exists functions $v_n(t)$ such that*

i) *$v_n(t)$ is in $L^2_{-\mu}((-\infty, \infty); H^1(R^n_+))$ and $\frac{\partial}{\partial t} v_n(t)$ in $L^2_{-\mu}((-\infty, \infty); L^2(R^n_+))$ and satisfies*

$$\frac{\partial}{\partial t} v_n(t) = -L^* v_n(t) - \mu e^{-2\mu t} \varphi_n(t) \qquad \text{in } R^n_+$$

$$B' v_n(t, 0) = e^{-2\mu t} \psi_n(t) \qquad \text{in } R^{n-1},$$

*for* $t\in(-\infty, \infty)$ *and* $\mu>\mu_0>0$, *where* $L^*$ *is a formal adjoint of* $L$.

ii) *It holds for* $\mu>\mu_0>0$,

$$\int_{-\infty}^{\infty}\mu\|e^{\mu t}v_n(t)\|^2+<e^{\mu t}v_n(t, 0)>^2 dt$$

$$\leqslant \text{const.}\int_{-\infty}^{\infty}\mu\|e^{-\mu t}\varphi_n(t)\|^2+<e^{-\mu t}\psi_n(t)>^2 dt,$$

iii) *and*

$$\|v_n(0)\|^2\leqslant \text{const.}\int_{-\infty}^{\infty}\mu\|e^{-\mu t}\varphi_n(t)\|^2+<e^{-\mu t}\psi_n(t)>^2 dt.$$

**Proof.** Let $\hat{\varphi}_n(\lambda)$ and $\hat{\psi}_n(\lambda)$ be Lapace transforms of $\varphi_n(t)$ and $\psi_n(t)$ respectively, that is,

$$\hat{\varphi}_n(\lambda)=\int_{-\infty}^{\infty}e^{-\mu t}\varphi_n(t)dt,$$

and

$$\hat{\psi}(\lambda)=\int_{-\infty}^{\infty}e^{-\lambda t}\psi_n(t)dt,$$

where $\lambda=\mu+i\sigma$, $\mu>0$.

Then we note that

$$\int_{-\infty}^{\infty}\|\hat{\varphi}_n(\lambda)\|^2 d\sigma=\int_{-\infty}^{\infty}\|e^{-\mu t}\varphi_n(t)\|^2 dt \tag{4.1}$$

and

$$\int_{-\infty}^{\infty}<\hat{\psi}_n(\lambda)>^2 d\sigma=\int_{-\infty}^{\infty}<e^{-\mu t}\psi_n(t)>^2 dt. \tag{4.2}$$

Let $w_n(\lambda)$ be the solution which satisfies

$$\begin{aligned}(\lambda-L^{(*)})w_n(\lambda)&=-\mu\hat{\varphi}_n(\lambda) \quad \text{in } R^n_+,\\ Bw_n(\lambda, 0)&=\hat{\psi}_n(\lambda) \quad \text{in } R^{n-1}.\end{aligned} \tag{4.3}$$

It follows from Theorem 3.1 and 3.2 that $w_n(\lambda)$ is sufficiently smooth and satisfies

$$\mu\|w_n(\lambda)\|^2+<w_n(\lambda, 0)>^2 \tag{4.4}$$

$$\leqq \text{const.} (\mu\|\hat{\varphi}_n(\lambda)\|^2 + <\psi_n(\lambda)>^2)\,.$$

We define $v_n(t)$ as follow for $\mu \geqq \mu_0$,

$$v_n(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\bar{\lambda}t} w_n(\lambda)\, d\sigma,$$

where $\lambda = \mu + i\sigma$.
Noting that

$$\int_{-\infty}^{\infty} \|e^{\mu t} v_n(t)\|^2 dt = \int_{-\infty}^{\infty} \|w_n(\lambda)\|^2 d\sigma$$

and

$$\int_{-\infty}^{\infty} \left\| e^{\mu t} \frac{\partial}{\partial t} v_n(t) \right\|^2 dt = \int_{-\infty}^{\infty} \|\bar{\lambda} w_n(\lambda)\|^2 d\sigma,$$

we obtain i) from (4.3). And integration of (4.4) with respect to $\sigma$ implies ii). To prove iii) we need introduce the symmetrizer of the operator $L$.

**Lemma 4.1.** *Suppose that* $a(x, y; \xi, \eta) = A\xi + \sum_{j=1}^{n-1} B_j \eta_j$ *has only real distinct eigen values for* $(x, y) \in \mathbf{R}^n$, $(\xi, \eta) \in \mathbf{R}^n$, $(\xi, \eta) \neq 0$. *Then there exists the matrix* $r(x, y; \xi, \eta)$ *which has the following properties*:

i) $r(x, y; \xi, \eta)$ *is a symmetric and positive definite matrix homogeneous of degree zero with respect to* $(\xi, \eta)$,

ii) $(ra)(x, y; \xi, \eta)$ *is symmetric.*

iii) $r(x, y; \xi, 0)$ *is independent of* $\xi$ *(denote by* $r_0(x, y)$*),*

iv) $r(x, y; \xi, \eta)$ *is sufficiently smooth for* $(x, y) \in \mathbf{R}^n$, $(\xi, \eta) \in \mathbf{R}^n$, $(\xi, \eta) \neq 0$ *and for any* $\alpha, j$, $D_x^j D_y^\alpha r(x, y; \xi, \eta)$ *is analytic with respect to* $\xi$ *in a strip*

$$|\mathrm{Im}\,\xi| \leqq \delta |\eta|, \qquad \eta \neq 0,$$

*where* $\delta$ *is a positive constant.*

We decompose $r(x, y; \xi, \eta) = r(x, y; \xi, \eta) - r_0(x, y) + r_0(x, y)$ and write $r_1(x, y; \xi, \eta) = r(x, y; \xi, \eta) - r_0(x, y)$. Let be $R$ and $R_1$ singular integral operators with it's symbol $r(x, y; \xi, \eta)$ and $r_1(x, y; \xi, \eta)$ respectively. Let $u$ be in $L^2(\mathbf{R}^n_+)$ and $u_0$ is equal to $u$ for $x>0$ and to zero for $x>0$.

**Lemma 4.2.** *There exists a partition of unity $\{\alpha_j\}$ in $\mathbf{R}^n$ such that $\sum \alpha_j^2 \equiv 1$ and for $u \in L^2(\mathbf{R}^n_+)$*

$$\sum_j \frac{1}{2} (\alpha_j (R+R^*) \alpha_j u_0, u) \geqq \delta \|u\|^2, \tag{4.5}$$

*where $\delta$ is a positive constant and $R^*$ is the adjoint operator of $R$.*

We write

$$H=\frac{1}{2} \sum_j \alpha_j (R+R^*) \alpha_j, \qquad H_1=\frac{1}{2} \sum_j \alpha_j R_1 \alpha_j$$

and

$$H_2=\frac{1}{2} \sum_j \alpha_j (R^*-R) \alpha_j.$$

Then it is obivous that $H_1 u_0$ and $H_2 u_0$ are in $H^1(\mathbf{R}_n)$ for $u \in H^1(\mathbf{R}^n_+)$. Hence integration by part gives

**Lemma 4.2.** *For $u \in H^1(\mathbf{R}^n_+)$ it holds*

$$\begin{aligned} &(H(Lu)_0, u)+(Hu_0, Lu) \\ &\quad =((HL+L^{(*)}H) u_0, u)-<(A^* r_0 u)(0), u(0)> \\ &\qquad -2\mathrm{Re}<(A^* H_1 u_0)(0), u(0)> -2\mathrm{Re}<(A^* H_2 u_0)(0), u(0)> \end{aligned} \tag{4.6}$$

**Remark.** We note that it holds for $u \in H^1(\mathbf{R}^n_+)$

$$<(H_2 u_0)(0)>^2 \leqq \text{const.} \|H_2 u_0\|^2_{H^1(R^n)} \leqq \text{const.} \|u\|^2 \tag{4.7}$$

Furthermore we can estimate roughly

$$<(H_1 u_0)(0)>^2 \leqq \text{const.} \|u\|^2_{H^1(R^n_+)},$$

because the symbol $r_1(x, y; \xi, \eta)$ has the following estimate for any multi integer $(j, \alpha)$

$$(4.8)\qquad |D_x^j D_y^\alpha r_1(x, y; \xi, \eta)| \leq \text{const.}\left|\frac{\eta}{\xi}\right|$$

for $\xi \neq 0$, $\eta \in \mathbf{R}^{n-1}$, $(x, y) \in \mathbf{R}^n$. And it follows from ii) in Lemma 3.1 that

$$(4.9)\qquad \|(HL + L^{(*)}H)u_0\| \leq \text{const.}\|u\|.$$

The proof of iii) of Proposition 4.1; since $v_n(t)$ satisfies i), we have

$$(Hv_n(0)_0, v_n(0)) = -\int_0^\infty \frac{\partial}{\partial t}(He^{\mu t}v_n(t)_0, e^{\mu t}v_n(t))dt$$

$$= \int_0^\infty \{2\mathrm{Re}\, e^{\mu t}(H(L^* v_n(t))_0, v_n(t)) - 2\mu(\mathrm{He}^{\mu t}v_n(t)_0, e^{\mu t}v_n(t))$$

$$-2\mu \mathrm{Re}(H\varphi_n(t)_0, v_n(t))\}\, dt.$$

According to Lemma 3.2, (4.7) and (4.9), we have

$$(4.10)\qquad (Hv_n(0)_0, v_n(0)) \leqslant \text{const.}\int_0^\infty \{\mu\|e^{\mu t}v_n(t)\|^2 + \mu\|e^{-\mu t}\varphi_n(t)\|^2$$

$$+ <e^{\mu t}v_n(t, 0)>^2 + <e^{\mu t}(H_1 v_n(t)_0)(0)>^2\}\, dt,$$

for $\mu \geq \mu_0 > 0$. The main point of this article is the proof given in the next section that for $\mu \geq \mu_0 > 0$,

$$(4.11)\qquad \int_{-\infty}^\infty <e^{\mu t}(H_1 v_n(t)_0)(0)>^2 dt$$

$$\leq \text{const.}\left\{\int_{-\infty}^\infty \{\mu\|e^{\mu t}v_n(t)\|^2 + \mu\|e^{-\mu t}\varphi_n(t)\|^2\right.$$

$$\left. + <e^{\mu t}v_n(t, 0)>^2\}\, dt\right\}.$$

The assumption A.IV) shall be used in order to derive this estimate. From (4.5), (4.10), (4.11) and the estimate ii) it follows that the estimate iii) holds. q.e.d.

**Theorem 4.1.** *Assume that* A.I, A.II, A.III *and* A.IV *are valid.*

*Then these exists a constant* $\mu_0$ *such that it holds for* $u(t)$, *a solution of* (1.1) *which is in* $L^2_\mu((0, \infty); H^1(\mathbf{R}^n_+))$ *and* $\frac{\partial}{\partial t} u(t)$ *in* $L^2_\mu((0, \infty)$: $L^2(\mathbf{R}^n_+))$ *for any* $\mu \geq \mu_0$,

$$\int_0^\infty \mu \| e^{-\mu t} u(t) \|^2 + < e^{-\mu t} u(t, 0) >^2 dt \tag{4.12}$$

$$\leq \text{const.} \{ \| g \|^2 + \int_0^\infty \frac{1}{\mu} \| e^{-\mu t} f(t) \|^2 + < e^{-\mu t} h(t) >^2 dt \}$$

**Proof.** Let $v_n(t)$ be a solution of the adjoint problem stated in Proposition 4.1. Then according to Lemma 3.3, we have

$$\int_0^\infty \Big( \Big( \frac{\partial}{\partial t} - L \Big) u(t), v_n(t) \Big) - \Big( u(t), \Big( -\frac{\partial}{\partial t} - L^{(*)} \Big) v_n(t) \Big) dt$$

$$= -(g, v_n(0)) + \int_0^\infty < Bu(t, 0), C' v_n(t, 0) >$$

$$+ < Cu(t, 0), B' v_n(t, 0) > dt.$$

By i) in Proposition 4.1, we obtain

$$\int_0^\infty \mu (e^{-\mu t} u(t), e^{-\mu t} \varphi_n(t)) + e^{-\mu t} Cu(t, 0), e^{-\mu t} \psi_n(t) > dt$$

$$= (g, v_n(0)) + \int_0^\infty (e^{-\mu t} f(t), e^{\mu t} v_n(t))$$

$$- < e^{-\mu t} h(t), e^{\mu t} C' v_n(t, 0) > dt$$

Hence, when we take $n \to \infty$, by use of Schwarz inequality, ii) and iii), we have

$$\int_0^\infty \mu \| e^{-\mu t} u(t) \|^2 + < e^{-\mu t} Cu(t, 0) >^2 dt$$

$$\leq \text{const.} \Big\{ \| g \|^2 + \int_0^\infty \frac{1}{\mu} \| e^{-\mu t} f(t) \|^2 + < e^{-\mu t} h(t) >^2 dt \Big\}$$

$$+ \frac{1}{2} \int_0^\infty \mu \| e^{-\mu t} u(t) \|^2 + < e^{-\mu t} Cu(t, 0) >^2 dt.$$

This estimate and (3.4) imply (4.12). q.e.d.

**Remark.** In the case of the homogeneous initial data, that is,

$g\equiv 0$, the estimate (4.12) was derived by O. K. Kreiss [4].

**Thorem 4.2.** *Under assumptions of Theorem* 4.1, *there exists a constant* $\mu_0>0$ *such that for* $\mu\geq\mu_0$

$$(4.13)\quad \|e^{-\mu t}u(t)\|^2\leq \text{const.}\left\{\|g\|^2+\int_0^\infty \frac{1}{\mu}\|e^{-\mu s}f(s)\|^2+<e^{-\mu s}h(s)>^2 ds\right\}$$

*for* $t$ *in* $[0, \infty)$.

**Proof.** By the same way as the proof of iii) in Proposition 4.1, we have

$$(4.14)\quad (Hu^{-\mu t}u(t)_0, e^{-\mu t}u(t))-(Hu(0)_0, u(0))$$
$$=\int_0^t \frac{\partial}{\partial s}(He^{-\mu s}u(s)_0, e^{-\mu s}u(s))ds$$
$$=\int_0^t 2\text{Re}(He^{-\mu s}(Lu(s))_0, e^{-\mu s}u(s))+2\text{Re}(He^{-\mu s}f(s)_0, e^{-\mu s}u(s))ds$$
$$-2\mu\int_0^t (He^{-\mu s}u(s)_0, e^{-\mu s}u(s))ds$$
$$\leq \text{const.}\int_0^t \mu\|e^{-\mu s}u(t)\|^2+\frac{1}{\mu}\|e^{-\mu s}f(t)\|^2+<e^{-\mu s}u(t, 0)>^2$$
$$+<e^{-\mu s}(H_1u(s)_0)(0)>^2 ds.$$

Then it holds

$$(4.15)\quad \int_0^\infty <e^{-\mu t}(H_1u(t)_0)(0)>^2 dt$$
$$\leq \text{const.}\left\{\|g\|^2+\int_0^\infty \mu\|e^{-\mu t}u(t)\|^2\right.$$
$$\left.+\frac{1}{\mu}\|e^{-\mu t}f(t)\|^2+<e^{-\mu t}u(t, 0)>^2 dt\right\}.$$

The main point of this article is to prove this inequality together with (4.11). It's proof shall be given in the next section. From (4.5), (4.14) and (4.15), we have

$$\|e^{-\mu t}u(t)\|^2<\text{const.}\left\{\|g\|^2+\int_0^\infty \mu\|e^{-\mu s}u(t)\|^2+\frac{1}{\mu}\|e^{-\mu s}f(s)\|^2\right.$$

$$+<e^{-\mu s}u(s,0)>^2 ds\Big\} .$$

This and (4.12) imply (4.13). q.e.d.

Noting that for $t<t_0$ the solution $u(t)$ of (1.1) is independent of $f(t)$ and $h(t)$ for $t>t_0$, we can obtain our main theorem stated in the introduction from Theorem 4.1 and 4.2.

## §5. Estimates of Mean Boundary Values

We shall prove the estimates (4.11) and (4.15). To do so we shall make use of the method which was introduced by Friedrichs and Lax [1].

**5.1 The case of constant coefficients.** Here we assume that the coefficients of $L$ are constant and that $L$ is strictly hyperbolic. Let $u(t)$ be in $L^2_\mu((0,\infty);H^1(\mathbf{R}^n_+))$ and $\frac{\partial}{\partial t}u(t)$ in $L^2_\mu((0,\infty);L^2(\mathbf{R}^n_+))$ for $\mu>0$. And $u(t)$ satisfies

$$\frac{\partial}{\partial t}u(t)=Lu(t)+f(t) \qquad \text{for } t>0, \tag{5.1}$$

where $f(t)$ is in $L^2_\mu((0,\infty);L^2(\mathbf{R}^n_+))$.

Then we have

**Theorem 5.1.** *Assume that the symbol* $r_1(\xi,\eta)$ *is homogeneous of degree zero with respect to* $(\xi,\eta)$ *in* $\mathbf{R}^n$, *analytic function of* $\xi$ *in a strip* $|\mathrm{Im}\,\xi|\leqq\delta|\eta|$, $\eta\neq0$, *and satisfies*

$$|r_1(\xi,\eta)|\leqq\text{const.}\left|\frac{\eta}{\xi}\right| \qquad \textit{for } \xi\neq0. \tag{5.2}$$

*Then it holds*

$$\int_0^\infty<e^{-\mu t}(R_1u(t)_0)(0)>^2dt \tag{5.3}$$

$$\leqq\text{const.}\Big\{\|g\|^2+\int_0^\infty\frac{1}{\mu}\|e^{-\mu t}f(t)\|^2+\mu\|e^{-\mu t}u(t)\|^2$$

$$+<e^{-\mu t}u(t, 0)>^2 dt\Big\},$$

*for* $u(t)$ *satisfying* (5.1), *where* $g=u(0)$.

**Proof.** We denote by $v(\lambda)$ the Laplace transform of $u(t)$, that is, $\mathrm{Re}\,\lambda=\mu>0$,

$$v(\lambda)=\int_0^\infty e^{-\lambda t}u(t)\,dt.$$

Then we can write

$$(R_1 v(\lambda)_0)(0, y)=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{iy\eta} r_1(\xi, \eta)\hat{v}(\lambda, \xi, \eta)\, d\xi\, d\eta, \tag{5.4}$$

where $\hat{v}(\lambda, \xi, \eta)$ stands for the Fourier transform of $v(\lambda)_0$ which is equal to $v(\lambda)$ for $x\geq 0$ and to zero for $x<0$. Hence $\hat{v}(\lambda, \xi, \eta)$ is analytic with respect to $\xi$ in $\mathrm{Im}\,\xi<0$. By assumption $r_1(\xi, \eta)$ is analytic in a strip $|\mathrm{Im}\,\xi|\leq\delta|\eta|$. So we may shift the line of integration with respect to $\xi$ in (5.4) into the complex plane, that is,

$$\int_{-\infty}^\infty r_1(\xi, \eta)\hat{v}(\lambda, \xi, \eta)d\xi=\int_{\mathrm{Im}\,\xi=-m} r_1(\xi, \eta)\hat{v}(\lambda, \xi, \eta)d\xi, \tag{5.5}$$

where $0<m<\delta\eta$.

**Lemma 5.1.** ([1], [3]) *There exists a positive constant* $k$ *such that*

i) *for* $|\lambda|\leq 2k|\eta|$ *there exists a* $m=m(\lambda, \eta)$ *in the range*

$$\frac{\delta}{2}|\eta|\leq m\leq\delta|\eta|, \tag{5.6}$$

*such that*

$$|(\lambda-\mathrm{i}a(\xi, \eta))^{-1}|\leq \text{const.}\frac{1}{|\xi|}, \tag{5.7}$$

*for* $\xi$ *with* $\mathrm{Im}\,\xi=-m(\lambda, \eta)$, *where* $a(\xi, \eta)=A\xi+\Sigma B_j\eta_j$,

ii) *for* $|\lambda|\geq\frac{k}{2}|\eta|$, $M(\lambda, \eta)=A^{-1}(\lambda-i\Sigma B_j\eta_j)$ *has distinct eigen values*

$\xi_j(\lambda, \eta)(j=1, \cdots, N)$ *and satisfies*

$$|\mathrm{Re}\,\xi_j(\lambda, \eta)| \leq \text{const. } \mu,$$

*for* $j=1, \cdots, N$.

In view of ii) we have

**Lemma 5.2.** *There exists a* $N\times N$ *matrix* $S(\lambda, \eta)$ *homogeneous of degree zero and symmetric positive definite such that* $S(\lambda, 0)$ *is a constant matrix* $(=S_0)$ *and*

$$|\mathrm{Re}(S(\lambda, \eta)M(\lambda, \eta))| \leq \text{const. } \mu, \tag{5.8}$$

*for* $|\lambda| \geq \frac{k}{2}|\eta|$.

By virtue of Lemma 5.1 and 5.2 we can estimate (5.5). We assume $|\lambda| \leq k|\eta|$. Since $v(\lambda, \xi, \eta)$ satisfies

$$(\lambda - \mathrm{ia}(\xi, \eta))v(\lambda, \xi, \eta) = \hat{g}(\xi, \eta) + \hat{f}(\lambda, \xi, \eta) + A\tilde{v}(\lambda, 0, \eta),$$

where $\hat{g}(\xi, \eta)$ is the Fourier transform of $g(x, y)_0$, $\hat{f}(\lambda, \xi, \eta)$ is the Fourier-Laplace transform of $f(t, x, y)_0$ and $\tilde{v}(\lambda, 0, \eta)$ is the Fourier transform of $v(\lambda, 0, y)$ with respect to $y$, we have by virtue of i) in Lemma 5.1 for $\mathrm{Im}\,\xi = -m$

$$|\hat{v}(\lambda, \xi, \eta)| < \text{const.} \frac{1}{|\xi|}\{|\hat{g}(\xi, \eta)| + |(\hat{f}(\lambda, \xi, \eta)| + |\tilde{v}(\lambda, 0, \eta)|\}. \tag{5.9}$$

We denote by $\widetilde{R_1 v(\lambda)}_0(0, \eta)$ the Fourier transform of $R_1 v(\lambda)_0(0, y)$ with respect to $y$. Then by virtue of (5.2), (5.9) and Schwarz inequality, we have for $|\lambda| \leq k|\eta|$,

$$\begin{aligned} |\widetilde{R_1 v(\lambda)}_0(0, \eta)|^2 &= \left|\int_{\mathrm{Im}\,\xi=-m} r_1(\xi, \eta)\hat{v}(\lambda, \xi, \eta)d\xi\right|^2 \\ &\leq \text{const.}|\eta| \int_{\mathrm{Im}\,\xi=-m} |\hat{v}(\lambda, \xi, \eta)|^2 d\xi \end{aligned} \tag{5.10}$$

$$\leq \text{const.}\left\{\frac{1}{|\eta|}\int(|\hat{g}(\xi,\eta)|^2+|\hat{f}(\lambda,\xi,\eta|^2)d\xi+|\tilde{v}(\lambda,0,\eta)|^2\right\}.$$

Hence we obtain

$$(5.11)\qquad \int_{|\lambda|\leq k|\eta|}|\widetilde{R_1 v(\lambda)}_0(0,\eta)|^2 d\sigma \leq \text{const.}\left\{\int|\hat{g}(\xi,\eta)|^2 d\xi + \frac{1}{|\eta|}\int|f(\lambda,\xi,\eta)|^2 d\xi\, d\sigma + \int|\tilde{v}(\lambda,0,\eta)|^2 d\sigma\right\}.$$

We next assume $|\lambda|\geq k|\eta|$. We denote by $\tilde{v}(\lambda, x, \eta)$ the Fourier transform of $v(\lambda, x, y)$ with respect to $y$. Then $\tilde{v}(\lambda, x, \eta)$ satisfies

$$\frac{d}{dx}\tilde{v}(\lambda,x,\eta)=M(\lambda,\eta)\tilde{v}(\lambda,x,\eta)+A^{-1}(\tilde{g}(x,\eta)+\tilde{f}(\lambda,x,\eta)).$$

Hence we have

$$\begin{aligned}(5.12)\qquad & S(\lambda,\eta)\tilde{v}(\lambda,0,\eta)\cdot\overline{\tilde{v}(\lambda,0,\eta)}\\ &=-\int_0^\infty\frac{d}{dx}\{e^{-2\delta|\eta|x}S(\lambda,\eta)\tilde{v}(\lambda,x,\eta)\cdot\overline{\tilde{v}(\lambda,x,\eta)}\}\,dx\\ &=2\delta|\eta|\int_0^\infty e^{-2\delta|\eta|x}S(\lambda,\eta)\tilde{v}(\lambda,x,\eta)\cdot\overline{\tilde{v}(\lambda,x,\eta)}dx\\ &\quad -2\text{Re}\int_0^\infty e^{-2\delta|\eta|x}S(\lambda,\eta)M(\lambda,\eta)\tilde{v}(\lambda,x,\eta)\cdot\overline{\tilde{v}(\lambda,x,\eta)}dx\\ &\quad -2\text{Re}\int_0^\infty e^{-2\delta|\eta|x}S(\lambda,\eta)A^{-1}(\tilde{g}(x,\eta)+\tilde{f}(\lambda,x,\eta))\cdot\overline{\tilde{v}(\lambda,x,\eta)}dx.\end{aligned}$$

Since $S(\lambda, \eta)$ is symmetric and positive definite, we have

$$|v(\lambda,x,\eta)|^2\leq\text{const.}\,(S(\lambda,x,\eta)\tilde{v}(\lambda,x,\eta)\cdot\overline{v(\lambda,x,\eta)}).$$

Hence by virtue of (5.8) and (5.12) we obtain

$$\begin{aligned}(5.13)\qquad & |\eta|\int_0^\infty|e^{-\delta|\eta|x}\tilde{v}(\lambda,x,\eta)|^2dx\\ &\leq\text{const.}\left\{\text{Re}\int_0^\infty e^{-2\delta|\eta|x}S(\lambda,\eta)A^{-1}\tilde{g}(x,\eta)\cdot\overline{\tilde{v}(\lambda,x,\eta)}dx\right.\\ &\quad+\int_0^\infty\frac{1}{\mu}|\tilde{f}(\lambda,x,\eta)|^2+\mu|\tilde{v}(\lambda,x,\eta)|^2dx\end{aligned}$$

$$+|\tilde{v}(\lambda, 0, \eta)|^2\Big\}, \qquad \text{for } |\lambda| \geq k|\eta|.$$

Considering that $S(\lambda, 0)$ is a constant matrix, that it holds

$$|S(\lambda, \eta) - S(\lambda, 0)| \leq \text{const.} \left|\frac{\eta}{\lambda}\right|$$

and that

$$\frac{1}{2\pi}\int \tilde{v}(\lambda, x, \eta) d\sigma = \frac{1}{2}\tilde{g}(x, \eta),$$

we have

$$\text{(5.14)} \quad \text{Re}\int_{|\lambda| \geq k|\eta|} d\sigma \int_0^\infty e^{-2\delta|\eta|x} S(\lambda, \eta) A^{-1} \tilde{y}(x, \eta) \cdot \overline{\tilde{v}(\lambda, x, \eta)} dx$$

$$\leq \text{const.} \Big\{\int_0^\infty |\tilde{g}(x, \eta)|^2 dx + |\eta| \int_{|\lambda| \leq k|\eta|} d\sigma \int_0^\infty |e^{-mx} \tilde{v}(\lambda, x, \eta)|^2 dx$$

$$+ \int_{|\lambda| \geq k|\eta|} e^{-2\delta|\eta|x} \left|\frac{\eta}{\lambda}\right| |\tilde{g}(x, \eta)| \cdot |\tilde{v}(\lambda, x, \eta)| d\sigma dx\Big\}.$$

Noting that

$$\text{const.} \int_{|\lambda| \geq k|\eta|} e^{-2\delta|\eta|x} \left|\frac{\eta}{\lambda}\right| |\tilde{g}(x \cdot \eta)| |\tilde{v}(\lambda, x, \eta)| d\sigma dx$$

$$\leq \frac{|\eta|}{2} \int_{|\lambda| \geq k|\eta|} d\sigma \int_0^\infty |e^{-\delta|\eta|x} \tilde{v}(\lambda, x, \eta)|^2 dx$$

$$+ \text{const.} \int_0^\infty |\tilde{g}(x, \eta)|^2 dn,$$

we obtain by (5.13), (5.14) and (5.10)

$$\text{(5.15)} \quad |\eta| \int_{|\lambda| \geq k|\eta|} d\sigma \int_0^\infty |e^{-\delta|\eta|x} \tilde{v}(\lambda, x, \eta)|^2 dx$$

$$\leq \text{const.} \Big\{\int_0^\infty |\tilde{g}(x, \eta)|^2 dx$$

$$+ \int \frac{1}{\mu} |\tilde{f}(\lambda, x, \eta)|^2 + \mu |\tilde{v}(\lambda, x, \eta)|^2 d\sigma dx$$

$$+ \int |\tilde{v}(\lambda, 0, \eta)|^2 d\sigma\Big\}.$$

Hence if we decompose

$$\int_{-\infty}^{\infty} < R_1 v(\lambda)_0(0) >^2 d\sigma = \int_{\mathbf{R}^{n-1}} d\eta \int |\widetilde{R_1 v(\lambda)}_0(0, \eta)|^2 d\sigma$$

$$\leq \text{const.} \int_{\mathbf{R}^{n-1}} d\eta \Big\{ |\eta| \int_{|\lambda| \leq k|\eta|} d\sigma \int_0^{\infty} |e^{-mx} \tilde{v}(\lambda, x, \eta)|^2 dx$$

$$+ |\eta| \int_{|\lambda| \geq k|\eta|} d\sigma \int_0^{\infty} |e^{-\delta|\eta|x} \tilde{v}(\lambda, x, \eta)|^2 dx \Big\},$$

in view of (5.11) and (5.15) we obtain (5.3). q.e.d.

**Corollary 5.1.** *Let* $v(t)$ *be in* $L^2_{-\mu}((-\infty, \infty): H^1(\mathbf{R}^n_+))$ *and* $\frac{\partial}{\partial t} v(t)$ *in* $L^2_{-\mu}((-\infty, \infty): L^2(\mathbf{R}^n_+))$ *for* $\mu > 0$. *If for* $\varphi(t)$ *in* $L^2_{-\mu}((-\infty, \infty): L^2(\mathbf{R}^n_+))$, $v(t)$ *satisfies for* $t \in (-\infty, \infty)$

(5.16) $$\frac{\partial}{\partial t} v(t) = -L^{(*)} v(t) + \varphi(t) \qquad \text{in } \mathbf{R}^n_+,$$

*it follows that it holds*

$$\int_{-\infty}^{\infty} < e^{\mu t}(R_1 v(t)_0)(0) >^2 dt \leq \text{const.} \Big\{ \int_{-\infty}^{\infty} \mu \| e^{\mu t} v(t) \|^2$$

$$+ \frac{1}{\mu} \| e^{\mu t} \varphi(t) \|^2 + < e^{\mu t} v(t, 0) >^2 dt \Big\}.$$

**5.2 The case of variable coefficients.** Here we assume that the coefficients of $L$ are constant outside a compact set in $\mathbf{R}^n$ and $A$ is independent of $y$ in $\mathbf{R}^{n-1}$. Let $R$ be a pseudo differential operator with it's symbol $r(x, y; \xi, \eta)$ which is homogeneous of degree zero with respect to $(\xi, \eta)$ and is independent of $(x, y)$ outside a compact set in $\mathbf{R}^n$. Moreover we assume that $r(x, y; \xi, \eta)$ is sufficiently smooth, $(\xi, \eta) \neq 0$, and that $D_x^j D_y^\alpha r(x, y; \xi, \eta)$ are analytic with respect to $\xi$ in $|\text{Im}\, \xi| \leq \delta |\eta|$, $\delta > 0$ and satisfy

(5.17) $$|D_x^j D_j^\alpha r(x, y; \xi, \eta)| \leq C_{j\alpha} \left| \frac{\eta}{\xi} \right|$$

for all $(j, \alpha)$.

Then we have

**Theorem 5.2.** *Let $u(t)$ be in $L^2_\mu((0, \infty); H^1(\mathbf{R}^n_+))$ and $\frac{\partial}{\partial t} u(t)$ in $L^2_\mu((0, \infty); L^2(\mathbf{R}^n_+))$ for $\mu>0$. If $u(t)$ satisfies (5.1) for $t>0$, it follows that there exists a positive number $\mu_0$ such that*

$$(5.18)\qquad \int_0^\infty < e^{-\mu t}(Ru(t)_0(0)>^2 dt$$

$$\leq \text{const.}\left\{\|g\|^2+\int_0^\infty \frac{1}{\mu}\|e^{-\mu t}f(t)\|^2+\mu\|e^{-\mu t}u(t)\|^2 + < e^{-\mu t}u(t, 0)>^2 dt\right\},$$

*for $\mu\geq\mu_0$.*

**Proof.** Since the symbol of $R$ is independent of $(x, y)$ outside a compact set in $\mathbf{R}^n$, we can decompose

$$R=R_1+R_\infty,$$

where the symbol of $R_\infty$ is independent of $(x, y)$ and the symbol $r_1(x, y: \xi, \eta)$ of $R_1$ has a compact support in $\mathbf{R}^n$ with respect to $(x, y)$. Hence it is sufficient to prove (5.18) for $R_1$. Moreover we assume without generality that the support of $u(t)$ is in $B_\varepsilon(0, y_0)=\{(x, y)$ in $\mathbf{R}^n: |x|^2+|y-y_0|^2\leq\varepsilon^2\}$.

Let $Q_j$ be the $j$-th path in $\mathbf{R}^{n-1}$ and $\rho_j$ be the distance of $Q_j$ from the origin such that for $\eta, \eta'$ in $Q_j$

$$(5.19)\qquad |\eta-\eta'|\leq\varepsilon|\eta|$$

and $\sum\frac{1}{\rho_j}$ is convergent.

Then we can construct a Garding-type partition unity such that, (c.f. [1])

**Lemma 5.3.** *There exists a set of $C^\infty(\mathbf{R}^{n-1})$ functions $p_j(\eta)$ where the support of $p_j(\eta)$ lies in $Q_j$ and such that for $\eta\in\mathbf{R}^{n-1}$*

$$\sum p_j^2(\eta)=1,$$

*and*

(5.20) $$|D_\eta p_j(\eta)|\leq\frac{\text{const.}}{1+|\eta|},$$

*where constant is independent of j.*

We denote by $p_j(D_y)$ the pseudo differential operator with symbol $p_j(\eta)$. Then $p_j(D_y)$ are bounded, real symmetric operators and satisfy

$$\sum p_j^2(D_y)=1.$$

We denote by $v(\lambda)$ the Laplace transform of $u(t)$. We write

(5.21) $$<(R_1 v(\lambda)_0(0)>^2=\sum_j<(p_j(D_y)R_1 v(\lambda)_0)(0)>^2$$
$$=\sum<(R_1 v_j(\lambda)_0)(0)>^2+\sum<(C_j v(\lambda)_0)(0)>^2,$$

where

$$v_j(\lambda)=p_j(D_y)v(\lambda) \quad\text{and}\quad C_j=[R_1, p_j(D_y)].$$

Then we have

**Lemma 5.4.** *It holds for all j*

(5.22) $$<C_j v(\lambda)_0(0)>^2\leq\text{const.}\frac{1}{\rho_j}\|v(\lambda)\|^2.$$

**Proof.** We denote by $\widetilde{C_j v_0}(0, \eta)$ the Fourier transform of $(C_j v_0)(0, y)$ with respect to $y$. Then we can express

(5.23) $$\widetilde{C_j v}(0, \eta)$$
$$=\frac{1}{2\pi}\int_{\mathbf{R}^{n-1}}d\eta'\int_{-\infty}^{\infty}\hat{r}_1(\eta-\eta': \xi, \eta')(p_j(\eta')-p_j(\eta))\hat{v}(\lambda, \xi, \eta)d\xi,$$

where

$$\hat{r}_1(\eta-\eta': \xi, \eta')=\int_{\mathbf{R}^{n-1}}e^{-iy\cdot(\eta-\eta')}r_1(0, y: \xi, \eta')dy,$$

which is analytic with respect to $\xi$ in $|\mathrm{Im}\,\xi|\leq\delta|\eta'|$ and in view of (5.17) satisfies for $\xi$ with $|\mathrm{Im}\,\xi|\leq\delta|\eta'|$,

$$(5.24)\qquad |\hat{r}_1(\eta-\eta'\colon \xi,\eta')|\leq\frac{\text{const.}}{(|\eta-\eta'|+1)^{n+1}}\frac{|\eta'|}{|\xi|}\,.$$

Hence by virtue of (5.20) we have for $\eta'\in Q_j$,

$$(5.25)\qquad |\hat{r}_1(\eta-\eta'\colon \xi,\eta')(p_j(\eta)-p_j(\eta'))|\leq\text{const.}\frac{1}{(|\eta-\eta'|+1)^n}\cdot\frac{1}{|\xi|+\rho_j},$$

with $|\mathrm{Im}\,\xi|=\delta|\eta'|$. On the other hand, since $\hat{r}_1(\eta-\eta';\xi,\eta')$ is analytic in $|\mathrm{Im}\,\xi|\leq\delta|\eta'|$ and $\hat{v}(\lambda,\xi,\eta')$ is analytic in $\mathrm{Im}\,\xi<0$, we may sift a line of integration of (5.23) into the complex plane, that is,

$$(5.26)\qquad \int_{-\infty}^{\infty}\hat{r}_1(\eta-\eta';\xi,\eta')\hat{v}(\lambda,\xi,\eta')d\xi$$
$$=\int_{\mathrm{Im}\,\xi=-\delta|\eta'|}\hat{r}_1(\eta-\eta';\xi,\eta')\hat{v}(\lambda,\xi,\eta')d\xi.$$

Therefore applying Hausdorff-Young's inequality to (5.23), we obtain (5.22) by mean of (5.25). q.e.d.

We shall estimate the first term in the right side of (5.21). To do so, we must give two different arguments for $|\lambda|\leq k\rho_j$ and for $|\lambda|\geq k\rho_j$.

**Lemma 5.5.** *It holds for* $|\lambda|\leq k\rho_j$

$$(5.27)\qquad \langle R_1 v_j(\lambda)_0(0)\rangle^2$$
$$\leq\text{const.}\left\{\frac{1}{\rho_j}(\|g_j\|^2+\|f_j(\lambda)\|^2+\|v(\lambda)\|^2)+\langle v_i(\lambda,0)\rangle^2\right\},$$

*where* $g_j=p_j(D_y)g$ *and* $f_j(\lambda)$ *are the Laplace transforms of* $p_j(D_y)f(t)$.

**Proof.** Let $\eta^{(j)}$ be in $Q_j$. Then we put $m_j=m(\lambda,\eta^{(j)})$. Since we have $|\lambda|\leq 2k|\eta'|$ for $|\lambda|\leq k\rho_j$ and for $\eta$ in $Q_j$, $\hat{r}_1(\eta-\eta';\xi,\eta')$ is anlaytic in $|\mathrm{Im}\,\xi|\leq m_j$. Hence we can express

$$(\widetilde{R_1 v_j(\lambda)_0})(0, \eta)=\frac{1}{2\pi}\int d\eta'\int_{\mathrm{Im}\,\xi=-m_j} \hat{r}_1(\eta-\eta';\ \xi, \eta')\hat{v}_j(\lambda, \xi, \eta')d\xi.$$

Therefore we can estimate by virtue of (5.24)

$$<R_1 v_j(\lambda)_0(0)>^2 \leq \text{const. } m_j\int_{\mathrm{Im}\,\xi=-m_j} <\hat{v}_j(\lambda, \xi)>^2 d\xi. \tag{5.28}$$

Applying the operator $p_j(D_y)$ to the relation $(\lambda-L)v(\lambda)=g+f(\lambda)$, we can write

$$(\lambda-L)v_j(\lambda)=h_j(\lambda) \tag{5.29}$$

where $h_j=p_j(D_y)(g+f(\lambda))+C^j v(\lambda)$ and where $C^j$ is the commutator of $p_j(D_y)$ and $L$.

Define the operator $L_j$ as

$$L_j=A(0)\frac{\partial}{\partial x}+i\sum_l B_l(0, y_0)\eta_l^{(j)},$$

and denote its difference from $L$ by $M_j$

$$M_j=L-L_j.$$

Then we have

$$||M_j v_j||^2 \leq \text{const.}\{(\varepsilon^2\rho_j^2+1)||v_j||^2+||h_j||^2\}. \tag{5.30}$$

For, by definition of $M_j$, we write

$$M_j v_j=(A(x)-A(0))\frac{\partial}{\partial x}v_j+\sum(B_l(x, y)-B_l(0, y_0))\frac{\partial}{\partial y_l}v_j.$$

$$+\sum B_l(0, y_0)\left(\frac{\partial}{\partial y_l}-i\eta_l^{(j)}\right)v_j+Kv_j.$$

Since $v_j$ satisfies

$$\frac{\partial}{\partial x}v_j=A^{-1}\left(\lambda-\sum B_l\frac{\partial}{\partial y_l}-K\right)v_j-A^{-1}h_j,$$

we can write

$$M_j v_j=(A(x)-A(0))\left\{A^{-1}\left(\lambda-\sum B_l\frac{\partial}{\partial y_l}-K\right)v_j-A^{-1}h_j\right\} \tag{5.31}$$

$$+\sum(B_l(x, y)-B_l(0, y_0))\frac{\partial}{\partial y_l}v_j$$

$$+\sum B_l(0, y_0)\left(\frac{\partial}{\partial y_l}-i\eta^{(j)}\right)v_j+Kv_j.$$

Since the support of $v_j(x, y)$ is sufficiently small, $|A(x)-A(0)|\leq\varepsilon$ and $|B_l-B_l(0, y_0)|\leq\varepsilon$. Hence for $|\lambda|\leq 2k\rho_j$ the norm of the first term and the second term in (5.31) is not greater than const. $(\varepsilon\rho_j\|v_j\|+\|h_j\|)$. And the norm of the third term is estimated by const. $\varepsilon\rho_j\|v_j\|$ from (5.19). Hence we have (5.30).

The relation (5.29) can be rewritted

$$(\lambda-L_j)v_j=M_jv_j+h_j. \tag{5.32}$$

We take it's Fourier transform with respect to $x$, we have

$$(\lambda-ia(\xi, \eta^{(j)})\hat{v}_j(\lambda, \xi)=\widehat{M_jv_j(\lambda)}(\xi)+\hat{h}_j(\xi)+A(0)v_j(\lambda, 0)\,, \tag{5.33}$$

where $a(\xi, \eta^{(j)})=A(0)\xi+\sum_l B_l(0, y_0)\eta_l^{(j)}$. According to (i) of Lemma 5.1, for any $\xi$ there exists $m(\lambda, \eta^{(j)})$, $\frac{1}{2}\delta|\eta^{(j)}|\leq m(\lambda, \eta^{(j)})\leq\delta|\eta^{(j)}|$ such that

$$|(\lambda-ia(\xi, \eta^{(j)}))^{-1}|\leq\text{const.}\frac{1}{|\xi|}\,, \tag{5.34}$$

for all $\xi$ with $\text{Im}\,\xi=-m(\lambda, \eta^{(j)})$ and $|\lambda|\leq k\rho_j$. Noting that

$$\frac{\delta}{2}\rho_j\leq m_j\leq\delta\rho_j, \tag{5.35}$$

we obtain from (5.33) and (5.34) for $\text{Im}\,\xi=-m_j$ and $|\lambda|\leq k\rho_j$

$$<\hat{v}_j(\lambda, \xi)>^2\leq\text{const.}\left\{\frac{1}{\rho_j^2}(<\widehat{M_jv_j}(\xi)>^2+<\hat{h}_j(\xi)>^2)\right. \tag{5.36}$$

$$\left.+\frac{1}{|\xi|^2}<v_j(\lambda, 0)>^2\right\}.$$

Noting that

$$\int_{\text{Im}\,\xi=-m_j}\frac{1}{|\xi|^2}\,d\xi\leq\text{const.}\frac{1}{\rho_j}\,,$$

and

$$(5.37)\qquad \int_{\mathrm{Im}\,\xi=-m_j} <\hat{v}_j(\xi)>^2 d\xi = \int_0^\infty <e^{-m_j x} v_j(x)>^2 dx,$$

we obtain by (5.36)

$$(5.38)\quad \int_{\mathrm{Im}\,\xi=-m_j} <\hat{v}_j(\lambda, \xi)>^2 d\xi \leq \text{const. } \varepsilon \int_{\mathrm{Im}\,\xi=-m_j} <\hat{v}_j(\lambda, \xi)>^2 d\xi$$
$$+\text{const.} \left\{\frac{1}{\rho_j^2} \|h_j\|^2 + \frac{1}{\rho_j} <v_j(\lambda, 0)>^2\right\},$$

for $|\lambda| \leq k\rho_j$. For $\varepsilon$ with const. $\varepsilon \leq \frac{1}{2}$ (5.37) implies

$$(5.40)\quad \rho_j \int <\hat{v}_j(\lambda, \xi)>^2 d\xi \leq \text{const.} \left\{\frac{1}{\rho_j} \|h_j\|^2 + <v_j(\lambda, 0)>^2\right\},$$

which and (5.28) lead us to (5.27) in view of (5.35) and of the fact that $\|h_j\| \leq \text{const.} (\|g_j\| + \|f_j(\lambda)\| + \|v(\lambda)\|)$. By integration of (5.27) with respect to $\sigma = \mathrm{Im}\,\lambda$, we have

$$(5.41)\qquad \int_{|\lambda| \leq k\rho_j} <(R_1 v_j(\lambda)_0)(0)>^2 d\sigma$$
$$\leq \text{const.} \left\{\|g_j\|^2 + \int_{-\infty}^\infty \frac{1}{\rho_j} (\|f_j(\lambda)\|^2 + \|v(\lambda)\|^2)\right.$$
$$\left. + <v_j(\lambda, 0)>^2 d\sigma\right\}.$$

Next we must consider the argument for $|\lambda| \geq k\rho_j$. In view of (5.28) and (5.37), we have

$$(5.42)\qquad <(R_1 v_j(\lambda)_0)(0)>^2 \leq \text{const. } m_j \int_0^\infty <e^{-m_j x} v_j(\lambda, x)>^2 dx.$$

**Lemma 5.6.** *There exists a positive* $\mu_0$ *such that*

$$(5.43)\qquad m_j \int_{|\lambda| \geq k\rho_j} d\sigma <e^{-m_j x} v_j(\lambda, x)>^2 dx$$
$$\leq \text{const.} \left\{\|g_j\|^2 + \int_{-\infty}^\infty \frac{1}{\mu} \|f_j(\lambda)\|^2 + \mu \|v_j(\lambda)\|^2 + <v_j(\lambda)>^2 d\sigma\right.$$

$$+\frac{1}{m_j}\int\|v(\lambda)\|^2 d\sigma+\rho_j\int_{|\lambda|\leq k\rho_j} d\sigma<e^{-m_j x}v_j(\lambda,\,x)>^2 dx\Big\}$$

for $\mu\geq\mu_0$.

**Proof.** The equation (5.29) can be rewritten

$$\frac{\partial}{\partial x}v_j=M(\lambda,\,D_y)v_j+h_j \tag{5.44}$$

where $M(\lambda,\,D_y)=A^{-1}\Big(\lambda-\sum B_l\frac{\partial}{\partial y_l}\Big)$ and $h_j=g_j+f_j(\lambda)+C_j v(\lambda)$, and where $g_j=p_j(A^{-1}g)$, $f_j(\lambda)=p_j(f(\lambda)+Kv(\lambda))$ and $C_j=[M(\lambda,\,D_y),\,p_j(D_y)]$. According to (ii) of Lemma 5.1, there exists a $N\times N$ matrix $s(x,\,y;\,\lambda,\,\eta)$ which is homogeneous of degree zero, symmetric, positive definite and satisfies

$$|\mathrm{Re}\,s(x,\,y;\,\lambda,\,\eta)\cdot M(x,\,y;\,\lambda,\,\eta)|\leq\text{const.}\,\mu \tag{5.45}$$

for $|\lambda|\geq\frac{1}{2}|\eta|$. We define the pseudo differential operator $S(\lambda,\,D_y)$ with it's symbol $s(x,\,y;\,\lambda,\,\eta)$. Since $s(x,\,y;\,\lambda,\,\eta)$ is symmetric and positive definite, we have for $v(x,\,y)$ which support is sufficiently small,

$$((S+S^*)v,\,v)\geq\text{const.}\|v\|^2, \tag{5.46}$$

where $S^*$ is the adjoint operator of $S(\lambda,\,D_y)$.

Since $v_j(\lambda)$ satisfies (5.44), we have

$$\begin{aligned}&<(S+S^*)v_j(\lambda,\,0),\,v_j(\lambda,\,0)> \\ &=-\int_0^\infty\frac{\partial}{\partial x}\{e^{-2m_j x}<(S+S^*)v_j(\lambda,\,x),\,v_j(\lambda,\,x)>\}\,dx \\ &=2m_j\int_0^\infty<(S+S^*)e^{-m_j x}v_j(\lambda,\,x),\,e^{-m_j x}v_j(\lambda,\,x)>dx \\ &\quad-2\mathrm{Re}\int_0^\infty e^{-2m_j x}<(S+S^*)Mv_j,\,v_j>dx \\ &\quad-2\mathrm{Re}\int_0^\infty e^{-2m_j x}<(S+S^*)h_j,\,v_j>dx \\ &\quad-2\mathrm{Re}\int_0^\infty e^{-2m_j x}<(S+S^*)_x v_j,\,v_j>dx.\end{aligned} \tag{5.47}$$

In view of (5.45), we obtain

$$2\mathrm{Re}\int_0^\infty < S+S^*) Me^{-m_j x} v_j(\lambda, x), e^{-m_j x} v_j(\lambda, x)> dx$$

$$\leq \text{const.}\ \mu \int_0^\infty < e^{-m_j x} v_j(\lambda, x)>^2 dx,$$

for $|\lambda| \geq k\rho_j$. Hence from this, (4.46) and (5.47) it follows that

$$(5.48) \qquad m_j \int_{|\lambda|\geq k\rho_j} d\sigma \int_0^\infty < e^{-m_j x} v_j(\lambda, x)>^2 dx$$

$$\leq \text{const.} \Big\{ 2\mathrm{Re} \int_{|\lambda|\geq k\rho_j} \int_0^\infty e^{2m_j x} <(S+S^*)h_j, v_j> dx$$

$$+ \int_{-\infty}^\infty < v_j(\lambda, 0)>^2 + \mu \| v_j(\lambda) \|^2 d\sigma \Big\}.$$

By definition of $h_j$, we can write

$$2\mathrm{Re}\int_0^\infty e^{-2m_j x} <(S+S^*)h_j, v_j(\lambda, x)> dx$$

$$= 2\mathrm{Re}\int_0^\infty e^{-2m_j x} (<(S+S^*)g_j, v_j(\lambda, x)>$$

$$+ <(S+S^*)f_j(\lambda), v_j(\lambda, x)> + <(S+S^*)C_j v(\lambda), v_j>) dx.$$

Since by assumption $A$ is independent of $y$, it follows that

$$C_j = [M(\lambda, D_y), p_j(D_y)]$$

$$= -\Big[ A^{-1} \sum B_l \frac{\partial}{\partial y_l}, \ p_j(D_y) \Big]$$

are bounded. Hence we have by Schwarz inequality

$$(5.49) \qquad \text{const.} \int_0^\infty \mathrm{Re}\, e^{-2m_j x} <(S+S^*) C_j v, v_j> dx$$

$$\leq \frac{1}{4} m_j \int_0^\infty < e^{-m_j x} v_j(\lambda, x)>^2 dx + \text{const.} \frac{1}{m_j} \| v_j(\lambda) \|^2.$$

We note that $s(x, y; \lambda, 0)$ is independent of $\lambda$. Denote it by $S_0$. Since $s(x, y; \lambda, \eta)$ is homogeneous of degree zero, it satisfies

$$|s(x, y; \lambda, \eta) - s(x, y; \lambda, 0)| \leq \text{const.} \left| \frac{\eta}{\lambda} \right|.$$

Hence we have

$$<(S-S_0)v_j> \leq \text{const.}\frac{\rho_j}{|\lambda|}<v_j>.$$

Moreover since $s(x, y; \lambda, \eta)$ is symmetric, the norm of the difference $S-S^*$ is not greater than const. $\frac{1}{|\lambda|}$. Hence we have

$$\text{(5.50)}\qquad \text{const.}\Big\{\text{Re}\int_{|\lambda|\geq k\rho_j} d\sigma\int_0^\infty e^{-2m_jx}<(S+S^*)g_j,\ v_j(\lambda,\ x)>dx\Big\}$$

$$\leq \text{const.}\Big\{\int_{|\lambda|\geq k\rho_j} d\sigma\int_0^\infty \text{Re}\, e^{-2m_jx}<S_0g_j,\ v_j)\lambda,\ x)>dx$$

$$+\int_{|\lambda|\geq k\rho_j} d\sigma\int_0^\infty \frac{e^{-2m_jx}}{|\lambda|}<g_j><v_j(\lambda,\ x)>$$

$$+\frac{\rho_j}{|\lambda|}e^{-m_jx}<g_j><v_j>dx\Big\}.$$

Considering that

$$\frac{1}{2\pi}\int_{-\infty}^\infty v_j((\lambda)\, d\sigma=\frac{1}{2}Ag_j,$$

we obtain

$$\int_{|\lambda|\geq k\rho_j} d\sigma\int_0^\infty \text{Re}\, e^{-2m_jx}<S_0 g_j,\ v_j(\lambda)>dx$$

$$\leqq \pi\int_0^\infty \text{Re}\, e^{-2m_jx}<S_0g_j,\ Ag_j>dx$$

$$+\int_{|\lambda|\leq k\rho_j} d\sigma\int_0^\infty \text{Re}\, e^{-2m_jx}<S_0 g_j,\ v_j(\lambda)>dx$$

$$\leq \text{const.}\Big\{||g_j||^2+\rho_j\int_{|\lambda|\leq k\rho_j} d\sigma\int_0^\infty <e^{-m_jx}v_j(\lambda,\ x))^2dx\Big\}.$$

From this, (5.48), (5.49) and (5.50), it follows that

$$m_j\int_{|\lambda|\geq k\rho_j} d\sigma\int_0^\infty e^{-m_jx}v_j(\lambda,\ x)>^2dx$$

$$\leq \text{const.}\Big\{||g_j||^2+\int\frac{1}{\mu}||f_j(\lambda)||^2+\mu||v_j(\lambda)||^2+\frac{1}{m_j}||v(\lambda)||^2d\sigma$$

$$+\rho_j\int_{|\lambda|\leq k\rho_j} d\sigma\int_0^\infty < e^{-m_j x} v_j(\lambda, x)>^2 dx\Big\}$$

$$+\frac{1}{2} m_j\int_{|\lambda|\geq k\rho_j} d\sigma\int_0^\infty < e^{-m_j x} v_j(\lambda, x)>^2 dx,$$

which implies (5.43). q.e.d.

In view of (5.40), we obtain our assertion (5.18) from (5.41), (5.42) and (5.43). This completes Theorem 5.2.

The analogous result for the solution of the adjoint problem holds, that is,

**Corollary 5.2.** *Let the symbol* $r_1$ *satisfy the same assumptions as Theorem* 5.2. *Then for* $v(t)$ *which have the same properties as Corollary* 5.1 *it holds*

$$\int_{-\infty}^\infty < e^{\mu t}(R_1 v(t)_0)(0)^2> dt$$

$$\leq \text{const.}\Big\{\int_{-\infty}^\infty \frac{1}{\mu}\|e^{\mu t}\varphi(t)\|^2+\mu\|e^{\mu t}v(t)\|^2+< e^{\mu t}v(t, 0)>^2 dt\Big\}.$$

## Appendix

Here we shall prove Lemma 3.2 and 3.3. Let us consider the following differential equation with parameters $(\lambda, \eta)$, $\mathrm{Re}\,\lambda>0$, $\eta\in\mathbf{R}^n$,

$$(A.1)\qquad \begin{cases} \dfrac{d}{dx} u(x)=M(\lambda, \eta)u(x)+f(x) \\ Bu(0)=g, \end{cases}$$

where $M(\lambda, \eta)=A^{-1}(\lambda-i\Sigma B_j\eta_j)$. We may assume without generality that $A$ has the form

$$A=\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where $A_1(A_2)$ has only negative (positive) eigen values. Hence $U(\lambda, 0)$

becomes the unit matrix. Here $U(\lambda, \eta)$ is the matrix constructed in the introduction, that is,

$$U^{-1}(\lambda, \eta) M(\lambda, \eta) U(\lambda, \eta)=\begin{pmatrix} M_1(\lambda, \eta) & * \\ 0 & M_2(\lambda, \eta) \end{pmatrix},$$

where $M_1(M_2)$ has only eigen values with negative (positive) real parts. If $B$ satisfies the Lopatinski's condition, i.e, $\det(B.U_1(\lambda, \eta))\neq 0$ for $(\lambda, \eta)$, $\mathrm{Re}\,\lambda>0$, $|\lambda|^2+|\eta|^2=1$, $B$ has the form $B=(B_1, B_2)$, $\det B_1\neq 0$. In fact, we have $BU_1(\lambda, 0)=B_1$ since $U(\lambda, 0)$ is the unit matrix. Hence we may assume that $B$ has the form $(I_l, B_2)$, where $I_l$ is the $l\times l$ unit matrix. Let $B'$ be a $(N-l)\times N$ matrix of which kernel is the orthogonal complement of $(A\,\mathrm{Ker}\,B)$ in $\mathbf{C}^N$. Then we have

**Lemma A.1.** *$B'$ has the following form,*

$$B'=(-(A_1 B_2 A_2^{-1})^*, I_{n-l}). \tag{A.2}$$

**Proof.** Assume that $B'$ has the form A.2). Let $v$ be in $\mathbf{C}^N$. Decompose $v={}^t(v_1, v_2)$, $v_1\in\mathbf{C}^l$, $v_2\in\mathbf{C}^{N-l}$. We prove that $\mathrm{Ker}\,B'$ is equal to $(A.\mathrm{Ker}\,B)^{\perp}$. Let $v$ be in Ker $B'$ and $u$ in $\mathrm{Ker}\,B$, that is, $v_2=(A_1 B_2 A_2^{-1})^* v_1$ and $u_1=-B_2 u_2$. Then we have

$$\begin{aligned}<Au, v> &= <A_1 u_1, v_1> + <A_2 u_2, v_2>\\ &= -<A_1 B_2 u_2, v_1> + <A_2 u_2, (A_1 B_2 A_2^{-1})^* v_1>\\ &=0,\end{aligned}$$

for all $u$ in $\mathrm{Ker}\,B$. Hence $v$ belongs to $(A\cdot\mathrm{Ker}\,B)^{\perp}$. Let $v$ be in $(A\,\mathrm{Ker}\,B)^{\perp}$. Then we can write for all $u\in\mathrm{Ker}\,B$

$$0=<Au, v>=<u_2, A_2^* v_2-(A_1 B_2)^* v_1>.$$

The above formula holds for all $u_2$ in $\mathbf{C}^{N-l}$. Hence $v$ belongs to $\mathrm{Ker}\,B'$.

**Lemma A.2.** *There exist matrices $C$ and $C'$ such that*

$$(A.3.) \qquad <Au,\ v>=<Bu,\ C'v>+<Cu,\ B'v>,$$

*for* $u$, $v$ *in* $\mathbf{C}^N$.

**Proof.** We can write

$$\begin{aligned}<Au,\ v> &= <A_1 u_1,\ v_1>+<A_2 u_2,\ v_2> \\ &= <(Bu-B_2 u_2),\ A_1^* v_1>+<A_2 u_2,\ (B'v+(A_1 B_2 A_2^{-1})^* v_1)> \\ &= <Bu,\ A_1^* v_1>+<A_2 u_2,\ B'v>.\end{aligned}$$

Putting $C'=(A_1^*, 0)$ and $C=(0, A_2)$, we have the formula (A.3).

q.e.d.

Since the $N\times N$ matrices $\begin{pmatrix}B\\C\end{pmatrix}=\begin{pmatrix}I_l & B_2\\0 & A_2\end{pmatrix}$ and $\begin{pmatrix}B'\\C'\end{pmatrix}=\begin{pmatrix}A_1^* & 0\\B_2' & I_{N-l}\end{pmatrix}$, where $B_2'=-(A_1 B_2 A_2^{-1})^*$, are non singular, we obtain (3.4).

Here we assume that $A\xi+\Sigma B_j\eta_j$ has real distinct eigen values and $A$ and $B_j$ are real. Then we have

**Lemma A.3.** [1] *If and only if* $B$ *satisfies the uniform Lopatinski's condition that is,* $\det(B\cdot U_1(\lambda, \eta))\neq 0$ *for* $(\lambda, \eta)$, $\mathrm{Re}\,\lambda\geqslant 0$, $|\lambda|^2+|\eta|^2=1$, *it follows that there exists the unique solution* $u(x)$ *of* (*A*.1) *for all* $(\lambda, \eta)$, $\mathrm{Re}\ \lambda>0$, *and satisfies*

$$(A.4) \qquad \mu\int_0^\infty |u(x)|^2 dx+|u(0)|^2\leq \text{const.}\left\{\int_0^\infty \frac{1}{\mu}|f(x)|^2 dx+|h|^2\right\},$$

*for all* $\mu=\mathrm{Re}\,\lambda>0$.

We prove that $B'$ satisfies the uniform Lopatinski's condition by use of Lemma A.3. Let us consider the adjoint equation

$$(A.1)^* \qquad \begin{cases}\dfrac{d}{dx}v(x)=-(A^*)^{-1}(\lambda+i\Sigma B_j^*\eta_j)\,v(x)+\varphi(x)\\ B'v(0)=h.\end{cases}$$

**Lemma A.4.** *If and only if* $u(x)$ *of* $u(x)$ *of the solution of* (A.1) *satisfies* (A.4), *it follows that it holds for* $v(x)$ *of the solution of* (A.1)*

(A.5) $$\mu\int_0^\infty |v(x)|^2dx+|v(0)|^2\leqslant \text{const.}\left\{\frac{1}{\mu}\int_0^\infty |\varphi(x)|^2+|\psi|^2\right\},$$

*for all* $(\lambda, \eta)$, $\mathrm{Re}\,\lambda=\mu>0$.

**Proof.** From (A.3) we have

(A.6) $$\int_0^\infty <Af(x),\ v(x)>dx-\int_0^\infty <u(x),\ A^*\varphi(x)>dx$$
$$=-<g,\ C'v(0)>-<Cu(0),\ \psi>.$$

We put $Af(x)=\mu v(x)$ and $g=C'v(0)$. Then $u(x)$ satisfies by virtue of (A.4),

(A.7) $$\mu\int_0^\infty |u(x)|^2dx+|u(0)|^2\leqslant \text{const.}\left(\int_0^\infty \mu|v(x)|^2dx+|C'v(0)|^2\right).$$

From (A.8) we obtain

$$\mu\int_0^\infty |v(x)|^2dx+|C'v(0)|^2$$
$$=\int_0^\infty <u(x),\ A^*\varphi(x)>dx-<Cu(0,\ \psi>.$$

Hence in view of (A.7) we have by Schwarz's inequality

$$\mu\int_0^\infty |v(x)|^2dx+|C'v(0)|^2\leqslant \text{const.}\left\{\int_0^\infty \frac{1}{\mu}|\varphi(x)|^2dx+|\psi|^2\right\},$$

which and (3.4) imply (A.5). q.e.d.

KYOTO UNIVERSITY

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