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Strongly regular graphs and finite permutation groups of rank 3

By

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§ 0. Introduction

Let (G, Ω) be a finite permutation group of rank 3. Then we can make Ω a strongly regular graph so that G is a subgroup of Aut Ω, the full automorphism group of the graph Ω. But, of course, the automorphism groups of strongly regular graphs are not always of rank 3. So it is interesting to determine strongly regular graphs with automorphism groups of rank 3.

Strongly regular graphs are also obtained from partial geometries, and such graphs are called geometric. Strongly regular graphs constructed from rank 3 permutation groups are often geometrizable. Therefore it is also interesting to determine partial geometries with automorphism groups of rank 3.

In § 2, we construct a partial geometry $\tilde{\Omega}_M$ associated with a finite group M (Theorem 2.6), and the full automorphism group of the graph Ω_M is determined (Theorem 2.7). We prove that (Aut Ω_M, Ω_M) is a primitive permutation group of rank 3 if and only if M is a cyclic group of order 5 or an elementary abelian 2-group of order greater than or equal to 4 (Theorem 2.10).

In § 3, we give some characterizations of Ω_M. Namely, Ω_M for $M=Z_5$ or E_{2^f} are the only geometric graphs with $r=3$ and $t=2$ whose automorphism groups are primitive of rank 3 (Theorem 3.1). Let Ω

be a strongly regular graph with $k=3(m-1)$ and $l=(m-1)(m-2)$, $m \geqq 4$. Then $\mu=6$ unless $\mu=9$ and $m=14$ or 352 (Theorem 3.7). As a corollary, we obtain that if (G, Ω) is a rank 3 permutation group of degree m^2 with subdegrees 1, $3(m-1)$ and $(m-1)(m-2)$, $m>23$, then the graph constructed from (G, Ω) is isomorphic to some Ω_M, $M=E_{2^f}$, unless $\mu=9$ and $m=352$ (Corollary 3.8).

The author would like to express his hearty thanks to Professor C. C. Sims for pointing out that Corollary 3.8 may be viewed as a special case of Conjecture A in [4], which is a consequence of Conjecture B, and that Conjecture B was proved by A. J. Hoffman.

§1. Notations and preliminary results

Graphs considered in this paper are finite undirected graphs without loops.

Definition 1.1. *A graph with n vertices is strongly regular if there exist integers k, l, λ and μ such that*

(i) *each vertex is adjacent to exactly k vertices and non-adjacent to exactly l other vertices, k and l positive, and*

(ii) *two adjacent vertices are both adjacent to exactly λ other vertices and two non-adjacent vertices are both adjacent to exactly μ vertices.*

Proposition 1.2. ([2], [3 I]). *Assume Ω is a strongly regular graph. Then*

(i) $\mu l=k(k-\lambda-1)$,

(ii) *the minimum polynomial of the adjacency matrix A of Ω is*

$$(x-k)(x^2-(\lambda-\mu)x-(k-\mu)),$$

(iii) *A has k as eigenvalue with multiplicity* 1, *and the multiplicities f, g of the roots r, s of $x^2-(\lambda-\mu)x-(k-\mu)$ as eigenvalues of A are respectively*

$$f=\frac{(k+l)s+k}{s-r} \quad \textit{and} \quad g=\frac{(k+l)r+k}{r-s}$$

with $f+g=k+l$, *and therefore*

(iv) *one of the following holds*:

(a) $k=l$, $\mu=\lambda+1=k/2$ *and* $f=g=k$, *or*

(b) $d=(\lambda-\mu)^2+4(k-\mu)$ *is a square, and* $\sqrt{d}$ *divides* $2k+(\lambda-\mu)(k+l)$.

If G is a rank 3 permutation group of even order on a finite set Ω, $|\Omega|=n$, and if Δ and Γ are the nontrivial orbits of G in $\Omega\times\Omega$, then the graphs (Ω, Δ) and (Ω, Γ) are a complementary pair of strongly regular graphs, each admitting G as a rank 3 automorphism group, the parameters k, l and λ, μ being respectively the subdegrees (other than 1) and the intersection numbers of G. G is primitive if and only if $0<\mu<k$. If $\mu=0$, then $k+1|n$, and if $\mu=k$, then $l+1|n$.

Definition 1.3. *A partial geometry with characteristic* (r, κ, t) *is a system of points and lines such that*

(i) *any two points are incident with not more than one line,*

(ii) *each point is incident with* r *lines,*

(iii) *each line is incident with* κ *points, and*

(iv) *if a point* x *is not incident with a line* L, *there pass through the point* x *exactly* t *lines* $(t\geqq 1)$ *intersecting* L.

The graph Ω *of a partial geometry* $\tilde{\Omega}$ *is defined as a graph whose vertices correspond to the points of the geometry, and in which two vertices are adjacent or non-adjacent according as the corresponding points are incident or non-incident with a common line.* Ω *is called a geometric graph with characteristic* (r, κ, t) *having geometric structure* $\tilde{\Omega}$.

Proposition 1.4. ([1] Theorem 4.1). *A geometric graph with characteristic* (r, κ, t) *is strongly regular with parameters*

$$k=r(\kappa-1),$$

$$l=(r-1)(\kappa-1)(\kappa-t)/t,$$

$$\lambda=(r-1)(t-1)+\kappa-2,$$

$\mu=rt,$

where $1\leqq t\leqq r$ *and* $1\leqq t\leqq\kappa$.

Definition 1.5. *A strongly regular graph with parameters of the form given in Proposition* 1.4 *is defined to be a pseudo-geometric graph with characteristic* (r, κ, t).

Proposition 1.6. ([1] Theorem 9.3). *A pseudo-geometric graph with characteristic* (r, κ, t) *has unique geometric structure*[1] *if*

$$\kappa>\frac{1}{2}[r(r-1)+t(r+1)(r^2-2r+2)].$$

The terminology and notation of [2] for rank 3 permutation groups are used throughout.

In a graph Ω, $\Delta(a)$ $(a\in\Omega)$ denotes the set of vertices adjacent to a, and $\Gamma(a)$ the set of vertices non-adjacent to a. A complete subgraph consisting of κ vertices is called a clique of order κ, that is, any two vertices of a clique are adjacent. $|X|$ denotes the cardinality of the set X. Finally,

$\mathfrak{S}_m$ = symmetric group of degree m,

Z_m = cyclic group of order m,

E_{p^f} = elementary abelian group of order p^f.

§2. Construction of geometric graphs with characteristic (3, m, 2)

Let M be a finite group of order $m\geqq 3$ with the identity element e, and $\Omega=\Omega_M=M\times M$ (direct product of sets). We call the elements of Ω vertices, and two vertices (a, b), (a', b') of Ω are, by definition, adjacent if and only if one of the following holds:

(i) $a=a'$,

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1) It is not always possible to define geometric structure in a pseudo-geometric graph. Also geometric structure in a geometric graph is not unique in general.

(ii) $b=b'$, or

(iii) $a^{-1}b=a'^{-1}b'$.

The set of vertices of Ω adjacent to (a, b) is denoted by $\Delta(a, b)$ instead of $\Delta((a, b))$, and put $\Gamma(a, b)=\Omega-\Delta(a, b)-\{(a, b)\}$, the set of vertices non-adjacent to (a, b). Now we shall define some automorphisms of Ω. First, define

$$(a, b)T_{x,y}=(ax, by), \qquad (a, b)\in\Omega$$

for $x, y\in M$, and put

$$\mathfrak{T}=\{T_{x,y}\,|\,x, y\in M\}.$$

(We write $(a, b)T_{x,y}$ instead of $((a, b))T_{x,y}$ for brevity.)

Lemma 2.1. *Every $T_{x,y}(x, y\in M)$ is an automorphism of Ω, and $\mathfrak{T}$, which is isomorphic to $M\times M$, is regular (in particular transitive) on Ω.*

Next, define

$$(a, b)S_1=(a^{-1}, a^{-1}b), \text{ and}$$

$$(a, b)S_2=(b^{-1}a, b^{-1}), \qquad (a, b)\in\Omega,$$

and put $\mathfrak{S}=<S_1, S_2>$, the group generated by S_1 and S_2.

Lemma 2.2. *S_1 and S_2 are automorphisms of Ω of order* 2, *and $\mathfrak{S}$ is isomorphic to the dihedral group of order* 6.

For any automorphism σ of M, we define

$$(a, b)X_\sigma=(a^\sigma, b^\sigma), \qquad (a, b)\in\Omega,$$

and put $\mathfrak{X}=\{X_\sigma\,|\,\sigma\in\mathrm{Aut}M\}$.

Lemma 2.3. *Every $X_\sigma(\sigma\in\mathrm{Aut}M)$ is an automorphism of Ω, and $\mathfrak{X}$ is isomorphic to* Aut M.

The proofs of Lemma 2.1 through Lemma 2.3 are automatic and we omit them.

Proposition 2.4. *Let $\mathfrak{G}$ be the group generated by $\mathfrak{T}$, $\mathfrak{X}$ and $\mathfrak{S}$. Then $\mathfrak{G}$ is the semidirect product of $<\mathfrak{T}, \mathfrak{X}>$ by $\mathfrak{S}$, and $<\mathfrak{T}, \mathfrak{X}>$ is the semidirect product of $\mathfrak{T}$ by $\mathfrak{X}$. Especially, the order of $\mathfrak{G}$ is equal to*

$$|\mathfrak{T}| \times |\mathfrak{X}| \times |\mathfrak{S}| = 6\,\mathrm{m}^2 \times |\mathrm{Aut}\, M|.$$

Proof. It is easily verified that $\mathfrak{T}$ is normal in $<\mathfrak{T}, \mathfrak{X}>$ and $\mathfrak{T} \cap \mathfrak{X} = \{e\}$, which means that $<\mathfrak{T}, \mathfrak{X}>$ is the semidirect product of $\mathfrak{T}$ by $\mathfrak{X}$. Also $<\mathfrak{T}, \mathfrak{X}>$ is normal in $\mathfrak{G}$ and $<\mathfrak{T}, \mathfrak{X}> \cap \mathfrak{S} = \{e\}$.

Define

$$K_1(a) = \{(a, b) \mid b \in M\},$$

$$K_2(a) = \{(b, a) \mid b \in M\}, \text{ and}$$

$$K_3(a) = \{(b, ba) \mid b \in M\}$$

for $a \in M$.

Lemma 2.5. *We have*

$$|K_i(a) \cap K_j(b)| = \begin{cases} m & i=j,\ a=b, \\ 0 & i=j,\ a \neq b, \\ 1 & i \neq j, \end{cases}$$

for $1 \leqq i, j \leqq 3$ and $a, b \in M$.

Theorem 2.6. *$\tilde{\Omega}_M$ with lines $K_i(a)$, $i=1, 2, 3$, $a \in M$, is a partial geometry with characteristic $(3, m, 2)$. Hence the graph Ω_M is a strongly regular graph with $n=m^2$, $k=3(m-1)$, $l=(m-1)(m-2)$, $\lambda=m$ and $\mu=6$.*

Proof. The first assertion is immediate from Lemma 2.5 and the fact that $K_i(a)$, $i=1, 2, 3$, $a\in M$, is a clique of order m. Then the second assertion follows from Proposition 1.4.

Now we shall determine the full automorphism group of Ω_M, namely, we prove the following theorem.

Theorem 2.7. *$\mathfrak{G}=<\mathfrak{T}, \mathfrak{X}, \mathfrak{S}>$ is the full automorphism group of the strongly regular graph Ω_M if $m\geqq 5$ or M is a cyclic group of order* 4. *If M is an elementary abelian group of order* 4, Aut Ω_M *is isomorphic to $\mathfrak{S}_4 \int \mathfrak{S}_2$, the wreath product of $\mathfrak{S}_4$ with $\mathfrak{S}_2$, and is of order* 1152. *If M is a cyclic group of order* 3, Aut Ω_M *is isomorphic to $\mathfrak{S}_3 \int \mathfrak{S}_3$ of order* 1296, *and acts imprimitively on Ω_M.*

Lemma 2.8. *If $m\geqq 5$, $K_i(a)\,(i=1, 2, 3,\ a\in M)$ are the only maximal cliques of order greater than* 4. *Hence the geometric structure in Ω_M is unique.*

Proof. Suppose K is a clique of order greater than 4. We may assume that (e, e) is contained in K because of the transitivity of $\mathfrak{G}$. Some $K_i(e)$ contains at least three vertices of K including (e, e), since

$$K\subseteq \Delta(e, e)\cup\{(e, e)\}$$
$$=K_1(e)\cup K_2(e)\cup K_3(e).$$

Suppose $(e, a), (e, b)\in K$, $a, b\in M-\{e\}$, $a\neq b$. Then the lemma is proved if we show that K is contained in $K_1(e)$. To the contrary suppose $(c, d)\in K-K_1(e)$. Then $(c, d)\in \Delta(e, a)$ implies $d=a$ or $c^{-1}d=a$ because $c\neq e$. In case $d=a$, $(c, a)\in \Delta(e, e)$ implies $c=a$, but this is impossible since $(a, a)\notin \Delta(e, b)$. Hence we must have $c^{-1}d=a$. Similarly we have $c^{-1}d=b$ from $(c, d)\in \Delta(e, b)$. But $c^{-1}d=a=b$ contradicts the assumption of $(e, a)\neq(e, b)$.

For any subsets K, L of Ω, we define their incidence number $i(K, L)$ by

$$i(K, L)=|\{(x, y)\in K\times L\,|\,x\in \Delta(y)\}|.$$

Also we put $M_a=M-\{e, a\}$ for $a\in M-\{e\}$, and then

$$L_i(a)=K_i(a)\cap\Gamma(e, e)=\begin{cases}\{(a, b)\,|\,b\in M_a\} & i=1,\\ \{(b, a)\,|\,b\in M_a\} & i=2,\\ \{(b, ba)\,|\,b^{-1}\in M_a\} & i=3.\end{cases}$$

Lemma 2.9. *For* $1\leqq i, j\leqq 3$ *and* $a, b\in M-\{e\}$, *we have*

$$i(L_i(a), L_j(b))=\begin{cases}(m-2)^2 & i=j,\ a=b,\\ 2(m-3) & i=j,\ a\neq b,\\ m-2 & i\neq j,\ a=b,\ a^2=e,\\ m-3 & i\neq j,\ a=b,\ a^2\neq e,\\ 3m-8 & i\neq j,\ a\neq b,\ ab=e,\\ 3m-9 & i\neq j,\ a\neq b,\ ab\neq e.\end{cases}$$

Proof of Theorem 2.7. Note that we have only to determine automorphisms of Ω_M which fix (e, e), since $\mathfrak{G}$ is transitive on Ω_M. First, we deal with the case $m\geqq 5$, and we prove that any automorphism θ of Ω fixing (e, e) is contained in $<\mathfrak{X}, \mathfrak{S}>$. From Lemma 2.8 θ is an automorphism of $\tilde{\Omega}_M$[1], and we may assume that θ fixes $K_i(e)$ $(i=1, 2, 3)$ since $\mathfrak{S}$ permutes $K_i(e)$ symmetrically. Then

$$(K_i(a))\,\theta=K_i(a_i),\qquad i=1, 2, 3$$

for some $a_i\in M$. This implies

$$(L_i(a))\,\theta=L_i(a_i),\qquad i=1, 2, 3,$$

but we know that a_i are the same element a^σ from Lemma 2.9, where σ is a permutation of M. Noting that $K_1(a)\cap K_2(b)=\{(a, b)\}\,(a, b\in M)$,

1) Any automorphism of a partial geometry $\tilde{\Omega}$ induces an automorphism of the graph Ω, but the converse is not true in general.

we have

$$(a, b)\theta=(a^{\sigma}, b^{\sigma}), \qquad (a, b)\in\Omega.$$

Now $(a, b)\in K_3(a^{-1}b)$ implies

$$(a, b)\theta=(a^{\sigma}, b^{\sigma})\in(K_3(a^{-1}b))\theta=K_3((a^{-1}b)^{\sigma}).$$

Hence we have

$$(a^{\sigma})^{-1}b^{\sigma}=(a^{-1}b)^{\sigma},$$

and σ is an automorphism of M. Then

$$\theta=X_{\sigma}\in\mathfrak{X},$$

and the theorem is proved in case $m\geqq 5$.

Next, suppose M is a cyclic group of order 4 generated by a. Then the maximal cliques of order four containing (e, e) are $K_i(e)$ $(i=1, 2, 3)$ and $K_0(e)=\{(e, e), (e, a^2), (a^2, e), (a^2, a^2)\}$. Therefore any automorphism θ of Ω fixing (e, e) also fixes $K_0(e)$, since

$$|K_0(e)\cap K_i(e)|=2, \qquad 1\leqq i\leqq 3, \text{ and}$$

$$|K_i(e)\cap K_j(e)|=1, \qquad 1\leqq i, j\leqq 3, \ i\neq j.$$

Therefore θ is an automorphism of $\tilde{\Omega}_M$. Then the same proof as in the case $m\geqq 5$ can be applied and we have $\theta\in<\mathfrak{X}, \mathfrak{S}>$ and $\mathrm{Aut}\,\Omega=\mathfrak{G}$.

If M is an elementary abelian group of order 4, the complementary graph of Ω is isomorphic to $\mathscr{L}_4$ (in Higman's notation) and $\mathrm{Aut}\,\Omega$ is isomorphic to $\mathfrak{S}_4 \int \mathfrak{S}_2$ by Lemma 1 of [31].

If M is a cyclic group of order 3, the complementary graph of Ω is not connected but consists of three cliques of order 3. It follows immediately that $\mathrm{Aut}\,\Omega$ is isomorphic to $\mathfrak{S}_3 \int \mathfrak{S}_3$.

Theorem 2.10. $(\mathrm{Aut}\,\Omega_M, \Omega_M)$ *is a primitive permutation group of rank* 3 *if and only if* M *is a cyclic group of order* 5 *or an elementary abelian* 2*-group of order greater than or equal to* 4.

Proof. $\mathrm{Aut}\,\Omega_M$ is of rank 3 if and only if the stabilizer of (e, e)

in Aut Ω_M is transitive both on $\Delta(e, e)$ and on $\Gamma(e, e)$. This is easily verified in case M is a cyclic group of order 5 or an elementary abelian 2-group of order greater than or equal to 4.

Conversely, suppose M is not an elementary abelian 2-group and the stabilizer $\mathfrak{H}=<\mathfrak{X}, \mathfrak{S}>$ of (e, e) in Aut $\Omega=\mathfrak{G}=<\mathfrak{T}, \mathfrak{X}, \mathfrak{S}>$ is transitive on $\Gamma(e, e)$. We can find some $a \in M-\{e\}$ whose order is not 2, since M is not an elementary abelian 2-group. Then (a, a^2) is a vertex in $\Gamma(e, e)$ and we consider the orbit containing (a, a^2) under the action of $\mathfrak{H}$. The orbit containing (a, a^2) under the action of $\mathfrak{S}$ is

$$U=\{(a, a^2), (a^{-1}, a), (a^{-1}, a^{-2}), (a^{-2}, a^{-1}), (a, a^{-1}), (a^2, a)\}.$$

For every $b \in M_a$, there exists some vertex $(a^i, a^j) \in U$ and some automorphism σ of M such that

$$(a^i, a^j)X_\sigma=(a, b),$$

since we assume that $\mathfrak{H}$ is transitive on $\Gamma(e, e)$. Then σ maps any power of a to some power of a, since σ maps $a^i(=a^{\pm 1}$ or $a^{\pm 2})$ to a. Hence b is some power of a, and this means that M is a cyclic group generated by a. Moreover, the intersection of $L_1(a)$ and the orbit containing (a, a^2) under the action of $\mathfrak{H}$ consists of at most three vertices: (a, a^2), (a, a^{-1}) and (a, a^i), where $2i\equiv 1 \pmod{m}$. It follows that

$$|L_1(a)|=m-2\leqq 3.$$

Hence M is a cyclic group of order smaller than or equal to 5. But if M is of order 3, Aut Ω is imprimitive, and it is easily verified that $\mathfrak{H}$ is not transitive on $\Delta(e, e)$ in case M is of order 4. Then the only possibility is that M is a cyclic group of order 5, and the theorem is proved.

§3. Some chracterizations

Theorem 3.1. *Let Ω be a geometric graph with characteristic* $(3, m, 2)$, $m\geqq 4$, $\tilde{\Omega}$ *being the geometric structure. If* (Aut Ω, Ω) *is a* *primitive permutation group of rank* 3, *then* Ω *is isomorphic to some* Ω_M, *where* $M=E_{2^f}$ *or* Z_5. *In particular*, m *must be equal to* 2^f *or* 5.

Remark. Sims [4] defined a family of geometric graphs $\mathscr{G}_2(f, q)$ with automorphism groups of rank 3 for any prime power q. Ω_M for $M=E_{2^f}$ is the special case of $q=2$. Also there exists a family of strongly regular graphs with $k=l=\frac{q-1}{2}$, $\lambda=\frac{q-5}{4}$ and $\mu=\frac{q-1}{4}$, where q is a prime power and $q\equiv 1 \pmod 4$. Ω_M for $M=Z_5$ is the special case of $q=5^2$.

Proof. It is not difficult to verify that there exist two non-isomorphic geometric graphs with characteristic (3, 4, 2) or (3, 5, 2), but only one of them (that is, Ω_M for $M=Z_2\times Z_2$ or Z_5) admits an automorphism group of rank 3.

We deal with the case $m>5$ in the following. We fix a vertex $a\in\Omega$, and count the number N of triangles contained in $\Delta(a)$, where a triangle is an ordered triple $(x, y, z)(x, y, z\in\Omega)$ such that $\{x, y, z\}$ is a clique of order 3. Let s be the number of edges in $\Delta(a)\cap\Delta(b)$, $b\in\Gamma(a)$. Note that s is independent of the choice of $b\in\Gamma(a)$, since we assume that $\mathrm{Aut}\,\Omega$ is of rank 3. Then we have

$$\begin{aligned}N&=k\lambda(\lambda-1)-2sl\\&=(m-1)\{3m(m-1)-2s(m-2)\}.\end{aligned}$$

On the other hand, we have

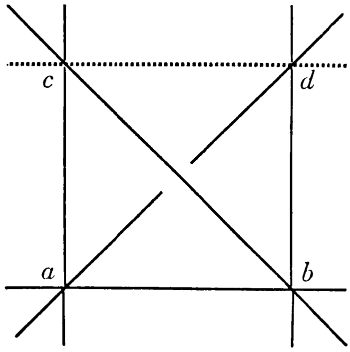
$$\begin{aligned}N&=k(\kappa-2)(\kappa-3)+2kv\\&=3(m-1)\{(m-2)(m-3)+2v\},\end{aligned}$$

where $v=0$ if every clique of order 4 is contained in a line, and $v=1$ otherwise. Therefore we have

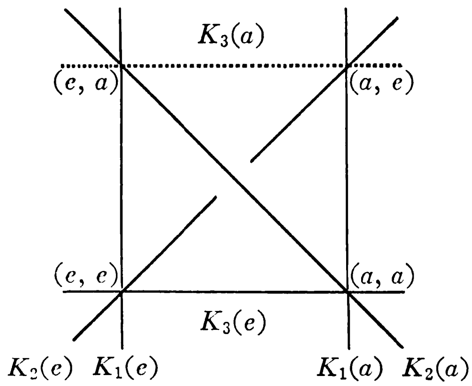
$$s=6+\frac{3-3v}{m-2},$$

and v must be equal to 1 since we assume that $m>5$. Note that $v=1$

means that two vertices c and d are adjacent if (a, b, c) and (a, b, d) are triangles not contained in a line.



Let M be a set with m elements and identify the vertices of Ω with $M\times M$ as follows. Choose an element of M and call it e, and choose a vertex of Ω and call it (e, e). Let $K_i(e)$ $(i=1, 2, 3)$ be the three lines through (e, e). Fix a bijective map f from M onto $K_3(e)$ such that $f(e)=(e, e)$, and let (a, a) be the image $f(a)$ of $a\in M$. Let (e, a) (resp. (a, e)) be the vertex in $\Delta(a, a)\cap K_1(e)$ (resp. $\Delta(a, a)\cap K_2(e)$)



different from (e, e) for $a\in M-\{e\}$. (Note that $|\Delta(a, a)\cap K_1(e)| = |\Delta(a, a)\cap K_2(e)|=2$ for $a\in M-\{e\}$.) Furthermore, let $K_1(a)$ (resp. $K_2(a)$) be the line through (a, a) and (a, e) (resp. the line through (a, a) and (e, a)). Note that (a, e) and (e, a) are adjacent, since $((e, e), (a, a), (a, e))$ and $((e, e), (a, a), (e, a))$ are triangles not contained in a line. Now let $K_3(a)$ be the line through (a, e) and (e, a). Then we have

$$|K_i(a)\cap K_j(b)| = \begin{cases} m & i=j,\ a=b, \\ 0 & i=j,\ a\neq b, \\ 1 & i\neq j. \end{cases}$$

Finally, let (a, b) be the unique vertex in $K_1(a)\cap K_2(b)$.

Now we define the *addition* in M by

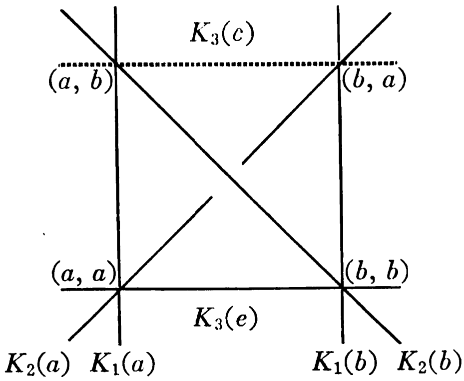
$$a+b=c \text{ if and only if } (a, b)\in K_3(c).$$

Lemma 3.2. $a+e=e+a=a$ *for* $a\in M$.

Proof. This is trivial since (e, a) and (a, e) are vertices in $K_3(a)$ by definition.

Lemma 3.3. $a+b=b+a$ *for* $a, b\in M$.

Proof. This is trivial if $a=b$. Therefore we may assume that $a\neq b$. Then $((a, a), (b, b), (a, b))$ and $((a, a), (b, b), (b, a))$ are triangles.



If one of them is contained in a line, the line must be $K_3(e)$, but this implies $a=b$, which is not the case. Hence there exists a line passing through (a, b) and (b, a), and this line must be of the form $K_3(c)$.

This means

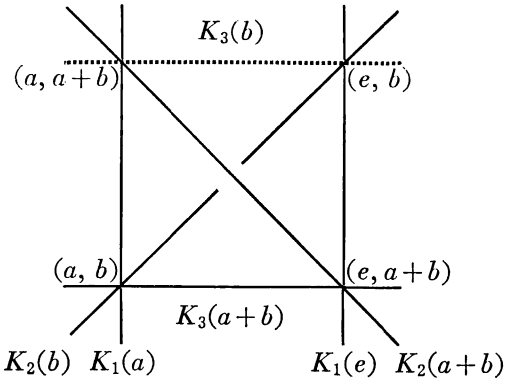
$$a+b=c=b+a,$$

and the lemma is proved.

Lemma 3.4. $a+(a+b)=b$ *for* $a, b \in M$. *In particular*

$$a+a=e \qquad \text{for } a \in M.$$

Proof. If $a=e$, this lemma is trivial from Lemma 3.2. Therefore assume $a \neq e$. Then $((a, b), (e, a+b), (e, b))$ and $((a, b), (e, a+b), (a, a+b))$ are triangles not contained in a line. Hence there exists a



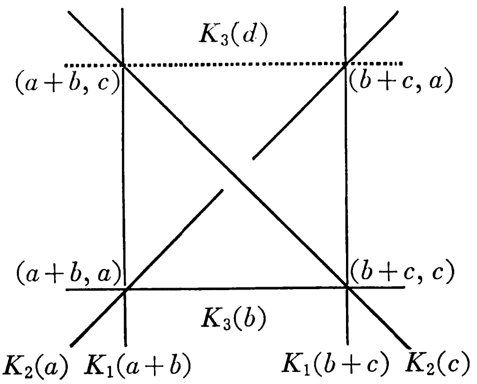
line passing through (e, b) and $(a, a+b)$, and this line must be $K_3(b)$. This means

$$a+(a+b)=b,$$

and the lemma is proved.

Lemma 3.5. $(a+b)+c=a+(b+c)$ *for* $a, b, c \in M$.

Proof. If $a=c$, this lemma is trivial from Lemmas 3.3 and 3.4. Therefore assume $a \neq c$. Then $((a+b, a), (b+c, c), (a+b, c))$ and $((a+b, a), (b+c, c), (b+c, a))$ are triangles not contained in a line. Hence there exists a line passing through $(a+b, c)$ and $(b+c, a)$, and



this line must be of the form $K_3(d)$. This means

$$(a+b)+c=d=(b+c)+a=a+(b+c),$$

and the lemma is proved.

From the above lemmas, we know that M is an abelian group of exponent 2 (that is, an elementary abelian 2-group) with respect to this addition, and the theorem is proved.

Corollary 3.6. *Let Ω be a strongly regular graph with $k=3(m-1)$, $l=(m-1)(m-2)$, $\lambda=m$ and $\mu=6$. If* $(\mathrm{Aut}\,\Omega, \Omega)$ *is a primitive permutation group of rank* 3 *and $m>23$, then Ω is isomorphic to some Ω_M, where $M=E_{2^f}$. In particular m must be a power of* 2.

Proof. From Proposition 1.6, Ω is geometrizable if $m>23$.

Remark: The restriction $m>23$ cannot be dropped, since there exists a strongly regular graph with $k=15$, $l=20$, $\lambda=6$ and $\mu=6$ whose automorphism group is primitive of rank 3. Of course, this graph is not geometrizable from Theorem 3.1.

Higman [3] proved that some families of rank 3 permutation groups are characterized by their subdegrees. Here we make similar consideration in the case $k=3(m-1)$ and $l=(m-1)(m-2)$.

Theorem 3.7. *Let Ω be a strongly regular graph with $k=3(m-1)$*

and $l=(m-1)(m-2)$, $m\geqq 4$. *Then* $\mu=6$ *unless* $\mu=9$ *and* $m=14$ *or* 352.

Proof. We may assume that the case (b) of Proposition 1.2 (iv) holds, since we have $m=5$ and $\mu=6$ if the case (a) occurs. From Proposition 1.2 (i), we have

$$\mu(m-2)=3(3m-4-\lambda).$$

We consider two cases according as 3 divides μ or not.

Case 1. $\mu=3\mu_0$. In this case $\mu_0(m-2)=3m-4-\lambda$, so that $(\mu_0-3)m=2\mu_0-4-\lambda$. We have $\lambda\geqq 0$ and $0\leqq\mu_0\leqq m-1$, since $0\leqq\mu\leqq k=3(m-1)$. Hence $(\mu_0-3)m\leqq 2\mu_0-4\leqq 2m-6$, so that $\mu_0\leqq 4$.

If $\mu_0=4$, we have $\lambda=4-m\geqq 0$, so that $m\leqq 4$. But then $\mu=3\mu_0=12>3(m-1)=k$, which is impossible.

If $\mu_0=3$, we have $\lambda=2$, $\mu=9$ and $d=12m+1$. Therefore $\sqrt{12m+1}$ is an integer and divides $2k+(\lambda-\mu)(k+l)=1+6m-7m^2$. Hence $\sqrt{12m+1}$ divides 65, because $\sqrt{12m+1}$ and m are relatively prime and $1+6m-7m^2=(1-6m)(12m+1)+65m^2$. Therefore $\sqrt{12m+1}=65$, 13, 5 or 1, and then $m=352$, 14, 2 or 0.

If $\mu_0=2$, we have $\lambda=m$ and $\mu=6$ as desired.

If $\mu_0=1$, we have $\lambda=2m-2$, $\mu=3$ and $d=4m^2-8m+1=(2m-2)^2-3$. But the only solution is $\sqrt{d}=1$ and $m=2$.

If $\mu_0=0$, $k+1=3m-2$ must divide $n=m^2$. Then $3m-2$ must divide $2m$. In particular, $3m-2\leqq 2m$, hence $m\leqq 2$.

Case 2. 3 does not divide μ. In this case 3 divides $m-2$ and we can write $m=3t+2$, $t\geqq 1$. Then $\mu t=9t+2-\lambda$, that is, $\lambda=(9-\mu)t+2$, so that $\mu\leqq 11$.

If $\mu=11$, we have $t=1$, $\lambda=0$ and $k=12$. Then $d=11^2+4(12-11)=125$ is not a square.

If $\mu=10$, we have $\lambda=-t+2\geqq 0$, so that $t=2$ or 1. If $t=2$, we have $\lambda=0$ and $k=21$, but it is easily proved that there exists no strongly regular graph with $k=21$, $l=42$, $\lambda=0$ and $\mu=10$. If $t=1$, we have $\lambda=1$ and $k=12$. Then $d=9^2+4(12-10)=89$ is not a square.

If $\mu=8$, we have $\lambda=t+2$ and $d=t^2+24t+16=(t+12)^2-128$. There exist two solutions, but $\sqrt{d}$ does not divide $2k+(\lambda-\mu)(k+l)$ in both cases.

If $\mu\leqq 7$, we have $\lambda=(9-\mu)t+2$ and $d=(9-\mu)^2t^2+(72-22\mu+2\mu^2)t+16-8\mu+\mu^2$, but there exists no solution with $t\geqq 1$, and the theorem is proved.

Corollary 3.8. *If (G, Ω) is a rank 3 permutation group of degree m^2 with subdegrees 1, $3(m-1)$ and $(m-1)(m-2)$, $m>23$, then the graph constructed from (G, Ω) is isomorphic to some Ω_M, $M=E_{2^f}$, and G is isomorphic to a subgroup of $\mathrm{Aut}\,\Omega_M=\mathfrak{G}=<\mathfrak{T}, \mathfrak{X}, \mathfrak{S}>$ unless $\mu=9$ and $m=352$.*

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