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A duality theorem

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The object of this note is to prove a certain duality relation between the functors Ext and Tor and apply it in particular to the case of Gorenstein rings of dimension $\leqq 2$. This is done in section 1. In section 2 we consider some questions on projectivity and homological dimensions of modules.

Throughout we consider only rings with unity which are both left and right noetherian; all modules will be assumed to be finitely generated and unitary, and the ring elements will be assumed to oprate on the right of the modules unless otherwise stated.

§1.

Let R be a ring and M an R-module. In what follows we choose a fixed resolution of M by means of finitely generated free R-modules:

$$\cdots F_i \to F_{i-1} \to \cdots \to F_0 \to M \to 0.$$

Let $\Omega^i M = \mathrm{Ker}(F_{i-1} \to F_{i-2})$. where by convention $F_{-1} = M$, and $\Omega^\circ M = M$.

Let $R \to S$ be a homomorphism of rings. If P is any S-module by P^* we mean the dual $Hom_S(P, S)$ considered as a left S-module. If Q is any R-module $Q \otimes_R S$ is considered as a right S-module in the usual way. The S-module P is said to be *torsionless* if the natural map $P \to P^{**}$ is injective.

Proposition 1. *For every integer* $n \geqq 0$, *there exist natural homomorphisms,*

$$f_n\colon \operatorname{Ext}_R^n(M, S) \to \operatorname{Tor}_n^R(M, S)^*.$$

For f_n *to be injective it is necessary and sufficient that* $\operatorname{Ext}_S^1(\Omega^{n-1}M \otimes_R S, S) = 0$; *for* f_n *to be surjective it is sufficient that* $\operatorname{Ext}_S^2(\Omega^{n-1}M \otimes_R S, S) = 0$. *If* f_{n+1} *is injective this condition is also necessary for the surjectivity of* f_n.

Propostion 2. *For every integer* $n \geqq 0$, *there exist natural homomorphisms*

$$g_n\colon \operatorname{Tor}_n^R(M, S) \to \operatorname{Ext}_R^n(M, S)^*.$$

g_n *is injective if and only if the natural map* $\lambda_n\colon (\Omega^n M \otimes S) \to (\Omega^n M \otimes S)^{**}$ *is injective, i.e.* $\Omega^n M \otimes_R S$ *is torsionless; for* g_n *to be surjective it is surfficient that* λ_n *is surjective. If* g_{n-1} *is injective this condition is also necessary for the surjectivity of* g_n.

Proof of Proposition 1.

If $n=0$, the result follows by the well-known isomorphism $\operatorname{Hom}_R(M, S) \cong \operatorname{Hom}_S(M \otimes_R S, S)$. Let us consider the case $n=1$. Applying $\operatorname{Tor}^R(\ , S)$ to the exact sequence

$$0 \to \Omega^1 M \to F_0 \to M \to 0. \tag{1}$$

We get the following exact sequence

$$0 \to \operatorname{Tor}_1^R(M, S) \to \Omega^1 M \otimes_R S \to F_0 \otimes_R S \to M \otimes_R S \to 0.$$

We spit this into two short exact sequences as follows:

$$0 \to \operatorname{Tor}_1^R(M, S) \to \Omega^1 M \otimes_R S \to X \to 0. \tag{2}$$

$$0 \to X \to F_0 \underset{R}{\otimes} S \to M \underset{R}{\otimes} S \to 0. \tag{3}$$

Taking S-duals in (2), we get the exact sequence

(4) $$0 \to X^* \to (\Omega^1 M \otimes_R S)^* \to \mathrm{Tor}_1^R(M, S)^* \to \mathrm{Ext}_S^1(X, S) \to \mathrm{Ext}_S^1(\Omega^1 M \otimes_R S, S).$$

Applying $\mathrm{Ext}_R(, S)$ to (1) and using the isomorphism stated in the beginning of the proof, we get the following exact sequences:

(5) $$0 \to (M \otimes S)^* \to (F_0 \otimes S)^* \to (\Omega^1 M \otimes S)^* \to \mathrm{Ext}_R^1(M, S) \to 0.$$

Taking S-duals in (3) we get the exact sequence:

(6) $$0 \to (M \otimes S)^* \to (F_0 \otimes S)^* \to X^* \to \mathrm{Ext}_S^1(M \otimes_R S, S) \to 0.$$

Consider the commutative diagram where the top line is exact.

$$\begin{tikzcd} 0 \arrow[r] & X^* \arrow[r, "\delta"] & (\Omega^1 M \otimes_R S)^* \arrow[r, "\chi"] & \mathrm{Tor}_1^R(M, S)^* \\ & (F_0 \otimes_R S)^* \arrow[u, "\alpha"] \arrow[ur, "\beta"'] & & \end{tikzcd}$$

Since image$(\beta) \subset$ image(δ) we get an induced homomorphism Coker(β) $\to$Cocker(δ); composing with the injection Coker$(\delta) \to \mathrm{Tor}_1^R(M, S)^*$ got from the top exact sequence and noting that Coker$(\beta) = \mathrm{Ext}_R^1(M, S)$ by (5), we get the homomorphism f_1: $\mathrm{Ext}_R^1(M, S) \to \mathrm{Tor}_1^R(M, S)^*$. Replacing M by $\Omega^{n-1}M$ and using the natural isomorphisms $\mathrm{Ext}_R^n(M, S) \cong \mathrm{Ext}_R^1(\Omega^{n-1}M, S)$, $\mathrm{Tor}_n^R(M, S) \cong \mathrm{Tor}_1^R(\Omega^{n-1}M, S)$ we get as in the case $n = 1$, natural homomorphisms f_n: $\mathrm{Ext}_R^n(M, S) \to \mathrm{Tor}_n^R(M, S)^*$. It is clear by the homotopy property of projective resolutions that these homomorphisms f_n's are independent of the resolution for M chosen. Clearly f_n are functorial in M and S. Now Ker$f_1 = X^*/$Image(β). Hence f_1 is injective$\Leftrightarrow X^* =$ image$(\beta) \Leftrightarrow \alpha$ is surjective$\Leftrightarrow \mathrm{Ext}_S^1(M \otimes_R S, S) = 0$ by (6). Also f_1 is surjective is equivalent to saying that χ is surjective or equivalently $\mathrm{Ext}_S^1(X. S) \to \mathrm{Ext}_S^1(\Omega^1 M \otimes_R S, S)$ is injective, by the exact sequence (4). In the general case we can therefore say the follwing: f_n is injective if and only if $\mathrm{Ext}_S^1(\Omega^{n-1}M \otimes_R S, S) = 0$; f_n is surjective if and only if the homomorphism $\mathrm{Ext}_S^1(X_n, S) \to$

$\mathrm{Ext}^1_S(\Omega^n M\otimes_R S, S)$ is injective where $X_n=\mathrm{Kernel}(F_{n-1}\otimes_R S\to\Omega^{n-1}M\otimes_R S)$. Now suppose $\mathrm{Ext}^2_S(\Omega^{n-1}M\otimes_R S, S)=0$. This clearly implies by the exact sequence $0\to X_n\to F_{n-1}\otimes_R S\to\Omega^{n-1}M\otimes_R S\to 0$, $\mathrm{Ext}^1_S(X_n, S)=0$. Hence f_n is surjective in this case. Suppose f_{n+1} is injective; then $\mathrm{Ext}^1_S(\Omega^n M\otimes_R S, S)=0$, so that f_n is surjective if and only if $\mathrm{Ext}^1_S(X_n, S)=0$, i.e. $\mathrm{Ext}^2_S(\Omega^{n-1}M\otimes_R S, S)=0$.

Corollary. *In order that all the f_n are isomorphisms it is necessary and sufficient that* $\mathrm{Ext}^i_S(\Omega^j M\otimes_R S, S)=0$ *for $i=1, 2$ and all $j\geq 0$.* This follows from Proposition 1 by using induction.

Proor of Proposition 2.

We split the exact sequence (5) into two parts as follows:

$$0\to(M\otimes S)^*\to(F_0\otimes S)^*\to Y\to 0. \tag{7}$$

$$0\to Y\to(\Omega^1 M\otimes_R S)^*\to\mathrm{Ext}^1_R(M, S)\to 0. \tag{8}$$

Taking duals with respect to S in (7) and comparing with (3) we get the commutative diagram:

$$\begin{array}{ccccccc} 0\to & Y^* & \to & (F_0\otimes_R S)^{**} & \to & (M\otimes_R S)^{**} & \\ & \uparrow\theta & & \uparrow\lambda & & \uparrow\mu & \\ 0\to & X & \longrightarrow & (F_0\otimes_R S) & \longrightarrow & (M\otimes_R S) & \to 0 \end{array} \tag{9}$$

The diagram gives rise to a homorphism $\theta\colon X\to Y^*$ preserving the commutativity. Taking duals in (8) we get another commutative diagram after making use of (2):

$$\begin{array}{ccccccc} 0\to & \mathrm{Ext}^1_R(M, S)^* & \to & (\Omega^1 M\otimes_R S)^{**} & \to & Y^* & \\ & \uparrow g_1 & & \uparrow q & & \uparrow\theta & \\ 0\to & \mathrm{Tor}^R_1(M, S) & \to & (\Omega^1 M\otimes_R S) & \to & X & \to 0 \end{array} \tag{10}$$

The map g_1 exists because of the commutativity of the right hand square. From (9) we find that θ is injective and $\mathrm{Ker}\mu\cong\mathrm{Coker}\theta$. Applying the snake lemma to (10) we get the following exact sequence:

$$0 \to \operatorname{Ker} g_1 \to \operatorname{Ker} q \to \operatorname{Ker} \theta \to \operatorname{Coker} g_1 \to \operatorname{Coker} q \to \operatorname{Coker} \theta.$$

This exact sequence gives the following facts: g_1 is injective$\Leftrightarrow q$ is injective, i.e. $\Omega^1 M \otimes_R S$ is a torsionless S-module; g_1 is surjective$\Leftrightarrow$ $\operatorname{Coker} q \to \operatorname{Coker} \theta = \operatorname{Ker} \mu$ is injective. Using a dimension shifting argument we can construct natural homomorphisms g_n: $\operatorname{Tor}_n^R(M, S) \to \operatorname{Ext}_R^n (M, S)^*$ and the following conclusions are valid: suppose λ_n denotes the natural mapping $\Omega^n M \otimes_R S \to (\Omega^n M \otimes_R S)^{**}$. For every integer n, there is a homomorphism $\operatorname{Coker} \lambda_n \xrightarrow{\varphi_n} \operatorname{Ker} \lambda_{n-1}$. g_n is injective if and only if λ_n is injective. A necessary and sufficient condition for g_n to be surjective is that φ_n should be injective. For g_n to be surjective it is sufficient that λ_n is surjective. Suppose g_{n-1} is injective. Then for g_n to be surjective it is also necessary that λ_n should be surjective.

Corollary. *In order that all the g_n's are isomorphisms it is necessary and sufficient that all the S-modules $\Omega^j M \otimes_R S$ for $j \geq 0$, are reflexive.*

Remarks. Since the homorphisms f_n, g_n are independent of the resolution for M chosen, we see that the conditions stated in Propositions 1 and 2 are also independent of the resolution. Given any module M over a ring R we can construct a module $D(M)$ associated with it as follows: take a finite presentation $F_1 \to F_0 \to M \to 0$ and define $D(M)$ to be the Cokernel of the map $F_0^* \to F_1^*$. This module $D(M)$ is not uniquely defined by M, but $\operatorname{Ext}_R^i(D(M), \mathrm{R})$ depend only on M for $i \geq 1$ [2]. It is well-known that M is torsionless$\Leftrightarrow \operatorname{Ext}_R^1(D(M), R) = 0$; M is reflexive$\Leftrightarrow \operatorname{Ext}_R^1(D(M), R) = 0$ and $\operatorname{Ext}_R^2(D(M), R) = 0$ [2]. Also it is clear that $D(D(M)) = M$. Hence the conditions $\operatorname{Ext}_S^i(\Omega^j M \otimes_R S, S) = 0$ for $i = 1, 2$ of Proposition 1 can also be interpreted as the reflexivity of the S-modules $D_S(\Omega^j M \otimes_R S)$. Similarly the conditions that $\Omega^j M \otimes_R S$ are reflexive stated in Proposition 2 can be interpreted as the vanishing of the groups $\operatorname{Ext}_S^i(D_S(\Omega^j M \otimes_R S), S)$ for $i = 1, 2$. Thus there is a clear duality in the assumptions and conclusions of Propositions 1 and 2.

Corollary. *Let R, S be commutative, S being an R-algebra. Suppose S is Gorenstein of dimention ≤ 2. Then if all the f_n's are isomorphisms, so are the g_n's and conversely.*

Proof. Observe that over S, a module N is reflexive if and only if $\mathrm{Ext}_S^i(N, S)=0$ for $i=1, 2$ [2]. Hence the corollary follows by applying Propositions 1 and 2.

§2.

In this section we discuss some criteria for projectivity of modules and consider some questions on homological dimensions.

Proposition 3. *R is regular local, S is an R-algebra finitely generated as an R-module such that right and left self injective dimensions of S are ≤ 1. Let M be an R-module. If there exists an integer $n\geq 2$ such that $\mathrm{Ext}_R^{n-1}(M, S)=\mathrm{Ext}_R^n(M, S)=0$, then $\mathrm{Ext}_R^i(M, S)=0$ for $i\geq n-1$ and $\mathrm{hd}_R M<n-1$.*

Proof. By a dimension shifting argument we can suppose that $n=2$. Let then $\mathrm{Ext}_R^1(M, S)=\mathrm{Ext}_R^2(M, S)=0$. We shall prove that M is R-projective. The proof uses the following simple fact [4]. If N is a module over S then N is torsionless $\Leftrightarrow N$ is reflexive $\Leftrightarrow \mathrm{Ext}_S^1(N, S)=0$. Now $\mathrm{Ext}_R^2(M, S)=0$ implies, in the notations of Proposition 1, that f_2 is injective, so that $\mathrm{Ext}_S^1(\Omega^1 M\otimes_R S, S)=0$. Hence by the result quoted above $\Omega^1 M\otimes_R S$ is a reflexive S-module. Proposition 2 gives therefore the isomorphism $\mathrm{Tor}_1^R(M, S)\cong \mathrm{Ext}_R^1(M, S)^*$. The hypothesis $\mathrm{Ext}_R^1(M, S)=0$ gives $\mathrm{Tor}_1^R(M, S)=0$. Since R is regular and M, S are finitely generated R-modules by a theorem of Licthenbaum [6] we get $\mathrm{Tor}_j^R(M, S)=0$ for $j\geq 1$. We assert that $\mathrm{Ext}_R^j(M, S)=0$ for $j\geq 1$. If $j=1, 2$ there is nothing to prove. So let $j\geq 3$. Now $\mathrm{Tor}_{j-1}^R(M, S)=0$ implies g_{j-1} is injective and so by Proposition 2,

$\Omega^{j-1}M\otimes_R S$ is torsionless. By applying the remark made in the beginning of the proof we get $\mathrm{Ext}_S^i(\Omega^{j-1}M\otimes_R S, S)=0$ for $i=1, 2$. Applying Proposition 1, we get an isomorphism $\mathrm{Ext}_R^j(M, S)\cong \mathrm{Tor}_j^R(M, S)^*$ and so we conclude $\mathrm{Ext}_R^j(M, S)=0$ for $j\geq 1$. If $\mathrm{hd}\, M=t$ then by a result of Auslander [1], we get $\mathrm{Ext}_R^t(M, S)\neq 0$. Hence we must have $t=0$, i.e. M is R-projective. The proposition is proved.

Proposition 4. *R is a regular local ring; $\mathfrak{A}$ is an ideal of R such that depth $R/\mathfrak{A}\leq 2$. Then a reflexive module M is projective if and only if* $\mathrm{Ext}_R^1(M, \mathfrak{A})=0$.

Proof. We have only to prove M reflexive and $\mathrm{Ext}_R^1(M, \mathfrak{A})=0$ implies M is projective. Applying $\mathrm{Ext}_R(M, \ )$ to the exact sequence $0\to\mathfrak{A}\to R\to R/\mathfrak{A}\to 0$ we arrive at the exact sequence $0\to\mathrm{Hom}(M, \mathfrak{A})\to\mathrm{Hom}(M, R)\to(\mathrm{Hom}\, M, R/\mathfrak{A})\to 0$. Tensoring with $\bar{R}=R/\mathfrak{A}$, and observing that $\mathrm{Hom}(M, \bar{R})$ is annihilated by $\mathfrak{A}$, we get the exact sequence

$$\mathrm{Hom}(M, R)\otimes\bar{R}\to\mathrm{Hom}(M, \bar{R})\to 0. \tag{11}$$

Now by [2] we have the exact sequence

$$0\to\mathrm{Tor}_2^R(DM, \bar{R})\to\mathrm{Hom}(M, R)\otimes_R\bar{R}\to \mathrm{Hom}_R(M, \bar{R})\to\mathrm{Tor}_1^R(DM, \bar{R})\to 0. \tag{12}$$

where DM is defined as in the remarks following propositions 1 and 2. From (11) and (12) we conclude $\mathrm{Tor}_1^R(DM, \bar{R})=0$; since R is regular by a theorem of Lichtenbaum [6] we get $\mathrm{Tor}_j^R(DM, \bar{R})=0$ for $j\geq 1$. Hence (12) reduces to an isomorphism

$$\mathrm{Hom}(M, R)\otimes\bar{R}\cong\mathrm{Hom}(M, \bar{R})\cong\mathrm{Hom}_{\bar{R}}(M\otimes\bar{R}, \bar{R}). \tag{13}$$

Now DM is defined by an exact sequence of the type:

$$0\to M^*\to F_0\to F_1\to DM\to 0$$

where F_0, F_1 are free R-modules. This gives $\mathrm{Tor}_j^R(M^*, \bar{R})\cong\mathrm{Tor}_{j+2}^R$

$(DM, \bar{R})$ for $j\geq 1$, i.e. $\mathrm{Tor}_j^R(M^*, \bar{R})=0$ for $j\geq 1$. Hence $M^*\otimes_R\bar{R}$ considered as an $\bar{R}$-module has finite projective dimension. By the isomorphism (13) this means $\mathrm{hd}_{\bar{R}}(M\otimes\bar{R})^*<\infty$. This implies by a well-known result $\mathrm{hd}_{\bar{R}}(M\otimes\bar{R})^*+\mathrm{depth}_{\bar{R}}(M\otimes\bar{R})^*=\mathrm{depth}\bar{R}$. Since depth $\bar{R}\leq 2$ by assumption, by a result of Auslander [1] we get $\mathrm{depth}_{\bar{R}}(M\otimes\bar{R})^*=\mathrm{depth}\bar{R}$. Hence by what precedes $(M\otimes\bar{R})^*$ is $\bar{R}$-projective, i.e. $M^*\otimes\bar{R}$ is $\bar{R}$-projective. Since we already know that $\mathrm{Tor}_1^R(M^*, \bar{R})=0$, by a proposition of Strooker [10] M^* is R-projective. M being reflexive this means M is R-projective.

Remarks. This generalises the Corollary to Proposition 4.7 of Auslander [1]. Using similar arguments we can prove the following: R is regular local, $\mathfrak{A}$ an ideal of R such that $\mathrm{depth}R/\mathfrak{A}\geq 2$. Then for any module M such that $M^*\neq 0$ and $\mathrm{Ext}_R^1(M, \mathfrak{A})=0$, we have $\mathrm{hd}_R M^*\leq\mathrm{depth}R/\mathfrak{A}-2$. In the previous proposition we can drop the regularity assumption on R provided we assume $\mathrm{hd}M^*<\infty$ and $R/\mathfrak{A}$ is a rigid module, i.e. whenever $\mathrm{Tor}_1^R(R/\mathfrak{A}, N)=0$ for a finitely generated module N, we should have $\mathrm{Tor}_j^R(R/\mathfrak{A}, N)=0$ for $j\geq 1$. For example if the ideal $\mathfrak{A}$ is generated by an R-sequence this latter condition is satisfied.

Next we note the following simple results whose proofs can be given on the lines of the previous proposition and hence omitted.

Proposition 5. *R is regular local, $\mathfrak{A}$ an ideal of definition and M is an R-module. Then M is projective* $\Leftrightarrow\mathrm{Ext}_R^1(M, \mathfrak{A})=0$.

Proposition 6. *R is a ring not necessarily commutative, $\mathfrak{A}$ is a 2-sided ideal contained in the radical and M is an R-module. Suppose* $\mathrm{Ext}_R^1(M, \mathfrak{A})=0$ *and $M/\mathfrak{A}M$ is $R/\mathfrak{A}$-projective. Then M is R-projective.*

Remark Proposition 6 generalises theorem 1.3 of Mark Ramras [9].

The following proposition is an analogue of the result of Strooker used in Propositiou 4, for the Ext functor:

Proposition 7. *Let R be a ring, $\mathfrak{A}$ a 2-sided ideal contained in the radical and M is an R-module. Let $0 \to K \to F \to M \to 0$ be exact with F free. Then M is projective if and only if* $\mathrm{Ext}_R^1(M, \bar{R})=0$ *and* $K \otimes \bar{R}$ *is* $\bar{R}$*-projective where* $\bar{R}=R/\mathfrak{A}$.

Proof. Let $K \otimes \bar{R}$ be $\bar{R}$-projective and $\mathrm{Ext}_R^1(M, \bar{R})=0$. By Proposition 2 we get an isomorphism $\mathrm{Tor}_1^R(M, \bar{R}) \cong \mathrm{Ext}_R^1(M, \bar{R})^*$ so that $\mathrm{Tor}_R^1(M, \bar{R})=0$. Applyiug $\mathrm{Tor}^R(, \bar{R})$ to the exact sequence:

$$0 \to K \to F \to M \to 0 \tag{13}$$

we get the exact sequence

$$0 \to K \otimes \bar{R} \to F \otimes \bar{R} \to M \otimes \bar{R} \to 0. \tag{14}$$

Applying Hom(, $\bar{R}$) to (13) and noting that $\mathrm{Ext}_R^1(M, \bar{R})=0$ we get the exact sequence;

$$0 \to (M \otimes \bar{R})^* \to (F \otimes \bar{R})^* \to (K \otimes \bar{R})^* \to 0$$

where $*$ denotes dual with respect to $\bar{R}$. This sequence splits since $K \otimes \bar{R}$, and hence $(K \otimes \bar{R})^*$ is $\bar{R}$-projective. Taking $\bar{R}$-duals again and noting that $K \otimes \bar{R}$, $F \otimes \bar{R}$ are reflexive we get the split exact sequence $0 \to (K \otimes \bar{R}) \to (F \otimes \bar{R}) \to (M \otimes \bar{R})^{**} \to 0$.
Comparing this with (14) we find that (14) is a split exact sequence i.e. $M \otimes \bar{R}$ is $\bar{R}$-projective. Since we already know that $\mathrm{Tor}_1^R(M, \bar{R})=0$, the result of Strooker implies that M is R-projective.

The next result is a generalisation of theorem 2.1 of Jans [7].

Proposition 8. *$R \to S$ is a ring homomorphism and M an R-module. Suppose* $\mathrm{hd}_R M=n<\infty$ *and* $\mathrm{Tor}_n^R(M, S)=0$. *Then* $\mathrm{Ext}_R^n(M, S)^*=0$.

Proof. In the notations of Proposition 1, $\Omega^n M$ is R-projective and so $\Omega^n M \otimes_R S$ is S-projective. Hence Proposition 2 gives the isomorphism $\mathrm{Tor}_n^R(M, S) \cong \mathrm{Ext}_R^n(M, S)^*$. This proves the proposition.

Proposition 9. *R, S are commutative rings, S being an R-algebra. Assume that S is Gorenstein of dimension ≤ 1. Let M be an R-module. Then if $\mathrm{Tor}_i^R(M, S)=0$ for $i \geq n$ we have $\mathrm{Ext}_R^i(M, S)=0$ for $i \geq n+1$ and conversely if $\mathrm{Ext}_R^i(M, S)=0$ for $i \geq n+1$, then $\mathrm{Tor}_i^R(M, S)=0$ for $i \geq n+1$.*

The proof of this is similar to that of Proposition 3 and hence omitted.

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