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Spherical functions on locally compact groups

By

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Introduction

In 1950, I. M. Gel'fand defined the generalized "spherical functions" and studied the connection with the unitary representations in [4]. He studied only the case when the representation of the given compact subgroup K is $k \to 1$.

After that, in [5], R. Godement defined the still more generalized spherical functions, and studied the connection with the representations on Banach spaces. For the given representation $\{\mathfrak{H}, T_x\}$ of the locally compact unimodular group G, we can define the representation $\{\mathfrak{H}, T_f\}$ of the algebra $L(G)$, which is the algebra of all continuous functions on G with compact supports. Then he said that $\{\mathfrak{H}, T_x\}$ is algebraically irreducible when non trivial T_f-invariant subspaces of $\mathfrak{H}$ do not exist, completely irreducible when every continuous linear operator T on $\mathfrak{H}$ can be strongly approximated by T_f, and topologically irreducible when non trivial closed T_f-invariant subspaces of $\mathfrak{H}$ do not exist.

Now let $\mathfrak{H}(\delta)$ be the set of vectors in $\mathfrak{H}$ which, under $k \to T_k$, transform according to δ, and $E(\delta)$ the continuous projection on $\mathfrak{H}(\delta)$, where δ is an irreducible representation of the given compact subgroup K. For the completely irreducible representation $\{\mathfrak{H}, T_x\}$ on a Banach space $\mathfrak{H}$, he defined the spherical function by

$$\phi_\delta(x) = \mathrm{Tr}[E(\delta) T_x]$$

when $\dim \mathfrak{H}(\delta) < +\infty$.

However, he studied only the case of completely irreducible representations on Banach spaces, and moreover he assumed on G that

(a) every δ is contained at most finite times in every completely irreducible representation of G.

This assumption is automatically satisfied for semi-simple Lie groups with faithful representations and the motion groups where K are maximal compact subgroups. But I feel it is rather restrictive for the general consideration.

The author generalizes the theory for every locally compact unimodular group and its representation on a Hausdorff, complete, locally convex topological vector space which is not completely irreducible in general but topologically irreducible. We study the topologically irreducible representations with the following property:

$$(*)\begin{cases} \text{there exists at least one pair } (K', \delta') \text{ of a compact subgroup} \\ K' \text{ of } G \text{ and its irreducible representation } \delta' \text{ such that } 0 < \\ \dim \mathfrak{H}(\delta') < +\infty, \end{cases}$$

and generalize the propositions of R. Godement for the completely irreducible representations. We define the spherical functions for the topologically irreducible representations with the property $(*)$, and obtain a necessary and sufficient condition that a given continuous function ϕ on G which satisfies $\chi_\delta * \phi = \phi$ and $\phi = \phi^\circ$, where $\chi_\delta = (\dim \delta)\mathrm{Tr}[\delta]$ and $\phi^\circ(x) = \int_K \phi(kxk^{-1})dk$, is a spherical function. In our case, the condition (a) on G is not assumed and the necessary and sufficient condition on ϕ is as follows; $\dim(L(\delta)/\mathfrak{p}) < +\infty$, where $\mathfrak{p} = \{f \in L(\delta);\ f' * \phi = 0\}$, and there exists a p-dimensional irreducible representation $f \to U_f$ of the algebra $L^\circ(\delta)$ such that $\phi(f) = (\dim \delta)\mathrm{Tr}[U_f]$. Here, $L(\delta) = \{f \in L(G);\ \bar{\chi}_\delta * f = f * \bar{\chi}_\delta = f\}$ ($\bar{\chi}_\delta$ is the complex conjugate of χ_δ), $f'(x) = f(x^{-1})$, $L^\circ(\delta) = \{f^\circ;\ f \in L(\delta)\}$ and $\phi(f) = \int_G \phi(x)f(x)dx$. This is one of the principal results of this paper. To show this, we construct a topologically irreducible representation which has ϕ as its spherical function, and this process gives the connection between spherical functions and representations. In the case of σ-compact G, all spherical functions are obtained from topologically irreducible representations on Fréchet spaces.

Some lemmas in this paper are very similar to those in [5] but proved under somewhat weaker assumptions, and for the sake of completeness the author does not omit them.

In §1, we give some definitions and prove some general lemmas on the irreducibilities.

In §2, we study a canonical irreducible subspace $\mathfrak{H}_0$ of $\mathfrak{H}$, and this is very important for the study of topologically irreducible representations with the property (*).

In §3, we study the multiplicity of δ in completely irreducible representations or in topologically irreducible representations with the property (*).

In §4, we define spherical functions and prove the necessary and sufficient condition that a given function ϕ on G is a spherical function. In general our results are rather weak, but in the case where G is σ-compact or the given function ϕ is positive-definite, they are satisfactory. For spherical functions of height 1, another characterization is possible.

In §5, analyzing the method of the construction of representations in §4, we obtain a connection between spherical functions and representations.

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§1. **Representations and their irreducibilities**

Let G be a locally compact unimodular group, and $\mathfrak{H}$ be a Hausdorff, complete, locally convex topological vector space.

A representation of G on $\mathfrak{H}$ is a homomorphism $x \rightarrow T_x$ of G in a group of non-singular continuous linear operators on $\mathfrak{H}$ such that

(a) for $a \in \mathfrak{H}$, $G \ni x \rightarrow T_x a \in \mathfrak{H}$ is continuous,

(b) for every compact subset C of G, $\{T_x;\ x \in C\}$ is equicontinuous.

If $\mathfrak{H}$ is "tonnelé", (a) implies (b) [1], hence in the case of a

Banach space or a Fréchet space, the condition (b) is not necessary. And in general, two conditions (a) and (b) are equivalent to

(c) $G\times\mathfrak{H}\ni(x, a)\rightarrow T_{x}a\in\mathfrak{H}$ is continuous.

Let $L(G)$ be the algebra of all continuous functions on G with compact supports (the product is convolution product). For every compact subset F of G, denote by $L_{F}(G)$ the space of all continuous functions on G whose supports are contained in F, with the supremum norm. We shall topologize $L(G)$ as the inductive limit of $L_{F}(G)$. On the other hand, we shall denote by $L_{s}(\mathfrak{H}, \mathfrak{H})$ the space of all continuous linear operators on $\mathfrak{H}$, topologized by the simple convergence, and by $L_{b}(\mathfrak{H}, \mathfrak{H})$ the same space, topologized by the uniform convergence on every bounded subset of $\mathfrak{H}$.

From the given representation $\{\mathfrak{H}, T_{x}\}$ of G, let's define the representation of the algebra $L(G)$ by

$$L(G)\ni f\rightarrow T_{f}=\int_{G}T_{x}f(x)dx,$$

where dx is a Haar measure on G. Then the following facts are known [3]; the representation $\{\mathfrak{H}, T_{f}\}$ of $L(G)$ satisfies

(i) $L(G)\ni f\rightarrow T_{f}\in L_{b}(\mathfrak{H}, \mathfrak{H})$ is a continuous homomorphism,

(ii) $\{T_{f}a; f\in L(G), a\in\mathfrak{H}\}$ spans a dense subspace of $\mathfrak{H}$,

(iii) for every compact subset C of G, $\{T_{f}; f\in E\}$ is equicontinuous, where $E=\{f\in L(G); \mathrm{supp}[f]\subset C, \|f\|_{L^{1}}\leqq 1\}$. Conversely, such representation of $L(G)$ is deduced by the above method from a representation of G.

In the following definitions, $\mathfrak{H}$ is not necessarily complete.

Definition 1. Let A be an associative algebra over the complex field $\mathbf{C}$. The representation $\{\mathfrak{H}, T_{x}\}$ of A on a vector space $\mathfrak{H}$ over $\mathbf{C}$ is called "algebraically irreducible" if its invariant subspaces are only $\{0\}$ and $\mathfrak{H}$.

Definition 2. Let A be an associative algebra over $\mathbf{C}$, and $\mathfrak{H}$ a locally convex vector space. The representation $\{\mathfrak{H}, T_x\}$ of A is called "completely irreducible" if $\{T_x;\ x \in A\}$ is dense in $L_s(\mathfrak{H}, \mathfrak{H})$.

Definition 3. Under the same situation as in Definition 2, $\{\mathfrak{H}, T_x\}$ is called "topologically irreducible" if its closed invariant subspaces are only $\{0\}$ and $\mathfrak{H}$.

In the case of finite-dimensional $\mathfrak{H}$, these three irreducibilities are equivalent by the Burnside's theorem [8]. And using a theorem on the extension of a continuous linear functional [9, p. 108], we know that the complete irreducibility implies topological irreducibility. If $\mathfrak{H}$ is a Banach space, algebraic irreducibility implies complete irreducibility [5]. We shall define the irreducibility of a representation of G by that of the corresponding representation of $L(G)$. The following lemma plays an important role in this paper.

Lemma 1. *Let A be an associative algebra over* $\mathbf{C}$, *and $\mathfrak{H}$ a locally convex vector space. The algebraically irreducible representation $\{\mathfrak{H}, T_x\}$ of A is completely irreducible, if every continuous linear operator which commutes with all T_x is a scalar multiple of the identity operator.*

Proof. Let's show more strong fact that,

(a) for arbitrarily given elements $a_1, a_2, \cdots, a_n \in \mathfrak{H}$ and continuous linear operator T on $\mathfrak{H}$, there exists an element $x \in A$ such that $T_x a_i = T a_i$ for $1 \leqq i \leqq n$.

We prove this by induction on n. For $n=1$, this is true. Suppose (a) is true for $n-1$, and let's prove it for n. Clearly we may assume that $a_1, a_2, \cdots, a_n$ are linearly independent. By the assumption of induction,

(b) for every $n-1$ elements $b_1, b_2, \cdots, b_{n-1} \in \mathfrak{H}$, there exists some $x \in A$ such that $T_x a_i = b_i$ for $1 \leqq i \leqq n-1$.

Denote by $\mathfrak{S}$ the subspace of $\mathfrak{H}$ spanned by $a_1, a_2, \cdots, a_{n-1}$. Let's show the following fact;

(c) suppose $T_x b=0$ for every $x\in\mathfrak{S}'=\{x\in A;\ T_x a_i=0,\ 1\leqq i\leqq n-1\}$. Then $b\in\mathfrak{H}$ is in $\mathfrak{S}$.

For every $(b_1, b_2, \cdots, b_{n-1})\in\mathfrak{H}\times\mathfrak{H}\times\cdots\times\mathfrak{H}=\mathfrak{H}^{n-1}$, take $x'\in A$ such that $T_{x'}a_i=b_i$ for $1\leqq i\leqq n-1$, and define a linear mapping

$$\Phi;\ \mathfrak{H}^{n-1}\to\mathfrak{H},$$

by $\Phi(b_1, b_2, \cdots, b_{n-1})=T_{x'}b$ (well-defined !). Let I_i be the imbedding from $\mathfrak{H}$ to the i-component of $\mathfrak{H}^{n-1}$, and set $F_i=\Phi I_i$ $(i=1, 2, \cdots, n-1)$. Then for any $x\in A$ and $a\in\mathfrak{H}$,

$F_i T_x a=\Phi I_i(T_x a)=T_{x_1}b$, and $T_x F_i a=T_x\Phi I_i a=T_x T_{x_2}b$, where x_1 and x_2 are elements in A such that $T_{x_1}a_i=T_x a$, $T_{x_1}a_j=0$ $(j\neq i)$ and $T_{x_2}a_i=a$, $T_{x_2}a_j=0$ $(j\neq i)$ respectively. Hence $T_{xx_2}a_i=T_x a=T_{x_1}a_i$ and $T_{xx_2}a_j=0=T_{x_1}a_j$ $(j\neq i)$. Therefore $xx_2-x_1\in\mathfrak{S}'$, hence $T_{x_1}b=T_{xx_2}b$. Thus $F_i T_x=T_x F_i$ for all $x\in A$. Hence we have $F_i=\lambda_i\cdot 1$ $(\lambda_i\in\mathbf{C},\ 1\leqq i\leqq n-1)$, and

$$T_{x'}b=\Phi(b_1, b_2, \cdots, b_{n-1})=\sum_{i=1}^{n-1}\lambda_i b_i$$
$$=\sum_{i=1}^{n-1}\lambda_i T_{x'}a_i=T_{x'}\left(\sum_{i=1}^{n-1}\lambda_i a_i\right).$$

Since $x'\in A$ in (b) can be arbitrarily chosen, $T_{x'}b=T_{x'}\sum_{i=1}^{n-1}\lambda_i a_i$ is true for all $x'\in A$. Hence $b=\sum_{i=1}^{n-1}\lambda_i a_i$, and therefore $b\in\mathfrak{S}$ by the algebraic irreducibility of $\{\mathfrak{H}, T_x\}$. Thus (c) is proved.

Since $a_n\notin\mathfrak{S}$, it follows from (c) that there exists some $x\in A$ such that

$$T_x a_i=0\,(1\leqq i\leqq n-1)\ \text{and}\ T_x a_n\neq 0.$$

Therefore $\{T_x a_n;\ x\in\mathfrak{S}'\}=\mathfrak{H}$. Let $x_0\in A$ be an element such that $TT_{x_0}a_i=Ta_i$, $1\leqq i\leqq n-1$. Then there exists some $x_1\in\mathfrak{S}'$ such that

$$T_{x_1}a_n=Ta_n-T_{x_0}a_n.$$

The element $x=x_1+x_0$ satisfies $T_x a_i=Ta_i\,(1\leqq i\leqq n)$. q.e.d.

Now, let K be a compact subgroup of G, and δ an (equivalence class of finite-dimensional) irreducible representation of K. We shall denote by $\mathrm{Tr}[\delta]$ the trace of (any element in) δ, and put $\bar{\chi}_\delta=(\dim\delta)\overline{\mathrm{Tr}[\delta]}$. For every $f\in L(G)$, we define

$$\bar{\chi}_\delta * f(x)=\int_K f(k^{-1}x)\bar{\chi}_\delta(k)dk,$$

$$f*\bar{\chi}_\delta(x)=\int_K f(xk^{-1})\bar{\chi}_\delta(k)dk,$$

and put $L(\delta)=\{f\in L(G);\ \bar{\chi}_\delta*f=f*\bar{\chi}_\delta=f\}$.

For a given representation $\{\mathfrak{H}, T_x\}$ of G,

$$E(\delta)=\int_K T_k\bar{\chi}_\delta(k)\,dk$$

is the continuous projection onto the subspace $\mathfrak{H}(\delta)=E(\delta)\mathfrak{H}$, and commutes with all $T_k (k\in K)$. And for an arbitrary $f\in L(\delta)$, T_f makes $\mathfrak{H}(\delta)$ invariant. Put

$$\tilde{T}_f=T_f|_{\mathfrak{H}(\delta)} \qquad \text{for } f\in L(\delta).$$

The following lemma is essentially due to R. Godement.

Lemma 2. *If the representation $\{\mathfrak{H}, T_x\}$ of G is algebraically, completely, or topologically irreducible, the corresponding representation $\{\mathfrak{H}(\delta), \tilde{T}_f\}$ of $L(\delta)$ is respectively algebraically, completely, or topologically irreducible too.*

§2. Maximal ideals in $L(G)$ and topologically irreducible representations with the property (*)

Let A be an associative algebra, and $\mathfrak{m}$ a left ideal in A. $\mathfrak{m}$ is called "regular" if there exists an element $u\in A$ such that $xu\equiv x (\mathrm{mod.}\ \mathfrak{m})$ for all $x\in A$. Similar definitions apply to right ideals and two-sided ideals.

Here we consider the associative algebra over $\mathbf{C}$ for a fixed δ such that

(a) the product of every element $f \in A$ and $\bar{\chi}_\delta$ is defined, and $f\,\bar{\chi}_\delta$, $\bar{\chi}_\delta f \in A$,

(b) $\bar{\chi}_\delta(f\bar{\chi}_\delta) = (\bar{\chi}_\delta f)\bar{\chi}_\delta$ for all $f \in A$,

(c) $(f\bar{\chi}_\delta)g = f(\bar{\chi}_\delta g)$ for all $f, g \in A$,

(d) $\bar{\chi}_\delta(\bar{\chi}_\delta f) = \bar{\chi}_\delta f$, $(f\bar{\chi}_\delta)\bar{\chi}_\delta = f\bar{\chi}_\delta$ for all $f \in A$.

Now we can prove the following lemma as in [5].

Lemma 3. *Let* $\mathfrak{a}$ *be a regular maximal left ideal in the subalgebra* $A(\delta) = \{f \in A;\ \bar{\chi}_\delta f = f\bar{\chi}_\delta = f\}$ *of* A, *and put*

$$\mathfrak{m} = \{f \in A;\ \bar{\chi}_\delta g f \bar{\chi}_\delta \in \mathfrak{a},\ \textit{for all}\ \ g \in A\},$$

then $\mathfrak{m}$ *is a regular maximal left ideal in* A, $\mathfrak{a} = \mathfrak{m} \cap A(\delta)$, *and we have* $f\bar{\chi}_\delta \equiv f(\text{mod. } \mathfrak{m})$ *for all* $f \in A$.

When a representation $\{\mathfrak{H}, T_x\}$ of G is given, for an arbitrarily chosen non zero element $a \in \mathfrak{H}$, $\mathfrak{m}_a = \{f \in L(G);\ T_f a = 0\}$ is a closed left ideal in $L(G)$. But in general we don't know whether it is maximal or not.

Theorem 1. *Let* G *be a locally compact unimodular group,* K *a compact subgroup of* G, $\{\mathfrak{H}, T_x\}$ *a topologically irreducible representation of* G, *and* δ *an irreducible representation of* K. *If we have* $0 < \dim \mathfrak{H}(\delta) < +\infty$, $\mathfrak{m}_a = \{f \in L(G);\ T_f a = 0\}$ *is a closed regular maximal left ideal in* $L(G)$ *for an arbitrary non zero element* $a \in \mathfrak{H}(\delta)$.

Proof. From Lemma 2 and the Burnside's theorem, there exists an element $u \in L(\delta)$ such that $\tilde{T}_u = 1$. As is easily seen, $\mathfrak{a} = \mathfrak{m}_a \cap L(\delta)$ is a closed regular maximal left ideal in $L(\delta)$ with the right identity u.

Next, $\mathfrak{m} = \{f \in L(G);\ \bar{\chi}_\delta * g * f * \bar{\chi}_\delta \in \mathfrak{a}$ for all $g \in L(G)\}$ is a regular maximal left ideal in $L(G)$ from Lemma 3, so we have only to prove

$\mathfrak{m} \subset \mathfrak{m}_a$. If f is in $\mathfrak{m}$, we have

$$E(\delta) T_g T_f a = T_h a = \tilde{T}_h a = 0, \quad h = \bar{\chi}_\delta * g * f * \bar{\chi}_\delta \in \mathfrak{a},$$

for all $g \in L(G)$. Now, $\{T_f a; f \in \mathfrak{m}\}$ is an invariant subspace of $\mathfrak{H}$, hence $\{T_f a; f \in \mathfrak{m}\} = \{0\}$ since $\mathfrak{H}(\delta) \neq \{0\}$. Namely f is in $\mathfrak{m}_a$.

q.e.d.

We shall consider a topologically irreducible representation $\{H, T_x\}$ of G with the following property:

$$(*) \begin{cases} \text{there exists at least one pair } (K', \delta') \text{ of a compact subgroup} \\ K' \text{ of } G \text{ and its irreducible representation } \delta' \text{ such that } 0 < \\ \dim \mathfrak{H}(\delta') < +\infty. \end{cases}$$

For an arbitrary non zero element $a' \in \mathfrak{H}(\delta')$, we define

$$\mathfrak{H}_0[K', \delta', a'] = \{T_f a'; f \in L(G)\}.$$

Lemma 4. *The space $\mathfrak{H}_0[K', \delta', a']$ is independent of K', δ' such that $0 < \dim \mathfrak{H}(\delta') < +\infty$ and $a' \in \mathfrak{H}(\delta')$ (denote it by $\mathfrak{H}_0$). The representation $\{\mathfrak{H}_0, T_f\}$ of $L(G)$ is algebraically irreducible.*

Proof. Let (K'', δ'') be another pair of compact subgroup K'' of G and its irreducible representation δ'' such that $0 < \dim \mathfrak{H}(\delta'') < +\infty$. Let $a'' \in \mathfrak{H}(\delta'')$. By the topological irreducibility of $\{\mathfrak{H}, T_x\}$, both $\mathfrak{H}_0[K', \delta', a']$ and $\mathfrak{H}_0[K'', \delta'', a'']$ are dense in $\mathfrak{H}$. If we denote by $E(\delta')$ the projection with respect to (K', δ'), we have $E(\delta')\, \mathfrak{H}_0[K'', \delta'', a''] \subset \mathfrak{H}_0[K'', \delta'', a'']$. On the other hand, $E(\delta')\, \mathfrak{H}_0[K'', \delta'', a''] = \mathfrak{H}(\delta')$. Hence we have $\mathfrak{H}_0[K', \delta', a'] \subset \mathfrak{H}_0[K'', \delta'', a'']$.

Next, $\mathfrak{m}_a = \{f \in L(G);\ T_f a = 0\}$ is a closed regular maximal left ideal in $L(G)$ by Theorem 1. By the closedness of $\mathfrak{m}_a$, we have $L(G) * h \not\subset \mathfrak{m}_a$ for an arbitrary non zero element $h \in L(G)$ (we have only to consider the "delta-sequence" attached to e; i.e., for every neighbourhood U of the unit e in G, we take a non negative function $e_U \in L(G)$ such that $\int_G e_U(x)\,dx = 1$ and $\operatorname{supp}[e_U] \subset U$), and hence $L(G) * h + \mathfrak{m}_a$

$=L(G)$. From this fact, we see easily $\{T_g(T_h a);\ g\in L(G)\}=\mathfrak{H}$. This implies the algebraic irreducibility of $\{\mathfrak{H}_0,\ T_f\}$. q.e.d.

Let (K, δ) be a pair which satisfies $0<\dim\mathfrak{H}(\delta)<+\infty$, and we put $\mathfrak{H}_0=H_0[K, \delta, a]$. Let T be a continuous linear operator on $\mathfrak{H}_0$ which commutes with all T_f on $\mathfrak{H}_0$. From the fact that there exists $u\in L(\delta)$ such that $\tilde{T}_u=T_u|_{\mathfrak{H}(\delta)}=1$, and that $\mathfrak{H}(\delta)\subset\mathfrak{H}_0$, we have

$$T_x Tb=T_x T(T_u b)=T_x T_u Tb=T_{L_x u}Tb=TT_{L_x}b=TT_x b,$$

for an arbitrary $b\in\mathfrak{H}(\delta)$, where $(L_x u)(y)=u(x^{-1}y)$. Noting the fact that $T_k b\in\mathfrak{H}(\delta)$ for every $k\in K$, we have

$$E(\delta)(Tb)=\int_K T_k(Tb)\bar{\chi}_\delta(k)dk=\int_K T(T_k b)\bar{\chi}_\delta(k)dk$$

$$=T\int_K T_k b\bar{\chi}_\delta(k)dk=T(E(\delta)b)=Tb,$$

i.e., $Tb\in\mathfrak{H}(\delta)$. Thus $\tilde{T}_f Tb=T\tilde{T}_f b$ is valid for all $f\in L(\delta)$, i.e., T commutes with every $\tilde{T}_f$ on $\mathfrak{H}(\delta)$. Hence T is a scalar operator on $\mathfrak{H}(\delta)$. Therefore the operator T is a scalar operator on $\mathfrak{H}_0$ too. By Lemma 1, $\{\mathfrak{H}_0,\ T_f\}$ is a completely irreducible representation of $L(G)$. Obviously the space $\mathfrak{H}_0$ is invariant under all T_x, so we can consider the "representation" $\{\mathfrak{H}_0,\ T_x\}$ of G. Of course $\mathfrak{H}_0$ is not complete in general. Clearly we have the next lemma.

Lemma 5. *The algebraically irreducible representation of G with the property $(*)$ is completely irreducible.*

§3. The multiplicities of irreducible representations of a compact subgroup

Throughout this section, G is a locally compact unimodular group, and K is a compact subgroup of G. For an arbitrary irreducible representation δ of K, we shall say that δ is contained p times in the representation $\{\mathfrak{H}, T_x\}$ of G if $\dim\mathfrak{H}(\delta)=p\cdot\dim\delta$.

Lemma 6. (See [5, p. 503, Lemma 1]) *If an associative algebra A over* **C** *has sufficiently many representations whose dimensions are not greater than n, the dimension of every completely irreducible representation of A is also not greater than n. (Here, the representation space is not assumed apriori to be complete).*

Let Ω be a set of representations of G. We shall say after R. Godement that Ω is "complete", if for every $f \in L(G)$ we can chose some representation $\{\mathfrak{H}, T_x\} \in \Omega$ such that $T_f \neq 0$.

Lemma 7. *Let Ω be a complete set of representations of G. If the irreducible representation δ of K is contained at most p times in every representation in Ω, we have*

(i) *δ is contained at most p times in every completely irreducible representation of G,*

(ii) *δ is contained at most p times in every topologically irreducible representation of G with the property* $(*)$,

(iii) *δ is contained at most p times or infinitely many times in every topologically irreducible representation of G. In the latter case, every irreducible representation of every compact subgroup of G is contained either no times or infinitely many times.*

Proof. For every representation $\{\mathfrak{H}', T'_x\}$ in Ω, we make the representation $\{\mathfrak{H}'(\delta), \tilde{T}'_f\}$ of $L(\delta)$. All such representations make a family containing sufficiently many representations of $L(\delta)$, and always $\dim \mathfrak{H}'(\delta) \leqq p \cdot \dim \delta$ by the assumption. So, by Lemma 6, we know that every completely irreducible representation of $L(\delta)$ has dimension $\leqq p \cdot \dim \delta$. Now (i) is clear by Lemma 2.

Let's prove (ii). Let $\{\mathfrak{H}, T_x\}$ be a topologically irreducible representation of G with the property $(*)$. If $\mathfrak{H}_0 = \mathfrak{H}_0[K', \delta', a']$, $\{\mathfrak{H}_0, T_f\}$ is a completely irreducible representation of $L(G)$. Repeating the proof of Lemma 2, we can easily see that $\{E(\delta)\mathfrak{H}_0, \tilde{T}_f\}$ is a completely irreducible representation of $L(\delta)$. Hence $\dim E(\delta)\mathfrak{H}_0 \leqq p \cdot \dim \delta$. But

$E(\delta)\mathfrak{H}_0$ is dense in $\mathfrak{H}(\delta)$. Therefore $\dim \mathfrak{H}(\delta) \leqq p \cdot \dim \delta$.

(iii) is clear from (i) and (ii). q.e.d.

Lemma 8. *Let G_0 be the intersection of the kernels of all finite-dimensional representations of G. Then the set of all finite-dimensional representations of G is complete if and only if $G_0 = \{e\}$.*

Proof. Assume $G_0 \neq \{e\}$. Then of course G_0 contains a non trivial closed abelian subgroup, say Z. If $z_0 \in Z$ is not the unit e, there exists a neighbourhood U of e in G such that

$$Uz_0 \cap U = \emptyset .$$

Now we chose a non zero function $\varphi \in L(G)$ such that $\text{supp}[\varphi] \subset U$, and define

$$f(x) = \begin{cases} \varphi(x) & \text{for } x = U, \\ -\varphi(xz_0) & \text{for } x \in Uz_0, \\ 0 & \text{otherwise,} \end{cases}$$

then $f \in L(G)$, $f \not\equiv 0$, and we can see

$$\int_Z f(xz)dz = 0 \qquad \text{for } x \in G.$$

From this, we have

$$\int_G \theta(x) f(x) dx = 0$$

for an arbitrary matrix element $\theta(x)$ of every finite-dimensional representation of G.

The converse is proved in [5, p. 506, Lemma 5]. q.e.d.

Lemma 9. *Let G be a connected semi-simple Lie group. The set of all finite-dimensional irreducible representations of G is complete if* *and only if G has a finite-dimensional faithful representation.*

Proof. Using the notation in Lemma 8, $G_0=\{e\}$ is equivalent to the fact that G has a finite-dimensional faithful representation [6]. On the other hand, every finite-dimensional representation of G is completely reducible. Therefore this lemma follows from Lemma 8.

q.e.d.

Using this lemma and Lemma 7, we obtain the following

Theorem 2. *Let G be a connected semi-simple Lie group with a finite-dimensional faithful representation, K a compact subgroup of G, and δ an irreducible representation of K. If δ is contained at most p times in every finite-dimensional irreducible representation of G, δ is contained at most p times in every completely irreducible representation of G and in every topologically irreducible representation of G with the property* $(*)$.

Let N be a closed subgroup of G, and $n\rightarrow\alpha(n)$ a one-dimensional representation of N. Let $\mathfrak{H}^{\alpha}$ be the set of all continuous functions f on G such that $f(nx)=\alpha(n)f(x)$ for $n\in N$, and T_x^{α} the operator on $\mathfrak{H}^{\alpha}$ such that $(T_x^{\alpha}f)(x')=f(x'x)$, and denote by Ω_N the set of all such representations. If $G=NK$ (not necessarily semi-direct), where K is a compact subgroup of G, we may consider $\mathfrak{H}^{\alpha}\subset L(K)$, and in this case every irreducible representation δ of K is contained at most $\dim\delta$ times in every representation in Ω_N.

If the closed subgroup N is connected and solvable, every finite-dimensional irreducible representation of N is one-dimensional (Lie's theorem, [5, p. 549]). Now we have the next lemma.

Lemma 10. (R, Godement [5]) *Let G be a locally compact unimodular group, and N a connected solvable subgroup of G. Then every finite-dimensional irreducible representation of G is contained in some representation in Ω_N.*

Let G be a connected semi-simple Lie group with a finite-dimensional faithful representation. It is well-known that there exists a connected solvable subgroup N such that $G=NK$, $N\cap K=\{e\}$, where K is a maximal compact subgroup of G. Therefore we know that Ω_N is complete from Lemmas 9 and 10, and also we immediately obtain the following theorem from Lemma 7.

Theorem 3. *Let G be a connected semi-simple Lie group with a faithful representation, K a maximal compact subgroup of G, and δ an irreducible representation of K. Then δ is contained at most $\dim \delta$ times in every completely irreducible representation of G and in every topologically irreducible representation of G with the property* (*).

Theorem 4. *Let G be a connected complex semi-simple Lie group, K a maximal compact subgroup of G, δ an irreducible representation of K, and Γ a maximal abelian subgroup of K. Then the multiplicity of δ in any completely irreducible representation of G and that of δ in any topologically irreducible representation of G with the property* (*) *are not greater than the maximum of the multiplicities of irreducible representations of Γ in δ.*

The proof of this theorem is essentially the same as for Theorem 3 in [5, p. 509].

Theorem 5. *Let G be a locally compact unimodular group, and K a compact subgroup of G. If there exists some abelian subgroup N such that $G=NK$, every irreducible representation δ of K is contained at most $\dim\delta$ times in every completely irreducible representation of G and in every topologically irreducible representation of G with the property* (*).

We have only to show that Ω_N is complete, but this is done in [5].

Lemma 11. *The set of all completely continuous linear operators on $\mathfrak{H}$ is closed in $L_b(\mathfrak{H}, \mathfrak{H})$.*

Proof. Let T_0 be an element of the closure of the above set. Assume the subset B of $\mathfrak{H}$ is bounded. For any given neighbourhood U of zero in $\mathfrak{H}$, there exists a neighbourhood U_1 of zero in $\mathfrak{H}$ such that $U_1 + U_1 \subset U$. Moreover, there exists a completely continuous linear operator T such that

$$T_0 b - Tb \in U_1 \qquad \text{for every } b \in B.$$

On the other hand, there exist some elements $a_1, a_2, \cdots, a_n \in \mathfrak{H}$ such that

$$Tb \in \bigcup_{i=1}^{n} (U_1 + a_i).$$

Thus $T_0 B \subset \bigcup_{i=1}^{n} (U + a_i)$. q.e.d.

Using this lemma, we can show the following theorem in the same way as [5, p. 515, Theorem 7].

Theorem 6. *Let G be a locally compact unimodular group, and K a compact subgroup of G. If every irreducible representation δ of K is contained at most finite times in the given representation $\{\mathfrak{H}, T_x\}$ of G, T_f is completely continuous for any $f \in L(G)$.*

§ 4. Spherical functions

The notations are same as in the preceding sections. Let $\{\mathfrak{H}, T_x\}$ be a topologically irreducible representation of G, and let $0 < \dim \mathfrak{H}(\delta) < +\infty$. Then we define

$$\phi_\delta(x) = \mathrm{Tr}[E(\delta) T_x] \qquad \text{for } x \in G.$$

If $\dim \mathfrak{H}(\delta) = p \cdot \dim \delta$, we call ϕ_δ the spherical function of type δ of height p. R. Godement treated the spherical functions only for the completely irreducible representations on Banach spaces.

We denote by $L^{\circ}(G)$ the space of all functions

$$f^{\circ}(x)=\int_K f(kxk^{-1})dk \qquad \text{for } f\in L(G),$$

and by $L^{\circ}(\delta)$ the space of all functions f° such that $f\in L(\delta)$. It is clear that $L^{\circ}(\delta)=\bar{\chi}_{\delta}*L^{\circ}(G)*\bar{\chi}_{\delta}$. From now on, we shall use the notation $f'(x)=f(x^{-1})$.

By the direct calculations, we have the following functional equations;

(1) $\phi_{\delta}=\phi_{\delta}^{\circ}$,

(2) $\chi_{\delta}*\phi_{\delta}=\phi_{\delta}*\chi_{\delta}=\phi_{\delta}$,

(3) $\int_K \phi_{\delta}(xky)\bar{\chi}_{\delta}(k)dk=\int_K \phi_{\delta}(ykx)\bar{\chi}_{\delta}(k)dk \qquad \text{for } x,\ y\in G$,

(4) $\int_K \phi_{\delta}(kxk^{-1}y)dk=\int_K \phi_{\delta}(kyk^{-1}x)dk \qquad \text{for } x,\ y\in G$,

(5) $f*\phi_{\delta}=\phi_{\delta}*f \qquad \text{for } f\in L^{\circ}(G)$.

Let ϕ be a continuous function on G such that $\phi^{\circ}=\phi$ and $\chi_{\delta}*\phi=\phi$. Are the following two statements on ϕ equivalent or not?

(I) ϕ is a spherical function of type δ of height p,

(II) $\dim(L(\delta)/\mathfrak{p})<+\infty$, where $\mathfrak{p}=\{f\in L(\delta);\, f'*\phi=0\}$,

and there exists a p-dimensional irreducible representation $f\to U_f$ of $L^{\circ}(\delta)$ such that $\phi(f)=(\dim\delta)\mathrm{Tr}[U_f]$ where

$$\phi(f)=\int_G \phi(x)f(x)dx.$$

The (I)$\Rightarrow$(II) part is proved in general. The (II)$\Rightarrow$(I) part is, in general, solved in somewhat a weaker form (cf. Proposition 1), but, for a σ-compact G, it is completely solved and moreover we know that every spherical function is obtained from topologically irreducible representation on Fréchet space. And for positive-definite ϕ, it is also completely solved and we know that every such spherical function is obtained from irreducible unitary representation of G. These are the main results in this paper.

First, we show (I)$\Rightarrow$(II). Let $\{\mathfrak{H}, T_x\}$ be a topologically irreducible representation of G such that $0<\dim\mathfrak{H}(\delta)<+\infty$ and $\phi=\phi_\delta$.

Lemma 12. *Considering only on* $\mathfrak{H}(\delta)$, *the set of the linear operators which commute with all* T_k *is*

$$\{\tilde{T}_f = T_f|_{\mathfrak{H}(\delta)};\ f\in L^\circ(\delta)\}.$$

Proof. For every linear operator A on $\mathfrak{H}(\delta)$, there exists at least one function $f\in L(\delta)$ such that $\tilde{T}_f=A$ by the Burnside's theorem. If A commutes with all T_k, we have

$$A=\int_K T_k\tilde{T}_f T_k^{-1}dk=\int_K\int_G T_k T_x T_k^{-1}f(x)dx\,dk=\tilde{T}_{f^\circ}.$$

The converse is clear. q.e.d.

Since the representation $\{\mathfrak{H}(\delta), T_k\}$ of K is equivalent to the p-times direct sum of δ, the set $\{\tilde{T}_f;\ f\in L^\circ(\delta)\}$ is identified with $\mathfrak{M}(p)$ by Lemma 12, where $\mathfrak{M}(p)$ is the set of all $p\times p$-complex matrices. And also we may write

$$\tilde{T}_f=U_f\otimes I \qquad (U_f\in\mathfrak{M}(p)),$$

where I is the unit matrix of degree $\dim\delta$. This representation $f\to U_f$ of $L^\circ(\delta)$ satisfies the condition in (II). The first part of (II) follows at once from

Lemma 13. $\mathfrak{p}$ *equals to* $\{f\in L(\delta);\ \tilde{T}_f=0\}$.

Now let's consider (II)$\Rightarrow$(I). We need some lemmas which are essentially same as those in [5].

Lemma 14. *The set* $\{L_k f;\ k\in K, f\in L^\circ(\delta)\}$ *is total in* $L(\delta)$.

Lemma 15. *Let ϕ be a function on G such that $\phi=\phi^{\circ}$, $\chi_{\delta}*\phi=\phi$. If a finite-dimensional irreducible representation $f\rightarrow U_{f}$ of $L^{\circ}(\delta)$ satisfies the relation $\phi(f)=(\dim\delta)\mathrm{Tr}[U_{f}]$, we have*

(a) $f'*\phi=\phi*f'$ *for every* $f\in L(\delta)$,

(b) $\mathfrak{p}=\{f\in L(\delta);\ f'*\phi=0\}$ *is a closed regular two-sided ideal in* $L(\delta)$,

(c) $L^{\circ}(\delta)\cap\mathfrak{p}=\{f\in L^{\circ}(\delta);\ U_{f}=0\}$.

The proofs of these two lemmas are essentially same as those of Lemmas 11, 12, and 13 in [5].

Of course we were very happy if we could show (II)$\Rightarrow$(I) without any more assumptions. But the author cannot do it.

A. General case

At first, we consider without any more assumptions. Let's denote by $\mathfrak{a}$ the maximal left ideal in $L(\delta)$ containing $\mathfrak{p}$. $\mathfrak{a}$ is regular and closed since $\dim(L(\delta)/\mathfrak{p})<+\infty$. The right identity modulo $\mathfrak{a}$ and the identity modulo $\mathfrak{p}$ is the function $u\in L^{\circ}(\delta)$ such that $U_{u}=1$. If we set

$$\mathfrak{m}=\{f\in L(G);\ \bar{\chi}_{\delta}*g*f*\bar{\chi}_{\delta}\in\mathfrak{a}\qquad \text{for every } g\in L(G)\},$$

$\mathfrak{m}$ is a closed regular maximal left ideal in $L(G)$ by Lemma 3, and the right identity modulo $\mathfrak{m}$ is also u. Considering the delta-sequence e_{xU} attached to x, we easily see that $L_{x}\mathfrak{m}\subset\mathfrak{m}$. Thus we can consider the "representation"

$$\{\mathfrak{H},\ L_{x}\},\ \text{where } \mathfrak{H}=L(G)/\mathfrak{m},$$

of G. We induce the quotient topology in $\mathfrak{H}$ from $L(G)$, and therefore $\mathfrak{H}$ is locally convex and Hausdorff, but we don't know whether $\mathfrak{H}$ is complete or not. Even if $\mathfrak{H}$ is not complete, we can define the representation $\{\mathfrak{H},\ L_{f}\}$ of $L(G)$ by integration, and we have $L_{f}=f*$. Moreover, since $\mathfrak{m}$ is invariant under the operation $\bar{\chi}_{\delta}*$, we can also define the projection

$$E(\delta)=\int_K L_k\bar{\chi}_\delta(k)dk,$$

and we have $E(\delta)\{f\}=\{\bar{\chi}_\delta * f\}$ for every $\{f\}\in\mathfrak{H}$, where $\{f\}=f+\mathfrak{m}$. Since $\mathfrak{m}$ is a maximal left ideal, the representation $\{\mathfrak{H}, L_f\}$ of $L(G)$ is algebraically irreducible.

From Lemma 3, we also have $f*\bar{\chi}_\delta - f\in\mathfrak{m}$ for every $f\in L(G)$. Thus $E(\delta)\{f\}=\{\bar{\chi}_\delta * f * \bar{\chi}_\delta\}$, and therefore we obtain that

$$\mathfrak{H}(\delta)=\{\{f\}\,;\, f\in L(\delta)\}.$$

Moreover we have $\mathfrak{a}=\mathfrak{m}\cap L(\delta)$ from Lemma 3. Thus

$$\{\{f\}\,;\, f\in L(\delta)\}\cong L(\delta)/\mathfrak{a}.$$

Since $\mathfrak{p}\subset\mathfrak{a}$, it follows from our assumption $\dim(L(\delta)/\mathfrak{p})>+\infty$ that $\dim\mathfrak{H}(\delta)<+\infty$. Therefore the finite-dimensional irreducible representation $\{\mathfrak{H}(\delta), \tilde{L}_f=L_f|_{\mathfrak{H}(\delta)}\}$ of $L(\delta)$ is equivalent to $\{L(\delta)/\mathfrak{a}, L_f\}$. Now, if we set $\phi_\delta(x)=\mathrm{Tr}[E(\delta)L_x]$, we have

$$\phi_\delta(f)=\int_G f(x)\,\mathrm{Tr}[E(\delta)L_x]\,dx=\mathrm{Tr}[\tilde{L}_f]=\mathrm{Tr}[L_f],$$

for every $f\in L(\delta)$. From Lemma 15 (c), we can see that $U_f=0$ implies $L_f=0$, if f is in $L^\circ(\delta)$. Let $\dim\mathfrak{H}(\delta)=\dim(L(\delta)/\mathfrak{a})$ be $q\cdot\dim\delta$, then in the proof of (I)$\Rightarrow$(II), we construct a q-dimensional irreducible representation $f\to V_f$ of $L^\circ(\delta)$ such that $V_f=0$ is equivalent to $\tilde{L}_f=0$ or $L_f=0$, and that $\phi_\delta(f)=(\dim\delta)\mathrm{Tr}[V_f]$ for every $f\in L^\circ(\delta)$. Therefore $U_f=0$ implies $V_f=0$. Thus the mapping $\psi\colon U_f\to V_f$ is well-defined and ψ is a homomorphism from the algebra $\mathfrak{M}(p)$ onto $\mathfrak{M}(q)$. From this, we have $p=q$ and ψ must be equivalent to the identity 1 [8, p. 429]. Consequently U_f is equivalent to V_f and $\phi(f)=\phi_\delta(f)$ for every $f\in L^\circ(\delta)$. This implies $\phi=\phi_\delta$.

Consider a topologically irreducible representation $\{\mathfrak{H}, T_x\}$ of G, where the space $\mathfrak{H}$ is not necessarily complete. If the integrals

$$T_f=\int_G T_x f(x)dx \qquad (f\in L(G)),\quad E(\delta)=\int_K T_k\bar{\chi}_\delta(k)dk$$

converge in $\mathfrak{H}$ and $\mathfrak{H}(\delta)=E(\delta)\mathfrak{H}$ is of finite-dimension, we set

$$\phi_\delta(x)=\mathrm{Tr}[E(\delta)T_x]$$

as before. And we shall call this "the spherical function in the generalized sense". Then the above consideration shows

Proposition 1. *Let G be a locally compact unimodular group, K a compact subgroup of G, and δ an irreducible representation of K. If the continuous function ϕ on G satisfies $\phi=\phi^\circ$ and $\chi_\delta*\phi=\phi$, the following two statements are equivalent;*

(I′) *ϕ is a spherical function in the generalized sence of type δ height p,*

(II) *$dim(L(\delta)/\mathfrak{p})<+\infty$, where $\mathfrak{p}=\{f\in L(\delta); f'*\phi=0\}$, and there exists a p-dimensional irreducible representation $f\to U_f$ of $L^\circ(\delta)$ such that $\phi(f)=(\dim\delta)\mathrm{Tr}[U_f]$.*

B. Case of σ-compact G

Assume that G is σ-compact, i.e., G is the union of the denombrable compact subsets K_n of G: $G=\cap K_n$. We may assume that $K_1\subset K_2\subset\cdots\subset K_n\subset\cdots$, and that every compact subset of G is contained in some K_n. Then,

$$\|f\|_n=\sup_{x\in K_n}\int_G|\phi(xy)|\cdot|f(y)|dy \qquad \text{for } n=1, 2, \cdots,$$

are semi-norms on $L(G)$, and $\|f\|_n=0$ for every n is equivalent to $f=0$. We shall denote by $\mathfrak{L}(G)$ the Fréchet space which is the completion of $L(G)$ by these semi-norms.

Lemma 16. *The linear operators L_x, $f*$, $\bar{\chi}_\delta*$, $*\bar{\chi}_\delta$, defined on $L(G)$ are continuous with respect to the topology in $\mathfrak{L}(G)$. Moreover $\{L_x; x\in C\}$ is equi-continuous for every compact subset C of G.*

Proof. By a simple calculation, we have

$$\|L_x f\|_n \leqq \|f\|_m \qquad (x \in C),$$

where m is an arbitrary integer such that $K_n C \subset K_m$. For $f*$, we have the inequality

$$\|f*h\|_n \leqq \|f\|_{L^1} \|h\|_m \qquad \text{for } h \in L(G),$$

where m is an arbitrary integer such that $K_n \cdot \mathrm{supp}[f] \subset K_m$. The same is true for $\bar{\chi}_\delta *$ and $* \bar{\chi}_\delta$. q.e.d.

By this lemma, we can extend L_x, $f*$, $\bar{\chi}_\delta *$, $* \bar{\chi}_\delta$ on the whole of $\mathfrak{L}(G)$ by continuity. Let's denote them by the same notations respectively. If we denote by $\mathfrak{L}(\delta)$ the completion of $L(\delta)$ in $\mathfrak{L}(G)$, we have $\bar{\chi}_\delta * \mathfrak{L}(G) * \bar{\chi}_\delta = \mathfrak{L}(\delta)$.

Lemma 17. *The linear operator $\phi(f)$ on $L(G)$ is continuous with respect to the topology in $\mathfrak{L}(G)$.*

This is clear.

Lemma 18. *Let $\hat{\mathfrak{p}}$ be the completion of $\mathfrak{p}$ in $\mathfrak{L}(G)$. Then,* (a) $\hat{\mathfrak{p}} \subsetneqq \mathfrak{L}(\delta)$, (b) *$\hat{\mathfrak{p}}$ is $L(\delta)$-invariant, i.e., $f*\hat{\mathfrak{p}} \subset \hat{\mathfrak{p}}$ for every $f \in L(\delta)$,* (c) *$u \notin \hat{\mathfrak{p}}$, where u is a function in $L^\circ(\delta)$ such that $U_u = 1$.*

Proof. (a) If $f \in \mathfrak{p}$, we have $\phi(f) = f' * \phi(e) = 0$. Thus $\phi = 0$ if $\mathfrak{p}$ is dense in $\mathfrak{L}(\delta)$. (b) Easy. (c) Since $f*u - f \in \mathfrak{p} \subset \hat{\mathfrak{p}}$ for every $f \in L(\delta)$, we have $f \in \hat{\mathfrak{p}}$ if $u \in \hat{\mathfrak{p}}$. This means $\hat{\mathfrak{p}} = \mathfrak{L}(\delta)$, and hence contradicts to (a). q.e.d.

As is easily seen, $\hat{\mathfrak{p}} \cap L(\delta) = \mathfrak{p}$. Therefore it can be considered that $L(\delta)/\mathfrak{p}$ is densely contained in $\mathfrak{L}(\delta)/\hat{\mathfrak{p}}$. Thus by the assumption $\dim(L(\delta)/\mathfrak{p}) < +\infty$, we have

$$\dim(\mathfrak{L}(\delta)/\hat{\mathfrak{p}}) < +\infty.$$

Therefore we can find a closed maximal $L(\delta)$-invariant subspace $\mathfrak{a}_0$ of $\mathfrak{L}(\delta)$ containing $\hat{\mathfrak{p}}$. $\mathfrak{a}_0$ does not contain u.

Lemma 19. *Set* $\mathfrak{m}_0=\{f\in\mathfrak{L}(G);\ \bar{\chi}_\delta*g*f*\bar{\chi}_\delta\in\mathfrak{a}_0$ *for every* $g\in L(G)\}$, *then*

(a) $\mathfrak{m}_0\not\ni u$, $\mathfrak{a}_0=\mathfrak{m}_0\cap\mathfrak{L}(\delta)$, *and* $f*u-f\in\mathfrak{m}_0$ *for every* $f\in L(G)$,

(b) $\mathfrak{m}_0$ *is closed and is maximal in the set of all closed* $L(G)$-*invariant subspaces of* $\mathfrak{L}(G)$.

Proof. (a) If $u\in\mathfrak{m}_0$, we have $\bar{\chi}_\delta*g*f*\bar{\chi}_\delta\in\mathfrak{a}_0$ for every $g\in L(G)$, and also $(\bar{\chi}_\delta*g*\bar{\chi}_\delta)*u\in\mathfrak{a}_0$, since $u\in L^\circ(\delta)$. Thus $L(\delta)\subset\mathfrak{a}_0$ and the completion $\mathfrak{L}(\delta)$ of $L(\delta)$ is contained in $\mathfrak{a}_0$. This is a contradiction, and hence $u\notin\mathfrak{m}_0$. Next,

$$\bar{\chi}_\delta*g*(f*u-f)*\bar{\chi}_\delta=(\bar{\chi}_\delta*g*f*\bar{\chi}_\delta)*u-(\bar{\chi}_\delta*g*f*\bar{\chi}_\delta)\in\mathfrak{a}_0$$

for every $g\in L(G)$ since $\mathfrak{p}\subset\mathfrak{a}_0$. Therefore $f*u-f\in\mathfrak{m}_0$. $\mathfrak{a}_0=\mathfrak{m}_0\cap\mathfrak{L}(\delta)$ is a consequence of the maximality of $\mathfrak{a}_0$.

(b) We have only to show the maximality of $\mathfrak{m}_0$. Let $\mathfrak{n}$ be a closed $L(G)$-invariant subspace of $\mathfrak{L}(G)$ such that $\mathfrak{m}_0\subset\mathfrak{n}\subsetneqq\mathfrak{L}(G)$. Since $L(G)\not\subset\mathfrak{n}$, u must not be contained in $\mathfrak{n}$. Therefore $\mathfrak{a}_0=\mathfrak{n}\cap\mathfrak{L}(\delta)$, from the maximality of $\mathfrak{a}_0$. For arbitrary $f\in\mathfrak{n}$, $g\in L(G)$,

$$\bar{\chi}_\delta*g*f*\bar{\chi}_\delta-\bar{\chi}_\delta*g*f=(\bar{\chi}_\delta*g)*(f*\bar{\chi}_\delta-f)\in\mathfrak{m}_0\subset\mathfrak{n},$$

since $f*\bar{\chi}_\delta-f\in\mathfrak{m}_0$. As $\bar{\chi}_\delta*g*f\in\mathfrak{n}$, we have $\bar{\chi}_\delta*g*f*\bar{\chi}_\delta\in\mathfrak{n}\cap\mathfrak{L}(\delta)=\mathfrak{a}_0$. Hence $f\in\mathfrak{m}_0$. This proves that $\mathfrak{n}\subset\mathfrak{m}_0$. q.e.d.

Take the delta-sequence e_{xU} attached to x, then $e_{xU}*f$ converges uniformly to L_xf on G when $U\to e$ for every $f\in L(G)$. Thus, for $f\in L(G)$, $e_{xU}*f$ also converges to L_xf in $\mathfrak{L}(G)$.

Lemma 20. *Let x be an arbitrary element of G. When $U\to e$, $e_{xU}*f$ converges to L_xf in $\mathfrak{L}(G)$ for every $f\in\mathfrak{L}(G)$. Therefore $\mathfrak{m}_0$ is invariant under the operation L_x.*

Proof. For $f\in\mathfrak{L}(G)$, there exists a sequence $f_i\in L(G)$ such that $f_i\rightarrow f$ in $\mathfrak{L}(G)$. In the inequality

$$||e_{xU}*f-L_xf||_n\leqq||e_{xU}*(f-f_i)||_n+||e_{xU}*f_i-L_xf_i||_n$$
$$+||L_xf_i-L_xf||_n,$$

we may assume that xU is always contained in a fixed compact subset C of G. Take i large enough, and then let $U\rightarrow e$, we see that the left hand side of the inequality becomes small enough. q.e.d.

To see $\{\mathfrak{L}(G), L_x\}$ is a representation of G, it rests only to show that $x\rightarrow L_xf$ is continuous for every $f\in\mathfrak{L}(G)$, since the space $\mathfrak{L}(G)$ is a Fréchet space. Take a sequence f_i in $L(G)$ which converges to f in $\mathfrak{L}(G)$. We have

$$||L_xf-f||_n\leqq||L_x(f-f_i)||_n+||L_xf_i-f_i||_n+||f_i-f||_n,$$

where we may assume that x belongs to a fixed compact neighbourhood C of e in G. If m is an integer such that $C\subset K_m$,

$$||L_x(f-f_i)||_n\leqq||f-f_i||_m.$$

Take i large enough at first, and let $x\rightarrow e$, then we see $||L_xf-f||_n\rightarrow 0$. Therefore $\{\mathfrak{L}(G), L_x\}$ is a representation of G.

The representation $\{\mathfrak{L}(G), L_f\}$ of $L(G)$ which corresponds to $\{\mathfrak{L}(G), L_x\}$ is given by $L_f=f*$. This can be seen by the direct calculation.

The representation $\{\mathfrak{L}(G), L_x\}$ of G is not irreducible in general. Now we set $\mathfrak{H}=\mathfrak{L}(G)/\mathfrak{m}_0$, and denote by T_x, T_f the continuous linear operators on $\mathfrak{H}$ induced by the natural way from L_x, L_f respectively. Then $\mathfrak{H}$ is a Fréchet space [2, p. 57], and the representation $\{\mathfrak{H}, T_x\}$ of G is topologically irreducible (Lemma 19 (b)). And as in general case A, $E(\delta)\{f\}=\{\bar{\chi}_\delta*f*\bar{\chi}_\delta\}$ for every $\{f\}\in\mathfrak{H}$. Therefore

$$\mathfrak{H}(\delta)=\{\{f\}\,;\,f\in\mathfrak{L}(\delta)\}\cong\mathfrak{L}(\delta)/\mathfrak{a}_0,$$

since $\mathfrak{a}_0=\mathfrak{m}_0\cap\mathfrak{L}(\delta)$. Thus the condition $\dim\mathfrak{H}(\delta)<+\infty$ is of course satisfied. Hence, by the same way as in general case A, we have the following

Theorem 7. *Let G be a locally compact, σ-compact, and unimodular group, K a compact subgroup of G, and δ an irreducible representation of K. If the continuous function ϕ on G satisfies $\phi^{\circ}=\phi$ and $\chi_{\delta}*\phi=\phi$, the following two statements are equivalent*;

(I) *ϕ is a spherical function of type δ of height p,*

(II) $\dim(L(\delta)/\mathfrak{p})<+\infty$, where $\mathfrak{p}=\{f\in L(\delta);\, f'*\phi=0\}$, and there exists a p-dimensional irreducible representation $f\to U_f$ of $L^{\circ}(\delta)$ such that $\phi(f)=(\dim\delta)\,\mathrm{Tr}[U_f]$.

Corollary. *In the case of σ-compact G, all spherical functions are obtained from topologically irreducible representations of G on Fréchet spaces.*

C. Case of positive-definite ϕ

If the given function ϕ is positive-definite.

$$\phi(e)\geqq 0,\qquad |\phi(x)|\leqq\phi(e),\qquad \phi(x^{-1})=\overline{\phi(x)}.$$

Set $f^*(x)=\overline{f(x^{-1})}$. Then $(f^*)^*=f$, $(f*g)^*=g^**f^*$, $(\bar{\chi}_{\delta}*f)^*=f^**\bar{\chi}_{\delta}$, $\phi(f^*)=\overline{\phi(f)}$. We define an "inner product" in $L(G)$ by

$$(f,\, g)=\phi(g^**f),$$

and set $\|f\|=\sqrt{(f,f)}$. Of course $\|f\|=0$ does not mean $f=0$ in general.

Lemma 21. *The inner product is invariant under the operations L_x. And $\|f*g\|\leqq\|g\|\,\|f\|_{L^1}$, $\|\bar{\chi}_{\delta}*f\|\leqq\sqrt{\dim\delta}\,\|f\|$ for every $f,\, g\in L(G)$.*

Proof. The first inequality follows from the positive-definiteness of the function

$$\int_G\int_G \phi(xzy)f^*(x)f(y)dxdy.$$

The other properties are clear. q.e.d.

Set $N=\{f\in L(G);\ \|f\|=0\}$, then N is invariant under the operations L_x, $f*$, $\bar{\chi}_\delta *$, and $*\bar{\chi}_\delta$. Thus we can consider L_x, $f*$, $\bar{\chi}_\delta *$, $*\bar{\chi}_\delta$, $(\ ,\ )$, and $\|\cdot\|$ on

$$H'(G)=L(G)/N.$$

Let $H(G)$ be the completion of $H'(G)$ with respect to the norm $\|\cdot\|$, and extend L_x, $f*$, $\bar{\chi}_\delta *$, and $\bar{\chi}_\delta *$ on $H(G)$ by continuity. The linear operators L_x are unitary on the Hilbert space $H(G)$. Since $L(\delta)\cap N=\bar{\chi}_\delta * N * \bar{\chi}_\delta$, the quotient space

$$H'(\delta)=L(\delta)/L(\delta)\cap N$$

is identified with $\bar{\chi}_\delta * H'(G)*\bar{\chi}_\delta$, and the completion $H(\delta)$ of $H'(\delta)$ is $\bar{\chi}_\delta * H(G)*\bar{\chi}_\delta$.

Lemma 22. $\mathfrak{p}=L(\delta)\cap N$.

Proof. If $f\in L(\delta)\cap N$,

$$\begin{aligned}|f'*\phi(x)| &= |\phi*f'(x)|=|\phi(L_xf)|=|\phi(u*L_xf)|\\ &=|(L_xf,\ u^*)|\leqq \|L_xf\|\|u^*\|=\|f\|\|u^*\|=0,\end{aligned}$$

where u is a function in $L^\circ(\delta)$ such that $U_u=1$. Conversely for every $f\in\mathfrak{p}$,

$$\|f\|^2=\phi(f^* * f)=[\phi * f' * (f^*)'](e)=[f' * \phi * (f^*)'](e)=0.$$

q.e.d.

By Lemma 22, the space $H'(\delta)$ is equal to $L(\delta)/\mathfrak{p}$ and is finite-dimensional from our assumption, hence $H'(\delta)$ coincides with $H(\delta)$. Let $\mathfrak{b}_0$ be a maximal $L(\delta)$-invariant proper subspace of $H(\delta)$. Of course $\mathfrak{b}_0$ may be equal to $\{0\}$. In any case, $u \notin \mathfrak{b}_0$.

Lemma 23. *Set* $\mathfrak{n}_0=\{f \in H(G);\ \bar{\chi}_\delta * g * f * \bar{\chi}_\delta \in \mathfrak{b}_0$ *for every* $g \in L(G)\}$, *then (a)* $u \notin \mathfrak{n}_0$, $\mathfrak{b}_0=\mathfrak{n}_0 \cap H(\delta)$, *and* $f*u-f \in \mathfrak{n}_0$ *for every* $f \in L(G)$, *(b)* $\mathfrak{n}_0$ *is closed and is maximal in the set of all closed* $L(G)$-*invariant subspaces of* $H(G)$.

The proof is formally same as that of Lemma 19.

Lemma 24. *When* $U \to e$, $e_{xU}*f$ *converges to* $L_x f$ *in* $H(G)$ *for every* $f \in H(G)$. *Hence* $\mathfrak{n}_0$ *is invariant under the operations* L_x.

Proof. If f is in $L(G)$,

$$\|e_{xU}*f-L_x f\|^2=\phi(f^* * e_U^* * e_U * f)-\phi(f^* * e_U * f)$$
$$-\phi(f^* * e_U * f)+\phi(f^* * f).$$

The right hand side converges to zero if $U \to e$. Therefore for every $f \in H'(G)$, $\|e_{xU}*f-L_x f\| \to 0$, when $U \to e$. Now, for any f in $H(G)$, there exists a sequence $f_i \in H'(G)$ such that $f_i \to f$ in $H(G)$, and we have

$$\|e_{xU}*f-L_x f\|=\|e_U*f-f\|$$
$$=\|e_U*f-e_U*f_i\|+\|e_U*f_i-f_i\|+\|f_i-f\|$$
$$\leqq 2\|f-f_i\|+\|e_U*f_i-f_i\|.$$

The last two terms become arbitrary small if we make i large enough and then $U \to e$. q.e.d.

To see $\{H(G), L_x\}$ is a representation of G, we have only to show that $x \to L_x f$ is continuous for every $f \in H(G)$, since $H(G)$ is a Hilbert space. Suppose that a sequence $f_i \in H'(G)$ converges to f. In the inequality

$$\|f-L_x f\| \leqq \|f_i - L_x f_i\| + 2\|f-f_i\|,$$

the right hand side can be small enough if i is large enough and x is in a sufficiently small neighbourhood of e in G. Thus $x \to L_x f$ is continuous.

The representation $\{H(G), L_x\}$ is not irreducible in general. Hence we consider the Hilbert space $\mathfrak{H} = \mathfrak{n}_0^{\perp}$ and the operators $T_x = L_x|_{\mathfrak{H}}$, where $\mathfrak{n}_0^{\perp}$ is the orthogonal complement of $\mathfrak{n}_0$. This representation $\{\mathfrak{H}, T_x\}$ is, by Lemma 23 (b), an irreducible unitary representation of G. As before, $E(\delta)f = \bar{\chi}_\delta * f * \bar{\chi}_\delta$ for $f \in \mathfrak{H}$, and

$$\mathfrak{H}(\delta) = H(\delta)/\mathfrak{b}_0,$$

and therefore $\dim \mathfrak{H}(\delta) < +\infty$. By the same argument as in general case A, we obtain the following

Theorem 8. *Let G be a locally compact unimodular group, K a compact subgroup, and δ an irreducible representation of K. If the continuous function ϕ on G satisfies $\phi = \phi^\circ$ and $\chi_\delta * \phi = \phi$, the following two statements are equivalent;*

(I$''$) *ϕ is a spherical function of type δ of height p obtained from an irreducible unitary representation of G,*

(II$''$) *ϕ is positive-definite and $\dim(L(\delta)/\mathfrak{p}) < +\infty$, where $\mathfrak{p} = \{f \in L(\delta); f' * \phi = 0\}$, and there exists a p-dimensional irreducible representation $f \to U_f$ of $L^\circ(\delta)$ such that $\phi(f) = (\dim\delta)\mathrm{Tr}[U_f]$.*

The following remark and corollary are applicable to the whole cases A, B, and C.

Corollary. *If the algebra $L^\circ(\delta)$ is commutative, δ is contained in every topologically irreducible representation of G either at most once or infinitely many times.*

Remark. In the case of $\delta=1$, we have $L^{\circ}(\delta)=L(\delta)=\{f\in L(G);\ f(k_1xk_2)=f(x)$ for $k_1, k_2\in K\}$. Therefore the condition $\dim(L(\delta)/\mathfrak{p})<+\infty$ is automatically satisfied by Lemma 15 (iii).

The next theorem in the case B is also verified in the cases A and C with some trivial modifications.

Theorem 9. *Let G be a locally compact, σ-compact and unimodular group, ϕ a continuous function on G such that* $\dim(L(\delta)/\mathfrak{p})<+\infty$, *where* $\mathfrak{p}=\{f\in L(\delta);\ f'*\phi=0\}$. *Suppose there exists a compact subgroup K of G and its irreducible representation δ such that $\chi_\delta*\phi\neq 0$, then the following two statements are equivalent*;

(i) *ϕ is proportional to a spherical function of height* 1,

(ii) $\phi(e)\int_K \phi(kxk^{-1}y)dk=\phi(x)\cdot\phi(y)$ *for every $x, y\in G$.*

§5. Correspondence between representations and spherical functions

It is well-known that the given two irreducible unitary representations are unitary equivalent if and only if the corresponding spherical functions coincide with each other [5]. But in general case, such a rigid correspondence does not exist.

Assume G is σ-compact. Let $\mathfrak{a}$ and $\mathfrak{a}_0$ be the same as in the cases A and B respectively, i.e., $\mathfrak{a}$ is a maximal left ideal in $L(\delta)$ containing $\mathfrak{p}$ and $\mathfrak{a}_0$ is a maximal $L(\delta)$-invariant subspace of $\mathfrak{L}(\delta)$ containing $\hat{\mathfrak{p}}$. Denote by $\mathfrak{A}$ the set of all such $\mathfrak{a}$, and by $\mathfrak{A}_0$ the set of all such $\mathfrak{a}_0$. Then, if $\mathfrak{a}\in\mathfrak{A}$, the completion $\hat{\mathfrak{a}}$ of $\mathfrak{a}$ in $\mathfrak{L}(\delta)$ belongs to $\mathfrak{A}_0$ and $\mathfrak{a}=\hat{\mathfrak{a}}\cap L(\delta)$, and conversely if $\mathfrak{a}_0\in\mathfrak{A}_0$, $\mathfrak{a}=\mathfrak{a}_0\cap L(\delta)$ belongs to $\mathfrak{A}$ and $\mathfrak{a}_0$ is the completion $\hat{\mathfrak{a}}$ of $\mathfrak{a}$ in $\mathfrak{L}(\delta)$. If $\mathfrak{a}\in\mathfrak{A}$ and $\hat{\mathfrak{a}}=\mathfrak{a}_0\in\mathfrak{A}_0$,

$$\mathfrak{m}=\{f\in L(G);\ \bar{\chi}_\delta*g*f*\bar{\chi}_\delta\in\mathfrak{a}\quad\text{for every } g\in L(G)\}$$

and

$$\mathfrak{m}_0=\{f\in\mathfrak{L}(G);\ \bar{\chi}_\delta*g*f*\bar{\chi}_\delta\in\mathfrak{a}_0\quad\text{for every } g\in L(G)\}$$

are combined by the relation $\mathfrak{m}_0 \cap L(G) = \mathfrak{m}$. Thus we know that $\mathfrak{L}(G)/\mathfrak{m}_0$ is the completion of $L(G)/\mathfrak{m}$ by the "suitable" topology. Similarly, in the case of positive-definite ϕ, we can see that $H(G)/\mathfrak{n}_0$ is the completion of $L(G)/\mathfrak{m}$.

Now, take a topologically irreducible representation $\{\mathfrak{H}, T_x\}$ of G and a corresponding spherical function ϕ_δ. By Lemma 13, $\mathfrak{p} = \{f \in L(\delta); f' * \phi_\delta = 0\} = \{f \in L(\delta); \check{T}_f = 0\}$. Moreover we have the following

Lemma 25. *For every non zero element $a \in \mathfrak{H}(\delta)$, $\mathfrak{a} = \{f \in L(\delta); \check{T}_f a = 0\}$ is a maximal left ideal in $L(\delta)$ containing $\mathfrak{p}$. Conversely, for every maximal left ideal $\mathfrak{a}$ in $L(\delta)$ containing $\mathfrak{p}$, there exists a unique non zero element $a \in \mathfrak{H}(\delta)$ up to scalar multiples, such that $\mathfrak{a} = \{f \in L(\delta); \check{T}_f a = 0\}$.*

Proof. The first half of the lemma is clear. Let's prove the latter half. Suppose, for every non zero element $a \in \mathfrak{H}(\delta)$, we can find an element $f \in \mathfrak{a}$ such that $\check{T}_f a \neq 0$, then the correspondence $f \to \check{T}_f$ is an irreducible representation on $\mathfrak{H}(\delta)$ of the algebra $\mathfrak{a}$. Then, by the Burnside's theorem, there exists an element $v \in \mathfrak{a}$ such that $\check{T}_v = 1$. It follows that $L(\delta) \subset \mathfrak{a}$, but this is impossible. Thus there exists some non zero element $a \in \mathfrak{H}(\delta)$ such that $\{f \in L(\delta); \check{T}_f a = 0\} \supset \mathfrak{a}$. This implies $\{f \in L(\delta); T_f a = 0\} = \mathfrak{a}$ by the maximality of $\mathfrak{a}$. There exists a $f \in L(\delta)$ such that $\check{T}_f a = 0$ and $\check{T}_f b \neq 0$ provided that a and b are linearly independent. Therefore the uniqueness of such a is proved.

q.e.d.

Now let $\mathfrak{a}$ be a maximal left ideal in $L(\delta)$ containing $\mathfrak{p}$, and a a corresponding element in $\mathfrak{H}(\delta)$, then

$$\mathfrak{m} = \{f \in L(G); \bar{\chi}_\delta * g * f * \bar{\chi}_\delta \in \mathfrak{a} \quad \text{for every } g \in L(G)\}$$

$$= \{f \in L(G); T_f a = 0\}.$$

Therefore the mapping φ: $T_f a \to \{f\}$ from $\mathfrak{H}_0 = \mathfrak{H}_0[K, \delta, a]$ onto $L(G)/\mathfrak{m}$ is linear, bijective, and

$$\varphi(T_x T_f a)=\{L_x f\}, \quad \varphi(E(\delta)T_f a)=\{\bar{\chi}_\delta * f * \bar{\chi}_\delta\}.$$

The given spherical function ϕ_δ is realized as a spherical function in the generalized sence on $\varphi(\mathfrak{H}_0)=L(G)/\mathfrak{m}$. This realization is exactly the one used in the proof of (II)$\Rightarrow$(I) for the case A. From this consideration, we obtain the following

Theorem 10. *Let G be a locally compact unimodular group, and K a compact subgroup of G. Suppose that two topologically irreducible representations $\{\mathfrak{H}, T_x\}$, $\{\mathfrak{H}', T'_x\}$ contain some δ at most finite times. Then $\phi_\delta=\phi'_\delta$ if and only if there exists a bijective linear mapping φ: $\mathfrak{H}_0 \to \mathfrak{H}'_0$ such that*

$$\varphi T_x = T'_x \varphi \qquad \textit{for every } x \in G, \textit{ and } \varphi E(\delta)=E'(\delta)\varphi,$$

where $\mathfrak{H}_0$ and $\mathfrak{H}'_0$ are the spaces given in Lemma 4.

Especially, if both of the two representation $\{\mathfrak{H}, T_x\}$, $\{\mathfrak{H}', T'_x\}$ are algebraically irreducible, φ maps $\mathfrak{H}$ onto $\mathfrak{H}'$.

We may say that two topologically irreducible representations $\{\mathfrak{H}, T_x\}$, $\{\mathfrak{H}', T'_x\}$ are "equivalent" if the corresponding spherical functions $\phi_\delta, \phi'_\delta$ coincide with each other for some δ. In the terminology of R. Godement [5], ϕ_δ is said to be "quasi-bounded" if there exists a positive function ρ on G such that ϕ_δ/ρ is bounded and that

(a) ρ is lower semi-continuous, (b) $\rho(xy) \leqq \rho(x)\rho(y)$,

(c) ρ is bounded on every compact subset of G.

An "equivalence" class of topologically irreducible representations contains a representation on a Banach space if and only if one of (or all of) the corresponding ϕ_δ is quasi-bounded, and moreover we can find a completely irreducible representation on a Banach space as a representative element (cf. [5]). And an "equivalence" class contains a unitary representation if and only if one of (or all of) the corresponding ϕ_δ is positive-definite. Particularly, if ϕ_δ is bounded, the corresponding "equivalence" class contains a representation on a Banach space such that the operator norms of all $T_x (x \in G)$ are equal to 1. If G is σ-compact, we can always find a representation on a Fréchet space as a representative element.

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