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The theorem of the cube for principal homogeneous spaces

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0. The statement of the theorem.

Let $f_i: X_i \to S$, $(i=1, 2, 3)$ be a proper flat S-prescheme of finite presentation such that f_i has a section e_i and that $f_{i*}(\mathcal{O}_{X_i}) \cong \mathcal{O}_S$ universally. Let G be a flat commutative S-group prescheme of finite presentation. For any subset I of $\{1, 2, 3\}$, we denote by $X_I = \prod_{i \in I} X_i$ the fibre product of X_i, $i \in I$, by s_I the immersion $X_I \to X_{\{1,2,3\}}$ defined by id_{X_i} for $i \in I$ and e_i for $i \in \{1, 2, 3\} - I$ and by $s_{J,I}$ the immersion $X_J \to X_I$ defined by id_{X_j} for $j \in J$ and e_j for $j \in I - J$ if $J \subset I$.

A trivialization of a $G_{X_{\{1,2,3\}}}$-torsor E with respect to e_i, $(i=1, 2, 3)$ is a set of isomorphisms $\alpha_I: s_I^*(E) \to G_{X_I}$ for any subset I of $\{1, 2, 3\}$ such that for $J \subset I$, $s_{J,I}^*(\alpha_I) = \alpha_J$. The set of isomorphism classes of trivializable $G_{X_{\{1,2,3\}}}$-torsors forms an abelian group which is denoted by $\mathrm{PH}_{(e_1,e_2,e_3)}(X_1 \times_S X_2 \times_S X_3, G)$.

We shall prove the following

The theorem of the cube. *Let* f_i: $X_i \to S$ $(i=1, 2, 3)$ *and* G *be as above. Then* $PH_{(e_1,e_2,e_3)}(X_1 \times_S X_2 \times_S X_3, G) = 0$ *if* G *satisfies moreover one of the following conditions*:

(1) *G is affine and smooth over S.*

(2) *G is finite and flat over S.*

(3) *G is an abelian scheme, S is quasi-compact and normal and f_i $(i=1, 2, 3)$ are geometrically normal.*

If G is the multiplicative group prescheme, $G_{m,S}$, this theorem is the ordinary theorem of the cube (cf. [1], [3], [6]). The notation and definitions are those of [4] and [5]. The cohomologies should be understood to be (f.p.q.c.)-cohomologies unless otherwise mentioned.

1. The formal non-ramifiedness of the functor $\mathbf{Corr}_S^G(X_1, X_2)$.

Let $f_i\colon X_i \to S$ $(i=1, 2)$ be a proper S-prescheme such that f_i has a section e_i and that $f_{i*}(\mathcal{O}_{X_i}) \cong \mathcal{O}_S$ univrsally and let G be a commutative affine flat S-group prescheme of finite presentation. We define a (f.p.q.c.)-sheaf of abelian groups, $\mathbf{Corr}_S^G(X_1, X_2)$, on the site $(\mathrm{Sch}/S)_{pq}$ by the following split exact sequence,

$$0 \longrightarrow \mathbf{PH}(X_1/S, G) \underset{S}{\times} \mathbf{PH}(X_2/S, G) \underset{(s_1^*,\, s_2^*)}{\overset{pr_2^* + pr_1^*}{\rightleftarrows}} \mathbf{PH}(X_1 \underset{S}{\times} X_2/S, G)$$

$$\longrightarrow \mathbf{Corr}_S^G(X_1, X_2) \longrightarrow 0.$$

$\mathbf{Corr}_S^G(X_1, X_2)$ is called the functor of divisorial correspondences of type G between X_1 and X_2 and satisfies the following properties;

(1) $\mathbf{Corr}_S^G(X_1, X_2) \underset{S}{\times} S' \cong \mathbf{Corr}_{S'}^{G'}(X_1', X_2')$, where $'$ on the shoulders denote the base change by $S' \to S$.

(2) $\mathrm{Corr}_S^G(X_1, X_2) \underset{\mathrm{def}}{\equiv} \mathbf{Corr}_S^G(X_1, X_2)(S)$ is a direct summand of $\mathrm{PH}(X_1 \underset{S}{\times} X_2/S, G) \underset{\mathrm{def}}{\equiv} \mathbf{PH}(X_1 \underset{S}{\times} X_2/S, G)(S)$ with the complement $\mathrm{PH}(X_1/S, G) \oplus \mathrm{PH}(X_2/S, G)$.

First of all, we shall prove

Lemma 1. $\mathbf{Corr}_S^G(X_1, X_2)$ *is formally non-ramified if G is a smooth affine commutative S-group prescheme of finite presentation and f_1 or f_2 is flat.*

Proof. We may assume that S is affine and that f_1 is flat. Let $S = \mathrm{Spec}(A)$, let I be a square zero ideal of A and let $\bar{S} = \mathrm{Spec}(A/I)$.

We have to show that the canonical morphism obtained from the base change by $\bar{S}\to S$,

$$i\ :\ \mathrm{Corr}_S^G(X_1,\ X_2)\longrightarrow \mathrm{Corr}_{\bar{S}}^{\bar{G}}(\bar{X}_1,\ \bar{X}_2)$$

is injective. Let ξ be an element of $\mathrm{Corr}_S^G(X_1,\ X_2)$ such that $i(\xi)=0$. By definition, ξ is representable by a $G_{X_1\times_S X_2}$-torsor E such that $s_1^*(E)$ (resp. $s_2^*(E)$) is a trivial G_{X_1} (resp. G_{X_2})-torsor and that $E\times_S\bar{S}$ is also a trivial $\bar{G}_{\bar{X}_1\times_{\bar{S}}\bar{X}_2}$-torsor. Then we should prove that E is itself a trivial $G_{X_1\times_S X_2}$-torsor.

Consider the following diagram,

$$\begin{array}{ccc} X_1\times_S X_2 & \xrightarrow{g_2} & X_2 \\ \downarrow{\scriptstyle g_1} & \searrow^{h} & \downarrow{\scriptstyle f_2} \\ X_1 & \xrightarrow[f_1]{} & S \end{array},$$

where g_1 and g_2 are canonical projections and where $h=f_1g_1=f_2g_2$. If $\mathscr{F}$ is a quasi-coherent $\mathcal{O}_{X_1\times_S X_2}$-Module, the Leray spectral sequence for the composite morphism $h=f_1g_1$ gives an exact sequence,

$$0\longrightarrow R^1f_{1*}(g_{1*}\mathscr{F})\longrightarrow R^1h_*(\mathscr{F})\longrightarrow f_{1*}R^1g_{1*}(\mathscr{F}).$$

If $\mathscr{F}=h^*\mathscr{G}$ for some quasi-coherent $\mathcal{O}_S$-Module $\mathscr{G}$, this sequence becomes

$$0\longrightarrow R^1f_{1*}(f_1^*\mathscr{G})\longrightarrow R^1h_*(h^*\mathscr{G})\longrightarrow R^1f_{2*}(f_2^*\mathscr{G}),$$

where we used the flat base change theorem for f_2 (cf. EGA, III (1.4.15)). Since S is affine, this sequence is equal to an sequence,

$$0\longrightarrow H^1(X_1,\ f_1^*\mathscr{G})\longrightarrow H^1(X_1\times_S X_2,\ h^*\mathscr{G})\longrightarrow H^1(X_2,\ f_2^*\mathscr{G}).$$

Moreover this sequence splits because X_1 and X_2 have sections from S.

On the other hand, we have the following commutative diagram from Lemma 2 below:

$$
\begin{array}{ccccc}
0\longrightarrow H^1(X_1 \text{ Lie } G\underset{\mathcal{O}_S}{\otimes} I\mathcal{O}_{X_1}) & \xrightarrow{j_1} & H^1(X_1, G) & \xrightarrow{i_1} & H^1(\bar{X}_1, \bar{G}) \\
\uparrow & & \uparrow s_1^* & & \uparrow \\
0\longrightarrow H^1(X_1 \underset{S}{\times} X_2, \text{Lie } G\underset{\mathcal{O}_S}{\otimes} I\mathcal{O}_{X_1\underset{S}{\times} X_2}) & \xrightarrow{j_{1,2}} & H^1(X_1 \underset{S}{\times} X_2, G) & \xrightarrow{i_{1,2}} & H^1(\bar{X}_1 \underset{\bar{S}}{\times} \bar{X}_2, \bar{G}) \\
\downarrow & & \downarrow s_2^* & & \downarrow \\
0\longrightarrow H^1(X_2, \text{Lie } G\underset{\mathcal{O}_S}{\otimes} I\mathcal{O}_{X_2}) & \xrightarrow{j_2} & H^1(X_2, G) & \xrightarrow{i_2} & H^1(\bar{X}_2, \bar{G})
\end{array}
$$

where the lines are exact and the left column splits. ξ defines an element ξ' of $H^1(X_1 \underset{S}{\times} X_2, G)$ such that $i_{1,2}(\xi')=0$ and $\xi=0$ if and only if $\xi'=0$.

Then the diagram chasing shows that $\xi=0$. q.e.d.

Lemma 2. *Let G be a smooth affine commutative S-group prescheme, let $f: X\to S$ be a S-prescheme quasi-compact and quasi-separated over S and let $\bar{S}$ be a closed subprescheme defined by a square-zero Ideal $\mathcal{I}$ of $\mathcal{O}_S$. Then we have an exact sequence.*

$$
\begin{aligned}
&0\longrightarrow f_*(\text{Lie } G\underset{\mathcal{O}_S}{\otimes} \mathcal{I}\mathcal{O}_X)\longrightarrow f_*(G)\longrightarrow \bar{f}_*(\bar{G})\longrightarrow \\
&R^1f_*(\text{Lie } G\underset{\mathcal{O}_S}{\otimes} \mathcal{I}\mathcal{O}_X)\longrightarrow R^1f_*(G)\longrightarrow R^1\bar{f}_*(\bar{G}).
\end{aligned}
$$

If $f_*(\mathcal{O}_X)\cong\mathcal{O}_S$ universally, then $f_*(G)\to\bar{f}_*(\bar{G})$ is surjective. Moreover, if S is affine, we have an exact sequence,

$$
0\longrightarrow H^1(X, \text{Lie } G\underset{\mathcal{O}_S}{\otimes} \mathcal{I}\mathcal{O}_X)\longrightarrow H^1(X, G)\longrightarrow H^1(\bar{X}, \bar{G}).
$$

Proof. We shall show that if S is affine, we have an exact sequence,

$$
\begin{aligned}
&0\longrightarrow \Gamma(X, \text{Lie } G\underset{\mathcal{O}_S}{\otimes} \mathcal{I}\mathcal{O}_X)\longrightarrow G(X)\longrightarrow \bar{G}(\bar{X})\longrightarrow \\
&H^1(X, \text{Lie } G\underset{\mathcal{O}_S}{\otimes} \mathcal{I}\mathcal{O}_X)\longrightarrow H^1(X, G)\longrightarrow H^1(\bar{X}, \bar{G}).
\end{aligned}
$$

The first exact sequence is obtained by localizing the above sequence.

An element ξ of $H^1(X, G)$ can be given by a Čech-cocycle. Since we are dealing with the (f.p.q.c.)-topology, ξ is given by a Čech-cocycle $g_{ij}\in G(U_{ij})$ for $\mathfrak{U}=\{U_i\}\in \mathrm{Cov}(X)$, where U_i is an affine scheme which is faithfully flat over an affine open set V_i of X, $\bigcup_i V_i=X$ and where $U_{ij}=U_i \underset{X}{\times} U_j$. The image of ξ in $H^1(\bar{X}, \bar{G})$ is zero if and only if $\{\bar{g}_{ij}\}$ is a Čech-coboundary. Then replacing $\mathfrak{U}$ by a finer open cover of X, we may assume that there exists $\bar{h}_i\in\bar{G}(\bar{U}_i)$ for all i such that $\bar{g}_{ij}=\bar{h}_i-\bar{h}_j$ on $\bar{U}_{ij}$ for all i, j.

Let $U_i=\mathrm{Spec}(A_i)$ and let $\bar{U}_i=\mathrm{Spec}(A_i/I_i)$, I_i being a square-zero ideal of A_i. Since G is smooth over S, there exists $h_i\in G(U_i)$ for all i such that $\bar{h}_i=h_i$ modulo I_i. Let $g'_{ij}=g_{ij}-h_i+h_j$. Then $\bar{g}'_{ij}=0$ modulo I_i.

Now we shall use the following

Sublemma. *Let G be an affine smooth T-prescheme and let $\bar{T}$ be a closed subprescheme of T defined by a square-zero Ideal $\mathscr{I}$ of $\mathcal{O}_T$. Then we have the following exact sequence,*

$$0\longrightarrow\Gamma(T, \mathrm{Lie}\, G\underset{\mathcal{O}_T}{\otimes}\mathscr{I})\longrightarrow G(T)\longrightarrow\bar{G}(\bar{T}).$$

Proof. Since G is affine over T, G is given by a quasi-coherent $\mathcal{O}_T$-Algebra $\mathscr{A}$ and $\mathscr{A}$ is the direct sum of $\mathcal{O}_T$ and the augmentation Ideal $\mathscr{J}$, i.e., $\mathscr{A}\cong\mathcal{O}_T\oplus\mathscr{J}$. Let g be an element of $G(T)$ such that $g=0$ modulo $\mathscr{I}$. Let g be defined by an $\mathcal{O}_T$-Algebra homomorphism $\varphi\colon \mathscr{A}\to\mathcal{O}_T$. Then φ sends $\mathscr{J}$ to $\mathscr{I}$ since the composite homomorphism $\mathscr{A}\xrightarrow{\varphi}\mathcal{O}_T\longrightarrow\mathcal{O}_{\bar{T}}$ factors through $\mathscr{A}/\mathscr{J}$. Since $\mathscr{I}$ is square-zero, $\varphi|\mathscr{J}$ defines an $\mathcal{O}_T$-Module homomorphism $\bar{\varphi}\colon\mathscr{J}/\mathscr{J}^2\to\mathscr{I}$. Conversely, if $\psi\colon\mathscr{J}/\mathscr{J}^2\to\mathscr{I}$ is any $\mathcal{O}_T$-Module homomorphism, we can construct an $\mathcal{O}_T$ -Algebra homomorphism $\tilde{\psi}$ by $\tilde{\psi}|_{\mathcal{O}_T}=\mathrm{id}_{\mathcal{O}_T}$ and $\tilde{\psi}|\mathscr{J}=\psi$ composed with the canonical projection $\mathscr{J}\to\mathscr{J}/\mathscr{J}^2$. Then it is easy to see that $\tilde{\bar{\varphi}}=\varphi$ and $\bar{\tilde{\psi}}=\psi$. On the other hand, $\mathrm{Hom}_{\mathcal{O}_T}(\mathscr{J}/\mathscr{J}^2, \mathscr{I})\cong\mathbf{Hom}_{\mathcal{O}_T}(\mathscr{J}/\mathscr{J}^2, \mathscr{I})(T)=\Gamma(T, \mathrm{Lie}\, G\underset{\mathcal{O}_T}{\otimes}\mathscr{I})$ since $\mathscr{J}/\mathscr{J}^2$ is locally free $\mathcal{O}_T$-Module.

q.e.d.

Now we shall go back to the proof of Lemma 2. From the sublemma, there exists an element η_{ij} of $\Gamma(U_{ij}, \mathrm{Lie}\, G \underset{O_S}{\otimes} \mathscr{I}\mathcal{O}_{U_{ij}})$ determined uniquely by g'_{ij}. Then η_{ij} is a Čech-cocycle of $C^1(\mathfrak{U}, \mathrm{Lie}\, G \underset{O_S}{\otimes} \mathscr{I}\mathcal{O}_X)$, hence defines an element ζ of $H^1(X, \mathrm{Lie}\, G \underset{O_S}{\otimes} \mathscr{I}\mathcal{O}_X)$ which goes to ξ.

ξ is a Čech-coboundary if and only if ζ comes from an element of $\bar{G}(\bar{X})$ by the following morphism δ: Let $\bar{g} \in \bar{G}(\bar{X})$ and let $\mathfrak{V} = \{V_i\}$ be an affine open cover of X. Then $\bar{g}|_{V_i}$ comes from g_i of $G(V_i)$, since G is smooth over S. Then for any i, j, $g_i - g_j$ corresponds with an element η_{ij} of $\Gamma(V_{ij}, \mathrm{Lie}\, G \underset{O_S}{\otimes} \mathscr{I}\mathcal{O}_X)$ and $\{\eta_{ij}\}$ is a Čech-cocycle of $C^1(\mathfrak{V}, \mathrm{Lie}\, G \underset{O_S}{\otimes} \mathscr{I}\mathcal{O}_X)$. Hence $\{\eta_{ij}\}$ defines an element ζ of $H^1(X, \mathrm{Lie}\, G \underset{O_S}{\otimes} \mathscr{I}\mathcal{O}_X)$. Then δ is a morphism which sends $\bar{g}$ to ζ. If ξ defines an element ζ which comes from $\bar{g} \in G(\bar{X})$ by the morphism δ above, we can see easily from the definition that ξ is a Čech-coboundary. Conversely, if ξ is a Čech-coboundary, replacing $\mathfrak{U}$ by finer cover, there exist $g'_i \in G(U_i)$ for all i, which is in turn coming from $\Gamma(U_i, \mathrm{Lie}(G) \underset{O_S}{\otimes} \mathscr{I}\mathcal{O}_X)$, such that $g'_{ij} = g'_i - g'_j$ on U_{ij} for all i, j. Since $g'_{ij}(= g'_i - g'_j) = 0$ modulo I, $\bar{g}'_i = \bar{g}_j$ for all i, j. Hence $\{\bar{g}'_i\}$ defines an element $\bar{g}'$ of $\bar{G}(\bar{X})$, which is easily seen to give ζ by δ. Here we note that $H^1_{pq}(X, \mathrm{Lie}\, G \underset{O_S}{\otimes} \mathscr{I}\mathcal{O}_X) \cong H^1_{\mathrm{Zar}}(X, \mathrm{Lie}\, G \underset{O_S}{\otimes} \mathscr{I}\mathcal{O}_X)$. $\bar{g}$ comes from an element of $G(X)$ if and only if $\delta(\bar{g}) = 0$. The remaining parts follows from the sublemma.

If $f_*(\mathcal{O}_X) \cong \mathcal{O}_S$ universally, $G(X) \to \bar{G}(\bar{X})$ is surjective since $G(X) = \mathrm{Hom}_S(\mathbf{Spec}(f_*\mathcal{O}_X), G)$, $\bar{G}(\bar{X}) \cong \mathrm{Hom}_{\bar{S}}(\mathbf{Spec}(\bar{f}_*\mathcal{O}_{\bar{X}}), \bar{G})$ and since $\mathbf{Spec}(\bar{f}_*(\mathcal{O}_{\bar{X}}))$ is a closed subprescheme of $\mathbf{Spec}(f_*(\mathcal{O}_X))$ defined by a square-zero Ideal. q.e.d.

Lemma 3. *Let G, X_1 and X_2 be as in Lemma 1. Then the unit section e of $\mathbf{Corr}^G_S(X_1, X_2)$ is representable by an open immersion.*

Proof. Let T be any S-prescheme and let $Z_T = (T, \alpha) \times_{\mathbf{Corr}^G_S(X_1, X_2)} (S, e)$ for any S-morphism $\alpha: T \to \mathbf{Corr}^G_S(X_1, X_2)$. We have to prove that Z_T is an open set of T. Namely, if t is a point of T such that $\alpha(t)$

$=0$, then $\operatorname{Spec}(\mathcal{O}_{T,t}) \subset Z_T$*). Since $Z_{\operatorname{Spec}(\mathcal{O}_{T,t})} \cong Z_T \times_T \operatorname{Spec}(\mathcal{O}_{T,t})$ and $t \in Z_{\operatorname{Spec}(\mathcal{O}_{T,t})}$, we may replace T by $\operatorname{Spec}(\mathcal{O}_{T,t})$. Let $A = \mathcal{O}_{T,t}$. Then α defines an element ξ of $\operatorname{Corr}_S^{G_T}(X_{1,T}, X_{2,T})$. Finally we may assume that $T = S$. By (f.p.q.c.)-descent, we may replace A by its completion $\hat{A}$ with respect to its maximal ideal $\mathfrak{m}$. In fact, if $\hat{S}$ is Spec $(\hat{A})$, the morphism

$$\mathbf{Corr}_S^G(X_1, X_2)(S) \longrightarrow \mathbf{Corr}_S^G(X_1, X_2)(\hat{S})$$

is injective because $\hat{S}$ is faithfully flat and quasi-compact over S and $\mathbf{Corr}_S^G(X_1, X_2)$ is a (f.p.q.c.)-sheaf. Let $A_n = A/\mathfrak{m}^{n+1}$ and let $S_n = \operatorname{Spec}(A_n)$. Then by virtue of Lemma 1, the canonical morphism

$$\operatorname{Corr}_{S_n}^{G_n}(X_{1,n}, X_{2,n}) \longrightarrow \operatorname{Corr}_{\bar{S}}^{\bar{G}}(\bar{X}_1, \bar{X}_2)$$

is injective, where $\bar{S} = \operatorname{Spec}(A/\mathfrak{m})$ and $\bar{G} = G \times_S \bar{S}$. Since ξ is zero, $\xi_n = \xi$ modulo $(\mathfrak{m}^{n+1})$ is zero.

ξ is representable uniquely up to isomorphisms by a $G_{X_1 \times_S X_2}$-torsor E such that $s_1^* E$ and $s_2^* E$ are trivial. Then $E = \varinjlim_n E \times_S S_n$ is trivial. The sections $\sigma_n : (X_1 \times_S X_2)_n \to E \times_S S_n$ which trivialize $E \times_S S_n$ can be chosen so that the following diagram is commutative for any $n \geqslant m$,

$$\begin{array}{ccc} (X_1 \times_S X_2)_n & \xrightarrow{\sigma_n} & E \times_S S_n \\ \uparrow & & \uparrow \\ (X_1 \times_S X_2)_m & \xrightarrow{\sigma_m} & E \times_S S_m \end{array} .$$

Then there exists a section $\sigma: X_1 \times_S X_2 \to E$ by virtue of EGA, III (5.4.1.). Therefore E is trivial. Thus $\operatorname{Spec}(A) \subset Z$. q.e.d.

Lemma 4. *Let $f_i: X_i \to S$ $(i=1, 2, 3)$ be a proper flat S-prescheme such that f_i has a section e_i and that $f_{i*}(\mathcal{O}_{X_i}) \cong \mathcal{O}_S$ universally and let G be a smooth affine commutative S-group prescheme of finite presenta-*

(*) In fact, $\mathbf{Corr}_S^G(X_1, X_2)$ is a functor of finite presentation since **PH**-functors are so and $\mathbf{Corr}_S^G(X_1, X_2)$ is a direct summand of a **PH**-functor, (cf. [4] or SGAD, Exp VI_B (10. 16)). Then the fact that $\operatorname{Spec}(0_{T,t}) \subset Z_T$ implies that there exists an affine open set U of t such that $U \subset Z_T$.

tion. Then any S-morphism $f: X_3 \to \mathbf{Corr}_S^G(X_1, X_2)$ *which sends the section* e_3 *to the unit section* e *of* $\mathbf{Corr}_S^G(X_1, X_2)$ *factors through the unit section, i.e.,* $f = e \cdot f_3$.

Proof. Let $Z = (X_3, f) \times_{\mathbf{Corr}_S^G(X_1, X_2)} (S, e)$. Then by Lemma 3, Z is an open subprescheme of X_3 which contains $e_3(S)$. To complete the proof, we have to show that $Z = X_3$. If $Z \neq X_3$, take any closed point x of $X_3 - Z$ and let $s = f_3(x)$. Then $f_s: X_{3,s} \to \mathbf{Corr}_{k(s)}^{G_s}(X_{1,s}, X_{2,s})$ does not factor through the unit section e_s of the latter. Therefore we are reduced to consider the case where $S = \mathrm{Spec}(k)$, where k is a field. By (f.p.q.c.)-descent, we may assume that k is algebraically closed. If G is connected, $\mathbf{Corr}_S^G(X_1, X_2)$ is representable by a S-group prescheme locally of finite type over S. In fact, $\mathbf{Corr}_S^G(X_1, X_2)$ is the kernel of the S-homomorphism $(s_1^*, s_2^*): \mathbf{PH}(X_1 \times_S X_2/S, G) \to \mathbf{PH}(X_1/S, G) \times_S \mathbf{PH}(X_2/S, G)$, where $\mathbf{PH}(T/S, G)$, $T = X_1 \times_S X_2$, X_1 or X_2 is representable by a S-group prescheme locally of finite type over S. If G is etale, $\mathbf{Corr}_S^G(X_1, X_2)$ is representable by an etale S-group prescheme. (For these results, see [4].) In general, G has a connected component G_0 such G/G_0 that is etale and satisfies the following exact sequence,

$$0 \longrightarrow \mathbf{Corr}_S^{G_0}(X_1, X_2) \longrightarrow \mathbf{Corr}_S^G(X_1, X_2) \longrightarrow \mathbf{Corr}_S^{G/G_0}(X_1, X_2),$$

whence the connected component $\mathbf{Corr}_S^G(X_1, X_2)^0$ of the unit section of $\mathbf{Corr}_S^{G_0}(X_1, X_2)$ is representable and coincides with $\mathbf{Corr}_S^G(X_1, X_2)^0$. Therefore $\mathbf{Corr}_S^G(X_1, X_2)^0$ is separated over S. Then the unit section e is a closed immersion. Then Z is a closed and open subprescheme of X. However since $f_{3*}(\mathcal{O}_{X_3}) \cong \mathcal{O}_S$, X_3 is connected by Zariski's connectedness theorem. Therefore $X_3 = Z$. q.e.d.

2. The proof of the theorem. The first case.

Let E be a $G_{X_{(1,2,3)}}$-torsor representing an element of $\mathrm{PH}_{(e_1,e_2,e_3)}(X_1 \times_S X_2 \times_S X_3/S, G)$. Then E defines a S-morphism

$$\xi: X_3 \longrightarrow \mathbf{Corr}_S^G(X_1, X_2)$$

which sends the section e_3 to the unit section e of $\mathbf{Corr}_S^G(X_1, X_2)$. Then ξ factors through the unit section e by virtue of Lemma 4. Moreover E considered as a $G_{(X_1 \times_S X_3) \times_{X_3} (X_2 \times_S X_3)}$-torsor defines an element η of $\mathrm{PH}((X_1 \times_S X_3) \times_{X_3} (X_2 \times_S X_3)/X_3, G)$ which is in turn isomorphic to the direct sum,

$$\mathrm{PH}(X_1 \times_S X_3/X_3, G) \oplus \mathrm{PH}(X_2 \times_S X_3/X_3, G) \oplus \mathbf{Corr}_S^G(X_1, X_2)(X_3).$$

The components of η by this decomposition are $s_{13}^*(E)$, $s_{23}^*(E)$ and ξ which are all zero. Hence η is zero. Then E is trivial. q.e.d.

As this proof shows, if $\mathbf{Corr}_S^G(X_1, X_2)=0$, the proof of the theorem becomes almost trivial. The following result shows that the difficulty of the proof of the theorem comes from the torus part of G.

Proposition 5. *Let f_i: $X_i \to S (i=1, 2)$ be a proper flat S-prescheme of finite presentation such that f_i has a section e_i and that $f_{i*}(\mathcal{O}_{X_i}) \cong \mathcal{O}_S$ and let G be a smooth affine commutative S-group prescheme of finite presentation. Suppose that the semi-simple rank of G is zero at every point of S. Then* $\mathbf{Corr}_S^G(X_1, X_2)=0$.

Proof. It is sufficient to prove that $\mathrm{Corr}_S^G(X_1, X_2)=0$. We may assume that S is affine, $S=\mathrm{Spec}(A)$. Since $\mathbf{Corr}_S^G(X_1, X_2)$ is a functor locally of finite presentation (cf. [4]), we may assume that A is a local ring. By (f.p.q.c.)-descent, we can replace A by its completion $\hat{A}$ with respect to the maximal ideal $\mathfrak{m}$. Let $k=A/\mathfrak{m}$ and let $s=\mathrm{Spec}(k)$. Suppose we have shown that $\mathbf{Corr}_s^{G_s}(X_{1,s}, X_{2,s})=0$. Then by Lemma 1, $\mathrm{Corr}_{S_n}^{G_n}(X_{1,n}, X_{2,n})=0$, whence one deduces $\mathbf{Corr}_S^G(X_1, X_2)=0$, using the argument of the proof of Lemma 3.

Now we shall show by induction on the unipotent rank of G_s that $\mathbf{Corr}_s^{G_s}(X_{1,s}, X_{2,s})=0$. By (f.p.q.c.)-descent, we may assume that k is perfect. Then G_s has a composition series,

$$0=G_0 \subset G_1 \subset \cdots \subset G_n=G_s$$

such that $G_{i+1}/G_i \cong G_{a,k}$ the additive group prescheme over k for $i=0, 1, \cdots, n-1$.

For an exact sequence, $0 \longrightarrow G_i \longrightarrow G_{i+1} \longrightarrow G_a \longrightarrow 0$, we have an exact sequence of group functors,

$$0 \longrightarrow \mathbf{Corr}_S^{G_i}(X_{1,s}, X_{2,s}) \longrightarrow \mathbf{Corr}_S^{G_{i+1}}(X_{1,s}, X_{2,s})$$

$$\longrightarrow \mathbf{Corr}_S^{G_a}(X_{1,s}, X_{2,s}).$$

Therefore if $\mathbf{Corr}_S^{G_a}(X_{1,s}, X_{2,s})=0$, we are done by induction on n. In the case where $G_s=G_a$, $\mathbf{Corr}_S^{G_a}(X_{1,s}, X_{2,s}) \cong \mathbf{PH}(X_{1,s} \underset{s}{\times} X_{2,s}/s, G_a)/\mathbf{PH}(X_{1,s}/s, G_a) \times \mathbf{PH}(X_{2,s}/s, G_a) \cong \mathbf{Lie}(\mathbf{Pic}(X_{1,s} \underset{S}{\times} X_{2,s}))/\mathbf{Lie}(\mathbf{Pic}(X_{1,s})) \times \mathbf{Lie}(\mathbf{Pic}(X_{2,s}))=0$ (cf. [4]). q.e.d.

Corollary 6. *Let G, X_1 and X_2 be as in Proposition 5. Then we have*

$$\mathrm{PH}(X_1/S, G) \underset{S}{\times} \mathrm{PH}(X_2/S, G) \cong \mathrm{PH}(X_1 \underset{S}{\times} X_2/S, G).$$

Therefore

$$H^1(X_1 \underset{S}{\times} X_2, G) \cong H^1(X_1, G) \oplus H^1(X_2, G)/H^1(S, G),$$

where $H^1(S, G)$ is considered as a subgroup of $H^1(X_1, G) \oplus H^1(X_2, G)$ by the injective homomorphism $E \rightarrow (f_1^* E, -f_2^* E)$.

Proof. Obvious by definition. See [4] and [5].

Corollary 7. *Let G be as in Proposition 5 and let A be an abelian scheme over S. Then we have*

$$H^1(A, G) \cong Ext^1_{S-gr}(A, G) \oplus H^1(S, G).$$

Proof. Let f_A and e_A be the structure morphism and the unit section of A respectively. Then we have,

$$H^1(A, G) \cong \mathrm{PH}(A/S, G) \oplus H^1(S, G).$$

Therefore it is sufficient to show that $\mathrm{PH}(A/S, G) \cong \mathrm{Ext}^1_{S-gr}(A, G)$. Take any element ξ of $\mathrm{PH}(A/S, G)$. ξ is representable by a G_A-torsor E such that $e_A^* E$ is trivial. Let π be the multiplication of A. Then the $G_{A\times A}$-torsor $\delta(E) = \pi^* E - pr_1^* E - pr_2^* E$ is trivial since $\mathrm{Corr}_S^G(A, A) = 0$ from Proposition 5. Then E has a structure of commutative group S-prescheme with a section of $e_A^* E$ as unit section E and is an extension of A by G (cf. [2], (1.3.5.)). The extension class of E is determined uniquely by ξ. Sending ξ to the extension class of E, one can define a homomorphism Φ which is the inverse of the canonical homomorphism i: $\mathrm{Ext}^1_{S-gr}(A, G) \to \mathrm{PH}(A/S, G)$. q.e.d.

3. The proof of the theorem. The second case.

Let f_i: $X_i \to S (i=1, 2)$ be a proper flat S-prescheme of finite presentation such that f_i has a section e_i and that $f_{i*}(\mathcal{O}_{X_i}) = \mathcal{O}_S$ universally, let G be a finite flat commutative S-group prescheme of finite presentation and let $D(G)$ be its Cartier dual.

We shall recall the following

Lemma 8. ([5]). $\mathbf{Corr}_S^G(X_1, X_2) \cong \mathbf{Hom}_{S-gr}(D(G), \mathbf{Corr}_S^{G_m}(X_1, X_2))$.

Lemma 9. $\mathbf{Corr}_S^G(X_1, X_2)$ *is formally non-ramified.*

Proof. Let S be an affine scheme and let $\bar{S}$ be a closed subscheme of defi S ned by a square-zero ideal. We have only to show that the canonical morphism

$$\mathrm{Corr}_S^G(X_1, X_2) \longrightarrow \mathrm{Corr}_{\bar{S}}^{\bar{G}}(\bar{X}_1, \bar{X}_2)$$

is injective. This follows from the commutativity of the diagram,

$$\begin{array}{ccc} \mathrm{Hom}_{S-gr}(D(G), \mathbf{Corr}_S(X_1, X_2)) & \longrightarrow & \mathrm{Hom}_{\bar{S}-gr}(D(\bar{G}), \mathbf{Corr}_{\bar{S}}(\bar{X}_1, \bar{X}_2)) \\ \downarrow & & \downarrow \\ \mathbf{Corr}_S(X_1, X_2)(D(G)) & \longrightarrow & \mathbf{Corr}(\bar{X}_1, \bar{X}_2)(D(\bar{G})), \end{array}$$

where $\mathbf{Corr}_S(X_1, X_2)=\mathbf{Corr}_S^{G_m}(X_1, X_2)$. q.e.d.

Lemma 10. *The unit section e of* $\mathbf{Corr}_S^G(X_1, X_2)$ *is representable by an open and closed immersion.*

Proof. It is sufficient to show that if $u: D(G)\to\mathbf{Corr}_S(X_1, X_2)$ is any homomorphism of S-groups and H is the kernel of u, then the set $Z=\{s\in S;\ H_S=D(G)_s\}$ is an open and closed set of S.

However since the unit section of $\mathbf{Corr}_S(X_1, X_2)$ is representable by an open and closed immersion (cf. [1]), H is an open and closed subgroup prescheme of $D(G)$, hence it is finite and flat. Then the rank of each fibre of H is locally constant, whence the required result follows easily. q.e.d.

Now Lemma 4 is an easy consequence of Lemma 10 if G is understood a finite flat commutative S-group prescheme of finite presentation in Lemma 4. Then one can prove the second case of the theorem following word for word the proof for the first case.

Proposition 11. *Let* G, X_1 *and* X_2 *be as above. If both* f_1 *and* f_2 *are geometrically normal*, $\mathbf{Corr}_S^G(X_1, X_2)=0$.

Proof. Since $\mathbf{Corr}_S^G(X_1, X_2)$ is a (f.p.q.c.)-sheaf, we have only to show that $\mathrm{Corr}_S^G(X_1, X_2)=0$ if S is an affine scheme. Moreover since $\mathbf{Corr}_S^G(X_1, X_2)$ is a functor locally of finite presentation over S (cf. [4]), we may assume that the affine ring A of S is a local ring. We may replace A by its completion with respect to the maximal ideal $\mathfrak{m}$. If we could prove that $\mathbf{Corr}_S^G(X_1, X_2)(k)=0$, where $k=A/\mathfrak{m}$, the proof will be completed, using the argument of the proof of Lemma 3. Therefore we shall show that $\mathrm{Corr}_S^G(X_1, X_2)=0$ if S is the spectrum of a field k. We may assume k algebraically closed. In this case $\mathbf{Corr}_S(X_1, X_2)=\mathbf{Hom}_{k\text{-}gr}(\mathbf{Alb}(X_1), \mathbf{Pic}^0_{X_2/k})$ which is torsion free (cf. [3], p. 155). Then $\mathrm{Corr}_S^G(X_1, X_2)=\mathrm{Hom}_{k\text{-}gr}(D(G), \mathbf{Corr}_S(X_1, X_2))=0$.

q.e.d.

Corollary 12. *Let* G, X_1 *and* X_2 *be as in Proposition* 11. *Then*

$$\mathrm{PH}(X_1 \times_S X_2/S, G) \cong \mathrm{PH}(X_1/S, G) \oplus \mathrm{PH}(X_2/S, G).$$

Therefore

$$H^1(X_1 \times_S X_2, G) \cong H^1(X_1, G) \oplus H^1(X_2, G)/H^1(S, G)$$

where $H^1(S, G)$ *is considered as a subgroup of* $H^1(X_1, G) \oplus H^1(X_2, G)$ *by the injective homomorphism defined as in Corollary 6.*

Corollary 13. *Let* G *be as above and let* A *be an abelian scheme over* S. *Then*

$$H^1(A, G) \cong \mathrm{Ext}^1_{S\text{-}gr}(A, G) \oplus H^1(S, G).$$

Proof. The same reasoning as for Corollary 7.

4. The proof of the theorem. The third case.

We shall prove the following

Proposition 14. *Let* S *be a quasi-compact normal prescheme,* A *be an abelian scheme over* S *and let* f_i: $X_i \to S$ $(i=1, 2)$ *be a proper flat geometrically normal* S*-prescheme such that* f_i *has a section* e_i *and that* $f_{i*}(\mathcal{O}_{X_i}) = \mathcal{O}_S$ *universally. Let* $\mathrm{Corr}^A_S(X_1, X_2)$ *be the set of all isomorphism classes of* $A_{X_1 \times_S X_2}$*-torsor* E *such that* s_1^*E *and* s_2^*E *are trivial. Then* $\mathrm{Corr}^A_S(X_1, X_2) = 0$.

Proof. From our assumptions on S and f_i, we have an inclusion,

$$\mathrm{Corr}^A_S(X_1, X_2) \subset H^1(X_1 \times_S X_2, A)_{\mathrm{rep}} = H^1(X_1 \times_S X_2, A)_{\mathrm{tor}}$$

(cf. [6]). If ξ is an element of $\mathrm{Corr}^A_S(X_1, X_2)$, there exists an integer $n > 0$ such that $n\xi = 0$.

Consider an exact sequence of (f.p.q.c.)-sheaves,

$$0 \longrightarrow {}_nA \longrightarrow A \xrightarrow{n} A \longrightarrow 0,$$

where ${}_nA$ is a finite flat commutative S-group prescheme.

Denote by $A_0(T)$, $H_0^1(T, {}_nA)$ and $H_0^1(T, A)$ the kernels of $A(T) \xrightarrow{e^*} A(S)$, $H^1(T, {}_nA) \xrightarrow{e^*} H^1(S, {}_nA)$ and $H^1(T, A) \xrightarrow{e^*} H^1(S, A)$ respectively, where T should be replaced by X_1, X_2 or $X_1 \underset{S}{\times} X_2$ and where e^* is the homomorphism canonically deduced from e_1 and e_2. Then we have the following commutative diagram.

$$\begin{array}{ccccccc}
0 & & 0 & & & & \\
\downarrow & & \downarrow & & & & \\
A_0(X_1) & \xrightarrow{j_1} & H_0^1(X_1, {}_nA) & \xrightarrow{i_1} & H_0^1(X_1, A) & \xrightarrow{n} & H_0^1(X_1, A) \\
s_1^* \uparrow \downarrow pr_1^* & & s_1^* \uparrow \downarrow pr_1^* & & \uparrow s_1^* & & \uparrow \\
A_0(X_1 \underset{S}{\times} X_2) & \xrightarrow{j_{12}} & H_0^1(X_1 \underset{S}{\times} X_2, {}_nA) & \xrightarrow{i_{12}} & H_0^1(X_1 \underset{S}{\times} X_2, A) & \xrightarrow{n} & H_0^1(X_1 \underset{S}{\times} X_2, A) \\
s_2^* \downarrow \uparrow pr_2^* & & s_2^* \downarrow \uparrow pr_2^* & & \downarrow s_2^* & & \downarrow \\
A_0(X_2) & \xrightarrow{j_2} & H_0^1(X_2, {}_nA) & \xrightarrow{i_2} & H_0^1(X_2, A) & \xrightarrow{n} & H_0^1(X_2, A) \\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}$$

where the lines are exact and two left columns are split exact. Since $n\xi = 0$, $\xi = i_{12}(\eta)$ for some element η of $H_0^1(X_1 \underset{S}{\times} X_2, {}_nA)$. Let $\eta_1 = s_1^*(\eta)$ and $\eta_2 = s_2^*(\eta)$. Then $i_1(\eta_1) = s_1^* i_{12}(\eta) = 0$. Also $i_2(\eta_2) = 0$. Therefore $\eta_1 = j_1(\zeta_1)$ and $\eta_2 = j_2(\zeta_2)$. Put $\zeta = pr_1^*(\zeta_1) + pr_2^*(\zeta_2)$. Then $j_{12}(\zeta) = \eta$. Hence $\xi = 0$. Thus $\mathrm{Corr}_S^A(X_1, X_2) = 0$. q.e.d.

Corollary 15. *Let S and f_i $(i=1, 2, 3)$ be as in the statement of the theorem and let A be an abelian scheme over S. Suppose moreover that S is a quasi-compact normal prescheme and that f_i $(i=1, 2, 3)$ is geometrically normal. Then*

$$\mathrm{PH}_{(e_1, e_2, e_3)}(X_1 \underset{S}{\times} X_2 \underset{S}{\times} X_3, A) = 0.$$

Proof. Easy from Proposition 14.

Corollary 16. *Let S be a quasi-compact regular prescheme and let A and B be abelian schemes over S. Then*

$$H^1(B, A)=H^1(B, A)_{\mathrm{rep}}\cong Ext^1_{S\text{-}gr}(B, A)\oplus H^1(S, A).$$

Proof. The first isomorphism is due to M. Raynaud ([6]). The second isomorphism is proved as in Corollary 7 and Corollary 13, using Corollary 15 and [2], Exp. VII, (1.3.5).

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